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Linearization of isometric embedding on Banach spaces

by

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Abstract. Let X, Y be Banach spaces, $f : X \to Y$ be an isometry with $f(0) = 0$, and $T : \overline{\text{span}}(f(X)) \to X$ be the Figiel operator with $T \circ f = \text{Id}_X$ and $\|T\| = 1$. We present a sufficient and necessary condition for the Figiel operator T to admit a linear isometric right inverse. We also prove that such a right inverse exists when $\overline{\text{span}}(f(X))$ is weakly nearly strictly convex.

1. Introduction. Let X, Y be real Banach spaces. A mapping $f : X \to Y$ is called an *isometry* if $\|f(u) - f(v)\| = \|u - v\|$ for any $u, v \in X$, and the isometry f is said to be *standard* if $f(0) = 0$.

There are many remarkable results concerning the properties of isometries and perturbations of isometries between Banach spaces (see, for instance, [15, 13, 11, 1, 10, 20] for the case of isometries; [3, 17, 12, 19, 22, 23, 21, 16] for perturbations of isometries; and [7, 8, 2, 5, 6, 9, 24] for recent developments on perturbations of isometries). Among all of them, Figiel [11] showed the following fundamental theorem, which guarantees the existence and uniqueness of a continuous linear left inverse for any standard isometric embedding.

THEOREM 1.1 (Figiel). *Let $f : X \to Y$ be a standard isometry. Then there exists a unique bounded linear operator $T(f) : \overline{\text{span}}(f(X)) \to X$ with $\|T(f)\| = 1$ such that*

$$T(f) \circ f = \text{Id}_X.$$

For convenience, the operator $T(f)$ as above will be called the *Figiel operator*, and if there is no ambiguity, we use T to denote $T(f)$.

For any non-surjective isometric embedding, Baker [1] showed the following important result.

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THEOREM 1.2 (Baker). *Let X, Y be Banach spaces, and $f : X \to Y$ be a standard isometry. If $\overline{\operatorname{span}}(f(X))$ is strictly convex, then f is linear.*

The problem whether there exists a linear isometric right inverse of the Figiel operator T has also attracted attention. For instance, this problem was investigated by Godefroy and Kalton [15]. Godefroy [13] further provided an elementary approach for the separable case, and in [14] presented more interesting details. We reformulate the deep results of [15] as follows.

THEOREM 1.3 (Godefroy–Kalton). *Let $f : X \to Y$ be a standard isometry, and T be the Figiel operator of f.*

(I) [15, Proposition 2.9 and Theorem 3.1] *If X is a separable Banach space, then there is a linear isometry $S : X \to \overline{\operatorname{span}}(f(X))$ such that $T \circ S = \mathrm{Id}_X$.*
(II) [15, p. 133] *If X is a non-separable weakly compactly generated Banach space, then there exists a Banach space Y such that there exists a non-linear isometry $f : X \to Y$ but X is not linearly isomorphic to any subspace of Y.*

(I) above shows that if X is separable, then the Figiel operator T admits a linear isometric right inverse. However, let H be a non-separable Hilbert space; then H is a non-separable weakly compactly generated space. (II) indicates that even for "the best" non-separable Banach space H, there exist a Banach space Y and a non-linear isometry $f : H \to Y$ such that the Figiel operator T does not admit a linear isometric right inverse. [15, p. 134] also provides more examples of pairs of Banach spaces (X, Y) such that X can be isometrically embedded into Y without being linearly isomorphic to a subspace of Y.

Therefore, the following problems deserve consideration.

PROBLEM 1.4. *Let $f : X \to Y$ be a standard isometry, and T be the Figiel operator of f.*

(I) *Find necessary and sufficient conditions for the Figiel operator T to admit a linear isometric right inverse $S : X \to \overline{\operatorname{span}}(f(X))$.*
(II) *What classes of non-separable Banach spaces of Y can guarantee the existence of a linear isometric right inverse S of the Figiel operator T?*

In this paper, we first give a necessary and sufficient condition for the Figiel operator to admit a linear isometric right inverse. We then prove that such an inverse exists when $\overline{\operatorname{span}}(f(X))$ is weakly nearly strictly convex.

In this paper, all symbols and notation are standard. All Banach spaces considered are real, and we use X to denote a real Banach space and X^*

its dual. For a subspace $M \subset X$, $M^\perp$ stands for the *annihilator* of M, i.e. $M^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in M\}$. If $M \subset X^*$, then ${}^\perp M$, the *pre-annihilator* of M, is defined as ${}^\perp M = \{x \in X : \langle x, x^*\rangle = 0$ for all $x^* \in M\}$. For Banach spaces X, Y, $\ell_\infty(X, Y)$ denotes the Banach space of all uniformly bounded mappings $m : X \to Y$, endowed with the supnorm. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$, $\overline{\mathrm{span}}(A)$ is the closed subspace linearly generated by A.

2. Main results. Suppose that X, Y are Banach spaces, $f : X \to Y$ is an isometry, and $T : \overline{\mathrm{span}}(f(X)) \to X$ is the Figiel operator. We first give a necessary and sufficient condition for the existence of a linear isometric right inverse of T.

THEOREM 2.1. *Let X, Y be two Banach spaces, $f : X \to Y$ be a standard isometry, and T be the Figiel operator with $\|T\| = 1$ and $T \circ f = \mathrm{Id}_X$.*

- (I) *If there exists a linear isometry $S : X \to \overline{\mathrm{span}}(f(X))$ such that $T \circ S = \mathrm{Id}_X$, then $T^* \circ S^* : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ is a w^*-to-w^* continuous projection with $\|T^* \circ S^*\| = 1$.*
- (II) *If there is a w^*-to-w^* continuous projection $P : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ with $\|P\| = 1$, then there is a unique linear isometric right inverse $S : X \to \overline{\mathrm{span}}(f(X))$ of T such that $T \circ S = \mathrm{Id}_X$ and $P = T^* \circ S^*$.*

Proof. (I) Since $T^* : X^* \to T(X^*)$ is a w^*-to-w^* continuous linear isometry and $S^* : \overline{\mathrm{span}}(f(X))^* \to X^*$ is a w^*-to-w^* continuous linear operator with $\|S^*\| = \|S\| = 1$, it follows that $T^* \circ S^* : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ is w^*-to-w^* continuous with $\|T^* \circ S^*\| = 1$. Furthermore,

$$\begin{aligned}(T^* \circ S^*) \circ (T^* \circ S^*) &= T^* \circ (S^* \circ T^*) \circ S^* \\ &= T^* \circ (T \circ S)^* \circ S^* \\ &= T^* \circ \mathrm{Id}_{X^*} \circ S^* = T^* \circ S^*.\end{aligned}$$

This shows that $T^* \circ S^*$ is a linear projection. Therefore, (I) holds.

(II) Suppose that $P : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ is a w^*-to-w^* continuous projection with $\|P\| = 1$. Then, for any $x \in X$, $\langle f(x), P(\cdot)\rangle$ is a w^*-continuous linear functional on $\overline{\mathrm{span}}(f(X))^*$, and this means $\langle f(x), P(\cdot)\rangle \in \overline{\mathrm{span}}(f(X))$. Moreover, we have

$$|\langle f(x), P(y^*)\rangle| \le \|P\|\,\|y^*\|\,\|f(x)\| = \|P\|\,\|x\|\,\|y^*\|. \tag{2.1}$$

Now, we define the desired linear isometry $S : X \to \overline{\mathrm{span}}(f(X))$ as follows:

$$S(x) = \langle f(x), P(\cdot)\rangle \quad \text{for all } x \in X. \tag{2.2}$$

Indeed, it is easy to see that S is a linear operator, and (2.1) implies

$$\|S\| \le \|P\| = 1. \tag{2.3}$$

For any $x^* \in X^*$ and $x \in X$,

$$\begin{aligned} \langle x^*, x - T\circ S(x)\rangle &= \langle x^*, x\rangle - \langle T^*(x^*), S(x)\rangle \\ &= \langle x^*, x\rangle - \langle f(x), P(T^*(x^*))\rangle \\ &= \langle x^*, x\rangle - \langle T^*(x^*), f(x)\rangle \\ &= \langle x^*, x\rangle - \langle x^*, x\rangle = 0. \end{aligned} \tag{2.4}$$

This shows that $T\circ S = \mathrm{Id}_X$. Thus,

$$\|x\| = \|T\circ S(x)\| \le \|T\|\cdot\|S(x)\| = \|S(x)\|. \tag{2.5}$$

(2.3) and (2.5) together imply that S is a linear isometry.

It is trivial that $T^*(X^*) \subset \overline{\mathrm{span}}(f(X))^*$ is w^*-closed. It follows from the w^*-closedness of $T^*(X^*)$ that

$$T^*(X^*) = [^\perp(T^*(X^*))]^\perp = (\overline{\mathrm{span}}(f(X))/^\perp(T^*(X^*)))^*. \tag{2.6}$$

Furthermore, for any $y^* \in \overline{\mathrm{span}}(f(X))^*$,

$$\begin{aligned} \langle T^*\circ S^*(y^*), f(x) + {}^\perp T^*(X^*)\rangle &= \langle T^*\circ S^*(y^*), f(x)\rangle \\ = \langle S^*(y^*), T\circ f(x)\rangle &= \langle S^*(y^*), x\rangle = \langle y^*, S(x)\rangle \\ = \langle f(x), P(y^*)\rangle &= \langle f(x) + {}^\perp T^*(X^*), P(y^*)\rangle. \end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that $P = T^*\circ S^*$.

Finally, if $S_i : X \to \overline{\mathrm{span}}(f(X))$, $i = 1, 2$, are two linear isometries such that $T\circ S_i = \mathrm{Id}_X$ and $P = T^*\circ S_i^*$, then necessarily $S_1 = S_2$. ∎

Proposition 2.4 below comes from [15, Proposition 2.6], which in fact goes back to Lindenstrauss' article [18]. For convenience, we present its proof. To state the proposition, we need the following definition of invariant mean on a semigroup and some related results, which are taken from Benyamini and Lindenstrauss's book [3, pp. 417–418].

Definition 2.2. Let G be a semigroup. A *left-invariant mean* on G is a linear functional μ on $\ell_\infty(G,\mathbb{R})$ such that:

- (I) $\mu(1) = 1$,
- (II) $\mu(f) \ge 0$ for every $f \ge 0$,
- (III) for all $f \in \ell_\infty(G)$ and $g \in G$, $\mu(f_g) = \mu(f)$, where f_g is the left translation of f by g, i.e., $f_g(h) = f(gh)$ for $h \in G$.

Analogously, we can define right-invariant means on G. An *invariant mean* is a linear functional on $\ell_\infty(G,\mathbb{R})$ which is both left-invariant and right-invariant. Note that (I) and (II) are equivalent to (I) and $\|\mu\| = 1$.

LEMMA 2.3. *Every Abelian semigroup G (in particular, every linear space) has an invariant mean.*

PROPOSITION 2.4. *Let X,Y be Banach spaces, $f : X \to Y$ be a standard isometry, and T be the Figiel operator such that $T \circ f = \mathrm{Id}_X$. Then there exists a continuous linear projection $Q : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ with $\|Q\| = 1$.*

Proof. We will first define a bounded linear operator $R : \overline{\mathrm{span}}(f(X))^* \to X^*$ with $\|R\| \le 1$.

Note that X is an Abelian group with respect to vector addition. By Lemma 2.3, there exists an invariant mean μ on X, which we denote also by μ_z or $\mu_z(\cdot)$, to emphasize that the mean is taken with respect to z. Since $f : X \to Y$ is a standard isometry, for any fixed $x \in X$ and $y^* \in \overline{\mathrm{span}}(f(X))^*$, and any $z \in X$,

$$|\langle f(x+z) - f(z), y^*\rangle| \le \|f(x+z) - f(z)\| \cdot \|y^*\| = \|x\| \cdot \|y^*\|.$$

Therefore, $((\langle f(x+z) - f(z), y^*\rangle)_{z\in X} \in \ell_\infty(X,\mathbb{R})$. For simplicity, we denote this map by $\langle f(x+z) - f(z), y^*\rangle_{z\in X}$. Making use of the invariant mean $\mu_z \in \ell_\infty(X,\mathbb{R})^*$, we define $R : \overline{\mathrm{span}}(X)^* \to X^*$ as follows. For any $z^* \in \overline{\mathrm{span}}(f(X))^*$, $x \in X$,

$$\langle R(z^*), x\rangle = \big\langle \mu_z, \langle f(x+z) - f(z), z^*\rangle_{z\in X}\big\rangle. \tag{2.8}$$

Indeed, it is trivial to show $R(z^*)$ is bounded linear functional on X with $\|R(z^*)\| \le \|z^*\|$. The linearity of $R : \overline{\mathrm{span}}(X)^* \to X^*$ is also obvious. In summary, $R : \overline{\mathrm{span}}(X)^* \to X^*$ is a bounded linear operator with $\|R\| \le 1$.

Next, we claim that $\|R\| = 1$ and $T^* \circ R : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ is a bounded linear projection with $\|T^* \circ R\| = 1$. In fact, this follows from the fact that $R \circ T^* = \mathrm{Id}_{X^*}$. Since T^* is a linear isometry, $\|T^* \circ R\| = \|R\| = 1$. Letting $Q = T^* \circ R$ completes the proof. ∎

REMARK 2.5. Theorem 2.1 shows that the existence of a linear isometric right inverse of the Figiel operator T depends on the existence of a w^*-to-w^* continuous linear projection P from $\overline{\mathrm{span}}(f(X))^*$ onto $T^*(X^*)$ with $\|P\| = 1$. Even though Proposition 2.4 shows that there always exist a bounded linear projection $Q : \overline{\mathrm{span}}(f(X))^* \to T^*(X^*)$ with $\|Q\| = 1$, we cannot claim that Q is w^*-to-w^* continuous in general. For example, let H be a non-separable Hilbert space. Then, by Godefroy–Kalton's Theorem 1.3, there exist a Banach space Y and a non-linear isometry $f : H \to Y$ such that H is not even linearly isomorphic to any subspace of Y. Thus, in this case, the projection Q is not w^*-to-w^* continuous by Theorem 2.1.

We continue to study (II) of Problem 1.4. We mainly show that if $\overline{\mathrm{span}}(f(X))$ is weakly nearly strictly convex, then the Figiel operator T admits a linear isometric right inverse. Note that Baker [1] proved (see Theorem 1.2) that any standard isometric embedding from a Banach space into a strictly convex Banach space is necessarily linear. Therefore, we need to consider the class of Banach spaces containing all strictly convex spaces.

The following definition of (weakly) nearly strictly convex space comes from Cabrera and Sadarangani [4].

A Banach space X is said to be (resp. *weakly*) *nearly strictly convex* if, given $x^* \in X^*$ with $|x^*| = 1$, the set $\{x \in X : \|x\| = 1,\, x^*(x) = 1\}$ is (resp. weakly) compact.

Clearly, strictly convex spaces are NSC, NSC spaces are WNSC, and WNSC spaces include all reflexive Banach spaces.

Finally, we prove the following main result.

THEOREM 2.6. *Let X, Y be Banach spaces, $f : X \to Y$ be a standard isometry, and T be the Figiel operator such that $\|T\| = 1$ and $T \circ f = \mathrm{Id}_X$. If $\overline{\mathrm{span}}(f(X))$ is a WNSC space, then there exists a linear isometry $S : X \to \overline{\mathrm{span}}(f(X))$ such that $T \circ S = \mathrm{Id}_X$.*

Proof. Recall that the bounded linear operator $R : \overline{\mathrm{span}}(f(X))^* \to X^*$ in (2.8) is defined as follows: for any $z^* \in \overline{\mathrm{span}}(f(X))^*$ and $x \in X$,

$$\langle R(z^*), x\rangle = \big\langle \mu_z, \langle f(x+z) - f(z), z^*\rangle_{z\in X}\big\rangle. \tag{2.9}$$

It follows from (2.9) that $|\langle R(z^*), x\rangle| \le \|z^*\|\,\|x\|$ and $\langle R(\cdot), x\rangle$ is a bounded linear functional on $\overline{\mathrm{span}}(f(X))^*$ with $\|\langle R(\cdot), x\rangle\| \le \|x\|$.

We define the desired linear isometry $S : X \to \overline{\mathrm{span}}(f(X))$ as follows: for any $x \in X$,

$$S(x) = \langle R(\cdot), x\rangle. \tag{2.10}$$

Indeed, we first assert $\langle R(\cdot), x\rangle \in \overline{\mathrm{span}}(f(X))$. Without loss of generality, we assume $\|x\| = 1$. Then there exists an $x^* \in X^*$ such that $\langle x^*, x\rangle = 1$ and $|x^*| = 1$. Since T^* is a w^*-to-w^* continuous linear isometry, $\|T^*(x^*)\| = 1$. For any $z \in X$,

$$T^*(x^*)(f(x+z) - f(z)) = \langle x^*, x + z - z\rangle = \langle x^*, x\rangle = 1. \tag{2.11}$$

Together with $\|f(x+z) - f(z)\| = \|x + z - z\| = 1$, this implies

$$\begin{aligned}&\{f(x+z) - f(z) : z \in X\}\\ &\qquad\subseteq \{y \in \overline{\mathrm{span}}(f(X)) : \|y\| = 1,\, T^*(x^*)(y) = 1\}.\end{aligned} \tag{2.12}$$

Since $\overline{\mathrm{span}}(f(X))$ is weakly nearly strictly convex, the set $\{y \in \overline{\mathrm{span}}(f(X)) : \|y\| = 1,\, T^*(x^*)(y) = 1\}$ is weakly compact. Let m be the locally convex Mackey topology on $\overline{\mathrm{span}}(f(X))^*$ (that is, the topology of uniform convergence on weakly compact subsets of $\overline{\mathrm{span}}(f(X))$). Then

$$(\overline{\mathrm{span}}(f(X))^*, m)^* = \overline{\mathrm{span}}(f(X)).$$

Suppose that $\{z^*_\alpha\}_{\alpha\in D} \subset \overline{\mathrm{span}}(f(X))^*$ is a net with $z^*_\alpha \xrightarrow{m} z^*_0$ for some $z^*_0 \in \overline{\mathrm{span}}(f(X))^*$. Since the set $\{y \in \overline{\mathrm{span}}(f(X)) : \|y\| = 1,\, T^*(x^*)(y) = 1\}$

is weakly compact, $\{z_\alpha^*\}_{\alpha\in D}$ is uniformly convergent to $\{z_0^*\}$ on that set. Consequently, $\{z_\alpha^*\}_{\alpha\in D}$ is uniformly convergent to $\{z_0^*\}$ on $\{f(x+z)-f(z) : z\in X\}$ by (2.12). According to (2.9), $\langle R(z_\alpha^*),x\rangle \to \langle R(z_0^*),x\rangle$. Therefore, $\langle R(\cdot),x\rangle$ is a Mackey continuous bounded linear functional on $\overline{\operatorname{span}}(f(X))^*$. Hence, $\langle R(\cdot),x\rangle \in \overline{\operatorname{span}}(f(X))$.

For any $u,v\in X$ and $z^*\in\overline{\operatorname{span}}(f(X))^*$, by (2.9) and (2.10),

$$\begin{aligned}\langle S(u+v),z^*\rangle &= \langle R(z^*),u+v\rangle\\ &= \big\langle \mu_z, \langle f(u+v+z)-f(z),z^*\rangle_{z\in X}\big\rangle\\ &= \big\langle \mu_z, \langle f(u+v+z)-f(u+z)+f(u+z)-f(x),z^*\rangle_{z\in X}\big\rangle\\ &= \big\langle \mu_z, \langle f(u+v+z)-f(u+z),z^*\rangle_{z\in X}\big\rangle\\ &\quad+\big\langle \mu_z, \langle f(u+z)-f(z),z^*\rangle_{z\in X}\big\rangle\\ &= \langle S(v),z^*\rangle+\langle S(u),z^*\rangle.\end{aligned}\tag{2.13}$$

Thus, the mapping $S: X\to\overline{\operatorname{span}}(f(X))$ is additive. Furthermore,

$$\begin{aligned}\langle S(u)-S(v),z^*\rangle &= \langle R(z^*),u\rangle-\langle R(z^*),v\rangle\\ &= \big\langle \mu_z, \langle f(u+z)-f(v+z),z^*\rangle_{z\in X}\big\rangle\\ &\le \|u-v\|\,\|z^*\|.\end{aligned}\tag{2.14}$$

Therefore, $\|S(u)-S(v)\|\le\|u-v\|$, i.e., S is 1-Lipschitz. This together with (2.13) shows that S is a bounded linear operator with $\|S\|\le 1$.

Finally, for any $x^*\in X^*$ and $x\in X$,

$$\begin{aligned}\langle x^*,T\circ S(x)\rangle &= \langle T^*(x^*),S(x)\rangle = \langle R(T^*(x^*)),x\rangle\\ &= \big\langle \mu_z, \langle f(x+z)-f(z),T^*(x^*)\rangle_{z\in X}\big\rangle\\ &= \big\langle \mu_z, \langle T(f(x+z)-f(z)),x^*\rangle_{z\in X}\big\rangle\\ &= \langle \mu_z, \langle x,x^*\rangle_{z\in X}\rangle = \langle x,x^*\rangle.\end{aligned}\tag{2.15}$$

So $T\circ S=\mathrm{Id}_X$. Therefore, S is a linear isometry. ∎

We originally proved Theorem 2.6 under the assumption that $\overline{\operatorname{span}}(f(X))$ is nearly strictly convex. Due to the referee's observations, the result was generalized to the case where $\overline{\operatorname{span}}(f(X))$ is weakly nearly strictly convex.

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