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# Incomparable families and maximal trees

by

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**Abstract.** We answer several questions of D. Monk by showing that every maximal family of pairwise incomparable elements of $\mathcal{P}(\omega)/\mathit{fin}$ has size continuum, while it is consistent with the negation of the Continuum Hypothesis that there are maximal subtrees of both $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega)/\mathit{fin}$ of size $\omega_1$.

**1. Introduction.** A *chain* in a partially ordered set $(\mathbb{P}, \leq)$ is a subset of $\mathbb{P}$ which is linearly ordered by $\leq$. On the other hand, the term *antichain* in $\mathbb{P}$ has two, quite different yet commonly used, meanings: in forcing terminology, an antichain is a set of elements of $\mathbb{P}$ any two of which are mutually *incompatible* (i.e. have no common lower bound); the other refers to families of pairwise incomparable elements. We shall call the former antichains and the latter *incomparable families*. We shall always assume that an incomparable family does not contain the maximal element of $\mathbb{P}$, which we require to exist.

Similarly, there are two distinct notions of a subtree of a partially ordered set $\mathbb{P}$ (for their connection with forcing "growing downward"). We call a partially ordered set $(\mathbb{T}, \leq)$ a *tree* if it has a largest element $\mathbf{1}$ and for every $t \in \mathbb{T}$ the set $\mathrm{pred}_{\mathbb{T}}(t) = \{s \in \mathbb{T} : s \geq t\}$ is well-ordered by the reverse order of $\leq$, i.e. $\mathrm{pred}_{\mathbb{T}}(t)$ is linearly ordered by $\leq$ with every strictly increasing chain being finite. Accordingly, $\mathbb{T} \subseteq \mathbb{P}$ is a *subtree* of a partially ordered set $(\mathbb{P}, \leq)$ with a maximal element $\mathbf{1}$ if $\mathbf{1} \in \mathbb{T}$ and $(\mathbb{T}, \leq\restriction(\mathbb{T} \times \mathbb{T}))$ is a tree. Note that we *do not* require that incomparable (equivalently, incompatible) elements of $\mathbb{T}$ are incompatible in the partial order $\mathbb{P}$ ([1]).

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([1]) Our notation for Boolean algebras differs from that of Monk [14] in that our trees are exactly images of trees according to Monk by the map which sends each element of the Boolean algebra to its complement.

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Incomparable families are naturally ordered by inclusion, and trees by end-extension: A tree $\mathbb{T}$ is an *end-extension* of a tree $\mathbb{S}$ if $\mathbb{S} \subseteq \mathbb{T}$ and $\mathrm{pred}_{\mathbb{T}}(s) = \mathrm{pred}_{\mathbb{S}}(s)$ for every $s \in \mathbb{S}$. Both orders obviously satisfy the hypothesis of the Kuratowski–Zorn lemma, hence (following [14]) we can talk about *maximal incomparable families* and *maximal trees* in $\mathbb{P}$. It is easy to see that a tree $\mathbb{T} \subseteq \mathbb{P}$ is maximal [14, Proposition 17.11] if and only if for every $p \in \mathbb{P}$ one of the following holds:

- there is a $q \in \mathbb{T}$ such that $q \leq p$, or
- there are incomparable elements $q_0, q_1$ of $\mathbb{T}$ such that $p \leq q_0$ and $p \leq q_1$.

We shall mostly consider the case when the underlying partial order is the set of positive (non-zero) elements of the Boolean algebra $\mathcal{P}(\omega)$ or the quotient Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$. Monk [14, Problems 157 and 158] asked what are the minimal sizes of maximal incomparable families and maximal trees in $\mathcal{P}(\omega)/\mathit{fin}$, whether these cardinal numbers are consistently below the size of the continuum, and whether they are consistently different. He also asked [14, Problem 156] whether $\omega$ and $2^\omega$ are the only possible cardinalities of maximal trees in $\mathcal{P}(\omega)$.

We answer these questions here by proving that every maximal family of pairwise incomparable elements of $\mathcal{P}(\omega)/\mathit{fin}$ has size continuum (Proposition 2.3), and that it is consistent with the negation of the Continuum Hypothesis that there are maximal subtrees of size $\omega_1$ of both $\mathcal{P}(\omega)/\mathit{fin}$ (Theorem 3.3) and $\mathcal{P}(\omega)$ (Theorem 4.1).

We conclude this introduction by fixing some notation. Given $f, g \in \omega^\omega$, we write $f \leq^* g$ if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. Similarly, given subsets $A$ and $B$ of $\omega$ we write $A \subseteq^* B$ to denote that $A \setminus B$ is finite. We say that $A$ and $B$ are *almost disjoint* if $A \cap B$ is finite, written $A \cap B =^* \emptyset$. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (AD) if any two of its elements are almost disjoint. It is *maximal almost disjoint* (MAD) if for every infinite $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$. Given an AD family $\mathcal{A}$, $\mathcal{I}(\mathcal{A})$ denotes the ideal generated by $\mathcal{A}$, that is, the family of all subsets of $\omega$ which can be almost covered by finitely many elements of $\mathcal{A}$, while $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ denotes the collection of $\mathcal{I}(\mathcal{A})$-positive sets. A family $\mathcal{I} \subseteq [\omega]^\omega$ is *independent* if given any two finite disjoint sets $F_0, F_1 \subseteq \mathcal{I}$,

$$\Big|\bigcap F_0 \setminus \bigcup F_1\Big| = \omega.$$

It is *maximal independent* if it cannot be extended to a larger independent family.

A family $\mathcal{R} \subseteq [\omega]^\omega$ is *reaping* if for every $A \subseteq \omega$ there is an $R \in \mathcal{R}$ such that $R \subseteq^* A$ or $R \cap A =^* \emptyset$.

We say that a set $R \in [\omega]^\omega$ *$\sigma$-reaps* a sequence $\langle X_n : n \in \omega \rangle$ of elements of $[\omega]^\omega$ if for every $n \in \omega$, $R \subseteq^* X_n$ or $R \subseteq^* \omega \setminus X_n$. A family $\mathcal{R} \subseteq [\omega]^\omega$ is *$\sigma$-reaping* if for every sequence $\langle X_n : n \in \omega \rangle$ of elements of $[\omega]^\omega$ there is an $R \in \mathcal{R}$ such that $R \in [\omega]^\omega$ $\sigma$-reaps $\langle X_n : n \in \omega \rangle$.

Given a tree $\mathbb{T}$, we shall call its elements *nodes*. A *branch* through a tree $\mathbb{T}$ is a maximal linearly ordered set, and the set of all branches of $\mathbb{T}$ is denoted by $[\mathbb{T}]$. The *height* of $\mathbb{T}$ is defined as $\mathrm{ht}(\mathbb{T}) = \sup\{\mathrm{o.t.}(B) : B \subseteq \mathbb{T} \text{ is a branch}\}$, where $\mathrm{o.t.}(B)$ is the order type of $B$. For an ordinal $\alpha \leq \mathrm{ht}(\mathbb{T})$, $\mathbb{T}_\alpha$, the $\alpha$th *level* of $\mathbb{T}$, is defined as the set of nodes $t \in \mathbb{T}$ such that $\mathrm{pred}_{\mathbb{T}}(t)$ has order type $\alpha$.

Finally, we shall be mentioning some standard cardinal invariants of the continuum (see [3] for more information). In particular, $\mathfrak{d}$ denotes the minimal size of a *dominating family* in $\omega^\omega$, i.e. a family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is an $f \in \mathcal{F}$ such that $g \leq^* f$; $\mathfrak{i}$ denotes the minimal size of a maximal independent family; $\mathsf{non}(\mathcal{M})$ denotes the minimal size of a set of reals which is not meager. The most relevant in our context is the *reaping number* $\mathfrak{r}$ defined as the minimal size of a reaping family, and its close relative $\mathfrak{r}_\sigma$, the minimal size of a $\sigma$-reaping family. Whether $\mathfrak{r} = \mathfrak{r}_\sigma$ is an open question (see [6]).

**2. Incomparable families.** Incomparable families in partial orders have been studied for a long time. One of the first applications of Ramsey's Theorem was to show that every infinite partial order contains an infinite chain or an infinite incomparable family. The analogous question with infinite replaced by uncountable turns out to be independent of the usual axioms of set theory ($\mathsf{ZFC}$): On the one hand, R. Bonnet and S. Shelah [4], and independently S. Todorčević [18], showed that assuming $\mathsf{CH}$ there is an uncountable Boolean algebra without an uncountable chain and an uncountable incomparable family, while, on the other hand, J. Baumgartner [2] showed that it follows from the Proper Forcing Axiom ($\mathsf{PFA}$) that every uncountable Boolean algebra contains an uncountable incomparable family. It was later shown by M. Losada and S. Todorčević [13] that Martin's Axiom $\mathsf{MA}_{\omega_1}$ suffices.

We will refer to (maximal) incomparable families in $\mathcal{P}(\omega)/\mathit{fin}$ simply as (maximal) incomparable families. We will also treat them as subfamilies of the partially (pre-)ordered set $([\omega]^\omega, \subseteq^*)$ rather than the Boolean algebra itself.

Special incomparable families have been studied extensively: every almost disjoint family and every independent family are incomparable. Note that neither a maximal almost disjoint family nor a maximal independent family can ever be maximal incomparable.

Every incomparable family (formally augmented by $\{\omega\}$) is a tree. As a warm up exercise, we show that there are maximal incomparable families $\mathcal{C}$ and $\mathcal{B}$ such that $\mathcal{C}$ is a maximal tree, while $\mathcal{B}$ is not.

PROPOSITION 2.1. *There is a maximal incomparable family $\mathcal{B}$ which is not a maximal tree.*

*Proof.* Let $A, B \subseteq \omega$ be infinite sets such that $B \subseteq A$ and $|\omega \setminus A| = |A \setminus B| = \omega$. Now, let $\mathcal{A} \subseteq [B]^\omega$ be an almost disjoint family of size $\mathfrak{c}$. For any $D \in [\omega \setminus A]^\omega$, choose $A_D \in \mathcal{A}$ so that the assignment $D \mapsto A_D$ is one-to-one. Let $\mathcal{B}_0 = \{A\} \cup \{D \cup A_D : D \in [\omega \setminus A]^\omega\}$. It is clear that $\mathcal{B}_0$ is an incomparable family. Let $\mathcal{B}$ be a maximal incomparable family containing $\mathcal{B}_0$.

CLAIM. *If $E \in \mathcal{B} \setminus \{A\}$, then $B \not\subseteq^* E$.*

Indeed, let $D = E \setminus A$. There are two cases: if $E = D \cup A_D$, then clearly $B \not\subseteq^* E$. On the other hand, $E \neq D \cup A_D$ yields $|A_D \setminus E| = \omega$ and, as $D \subseteq E$, $B \not\subseteq^* E$ follows.

It is clear that $B \notin \mathcal{B}$ and the above claim shows that $\mathcal{B} \cup \{B\}$ is a tree, thus $\mathcal{B}$ is not a maximal tree. ∎

A simple proof that there is a maximal incomparable family which is also a maximal tree uses the notion of a completely separable MAD family (see [9, 8]). A MAD family $\mathcal{A}$ is *completely separable* if for every $X \in \mathcal{I}^+(\mathcal{A})$, there are $\mathfrak{c}$-many $A \in \mathcal{A}$ such that $A \subseteq^* X$. Unfortunately it is not known whether completely separable MAD families exist. They do exist in all known models of ZFC, and in particular they exist if $2^\omega < \aleph_\omega$ [17].

PROPOSITION 2.2. *Assuming the existence of a completely separable MAD family, there is a maximal incomparable family which is also a maximal tree.*

*Proof.* Let $\mathcal{A}$ be a completely separable MAD family, and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$.

We recursively construct an increasing chain $\{\mathcal{A}_\alpha : \alpha < \mathfrak{c}\}$ of subfamilies of $\mathcal{A}$ such that $|\mathcal{A}_\alpha| \leq |\omega \cdot \alpha|$, and an increasing chain $\{\mathcal{B}_\alpha : \alpha < \mathfrak{c}\}$ of incomparable families such that $\mathcal{B}_\alpha \subseteq \mathcal{I}(\mathcal{A})$ and $|\mathcal{B}_\alpha| \leq |\omega \cdot \alpha|$, so that the following hold for any $\alpha < \mathfrak{c}$:

(1) for every $B \in \mathcal{B}_\alpha$, there exists $F \subseteq [\mathcal{A}_\alpha]^{<\omega}$ such that $B \subseteq^* \bigcup F$,
(2) either there is $B \in \mathcal{B}_{\alpha+1}$ with $B \subseteq^* C_\alpha$, or there are $B_0, B_1 \in \mathcal{B}_{\alpha+1}$ such that $B_0 \neq B_1$ and $C_\alpha \subseteq^* B_0 \cap B_1$.

At step $\alpha$, let $\overline{\mathcal{A}} = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$ and $\overline{\mathcal{B}} = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. If $\alpha$ is a limit ordinal, set $\mathcal{A}_\alpha = \overline{\mathcal{A}}$ and $\mathcal{B}_\alpha = \overline{\mathcal{B}}$. In the successor step, we consider two cases:

- If $C_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there is $A \in \mathcal{A} \setminus \overline{\mathcal{A}}$ contained in $C_\alpha$. We then define $\mathcal{A}_{\alpha+1} = \overline{\mathcal{A}} \cup \{A\}$ and $\mathcal{B}_{\alpha+1} = \overline{\mathcal{B}} \cup \{A\}$.

- If $C_\alpha \in \mathcal{I}(\mathcal{A})$ and there is $X \in \overline{\mathcal{B}}$ with $X \subseteq^* C_\alpha$, then we set $\mathcal{A}_{\alpha+1} = \overline{\mathcal{A}}$ and $\mathcal{B}_{\alpha+1} = \overline{\mathcal{B}}$. If for every $X \in \overline{\mathcal{B}}$, $X \not\subseteq^* C_\alpha$, we choose $A_0, A_1 \in \mathcal{A} \setminus (\overline{\mathcal{A}} \cup F)$, where $F \in [\mathcal{A}]^{<\omega}$ is such that $C_\alpha \subseteq^* \bigcup F$, and define $\mathcal{A}_{\alpha+1} = \overline{\mathcal{A}} \cup F \cup \{A_0, A_1\}$ and $\mathcal{B}_{\alpha+1} = \overline{\mathcal{B}} \cup \{A_0 \cup C_\alpha,\ A_1 \cup C_\alpha\}$.

Let $\mathcal{B} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{B}_\alpha$. From the recursive definition above it is straightforward that $\mathcal{B}$ is maximal incomparable. Moreover, for each $C \subseteq \omega$ one of the following conditions holds:

- there is $B \in \mathcal{B}$ such that $B \subseteq^* C$; or
- there are $B_0, B_1 \in \mathcal{B}$ such that $B_0 \neq B_1$ and $C \subseteq^* B_0 \cap B_1$.

Therefore, $\mathcal{B}$ is also a maximal tree. ∎

We have not been able to prove this proposition without the extra assumption. There should be a ZFC construction (2).

Finally, we state the main result of this section in a more general setting. Let $\mathbb{B}$ be a Boolean algebra. Denote by $\mathbb{B}^+$ the set of all non-zero elements of $\mathbb{B}$. Recall that the *density* of $\mathbb{B}$ is defined as

$$d(\mathbb{B}) = \min\{|D| : D \subseteq \mathbb{B},\ D \text{ is dense in } \mathbb{B}^+\},$$

and the *factor* algebra of $\mathbb{B}$ below $b$ is

$$\mathbb{B}{\restriction}b = \{x \in \mathbb{B} : x \leq b\}.$$

PROPOSITION 2.3. *Every maximal incomparable family in a Boolean algebra $\mathbb{B}$ has size at least* $\min\{d(\mathbb{B}{\restriction}b) : b \in \mathbb{B}^+\}$.

*Proof.* Let $A \subseteq \mathbb{B}^+$ be an infinite incomparable family in a Boolean algebra $\mathbb{B}$, and let $\mu = \min\{d(\mathbb{B}{\restriction}b) : b \in \mathbb{B}^+\}$. Assume that $|A| < \mu$, and let $C$ be the Boolean algebra generated by $A$. Since $A$ is infinite, $|C| = |A|$. Let $a \in A$ be arbitrary. As $|C| < \mu$, $C$ is dense neither in $\mathbb{B}{\restriction}a$ nor in $\mathbb{B}{\restriction}{-a}$, so there exist $c_0, c_1 \in \mathbb{B}^+$ with $c_0 \leq -a$, $c_1 \leq a$ and such that no $b \in C^+$ is below either $c_0$ or $c_1$.

Let $d = (a - c_1) \vee c_0$. Clearly $d \notin A$, for otherwise $d \wedge -a = c_0 \in C^+$, contradicting the fact that no element of $C^+$ is below $c_0$. Let us see that $d$ is incomparable with every $x \in A$. If $d \leq x$, then $-x \wedge a \leq c_1$, which is absurd for $-x \wedge a \in C^+$. On the other hand, $x \leq d$ yields $x \wedge -a \leq c_0$, but this is not possible as then $x \wedge -a \in C^+$. Therefore, $A \cup \{d\}$ is an incomparable family, thus $A$ is not maximal. ∎

COROLLARY 2.4. *Every maximal incomparable family in $\mathcal{P}(\omega)/fin$ has size $\mathfrak{c}$.*

(2) Added in proof: There is. The proof will appear elsewhere.

**3. Maximal trees in $\mathcal{P}(\omega)/\mathit{fin}$.** In this section we consider maximal subtrees of the Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$ or rather maximal subtrees of the partially (pre-)ordered set $([\omega]^\omega, \subseteq^*)$. In this context $\mathcal{T} \subseteq [\omega]^\omega$ is a *maximal tree* if $\omega \in \mathcal{T}$, for every $t \in \mathcal{T}$ the set $\mathrm{pred}_{\mathcal{T}}(t) = \{s \in \mathcal{T} : t \subseteq^* s\}$ is well-ordered by $\supseteq^*$, and for every $C \subseteq \omega$ either

- there is a $t \in \mathcal{T}$ such that $t \subseteq^* C$, or
- there are incomparable $t_0, t_1 \in \mathcal{T}$ such that $C \subseteq^* t_0 \cap t_1$.

Again, we shall refer to maximal trees in $([\omega]^\omega, \subseteq^*)$ simply as maximal trees.

The main theorem of this section will show that it is consistent with $\neg\mathsf{CH}$ that there is a maximal tree of size $\omega_1$. We shall in fact show that this happens in the Sacks model. This could probably be done directly by using $\mathsf{CH}$ or $\diamondsuit$ to construct a maximal tree whose maximality is indestructible by any countable support iteration of Sacks forcing. We choose to take advantage of the so called *parametrized $\diamondsuit$-principles* as introduced in [15]. Following Vojtáš [19], we shall call a triple $(A, B, \to)$ an *invariant* if

(1) $\to \subseteq A \times B$,
(2) for every $a \in A$, there is $b \in B$ such that $a \to b$, and
(3) there is no $b \in B$ such that $a \to b$ for all $a \in A$.

We say that $D \subseteq B$ is *dominating* if for every $a \in A$, there is a $d \in D$ such that $a \to d$. Given an invariant $(A, B, \to)$ we define its *evaluation* by

$$\langle A, B, \to \rangle = \min\{|D| : D \subseteq B \text{ and } D \text{ is dominating}\}.$$

An invariant $(A, B, \to)$ is *Borel* if $A, B$ and $\to$ are Borel subsets of Polish spaces. Most (but not all) of the usual cardinal invariants of the continuum can be represented as evaluations of Borel invariants.

It is shown in [15] that to any Borel invariant $(A, B, \to)$ one can naturally associate a guessing principle $\diamondsuit(A, B, \to)$, which in turn implies that the evaluation $\langle A, B, \to \rangle$ is $\leq \omega_1$. It is also shown there that $\diamondsuit(A, B, \to)$ holds in most of the natural models where this inequality holds. For our purposes, we need to work in a slightly more general framework than the one in [15].

Definition 3.1. We say that an invariant $(A, B, \to)$ is an $L(\mathbb{R})$-*invariant* if $A, B$ and $\to$ are subsets of Polish spaces and all three belong to $L(\mathbb{R})$ ([3]).

Following [15], given an $L(\mathbb{R})$-invariant $(A, B, \to)$, $\diamondsuit_{L(\mathbb{R})}(A, B, \to)$ denotes the following principle:

*For every $F : 2^{<\omega_1} \to A$ such that $F \restriction 2^\alpha \in L(\mathbb{R})$ for all $\alpha < \omega_1$, there is a $g : \omega_1 \to B$ such that for every $f \in 2^{\omega_1}$ the set $\{\alpha : F(f \restriction \alpha) \to g(\alpha)\}$ is stationary.*

([3]) Informally, this means that each of the sets $A$, $B$, and $\to$ has a definition involving only a subset of $\omega$ as a parameter.

The witness $g$ will be called a *guessing sequence* for $F$. Also, when $F(f\restriction\alpha) \to g(\alpha)$, we say that $g$ *guesses* $f$ (via $F$) at $\alpha$.

A secondary purpose of this paper is to provide further evidence that parametrized $\Diamond$ principles are useful instruments to have in one's mathematical toolbox.

Following the notation of [3], given two $L(\mathbb{R})$-invariants $\mathbb{A} = (A_-, A_+, {}_{\mathbb{A}}\to)$ and $\mathbb{B} = (B_-, B_+, {}_{\mathbb{B}}\to)$ we define the *sequential composition*

$$\mathbb{A};\mathbb{B} = (A_- \times \mathrm{Borel}(B_-^{A_+}), A_+ \times B_+, \to),$$

where $\mathrm{Borel}(B_-^{A_+})$ denotes the set of all Borel functions from $A_+$ to $B_-$, and $(a_-, f) \to (a_+, b_+)$ if $a_{-\mathbb{A}} \to a_+$ and $f(a_-)_{\mathbb{B}} \to b_+$.

It is easy to see that $\mathbb{A};\mathbb{B}$ is an $L(\mathbb{R})$-invariant, and in [3] it is proved that $\langle \mathbb{A};\mathbb{B}\rangle = \max\{\langle\mathbb{A}\rangle, \langle\mathbb{B}\rangle\}$. As usual, we will identify an invariant with its evaluation. In particular, we shall denote by $\mathfrak{d}$ both the invariant $(\omega^\omega, \omega^\omega, <)$ and its evaluation ([4]), and by $\mathfrak{r}_\sigma$ both the invariant $(([\omega]^\omega)^\omega, [\omega]^\omega, \text{is } \sigma\text{-reaped})$ and its evaluation.

Now we are ready to state and prove the main result of the paper.

Theorem 3.2. *$\Diamond_{L(\mathbb{R})}(\mathfrak{r}_\sigma;\mathfrak{d})$ implies that there is a maximal tree of size $\omega_1$.*

*Proof.* First, for every infinite $\alpha \in \omega_1$, fix a bijection $e_\alpha : \omega \to \alpha$, and for a set $A \in [\omega]^\omega$ and a countable $\subseteq^*$-decreasing sequence $\vec{X}$ of subsets of $\omega$ such that $X_0 \subseteq^* A$, denote by $P(A, \vec{X}) \subseteq A$ a pseudo-intersection found in a Borel way ([5]).

By a suitable coding, we may assume that the domain of the function $F$ is $\bigcup_{\alpha\in\omega_1}(([\omega]^\omega)^\alpha)^\alpha \times [\omega]^\omega$. We shall define $F$ by recursion on $\alpha$. For $\beta < \alpha$, let $\vec{X}_\beta = \langle X_{\beta,\gamma} : \gamma < \alpha\rangle$. For $(\langle \vec{X}_\beta : \beta < \alpha\rangle, Y) \in (([\omega]^\omega)^\alpha)^\alpha \times [\omega]^\omega$, we define $F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Y)$ as follows:

(1) If $\langle X_{\beta,0} : \beta < \alpha\rangle$ is not an AD family or does not cover $\omega$, or if one of the sequences $\vec{X}_\beta$ is not a $\subseteq^*$-decreasing, let

$$F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Y) = \langle \bar{\omega}, \overline{\mathrm{Id}}\rangle.$$

Here $\bar{\omega}$ denotes the sequence which takes constant value $\omega$, and $\overline{\mathrm{Id}}$ the function from $[\omega]^\omega$ to $\omega^\omega$ which takes every set to the identity function. In other words, this is the irrelevant case.

(2) If $\langle X_{\beta,0} : \beta < \alpha\rangle$ is an AD family and covers $\omega$, and every $\vec{X}_\beta$ is a $\subseteq^*$-decreasing sequence, define $A_0 = X_{e_\alpha(0),0}$ and $A_n = X_{e_\alpha(n),0} \setminus \bigcup_{i<n} X_{e_\alpha(i),0}$

([4]) It is customary to denote by $\mathfrak{d}$ the invariant $(\omega^\omega, \omega^\omega, <^*)$. In fact, it is an open problem whether the corresponding weak diamonds are equivalent.

([5]) More precisely, for every $\alpha < \omega_1$ there is a Borel function $P_\alpha : [\omega]^\omega \times ([\omega]^\omega)^\alpha \to [\omega]^\omega$ such that if $A$ is an infinite subset of $\omega$ and $\vec{X}$ is a $\subseteq^*$-decreasing sequence of subsets of $\omega$ such that $X_0 \subseteq^* A$ of length $\alpha$, then $P_\alpha(A, \vec{X}) = P(A, \vec{X}) \subseteq A$ is a pseudo-intersection of $\vec{X}$.

for $n > 0$. Let $H_n : \omega \to P(A_n, \vec{X}_{e_\alpha(n)})$ be the increasing enumeration of $P(A_n, \vec{X}_{e_\alpha(n)})$. Then let $\vec{Z} = \langle Z_n : n \in \omega \rangle$, where

$$Z_n = H_n^{-1}[P(A_n, \vec{X}_{e_\alpha(n)}) \cap Y].$$

Now, define a function $\varphi_{\vec{Z}} : [\omega]^\omega \to \omega^\omega$ as follows:

(a) If $A \in [\omega]^\omega$ does not $\sigma$-reap $\vec{Z}$, then define $\varphi_{\vec{Z}}(A) = \mathrm{Id}$.

(b) If $A \in [\omega]^\omega$ $\sigma$-reaps $\vec{Z}$, then define $\varphi_{\vec{Z}}$ by

$$\varphi_{\vec{Z}}(A)(n) = \min\{k \in \omega : A \setminus Z_n \subseteq k \text{ or } Z_n \cap A \subseteq k\}.$$

Finally, define $F(\langle \vec{X}_\beta : \beta < \alpha \rangle, Y) = (\vec{Z}, \varphi_{\vec{Z}})$.

Let $g : \omega_1 \to [\omega]^\omega \times \omega^\omega$ be a guessing sequence for the function $F$. Let $D_\alpha \in [\omega]^\omega$ and $h_\alpha \in \omega^\omega$ be such that $g(\alpha) = (D_\alpha, h_\alpha)$. We shall define $\langle X_{\alpha,\gamma} : \alpha, \gamma \in \omega_1 \rangle$ recursively so that $\vec{X}_\alpha = \langle X_{\alpha,\gamma} : \gamma \in \omega_1 \rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of $\omega$ and such that $\{X_{\alpha,0} : \alpha \in \omega_1\}$ is an AD family. The construction is as follows:

(1) Start with a family $\langle X_{n,m} : n, m \in \omega \rangle$ such that $\vec{X}_n = \langle X_{n,m} : m \in \omega \rangle$ is a $\subseteq$-decreasing sequence, and $\{X_{n,0} : n \in \omega\}$ is a partition of $\omega$ into infinite sets.

(2) Suppose $\vec{X}_\beta = \langle X_{\beta,\gamma} : \gamma < \alpha \rangle$ has been constructed for all $\beta < \alpha$, where $\alpha$ is an even ordinal. Define $A_0^\alpha = X_{e_\alpha(0),0}$, and for $n > 0$, $A_n^\alpha = X_{e_\alpha(n),0} \setminus \bigcup_{i<n} X_{e_\alpha(i),0}$. Let $H_n^\alpha$ be the increasing enumeration of $P(A_n^\alpha, \vec{X}_{e_\alpha(n)})$. Then define $X_{e_\alpha(n),\alpha} = X_{e_\alpha(n),\alpha+1} = H_n^\alpha[D_\alpha]$. Now, for $n \in \omega$, let $a_n^0, a_n^1 \in H_n^\alpha[D_\alpha \setminus h_\alpha(n)]$ be distinct natural numbers, and define $X_{\alpha,0} = \{a_n^0 : n \in \omega\}$ and $X_{\alpha+1,0} = \{a_n^1 : n \in \omega\}$. Finally, let $\vec{X}_\alpha$ and $\vec{X}_{\alpha+1}$ be $\subseteq^*$-decreasing sequences of length $\alpha + 2$ whose first elements are $X_{\alpha,0}$ and $X_{\alpha+1,0}$, respectively.

This concludes the recursive construction.

To define the tree, for every infinite even $\alpha \in \omega_1$, define two sets $B_\alpha, B_{\alpha+1}$ as follows:

$$B_\alpha = X_{\alpha,0} \cup \bigcup_{n\in\omega} A_n^\alpha \setminus H_n^\alpha[D_\alpha \setminus h_\alpha(n)] = X_{\alpha,0} \cup \Big(\omega \setminus \bigcup_{n\in\omega} H_n^\alpha[D_\alpha \setminus h_\alpha(n)]\Big),$$

$$\begin{aligned} B_{\alpha+1} &= X_{\alpha+1,0} \cup \bigcup_{n\in\omega} A_n^\alpha \setminus H_n^\alpha[D_\alpha \setminus h_\alpha(n)] \\ &= X_{\alpha+1,0} \cup \Big(\omega \setminus \bigcup_{n\in\omega} H_n^\alpha[D_\alpha \setminus h_\alpha(n)]\Big). \end{aligned}$$

The tree $\mathcal{T}$ is then defined by

$$\mathcal{T} = \{X_{\alpha,\beta} : \alpha, \beta \in \omega_1\} \cup \{B_\alpha : \alpha \in [\omega, \omega_1)\}.$$

Claim 1. *$\mathcal{T}$ is a tree.*

First we show that $\{B_\alpha : \alpha \in [\omega, \omega_1)\}$ is an incomparable family. For a fixed $\alpha$ it is clear that $B_\alpha$ and $B_{\alpha+1}$ are incomparable for every even $\alpha$. For even $\beta < \alpha$, note that $X_{\beta,0} \subseteq^* B_\beta$ and $X_{\beta+1,0} \subseteq^* B_{\beta+1}$, while $X_{\beta,0} \not\subseteq^* B_\alpha$ and $X_{\beta,0} \not\subseteq^* B_{\alpha+1}$. On the other hand, note that $X_{\alpha,0} \not\subseteq^* B_\beta$, $X_{\alpha,0} \not\subseteq^* B_{\beta+1}$, $X_{\alpha+1,0} \not\subseteq^* B_\beta$, and $X_{\alpha+1,0} \not\subseteq^* B_{\beta+1}$. The same argument shows that $\mathcal{T}$ is, in fact, a tree.

Claim 2. *$\mathcal{T}$ is a maximal tree.*

Let $Y \in [\omega]^\omega$ be an arbitrary set, and let $\alpha \in \omega_1$ be an even ordinal such that $g$ guesses the branch $(\langle \vec{X}_\alpha : \alpha \in \omega_1\rangle, Y)$ at $\alpha$. Then $F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Y) = (\vec{Z}, \varphi_{\vec{Z}})$ is dominated by $g(\alpha) = (D_\alpha, h_\alpha)$, which means that for all $n \in \omega$, $D_\alpha$ $\sigma$-reaps $\vec{Z}$, and $\varphi_{\vec{Z}}(D_\alpha) \le h_\alpha$. There are two cases:

Case 1: *There is $n \in \omega$ such that $D_\alpha \subseteq^* Z_n$.* Then

$$X_{e_\alpha(n),\alpha} = H^\alpha_n[D_\alpha] \subseteq^* H^\alpha_n[Z_n] = Y \cap P(A^\alpha_n, \vec{X}_{e_\alpha(n)}) \subseteq Y.$$

Case 2: *For all $n \in \omega$, $D_\alpha \cap Z_n$ is finite.* In this case, for all $n \in \omega$,

$$Z_n \cap D_\alpha \setminus h_\alpha(n) = \emptyset.$$

Hence, for all $n \in \omega$,

$$H^\alpha_n[D_\alpha \setminus h_\alpha(n)] \cap Y \cap A^\alpha_n = \emptyset,$$

which yields $Y \subseteq \bigcup_{n\in\omega} A^\alpha_n \setminus H^\alpha_n[D_\alpha \setminus h_\alpha(n)]$. Thus, $Y \subseteq B_\alpha \cap B_{\alpha+1}$. ∎

Corollary 3.3. *It is consistent with $\neg\mathsf{CH}$ that there is a maximal tree of size $\omega_1$.*

*Proof.* It is well known that in the Sacks model, both $\mathfrak{r}_\sigma = \omega_1$ and $\mathfrak{d} = \omega_1$. Hence, in this model we also have $\mathfrak{r}_\sigma ; \mathfrak{d} = \omega_1$. It follows [15, 7] that $\diamondsuit(\mathfrak{r}_\sigma ; \mathfrak{d})$ holds in the Sacks model, and so we may deduce the result. ∎

We denote ([6])

$$\mathfrak{tr} = \min\{|\mathcal{T}| : \mathcal{T} \subseteq [\omega]^\omega \text{ is a maximal tree}\}.$$

It is easy to see that all maximal trees have size at least $\mathfrak{r}$ (the subalgebra of $\mathcal{P}(\omega)$ generated by a maximal tree $\mathcal{T}$ is a reaping family), i.e. $\mathfrak{r} \le \mathfrak{tr}$. Theorem 3.2 seems to suggest that perhaps also $\mathfrak{d} \le \mathfrak{tr}$. For this we have only partial evidence. Before presenting it, let us call a tree $\mathcal{T} \subseteq [\omega]^\omega$ an *ideal-tree* if for any $A \in \mathcal{T}$, the family $\{A \cap B : B \in \mathcal{T} \wedge A \not\subseteq^* B\}$ generates a proper ideal $\mathcal{I}_A$ on $A$. This notation will also be used in the next section.

Proposition 3.4.

(1) *If $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree with a terminal node, i.e. a branch whose length is a succesor ordinal, then $|\mathcal{T}| = \mathfrak{c}$.*

([6]) In the language of Monk [14], $\mathfrak{tr} = Inc^{tree}_{mm}(\mathcal{P}(\omega)/fin)$.

(2) *If* $\mathcal{T} \subseteq [\omega]^\omega$ *is a maximal tree with a branch whose length is a limit ordinal of countable cofinality, then* $|\mathcal{T}| \geq \mathfrak{d}$.
(3) *If* $\mathcal{T} \subseteq [\omega]^\omega$ *is a maximal tree containing an infinite AD family, then* $|\mathcal{T}| \geq \mathfrak{d}$.
(4) *If* $\mathcal{T} \subseteq [\omega]^\omega$ *is a maximal ideal-tree, then* $|\mathcal{T}| \geq \mathfrak{d}$.

*Proof.* For (1), let $b \subseteq \mathcal{T}$ be such a branch and let $X \in b$ be its last element. Then for any $Y \subseteq X$, there are $t, s \in \mathcal{T}$ such that $X \setminus Y \subseteq t \cap s$, so $X \setminus t \cap s \subseteq Y$. This means that $\{X \setminus t \cap s : t, s \in \mathcal{T}\}$ is a dense subset in $\mathcal{P}(X)/\mathit{fin}$, which implies that $\mathcal{T}$ has cardinality $\mathfrak{c}$.

For (2), let $\mathcal{T}$ be a tree of size less than $\mathfrak{d}$ having a branch of countable cofinality. Let $\langle A_n : n \in \omega\rangle$ be a cofinal sequence in such branch. Define $B_n = \bigcap_{m \leq n} A_m \setminus A_{n+1}$. For every pair $X, Y \in \mathcal{T}$ of incomparable nodes, define the following function:

$$\varphi_{X,Y}(n) = \min\{k \in \omega : (\exists l \geq n)(k \cap B_l \setminus (X \cap Y) \neq \emptyset)\}.$$

Since $\mathcal{T}$ is a tree, and $X$ and $Y$ are incomparable in $\mathcal{T}$, this function is well-defined. Also, since $|\mathcal{T}| < \mathfrak{d}$, there is an increasing function $h$ not dominated by any function in the family $\{\varphi_{X,Y} : X, Y \in \mathcal{T} \text{ incomparable}\}$. Let

$$Z = \bigcup_{n \in \omega} B_n \cap h(n).$$

Note that there is no set $X \in \mathcal{T}$ almost contained in $Z$, since such a set would be a pseudo-intersection of $\{A_n : n \in \omega\}$, and would contradict the fact that $\{A_n : n \in \omega\}$ is a cofinal sequence in the branch in question. Also, there is no pair of incomparable elements $X, Y$ such that $Z \subseteq^* X \cap Y$; for given $n \in \omega$ with $\varphi_{X,Y}(n) \leq h(n)$, there is $l \geq n$ such that

$$h(n) \cap B_l \setminus (X \cap Y) \neq \emptyset,$$

and this happens infinitely many times, hence, since the sets $B_n$ are disjoint, we conclude that $Z \setminus (X \cap Y)$ is infinite.

For (3), let $\mathcal{T}$ be a tree of cardinality less than the dominating number, and let $\{A_n : n \in \omega\}$ be an almost disjoint family contained in $\mathcal{T}$.

Let $B_0 = A_0$ and $B_n = A_n \setminus \bigcup_{i<n} A_i$ for $n > 0$. For $X \in \mathcal{T}$ such that there are infinitely many $n \in \omega$ with $X \cap B_n \neq \emptyset$, define the following function:

$$\varphi_X(n) = \min\{k \in \omega : (\exists l \geq n)(k \cap B_l \cap X \neq \emptyset)\}.$$

Let $h_0 \in \omega^\omega$ be an increasing function which is not dominated by any function in the family $\{\varphi_X : X \text{ in } \mathcal{T} \text{ such that } \exists^\infty n \in \omega\ X \cap B_n \neq \emptyset\}$. Now, for incomparable $X, Y \in \mathcal{T}$, note that for all $n \in \omega$, $B_n \setminus X \cap Y$ is infinite (otherwise, $A_n$ would be almost contained in $X \cap Y$). Define the following function:

$$\phi_{X,Y}(n) = \min\{k \in \omega : B_{n+1} \cap [h_0(n+1), k) \setminus (X \cap Y) \neq \emptyset\}.$$

Let $h_1 > h_0$ be an increasing function not dominated by any element of the family $\{\phi_{X,Y} : X, Y \in \mathcal{T}, X, Y \text{ incomparable}\}$. Let

$$Z = \bigcup_{n\in\omega} B_n \cap [h_0(n), h_1(n)).$$

CLAIM. *$\mathcal{T} \cup \{Z\}$ is a tree.*

(i) *For all $X \in \mathcal{T}$, $X \not\subseteq^* Z$.* If $X$ intersects only finitely many $B_n$'s, then $X \cap Z$ is finite, hence, $X \not\subseteq^* Z$. If there are infinitely many $n \in \omega$ with $X \cap B_n \neq \emptyset$, then $X \setminus Z$ is infinite, since $h_0$ is not dominated by $\varphi_X$. To see this, let $n$ be such that $\varphi_X(n) < h_0(n)$. By the definition of $\varphi_X$, this means that there is $k \geq n$ such that $X \cap B_k \cap \varphi_X(n) \neq \emptyset$, which implies that $X \cap B_k \cap h_0(k) \neq \emptyset$. Since this happens infinitely often and $B_k \cap h_0(k) \cap Z = \emptyset$ for all $k$, $X \setminus Z$ is infinite.

(ii) *For all incomparable $X, Y \in \mathcal{T}$, $Z \not\subseteq^* X \cap Y$.* Let $n \in \omega$ be such that $\phi_{X,Y}(n) < h_1(n)$. By the definition of $\phi_{X,Y}$, this implies that $[h_0(n+1), h_1(n+1)) \cap B_{n+1} \setminus (X \cap Y) \neq \emptyset$. Since this happens for infinitely many $n$, $Z \setminus (X \cap Y)$ is infinite.

To see (4), it suffices to prove that every maximal ideal-tree $\mathcal{T}$ contains an infinite almost disjoint family. We construct such a family $\{A_n : n \in \omega\}$ recursively. Let $T \in \mathcal{T}$ be an arbitrary non-terminal node. Let $A_0 \in \mathcal{T}$ be such that $A_0 \subsetneq^* T$. Having defined $\{A_m : m < n\}$, consider

$$Y = T \setminus \bigcup_{m<n} A_m.$$

As $\mathcal{T}$ is an ideal-tree, $Y \notin \mathcal{T}$. That is: either there is $B \in \mathcal{T}$ such that $B \subseteq^* Y$, in which case we define $A_n$ to be such $B$, or $Y \subseteq^* T_0 \cap T_1$ for two incomparable elements $T_0$ and $T_1$ of $\mathcal{T}$. To finish the proof we shall see that the second case leads to a contradiction. Suppose there are $T_0$ and $T_1$ incomparable elements of $\mathcal{T}$ such that $Y \subseteq^* T_0 \cap T_1$. Now, there is $i \in \{0, 1\}$ with $T_i \not\supseteq^* T$, and by the definition of an ideal-tree, $\bigcup_{m<n} A_m \cup T_i \not\supseteq^* T$, which leads to a contradiction. Hence, the family $\{A_n : n \in \omega\}$ is an infinite almost disjoint family we need. ∎

The argument given above allows stating the following consistency result.

THEOREM 3.5. *It is consistent with* ZFC *that* $\mathfrak{tr} < \mathsf{non}(\mathcal{M})$. *In particular, it is consistent with* ZFC *that* $\mathfrak{tr} < \mathfrak{i}$.

*Proof.* The proof is as in Corollary 3.3, only with the Sacks forcing replaced by any encarnation of the *fat tree* forcing (see [20, 12, 10]). The forcing is $\omega^\omega$-bounding and preserves selective ultrafilters, hence the model obtained by a countable support iteration of the forcing of length $\omega_2$ produces a model of $\mathfrak{r}_\sigma = \omega_1$ and $\mathfrak{d} = \omega_1$. Therefore, in this model, we also get $\mathfrak{r}_\sigma; \mathfrak{d} = \omega_1$. Again, it follows [15, 7] that $\diamondsuit(\mathfrak{r}_\sigma; \mathfrak{d})$ holds here.

On the other hand, the forcing naturally adds an *eventually different* real, hence $\mathsf{non}(\mathcal{M}) = \mathfrak{c}$ in the resulting model. The fact that $\mathsf{non}(\mathcal{M}) \leq \mathfrak{i}$ was proved in [1]. ∎

QUESTION 3.6.

(1) *Is* $\mathfrak{tr} \geq \mathfrak{d}$*?*
(2) *Is* $\mathfrak{tr} > \max\{\mathfrak{r}, \mathfrak{d}\}$ *consistent with* $\mathsf{ZFC}$*?*

The tree constructed in Theorem 3.2, assuming $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$, is of height $\omega_1$. In the next section we show that, under the same assumption, there is also a maximal tree of size $\omega_1$ and height $\omega$.

**4. Maximal trees in $\mathcal{P}(\kappa)$.** Monk [14, Proposition 17.9] notes that for each infinite cardinal $\kappa$ there are maximal trees in the Boolean algebra $\mathcal{P}(\kappa)$ of size $\kappa$ and of size $2^\kappa$, and asks if these are the only values. Here we shall show that it is consistent that there are also other values. The construction also provides the example promised at the end of the previous section.

THEOREM 4.1. $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ *implies that there is a tree* $\mathcal{T} \subseteq [\omega]^\omega$ *of height* $\omega$ *and of size* $\omega_1$*, which is maximal both as a subtree of* $\mathcal{P}(\omega)$ *and as a subtree of* $([\omega]^\omega, \subseteq^*)$.

*Proof.* Recall that a tree $\mathcal{T} \subseteq [\omega]^\omega$ is an ideal-tree if for any $A \in \mathcal{T}$, the family $\{A \cap B : B \in \mathcal{T} \wedge A \not\subseteq^* B\}$ generates a proper ideal $\mathcal{I}_A$ on $A$. We shall, in fact, construct an ideal-tree.

Given $f \in 2^{<\omega_1}$, let us say that $f$ *codes a family of sets* $\mathcal{F}$ if for all $X \in \mathcal{F}$ there is a limit ordinal $\alpha \in \mathrm{dom}(f)$ such that for $n \in \omega$, $n \in X$ if and only if $f(\alpha+n) = 1$. For each limit $\alpha \in \omega_1$, $\alpha > \omega^2$, fix a bijection $e_\alpha : \omega \to \lim(\alpha)$. If $\alpha$ is a limit ordinal, and $f \in 2^\alpha$ codes an ideal-tree $\mathcal{T}$, let $\{A_n : n \in \omega\}$ be the enumeration of $\mathcal{T}$ given by

$$A_n = \{m : f(e_\alpha(n) + m) = 1\}.$$

Also, in this proof, for a given set $X \in [\omega]^\omega$, the symbol $X$ plays two roles according to the context: it denotes the set $X$, and also the increasing enumeration of $X$, that is, for $n \in \omega$, $X(n)$ is the $n$th element of $X$.

Now, it is easy, yet tedious, to show that there is a Borel function $H : 2^\alpha \to ([\omega]^\omega)^\omega$ such that if $f$ codes an ideal-tree, then $H(f) = \langle Z_n : n \in \omega\rangle$, where

- $\{Z_n : n \in \omega\}$ is pairwise disjoint,
- for every $n \in \omega$, $Z_n \subseteq A_n$, and
- for every $n \in \omega$, $Z_n \cap I$ is finite whenever $I \in \mathcal{I}_{A_n}$.

Having fixed all this, define a function $F : [\omega]^\omega \times 2^{<\omega_1} \to ([\omega]^\omega)^\omega \times (\omega^\omega)^{[\omega]^\omega}$ as follows ([7]):

(1) If $f \in 2^\alpha$ does not code an ideal-tree, or if $\alpha$ is not a limit ordinal, let $F(X, f) = (\bar{\omega}, \overline{\mathrm{Id}})$. Here, as in the proof of Theorem 3.2, $\bar{\omega}$ denotes the sequence which takes constant value $\omega$, and $\overline{\mathrm{Id}}$ the function from $[\omega]^\omega$ to $\omega^\omega$ which takes every set to the identity function.

(2) If $\alpha$ is a limit ordinal, and $f \in 2^\alpha$ codes an ideal-tree $\mathcal{T}$, let $\{A_n : n \in \omega\}$ be the enumeration of $\mathcal{T}$ given above, and let $\{Z_n : n \in \omega\}$ be the pairwise disjoint refinement of $\{A_n : n \in \omega\}$ given by $H(f)$. Furthermore, let $Y_n = Z_n^{-1}[Z_n \cap X]$, and define a function $\varphi_{(X,f)} : [\omega]^\omega \to \omega^\omega$ in the following way:

(a) if $W \in [\omega]^\omega$ does not $\sigma$-reap $\langle Y_n : n \in \omega\rangle$, let $\varphi_{(X,f)}(W) = \mathrm{Id}$;
(b) if $W \in [\omega]^\omega$ does $\sigma$-reap $\langle Y_n : n \in \omega\rangle$, let

$$\varphi_{(X,f)}(W)(n) = \min\{k \in \omega : W \setminus Y_n \subseteq k \text{ or } W \cap Y_n \subseteq k\}.$$

Then define $F(X, f) = (\langle Y_n : n \in \omega\rangle, \varphi_{(X,f)})$.

Let $g : \omega_1 \to [\omega]^\omega \times \omega^\omega$ be a guessing function for $F$. For $\alpha \in \omega_1$, let $X_\alpha \in [\omega]^\omega$ and $h_\alpha$ be such that $g(\alpha) = (X_\alpha, h_\alpha)$. For every $\alpha \in \omega_1$, let $D_\alpha \subseteq X_\alpha$ be an infinite co-infinite subset in $X_\alpha$. Recursively construct three sequences $\langle \mathcal{T}_\beta : \beta \in \omega_1\rangle$, $\langle f_\beta : \beta \in \omega_1\rangle$ and $\langle \alpha_\beta : \beta \in \omega_1\rangle$ such that:

- $\langle \mathcal{T}_\beta : \beta \in \omega_1\rangle$ is a sequence of countable ideal-trees.
- For all $\beta$, $f_\beta \in 2^{\alpha_\beta}$ and codes the tree $\mathcal{T}_\beta$.
- $\langle \alpha_\beta : \beta \in \omega_1\rangle$ is an increasing continuous sequence of countable ordinals.
- For all $\beta \in \omega_1$ we have $f_\beta \subseteq f_{\beta+1}$.

The construction is as follows:

*Base step*: Start with a countable ideal-tree $\mathcal{T}_0$ of height $\omega$ with $\omega \in \mathcal{T}_0$ and with the following property: the successors of every $A \in \mathcal{T}_0$ form an almost disjoint family of infinite subsets of $A$ such that for any finite $F \subseteq \omega$, there are incomparable $t_0, t_1 \in \mathcal{T}_0$ with $F \subseteq t_0 \cap t_1$. Let $f_0 \in 2^{\alpha_0}$ code $\mathcal{T}_0$.

*Successor step*: Suppose that the ideal-tree $\mathcal{T}_\beta$ has been defined, enumerated as above by $\{A_n : n \in \omega\}$, and coded by an $f_\beta \in 2^{\alpha_\beta}$. Let

$$B_\beta = \omega \setminus \bigcup_{n\in\omega} Z_n[X_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)].$$

Let $m \in \omega$ be such that $A_m = \omega$ (in the fixed enumeration of $\mathcal{T}_\beta$). Let $C_0, C_1 \subseteq Z_m \setminus Z_m[D_{\alpha_\beta}]$ be disjoint sets such that $\omega \setminus (C_0 \cup C_1 \cup B_\beta \cup Z_m[D_{\alpha_\beta}])$

([7]) A very simple coding turns such a function into a function with domain $2^{<\omega_1}$; use the first $\omega$ bits to code the first coordinate of $F$, and then attach the second coordinate.

is infinite. Then define

$$\mathcal{T}_{\beta+1} = \mathcal{T}_\beta \cup \{Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] : n \in \omega\} \cup \{B_\beta \cup C_0, B_\beta \cup C_1\},$$

and let $f_{\beta+1} \in 2^{\alpha_{\beta+1}}$ be an extension of $f_\beta$ that codes $\mathcal{T}_{\beta+1}$.

*Limit step*: If $\beta$ is a limit ordinal and the trees $\mathcal{T}_\gamma$ have been defined for all $\gamma < \beta$, let $\mathcal{T}_\beta = \bigcup_{\gamma<\beta} \mathcal{T}_\gamma$, $f_\beta = \bigcup_{\gamma<\beta} f_\gamma$ and $\alpha_\beta = \sup\{\alpha_\gamma : \gamma < \beta\}$. Note that, in this way, $f_\beta$ codes $\mathcal{T}_\beta$.

Above, the sets $D_\alpha$ are used to prove that in every step of the recursion the trees $\mathcal{T}_{\alpha+1}$ are ideal-trees.

Finally, let $\mathcal{T} = \bigcup_{\alpha<\omega_1} \mathcal{T}_\alpha$, and let $f = \bigcup_{\alpha\in\omega_1} f_\alpha \in 2^{\omega_1}$ be the branch that codes all of $\mathcal{T}$.

Obviously, $\mathcal{T}$ is a tree, being an increasing union of trees. Also, no finite set can be added to $\mathcal{T}$ due to the way we constructed $\mathcal{T}_0$.

Claim. *$\mathcal{T}$ is a maximal tree in $\mathcal{P}(\omega)$.*

Let $X \in [\omega]^\omega$ be arbitrary. Since $g$ guesses every branch stationarily often, there is $\beta$ such that $g$ guesses $(X, f)$ in $\alpha_\beta$. Then $\mathcal{T}_\beta$ is coded by $f{\restriction}\alpha_\beta = f_\beta$. Consequently, $D_\beta$ $\sigma$-reaps $\langle Z_n^{-1}[Z_n \cap X] : n \in \omega\rangle$, and $\varphi_{(X, f\restriction\alpha_\beta)}(X_{\alpha_\beta})(n) \le h_{\alpha_\beta}(n)$ for every $n \in \omega$.

If there is $n \in \omega$ such that $X_{\alpha_\beta} \subseteq^* Z_n^{-1}[Z_n \cap X]$, then

$$Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \subseteq Z_n \cap X$$

(recall that $Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \in \mathcal{T}$).

If for all $n \in \omega$, $X_{\alpha_\beta} \cap Z_n^{-1}[Z_n \cap X]$ is finite, then for all $n \in \omega$,

$$D_{\alpha_\beta} \cap Z_n^{-1}[Z_n \cap X] \subseteq h_{\alpha_\beta}(n).$$

This implies that for all $n \in \omega$,

$$Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \cap X = \emptyset,$$

which in turn yields

$$X \subseteq \omega \setminus \bigcup_{n\in\omega} Z_n[D_\beta \setminus h_{\alpha_\beta}(n)] = B_\beta.$$

By the construction of $\langle \mathcal{T}_\beta : \beta \in \omega_1\rangle$, we have $X \subseteq (B_\beta \cup C_0) \cap (B_\beta \cup C_1)$, both of which are elements of $\mathcal{T}$. Therefore, $\mathcal{T}$ is maximal.

To finish the proof note that, by the construction, $\mathcal{T}$ has height $\omega$. ∎

This theorem has the following two immediate corollaries.

Corollary 4.2. *It is consistent with $\neg\mathsf{CH}$ that there is a maximal tree of size $\omega_1$ and height $\omega$.*

Corollary 4.3. *It is consistent with $\neg\mathsf{CH}$ that there is a maximal tree in $\mathcal{P}(\omega)$ of size $\omega_1$.*

Next we will show that an analogous result is also true for uncountable cardinals. First, recall that $\clubsuit$ denotes the following principle introduced by Ostaszewski [16]:

*There is a sequence* $\{C_\alpha : \alpha \in \lim(\omega_1)\}$ *such that the order type of each* $C_\alpha$ *is* $\omega$, $\sup C_\alpha = \alpha$, *and for every uncountable* $X \subseteq \omega_1$ *there is an* $\alpha \in \lim(\omega_1)$ *such that* $C_\alpha \subseteq X$.

Theorem 4.4. *If* $V \models \mathsf{GCH}$ *and* $\omega_1 < \kappa < \lambda$ *are regular cardinals, then there is a forcing notion* $\mathbb{P} \in V$ *which does not collapse cardinals such that if* $G$ *is* $\mathbb{P}$*-generic over* $V$, *then* $V[G] \models 2^\omega = \kappa$, $2^{\omega_1} = \lambda$, *and there are maximal trees in* $\mathcal{P}(\omega_1)$ *of size* $\omega_1$, $\kappa$ *and* $\lambda$.

*Proof.* Let $V$ be a model of $\mathsf{GCH}$ and let $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\lambda, 2) \times \mathbb{S}^\kappa$, where $\mathrm{Fn}_{\omega_1}(\lambda, 2)$ denotes the forcing for adding $\lambda$-many subsets of $\omega_1$ with countable conditions, and $\mathbb{S}^\kappa$ is the countable support product of $\kappa$-many copies of the Sacks forcing. Standard arguments show that $\mathbb{P}$ is proper and has the $\omega_2$-c.c., hence does not collapse cardinals. Equally standard arguments involving counting of names show that $V[G] \models 2^\omega = \kappa$, $2^{\omega_1} = \lambda$. In fact, the product can be seen as an iteration, first forcing with $\mathrm{Fn}_{\omega_1}(\lambda, 2)$, and then with $\mathbb{S}^\kappa$, where the intermediate model is a model of $\diamondsuit$. It follows by an unpublished result of Baumgartner (written up in [11]) that $V[G] \models \clubsuit$.

By Monk's results there are, in $V[G]$, maximal trees in $\mathcal{P}(\omega_1)$ of size $\omega_1$ and $\lambda = 2^{\omega_1}$, so it suffices to show that there is also a maximal tree of size $\kappa$. This, however, is easy now. Let $\mathcal{C} = \{C_\alpha : \alpha \in \lim(\omega_1)\}$ be a $\clubsuit$-sequence. In particular, $\mathcal{C}$ is an incomparable family in $\mathcal{P}(\omega_1)$. Let $\mathcal{T}_0$ be a tree of size $2^\omega$ extending $\mathcal{C}$, which is easy to find: for instance, add to $\mathcal{C}$ an AD family of size $2^\omega$ inside one of the $C_\alpha$'s. Note that every tree extending $\mathcal{C}$ in $\mathcal{P}(\omega_1)$ consists only of countable sets as every uncountable set contains one of the $C_\alpha$'s. Hence, such a tree has size at most $|[\omega_1]^\omega| = 2^\omega = \kappa$. Therefore, any maximal tree extending $\mathcal{T}_0$ has size $2^\omega = \kappa$. ∎

Corollary 4.5. *It is consistent with* $\mathsf{ZFC}$ *that there are maximal trees in* $\mathcal{P}(\omega_1)$ *of size* $\omega_1$, $\omega_{17}$ *and* $\omega_{1789}$.

**5. Final remarks.** Our interest in this subject stems from a question of S. Geschke and N. Bowler [5], who, assuming $\mathfrak{r} = \mathfrak{c}$, have constructed a *self-dual uniform matroid* on $\omega$ and asked whether such an object exists in $\mathsf{ZFC}$. It is their result that the existence of a self-dual uniform matroid (see [5] for the actual definition) is equivalent to the existence of a maximal incomparable family $\mathcal{A}$ such that

(1) if $A \in \mathcal{A}$, then $\omega \setminus A \in \mathcal{A}$, and
(2) given two disjoint infinite sets $I, J \subseteq \omega$, there is an $A \in \mathcal{A}$ such that either $A \subseteq^* I$, $A \subseteq^* J$, or $I \subseteq^* A \subseteq^* \omega \setminus J$.

It seems that knowing that every maximal incomparable family has size $\mathfrak{c}$ should help in constructing such a family; however, their question remains open.

QUESTION 5.1 (Geschke–Bowler [5]). *Is there a maximal incomparable family satisfying* (1) *and* (2)*?*

We have seen that maximal trees can have quite different shapes (height 2, height $\omega$, height $\omega_1$).

QUESTION 5.2. *Can there be a maximal tree of countable width, i.e. a tree having each level countable? If so, is the existence of such a tree consistent with* $\neg\mathsf{CH}$*?*

QUESTION 5.3. *Is it consistent that* $\omega_1 < \mathfrak{tr} < \mathfrak{c}$*?*

Monk [14] defines the *tree spectrum* of a Boolean algebra $\mathbb{B}$ as the set of all possible cardinalities of maximal subtrees of $\mathbb{B}$.

QUESTION 5.4. *What are the posible sizes of the tree spectrum of* $\mathcal{P}(\omega)/\mathit{fin}$*?*

QUESTION 5.5. *Is the only difference between the tree spectrum of* $\mathcal{P}(\omega)/\mathit{fin}$ *and that of* $\mathcal{P}(\omega)$ *the fact that* $\omega$ *belongs to the tree spectrum of* $\mathcal{P}(\omega)$*? In other words, is it consistent for some uncountable cardinal* $\kappa$ *to belong to one but not the other?*

QUESTION 5.6. *Does every uncountable maximal incomparable family in* $\mathcal{P}(\omega)$ *have size* $\mathfrak{c}$*?*

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