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On uniqueness of distribution of a random variable whose independent copies span a subspace in L_p

by

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Abstract. Let $1 \le p < 2$ and let $L_p = L_p[0,1]$ be the classical L_p-space of all (classes of) p-integrable functions on $[0,1]$. It is known that a sequence of independent copies of a mean zero random variable $f \in L_p$ spans in L_p a subspace isomorphic to some Orlicz sequence space l_M. We give precise connections between M and f and establish conditions under which the distribution of a random variable $f \in L_p$ whose independent copies span l_M in L_p is essentially unique.

1. Introduction. Let us recall the following well-known dichotomy principle for subspaces of the space $L_p(0,1)$, $2 < p < \infty$: every infinite-dimensional closed subspace of $L_p(0,1)$ with $2 < p < \infty$, either is isomorphic to a Hilbert space and complemented in $L_p(0,1)$, or contains a subspace isomorphic to l_p complemented in $L_p(0,1)$ [10, Corollary 2]. This implies easily that if l_q embeds isomorphically into $L_p(0,1)$, $0 < q < \infty$ and $2 < p < \infty$, then either $q = p$ or $q = 2$.

At the same time, the subspace structure of $L_p(0,1)$, $1 \le p < 2$, is much more complicated. In particular, the class of all subspaces of $L_1 = L_1(0,1)$ is very rich and does not have any reasonable description yet. If we consider only symmetric subspaces of L_1, that is, subspaces with a symmetric basis or isomorphs of some symmetric function spaces, then these subspaces are known to be isomorphic to averages of Orlicz spaces [7, 14].

More information is available on subspaces of L_1 isomorphic to Orlicz spaces. First of all, an isomorph of an Orlicz sequence space $l_M \neq l_1$ in L_1 can always be given by the span of a sequence of independent identically

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distributed (i.i.d.) random variables. This was discovered by M. I. Kadec [9] in 1958, who proved that for arbitrary $1 \le p < q < 2$ there exists a symmetrically distributed function $f \in L_p$ (a q-stable random variable) such that the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f spans in L_p a subspace isomorphic to l_q.

This direction of study was taken further by J. Bretagnolle and D. Dacunha-Castelle (see [5, 6, 7]). In particular, D. Dacunha-Castelle showed that for every given mean zero $f \in L_p = L_p(0,1)$, the sequence $\{f_k\}_{k=1}^\infty$ of its independent copies is equivalent in L_p to the unit vector basis of some Orlicz sequence space l_M [7, Theorem 1, p. X.8]. Moreover, J. Bretagnolle and D. Dacunha-Castelle proved that an Orlicz function space $L_M = L_M[0,1]$ can be isomorphically embedded into the space L_p, $1 \le p < 2$, if and only if M is equivalent to a p-convex and 2-concave Orlicz function on $[0,\infty)$ [6, Theorem IV.3]. Later on some of these results were independently rediscovered by M. Braverman [2, 4].

Note that the methods used in [5, 6, 7, 2, 4] depend heavily on the techniques related to the theory of random processes. In a recent paper [1], the first two of the present authors suggested a different approach to this problem, based on methods and ideas from the interpolation theory of operators. In addition, it should be pointed out that papers [5, 6, 7, 2, 4] only give the existence of a function f such that the sequence of its independent copies is equivalent in L_p to the unit vector basis in some Orlicz sequence space and do not address the determination of f, whereas [1] focuses on revealing precise connections between the Orlicz function and the distribution of the corresponding random variable f. Among other results, the following is shown in [1]. Let $1 \le p < 2$ and let M be a p-convex and 2-concave Orlicz function on $[0,\infty)$ such that $M(t) \not\approx t^p$ for small $t > 0$ and the function

$$S(u) := -2pM(u) + (p+1)uM'(u) - u^2M''(u)$$

is positive on $(0,\infty)$, increasing and bounded on $(0,1)$. Then, under some technical conditions on M (see [1, Proposition 12 and Theorem 15]) the unit vector basis in l_M is equivalent in L_p to the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of an arbitrary mean zero function $f \in L_p$ whose distribution function

$$n_f(\tau) := \lambda\{u : |f(u)| > \tau\}, \qquad \tau > 0$$

(λ is the Lebesgue measure) is equivalent to the function $S(1/\tau)$ for $\tau \ge 1$.

The present paper continues this direction of research. Our main result (Theorem 1.1) states that when an Orlicz function M is 'far' from the extreme functions t^p and $t^2, 1 \le p < 2$, the distribution of a random variable

$f \in L_p$ whose independent copies span l_M is essentially equivalent to that of the function

$$\mathfrak{m}(t) = \frac{1}{M^{-1}(t)}, \qquad t > 0.$$

THEOREM 1.1. *Let $1 \le p < 2$ and let M be an Orlicz function. The following conditions are equivalent:*

(i) *M is $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave for some $\varepsilon > 0$.*

(ii) *If a sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of a mean zero random variable $f \in L_p$ satisfies*

$$C^{-1}\Big\|\sum_{k=1}^{n} e_k\Big\|_{l_M} \le \Big\|\sum_{k=1}^{n} f_k\Big\|_p \le C\Big\|\sum_{k=1}^{n} e_k\Big\|_{l_M} \tag{1.1}$$

for some constant $C > 0$ independent of $n \in \mathbb{N}$, then the distribution function $n_f(\tau)$ is equivalent to that of $\mathfrak{m}$ for large τ. Here, $\{e_k\}_{k=1}^{\infty}$ is the unit vector basis in l_M.

(iii) *If a sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of a mean zero random variable $f \in L_p$ is equivalent in L_p to the unit vector basis $\{e_k\}_{k=1}^{\infty}$ in l_M, then the distribution function $n_f(\tau)$ is equivalent to that of $\mathfrak{m}$ for large τ.*

(iv) *The function $\mathfrak{m}$ is in L_p and any sequence of independent copies of a mean zero random variable equimeasurable with $\mathfrak{m}$ is equivalent in L_p to the unit vector basis in l_M.*

Observe that in the special case $M(t) = t^q$, where $1 \le p < q < 2$, assertion (ii) of Theorem 1.1 was proved in [4, Ch. 3, Theorem 2] (see also [3, Theorem 2]) by using a completely different (and more complicated) approach.

It is worth noting that the assertion of Theorem 1.1 is in a sense sharp. Namely, in Proposition 5.3 we show that there exist two random variables x and y with non-equivalent distribution for large τ whose independent copies span in L_1 the same Orlicz space l_M, where M is equivalent to the function $t/\log(e/t)$ for small $t > 0$.

Note that in the special case $p = 1$, another attempt to describe the connection between the distribution of a random variable $f \in L_p$ and the corresponding Orlicz function M can be found in [16]. However, the methods used in [16] have a strong combinatorial flavor and formulas obtained there seem to be less accessible. Moreover, in [16] the question of uniqueness of distribution of f is not raised at all.

The proof of Theorem 1.1 is given in Section 4. Two important components of the proof are Proposition 2.4 and Theorem 3.3.

2. Preliminaries and auxiliary results

2.1. Orlicz functions and spaces. For the theory of Orlicz spaces we refer to [11, 13].

Let M be an *Orlicz function*, that is, an increasing convex function on $[0,\infty)$ such that $M(0)=0$. To any Orlicz function M we associate the *Orlicz sequence space* l_M of all sequences of scalars $a=(a_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^\infty M\left(\frac{|a_n|}{\rho}\right)<\infty$$

for some $\rho>0$. When equipped with the norm

$$\|a\|_{l_M}:=\inf\left\{\rho>0:\ \sum_{n=1}^\infty M\left(\frac{|a_n|}{\rho}\right)\le 1\right\},$$

l_M is a Banach space. Clearly, if $M(t)=t^p$, $p\ge 1$, then the Orlicz space l_M is the familiar space l_p. Moreover, the sequence $\{e_n\}_{n=1}^\infty$ given by

$$e_n=(\underbrace{0,\dots,0,}_{n-1\text{ times}}1,0,\dots)$$

is a Schauder basis in every Orlicz space l_M provided that M satisfies the Δ_2-*condition at zero*, i.e., there are $u_0>0$ and $C>0$ such that $M(2u)\le CM(u)$ for all $0<u<u_0$.

Similarly, if M is an Orlicz function, then the *Orlicz function space* $L_M=L_M[0,1]$ consists of all measurable functions x on $[0,1]$ such that the norm

$$\|x\|_{L_M}=\inf\left\{u>0:\int_0^1 M(|x(t)|/u)\,dt\le 1\right\}$$

is finite.

Let $1\le p<q<\infty$. Given an Orlicz function M, we say that M is p-*convex* if the map $t\mapsto M(t^{1/p})$ is convex, and q-*concave* if $t\mapsto M(t^{1/q})$ is concave. Throughout this paper, we assume that $M(1)=1$ and $M:[0,\infty)\to[0,\infty)$ is a bijection.

Careful inspection of the proof of [1, Lemma 5] yields the following two lemmas.

LEMMA 2.1. *Let $1\le p<\infty$. An Orlicz function $M:[0,\infty)\to[0,\infty)$ satisfying the Δ_2-condition at 0 is equivalent to a p-convex Orlicz function on $[0,1]$ if and only if there exists a constant $C>0$ such that for all $0<s<1$ and all $0<t\le 1$ we have*

$$M(st)\le Cs^pM(t).$$

LEMMA 2.2. *Let $1<q<\infty$. An Orlicz function $M:[0,\infty)\to[0,\infty)$ is equivalent to a q-concave Orlicz function on $[0,1]$ if and only if there exists* *a constant* $C > 0$ *such that for all* $0 < s < 1$ *and all* $0 < t \leq 1$ *we have*

$$s^q M(t) \leq CM(st).$$

In what follows, we will denote by f^* the non-increasing right-continuous rearrangement of a random variable f, that is,

$$f^*(s) := \inf\{t : n_f(t) \leq s\},$$

where n_f is the distribution function of the random variable f. One says that random variables f and g are *equimeasurable* if $f^*(t) = g^*(t)$, $0 < t \leq 1$ (equivalently, $n_f(\tau) = n_g(\tau)$, $\tau > 0$). Finally, given two positive functions (quasinorms) f and g are said to be *equivalent* (we write $f \sim g$) if there exists a positive finite constant C such that $C^{-1}f \leq g \leq Cf$. Sometimes, we say that these functions are equivalent for large (or small) values of the argument, meaning that the preceding inequalities hold only for the specified values.

2.2. A condition for independent copies of a mean zero f to be equivalent in L_p to the unit vector basis of l_M. For $f \in L_1(0,1)$, $k \in \mathbb{N}$, and $t > 0$ we set

$$\bar{f}_k(t) := \begin{cases} f(t-k+1), & t \in [k-1, k), \\ 0, & \text{otherwise.} \end{cases}$$

The following assertion is an immediate consequence of the famous Rosenthal inequality [15] (or its more general version due to Johnson and Schechtman [8]). It establishes a connection between the behavior in L_p of an arbitrary sequence $\{f_k\}_{k=1}^\infty$ of independent copies of a mean zero random variable $f \in L_p$ and that of the corresponding sequence $\{\bar{f}_k\}_{k=1}^\infty$ in the Banach sum $(L_p + L_2)(0,\infty)$ of the Lebesgue spaces $L_p(0,\infty)$ and $L_2(0,\infty)$.

Lemma 2.3. *Let* $1 \leq p \leq 2$. *For every finitely supported* $a = (a_k)_{k=1}^\infty$ *and for a mean zero random variable* $f \in L_p(0,1)$ *we have*

$$\Big\| \sum_{k=1}^\infty a_k f_k \Big\|_p \sim \Big\| \sum_{k=1}^\infty a_k \bar{f}_k \Big\|_{L_p+L_2}.$$

Lemma 2.3 allows us to investigate sequences of independent identically distributed mean zero random variables in $L_p = L_p(0,1)$.

Proposition 2.4. *Let* $1 \leq p \leq 2$ *and let* $f \in L_p$ *be a mean zero random variable. The following conditions are equivalent:*

(a) *A sequence* $\{f_k\}_{k=1}^\infty$ *of independent copies of* f *is equivalent* (*in* L_p) *to the unit vector basis in* l_M.
(b) *A sequence* $\{f_k\}_{k=1}^\infty$ *of independent copies of* f *satisfies* (1.1) *with some constant* $C > 0$ *independent of* $n \in \mathbb{N}$.

(c) *The following equivalence holds:*

$$\frac{1}{M^{-1}(t)} \sim \Big(\frac{1}{t}\int_0^t f^*(s)^p\,ds\Big)^{1/p} + \Big(\frac{1}{t}\int_t^1 f^*(s)^2\,ds\Big)^{1/2}, \qquad 0 < t \le 1. \tag{2.1}$$

Proof. The implication (a)⇒(b) is obvious.

(b)⇒(c). We have

$$\Big\|\sum_{k=1}^n e_k\Big\|_{l_M} \sim \Big\|\sum_{k=1}^n f_k\Big\|_p \overset{\text{Lemma 2.3}}{\sim} \Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{L_p+L_2}.$$

Since $1 \le p \le 2$, it follows that

$$\|x\|_{L_p+L_2} \sim \Big(\int_0^1 x^*(s)^p\,ds\Big)^{1/p} + \Big(\int_1^\infty x^*(s)^2\,ds\Big)^{1/2}.$$

Therefore, from the equalities

$$\Big(\sum_{k=1}^n \bar{f}_k\Big)^*(s) = f^*\Big(\frac{s}{n}\Big), \qquad s > 0,$$

and

$$\Big\|\sum_{k=1}^n e_k\Big\|_{l_M} = \inf\Big\{\rho > 0 : nM\Big(\frac{1}{\rho}\Big) \le 1\Big\} = \frac{1}{M^{-1}(1/n)}, \qquad n \ge 1,$$

it follows that

$$\begin{aligned}\frac{1}{M^{-1}(1/n)} &\sim \Big(\int_0^1 f^*\Big(\frac{s}{n}\Big)^p ds\Big)^{1/p} + \Big(\int_1^n f^*\Big(\frac{s}{n}\Big)^2 ds\Big)^{1/2}\\ &= \Big(n\int_0^{1/n} f^*(s)^p\,ds\Big)^{1/p} + \Big(n\int_{1/n}^1 f^*(s)^2\,ds\Big)^{1/2}, \qquad n \ge 1.\end{aligned}$$

Let $t \in (1/(n+1), 1/n)$ for some $n \ge 1$. We clearly have $M^{-1}(1/n) \sim M^{-1}(t)$ and

$$\begin{aligned}&\Big(n\int_0^{1/n} f^*(s)^p\,ds\Big)^{1/p} + \Big(n\int_{1/n}^1 f^*(s)^2\,ds\Big)^{1/2}\\ &\qquad\qquad \sim \Big(\frac{1}{t}\int_0^t f^*(s)^p\,ds\Big)^{1/p} + \Big(\frac{1}{t}\int_t^1 f^*(s)^2\,ds\Big)^{1/2}.\end{aligned}$$

The assertion (2.1) follows immediately from the equivalences above.

(c)⇒(a). By [7, Theorem 1, p. X.8] (see also [1, Theorem 9]), for every mean zero $f \in L_p(0,1)$ the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f is equivalent in L_p to the unit vector basis in some Orlicz sequence space l_N.

Arguing as in the proof of (b)⇒(c). we conclude that

$$\frac{1}{N^{-1}(t)} \sim \Big(\frac{1}{t}\int_0^t f^*(s)^p\,ds\Big)^{1/p} + \Big(\frac{1}{t}\int_t^1 f^*(s)^2\,ds\Big)^{1/2}, \qquad t \in (0,1).$$

Together with (2.1) the equivalence above show that the Orlicz functions M and N are equivalent on $[0,1]$, and thus $l_N = l_M$. This completes the proof. ∎

3. When does (2.1) hold for $f = \mathfrak{m}$? The following proposition provides necessary and sufficient conditions for $\mathfrak{m}^p$ to be equivalent to its Cesàro transform.

PROPOSITION 3.1. *Let $1 \le p < \infty$ and let M be a p-convex Orlicz function satisfying the Δ_2-condition at 0. The following conditions are equivalent:*

(i) *M is equivalent on $[0,1]$ to a $(p+\varepsilon)$-convex Orlicz function for some $\varepsilon > 0$.*

(ii)
$$\frac{1}{t}\int_0^t \mathfrak{m}(s)^p\,ds \le \mathrm{const}\cdot \mathfrak{m}(t)^p, \qquad t \in (0,1).$$

Proof. Define

$$\varphi(t) = t\mathfrak{m}(t)^p, \qquad t \in (0,1).$$

(i)⇒(ii). It suffices to show that

$$\int_0^t \frac{\varphi(s)\,ds}{s} \le \mathrm{const}\cdot\varphi(t), \qquad t \in (0,1). \tag{3.1}$$

It follows directly from the definitions that, for all $s \in (0,1)$,

$$\sup_{0<t\le 1}\frac{\varphi(st)}{\varphi(t)} = s\sup_{0<t\le 1}\Big(\frac{(M^{-1}(t))^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}}\Big)^{\frac{p}{p+\varepsilon}}.$$

Since M is $(p+\varepsilon)$-convex, the mapping

$$t \mapsto (M^{-1}(t))^{p+\varepsilon}, \qquad t \in (0,1],$$

is concave. In particular, we have

$$\frac{(M^{-1}(t))^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}} \le s^{-1}, \qquad 0 < s,t \le 1.$$

Therefore,

$$\sup_{t\in(0,1)}\frac{\varphi(st)}{\varphi(t)} \le s^{\frac{\varepsilon}{p+\varepsilon}}, \qquad 0 < s \le 1.$$

Applying now Lemma II.1.4 from [12], we infer (3.1).

(ii)⇒(i). Since M is p-convex, it follows that

$$\frac{M(s)}{s^p} \le \frac{M(t)}{t^p}, \qquad 0 < s \le t \le 1.$$

Replacing s with $M^{-1}(s)$ and t with $M^{-1}(t)$, we infer that φ is increasing.

By the assumption, we have

$$\int_0^t \frac{\varphi(s)}{s}\,ds \le C\varphi(t), \qquad t \in (0,1),$$

for some $C > 0$. Take $s_0 < e^{-2C}$. We claim that

$$\sup_{t\in(0,1)} \frac{\varphi(s_0 t)}{\varphi(t)} < 1. \tag{3.2}$$

Indeed, suppose that the supremum in (3.2) equals 1. In particular, there exists $t \in (0,1)$ such that $\varphi(s_0 t) > \varphi(t)/2$. Since φ is increasing and since $\log(s_0^{-1}) > 2C$, it follows that

$$\int_0^t \frac{\varphi(s)}{s}\,ds \ge \int_{s_0 t}^t \frac{\varphi(s)}{s}\,ds \ge \varphi(s_0 t)\log\left(\frac{t}{s_0 t}\right) > C\varphi(t).$$

This contradiction proves the claim.

According to (3.2), we can fix $a \in (0,1)$ such that

$$\varphi(s_0 t) \le a\varphi(t), \qquad t \in (0,1). \tag{3.3}$$

Without loss of generality, we can assume $a > s_0^{1/(1+p)}$. Hence, there exists $\varepsilon \in (0,1)$ such that $a = s_0^{\varepsilon/(p+\varepsilon)}$.

For every $s \in (0,1]$ there exists $n \in \mathbb{N}$ such that $s \in (s_0^{n+1}, s_0^n)$. Since φ is increasing, it follows that

$$\varphi(st) \le \varphi(s_0^n t) \overset{(3.3)}{\le} s_0^{\frac{n\varepsilon}{p+\varepsilon}}\varphi(t) \le s_0^{-\frac{\varepsilon}{p+\varepsilon}} s^{\frac{\varepsilon}{p+\varepsilon}}\varphi(t), \qquad t \in (0,1).$$

Hence,

$$\varphi(st) \le \text{const} \cdot s^{\frac{\varepsilon}{p+\varepsilon}}\varphi(t), \qquad s,t \in (0,1),$$

or equivalently

$$(st)^{-\frac{\varepsilon}{p+\varepsilon}}\varphi(st) \le \text{const} \cdot t^{-\frac{\varepsilon}{p+\varepsilon}}\varphi(t), \qquad s,t \in (0,1).$$

Therefore, it follows from the definition of φ that

$$M(st) \le \text{const} \cdot s^{p+\varepsilon} \cdot M(t), \qquad s,t \in (0,1).$$

The argument is completed by referring to Lemma 2.1. ∎

Now, we prove a dual result.

Proposition 3.2. *Let M be a q-concave Orlicz function for some $1 < q < \infty$. The following conditions are equivalent:*

(i) *M is equivalent on $[0,1]$ to a $(q-\varepsilon)$-concave Orlicz function for some $\varepsilon > 0$.*

(ii)

$$\frac{1}{t}\int_t^1 \mathfrak{m}(s)^q\,ds \le \mathrm{const}\cdot \mathfrak{m}(t)^q, \qquad t\in(0,1). \tag{3.4}$$

Proof. Define

$$\psi(t) := t\mathfrak{m}(t)^q, \qquad t\in(0,1).$$

(i)⇒(ii). It suffices to verify that

$$\int_t^1 \frac{\psi(s)}{s}\,ds \le \mathrm{const}\cdot\psi(t), \qquad t\in(0,1).$$

We have

$$\sup\frac{\psi(st)}{\psi(t)} = s\cdot\sup\biggl(\frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}}\biggr)^{\frac{q}{q-\varepsilon}},$$

where the supremums are taken over all $t\in(0,1)$ and $s>1$ such that $0<st\le 1$. Since M is $(q-\varepsilon)$-concave, it follows that the mapping

$$t\mapsto (M^{-1}(t))^{q-\varepsilon}, \qquad t\in(0,1),$$

is convex. In particular,

$$\frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}} \le s^{-1}, \qquad s>1,\ 0<st\le 1.$$

Therefore,

$$\sup\frac{\psi(st)}{\psi(t)} \le s^{-\frac{\varepsilon}{q-\varepsilon}} < 1,$$

where again the supremum is taken over all $t\in(0,1)$ and $s>1$ such that $0<st\le 1$. Applying now Lemma II.1.5 from [12], we infer (3.4).

(ii)⇒(i). Since M is q-concave, it follows that

$$\frac{M(s)}{s^q} \ge \frac{M(t)}{t^q}, \qquad 0<s\le t\le 1.$$

Replacing s with $M^{-1}(s)$ and t with $M^{-1}(t)$, we infer that ψ is decreasing.

By the assumption, we have

$$\int_t^1 \frac{\psi(s)}{s}\,ds \le C\psi(t), \qquad t\in(0,1),$$

for some $C>0$. Take $s_0>e^{2C}$. We claim that

$$\sup_{t\in(0,s_0^{-1})}\frac{\psi(s_0t)}{\psi(t)} < 1. \tag{3.5}$$

Indeed, suppose that the supremum in (3.5) equals 1. In particular, there exists $t \in (0, s_0^{-1})$ such that $\psi(s_0 t) \ge \psi(t)/2$. Since ψ is decreasing, it follows that

$$\int_t^1 \frac{\psi(s)}{s}\, ds \ge \int_t^{s_0 t} \frac{\psi(s)}{s}\, ds \ge \psi(s_0 t) \log\left(\frac{s_0 t}{t}\right) > C\psi(t).$$

This contradiction proves the claim.

According to (3.5), we can fix $b \in (0,1)$ such that

$$\psi(s_0 t) \le b\psi(t), \qquad t \in (0, s_0^{-1}). \tag{3.6}$$

Without loss of generality, $b > s_0^{-1}$. Hence, there exists $\varepsilon > 0$ such that $b = s_0^{-\varepsilon/(q-\varepsilon)}$.

Let $s > 1$ and $0 < t < s^{-1}$. We can find $n \in \mathbb{N}$ such that $s \in (s_0^n, s_0^{n+1})$. Again appealing to the fact that ψ is decreasing, we have

$$\psi(st) \le \psi(s_0^n t) \overset{(3.6)}{\le} s_0^{-\frac{n\varepsilon}{q-\varepsilon}} \psi(t) \le s_0^{\frac{\varepsilon}{q-\varepsilon}} s^{-\frac{\varepsilon}{q-\varepsilon}} \psi(t).$$

It follows that

$$\psi(st) \le \mathrm{const} \cdot s^{-\frac{\varepsilon}{q-\varepsilon}} \psi(t), \qquad s > 1,\ t \in (0, s^{-1}),$$

or equivalently

$$s^{\frac{\varepsilon}{q-\varepsilon}} \psi(s) \le \mathrm{const} \cdot t^{\frac{\varepsilon}{q-\varepsilon}} \psi(t), \qquad 0 < t \le s \le 1.$$

Therefore, from the definition of ψ, we have

$$\frac{s}{(M^{-1}(s))^{q-\varepsilon}} \le \mathrm{const} \cdot \frac{t}{(M^{-1}(t))^{q-\varepsilon}}, \qquad 0 < t \le s \le 1,$$

or

$$\mathrm{const} \cdot s^{q-\varepsilon} \cdot M(t) \le M(st), \qquad \forall t, s \in (0,1].$$

Applying Lemma 2.2, we complete the proof. ∎

The following theorem answers the question stated in the title of the present section.

THEOREM 3.3. *Let $1 \le p < 2$ and let M be a p-convex and 2-concave Orlicz function. The following conditions are equivalent:*

(i) *The equivalence (2.1) holds for $f = \mathfrak{m}$.*
(ii) *M is $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave for some $\varepsilon > 0$.*

Proof. (ii)⇒(i). If M is $(p+\varepsilon)$-convex for some $\varepsilon > 0$, then it follows from Proposition 3.1 that

$$\left(\frac{1}{t}\int_0^t \mathfrak{m}(s)^p\, ds\right)^{1/p} \le \mathrm{const} \cdot \mathfrak{m}(t), \qquad t \in (0,1). \tag{3.7}$$

If M is $(2-\varepsilon)$-concave for some $\varepsilon > 0$, then Proposition 3.2 implies

$$\left(\frac{1}{t}\int_t^1 \mathfrak{m}(s)^2\,ds\right)^{1/2} \le \mathrm{const}\cdot \mathfrak{m}(t), \qquad t\in(0,1). \tag{3.8}$$

Observe now that the inequality

$$\mathfrak{m}(t) \le \left(\frac{1}{t}\int_0^t \mathfrak{m}(s)^p\,ds\right)^{1/p}, \qquad t\in(0,1) \tag{3.9}$$

holds trivially, due to the fact that $\mathfrak{m}$ is decreasing.

The equivalence (2.1) for $f=\mathfrak{m}$ follows immediately from (3.7)–(3.9).

(i)⇒(ii). Suppose that (2.1) holds for $f=\mathfrak{m}$. Then, we have (3.7) and (3.8). Applying Propositions 3.1 and 3.2, we find that M is $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave for some $\varepsilon>0$, and the proof is complete. ∎

4. When does (2.1) hold for a unique f (up to equivalence near 0)? This section contains the proof of Theorem 1.1.

Proof of Theorem 1.1. The implications (ii)⇒(iii) and (iii)⇒(iv) are obvious. In view of [6, Theorem IV.3] or [1, Theorem 9], the implication (iv)⇒(i) follows by combining Proposition 2.4 and Theorem 3.3.

(i)⇒(ii). We begin with the following technical lemma.

Lemma 4.1. *Let $1\le p<\infty$, $1<q<\infty$ and let M be an Orlicz function.*

(i) *If M is $(q-\varepsilon)$-concave for some $\varepsilon>0$, then*

$$N\sup_{t>0}\frac{\mathfrak{m}(Nt)^q}{\mathfrak{m}(t)^q}\to 0, \qquad N\to\infty.$$

(ii) *If M is $(p+\varepsilon)$-convex for some $\varepsilon>0$, then*

$$\frac{1}{N}\sup_{t>0}\frac{\mathfrak{m}(t/N)^p}{\mathfrak{m}(t)^p}\to 0, \qquad N\to\infty.$$

Proof. The proofs of (i) and (ii) are very similar. So, we prove (i) only.

Since M is $(q-\varepsilon)$-concave, it follows that the mapping

$$t\mapsto \frac{M(t)}{t^{q-\varepsilon}}, \qquad t>0,$$

is decreasing. Hence, the mapping

$$t\to t\mathfrak{m}(t)^{q-\varepsilon} = \frac{t}{(M^{-1}(t))^{q-\varepsilon}}, \qquad t>0,$$

is also decreasing. Therefore,

$$N^{\frac{q}{q-\varepsilon}}\sup_{t>0}\frac{\mathfrak{m}(Nt)^q}{\mathfrak{m}(t)^q} = \left(\sup_{t>0}\frac{Nt\mathfrak{m}(Nt)^{q-\varepsilon}}{t\mathfrak{m}(t)^{q-\varepsilon}}\right)^{\frac{q}{q-\varepsilon}} \le 1,$$

whence

$$N \sup_{t>0} \frac{\mathfrak{m}(Nt)^q}{\mathfrak{m}(t)^q} \le N^{-\varepsilon/(q-\varepsilon)} \to 0 \quad \text{as } N \to \infty. \ \blacksquare$$

Now, let M be a $(p+\varepsilon)$-convex and $(2-\varepsilon)$-concave Orlicz function and let f be a mean zero L_p function. Suppose the sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of f satisfies inequality (1.1) with some constant $C > 0$ independent of $n \in \mathbb{N}$. It suffices to show that the functions f^* and $\mathfrak{m}$ are equivalent for small values of the argument. For simplicity we abuse the notation assuming that $f = f^*$. By Proposition 2.4 we know that the equivalence (2.1) holds for f, that is,

$$\mathfrak{m}(t) \sim \left(\frac{1}{t}\int_0^t f(s)^p\,ds\right)^{1/p} + \left(\frac{1}{t}\int_t^1 f(s)^2\,ds\right)^{1/2}, \quad t \in (0,1). \tag{4.1}$$

By Theorem 3.3, we also have

$$\mathfrak{m}(t) \sim \left(\frac{1}{t}\int_0^t \mathfrak{m}(s)^p\,ds\right)^{1/p} + \left(\frac{1}{t}\int_t^1 \mathfrak{m}(s)^2\,ds\right)^{1/2}, \quad t \in (0,1). \tag{4.2}$$

Observe now that the estimate

$$f(t) \le C_1 \mathfrak{m}(t), \quad t \in (0,1), \tag{4.3}$$

for some $C_1 > 0$ follows immediately from (4.1) and the (already used) inequality

$$f(t) \le \left(\frac{1}{t}\int_0^t f(s)^p\,ds\right)^{1/p}, \quad t \in (0,1).$$

Thus, we need to show that the estimate

$$\mathfrak{m}(t) \le \mathrm{const}\cdot f(t) \tag{4.4}$$

holds for all sufficiently small $t \in (0,1)$.

By Propositions 3.1 and 3.2, there exists a constant $C_0 > 0$ such that

$$\frac{1}{t}\int_0^t \mathfrak{m}(s)^p\,ds \le C_0^p \mathfrak{m}(t)^p, \quad t \in (0,1), \tag{4.5}$$

$$\frac{1}{t}\int_t^1 \mathfrak{m}(s)^2\,ds \le C_0^2 \mathfrak{m}(\mathrm{t})^2, \quad t \in (0,1). \tag{4.6}$$

Moreover, there is a constant $C > 0$ such that for a given $t \in (0,1)$, from (4.1) it follows that either

$$\left(\frac{1}{t}\int_t^1 f(s)^2\,ds\right)^{1/2} \ge \frac{1}{2C}\mathfrak{m}(t), \tag{4.7}$$

or

$$(4.8)\qquad \left(\frac{1}{t}\int_0^t f(s)^p\,ds\right)^{1/p} \ge \frac{1}{2C}\mathfrak{m}(t).$$

By Lemma 4.1, we can fix N so large that

$$(4.9)\qquad \sup_{t>0}\frac{\mathfrak{m}(Nt)^2}{\mathfrak{m}(t)^2} \le \frac{1}{8NC^2C_1^2}, \qquad \sup_{t>0}\frac{\mathfrak{m}(t/N)^p}{\mathfrak{m}(t)^p} \le \frac{N}{2^{p+1}C_1^pC^p}.$$

Let $t\in(0,1/N)$. Firstly, we consider the situation when (4.7) holds. Squaring this inequality and then applying (4.3), we obtain

$$\begin{aligned}\frac{1}{4C^2}\mathfrak{m}(t)^2 &\le \frac{1}{t}\int_t^1 f(s)^2\,ds = \frac{1}{t}\int_t^{Nt} f(s)^2\,ds + \frac{1}{t}\int_{Nt}^1 f(s)^2\,ds\\ &\le (N-1)f(t)^2 + \frac{NC_1^2}{Nt}\int_{Nt}^1 \mathfrak{m}(s)^2\,ds.\end{aligned}$$

Hence, by (4.6), we have

$$\frac{1}{4C^2}\mathfrak{m}(t)^2 \le (N-1)f(t)^2 + NC_1^2C_0^2\mathfrak{m}(Nt)^2.$$

Combining the latter estimate with the first inequality in (4.9), we obtain

$$\begin{aligned}(N-1)f(t/N)^2 &\ge (N-1)f(t)^2\\ &\ge \frac{1}{4C^2}\mathfrak{m}(t)^2 - NC_1^2C_0^2\mathfrak{m}(Nt)^2 \overset{(4.9)}{\ge} \frac{1}{8C^2}\mathfrak{m}(t)^2.\end{aligned}$$

If (4.8) holds, then

$$\frac{1}{2^pC^p}\mathfrak{m}(t)^p \le \frac{1}{t}\int_0^t f(s)^p\,ds = \frac{1}{t}\int_0^{t/N} f(s)^p\,ds + \frac{1}{t}\int_{t/N}^t f(s)^p\,ds.$$

Taking (4.3) and (4.5) into account, we obtain

$$\begin{aligned}\frac{1}{2^pC^p}\mathfrak{m}(t)^p &\le \frac{C_1^p/N}{t/N}\int_0^{t/N} \mathfrak{m}(s)^p\,ds + \left(1-\frac{1}{N}\right)f\left(\frac{t}{N}\right)^p\\ &\le \frac{1}{N}C_1^pC_0^p\mathfrak{m}\left(\frac{t}{N}\right)^p + \left(1-\frac{1}{N}\right)f\left(\frac{t}{N}\right)^p.\end{aligned}$$

We infer from this estimate and the second inequality in (4.9) that

$$\left(1-\frac{1}{N}\right)f\left(\frac{t}{N}\right)^p \ge \frac{1}{2^pC^p}\mathfrak{m}(t)^p - \frac{1}{N}C^pC_0^p\mathfrak{m}\left(\frac{t}{N}\right)^p \overset{(4.9)}{\ge} \frac{1}{2^{p+1}C^p}\mathfrak{m}(t)^p.$$

In either case, we have

$$f(t/N) \ge \mathrm{const}\cdot\mathfrak{m}(t), \qquad t\in(0,1/N),$$

for a universal constant. Since $\mathfrak{m}(t) \sim \mathfrak{m}(t/N)$, it follows that

$$f(t) \geq \text{const} \cdot \mathfrak{m}(t), \qquad t \in (0, 1/N^2).$$

The latter inequality together with (4.3) suffices to conclude the proof of the implication (i)⇒(ii). ■

5. Sharpness of Theorem 1.1. Let $\{h_k\}_{k=1}^\infty$ (respectively, $\{g_k\}_{k=1}^\infty$) be a sequence of pairwise disjoint measurable subsets of $(0,1)$ such that $\lambda(h_k) = 2^{-k-2^k}$ (respectively, $\lambda(g_k) = 4^{-k-4^k}$), $k \geq 1$. We define functions $x, y \in L_1(0,1)$ by setting

$$x = \sum_{k=1}^\infty 2^{2^k} \chi_{h_k}, \qquad y = \sum_{k=1}^\infty 4^{4^k} \chi_{g_k} \tag{5.1}$$

(χ_c is the indicator function of a set c).

Lemma 5.1. *We have*

$$\int_0^1 \min\{x(s), tx(s)^2\}\, ds \sim \int_0^1 \min\{y(s), ty(s)^2\}\, ds \sim \frac{1}{\log(e/t)}, \quad 0 < t \leq 1.$$

Proof. It is clear that

$$\int_0^1 \min\{x(s), tx(s)^2\}\, ds = \sum_{2^{2^k} \geq 1/t} 2^{2^k} \cdot 2^{-k-2^k} + t \cdot \sum_{2^{2^k} < 1/t} 2^{2^{k+1}} \cdot 2^{-k-2^k}.$$

Let $t < 1/4$. If m is the maximal positive integer such that $2^{2^m} < 1/t$, then

$$\int_0^1 \min\{x(s), tx(s)^2\}\, ds = \sum_{k=m+1}^\infty 2^{-k} + t \cdot \sum_{k=1}^m 2^{2^k - k} = 2^{-m} + t \cdot \sum_{k=1}^m 2^{2^k - k}.$$

Also,

$$\sum_{k=1}^m 2^{2^k - k} \leq 2^{2^m - m} + (m-1) \cdot 2^{2^{m-1} - m + 1} \leq 2^{2^m - m} + 2^{2^{m-1}} \leq 2 \cdot 2^{2^m - m}.$$

Therefore,

$$2^{-m} \leq \int_0^1 \min\{x(s), tx(s)^2\}\, ds \leq 2^{-m} + 2t \cdot 2^{2^m - m} \leq 3 \cdot 2^{-m}.$$

It now follows from the definition of m that

$$\frac{1}{\log_2(1/t)} \leq \int_0^1 \min\{x(s), tx(s)^2\}\, ds \leq \frac{6}{\log_2(1/t)}.$$

The similar equivalence for y follows *mutatis mutandis.* ■

LEMMA 5.2. *The distributions of the functions x and y are not equivalent.*

Proof. Suppose that $n_x(Ct) \le Cn_y(t)$, $t > 0$. Fix k such that

$$2^{2k+1} > \log_2 C + 1$$

and select t such that both t and Ct belong to the interval $(2^{2^{2k+1}}, 2^{2^{2k+2}})$. Then

$$n_x(Ct) = n_x(2^{2^{2k+1}}) \ge 2^{-(2k+2)-2^{2k+2}}$$

and

$$n_y(t) = n_y(4^{4^k}) \le 2 \cdot 4^{-(k+1)-4^{k+1}} = 2^{-2k-1-2^{2k+3}}.$$

It follows that

$$2^{2k+2+2^{2k+2}} \ge \frac{1}{C} \cdot 2^{2k+1+2^{2k+3}},$$

or equivalently

$$2k + 2 + 2^{2k+2} \ge -\log_2(C) + 2k + 1 + 2^{2k+3}.$$

Clearly, this contradicts the choice of k. ∎

Let $\{x_k\}_{k=1}^{\infty}$ (respectively, $\{y_k\}_{k=1}^{\infty}$) be a sequence of independent copies of a mean zero random variable equimeasurable with x (respectively, y), where x and y are defined in (5.1). Let us show that the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ span in L_1 the same Orlicz space l_M, where M is equivalent to the function $t/\log(e/t)$ for small $t > 0$. Note that M does not satisfy condition (i) of Theorem 1.1; more precisely, M is not $(1+\varepsilon)$-convex for any $\varepsilon > 0$. Taking into account Lemma 2.3, it suffices to prove the following proposition.

PROPOSITION 5.3. *For every finitely supported $a = (a_k)_{k=1}^{\infty}$, we have*

$$\Big\| \sum_{k=1}^{n} a_k \overline{x}_k \Big\|_{L_1+L_2} \sim \Big\| \sum_{k=1}^{n} a_k \overline{y}_k \Big\|_{L_1+L_2} \sim \|(a_k)_{k=1}^{\infty}\|_{l_M}.$$

Proof. Define an Orlicz function N by setting

$$N(t) = \begin{cases} t^2, & t \in (0,1) \\ 2t-1, & t \ge 1. \end{cases}$$

It is easy to check that $\|z\|_{L_1+L_2} \sim \|z\|_{L_N}$ for every $z \in L_1 + L_2$, where L_N is the Orlicz function space on $[0,1]$.

Setting

$$M(t) = \int_0^1 N(tx(s))\, ds, \qquad t > 0,$$

we obtain

$$\Big\|\sum_{k=1}^{\infty} a_k \overline{x}_k\Big\|_{L_N} \le 1 \;\Leftrightarrow\; \int_0^{\infty} N\Big(\sum_{k=1}^{\infty} |a_k|\,|\overline{x}_k(s)|\Big)\,ds \le 1$$
$$\Leftrightarrow\; \sum_{k=1}^{\infty}\int_0^1 N(|a_k|\,|x_k(s)|)\,ds \le 1$$
$$\Leftrightarrow\; \sum_{k=1}^{\infty} M(a_k) \le 1 \;\Leftrightarrow\; \|a\|_{l_M} \le 1.$$

Therefore,

$$\Big\|\sum_{k=1}^{\infty} a_k \overline{x}_k\Big\|_{L_1+L_2} \sim \|a\|_{l_M}.$$

Since $N(t) \sim \min\{t, t^2\}$ $(t > 0)$, it follows that

$$M(t) \sim \int_0^1 \min\{tx(s), (tx(s))^2\}\,ds,$$

and Lemma 5.1 yields

$$M(t) \sim \frac{t}{\log(e/t)}, \qquad 0 < t \le 1.$$

This proves the assertion for the sequence $\{x_k\}$. The proof of the similar assertion for $\{y_k\}$ is the same. ∎

Remark 5.4. It is natural to ask what happens when $M(t)$ is close to t^2. Our example (Lemma 5.2 and Proposition 5.3 above) is in sharp contrast with Theorem 4.2 in [2]. The latter theorem states that if a sequence of independent copies of a mean zero random variable f spans l_M where $M(t) = t^2 \log(1/t)$ near 0, then the distribution function n_f is unique (up to equivalence for large arguments).

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