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Weak amenability of weighted group algebras on some discrete groups

by

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Abstract. Weak amenability of $\ell^1(G,\omega)$ for commutative groups G was completely characterized by N. Gronbaek in 1989. In this paper, we study weak amenability of $\ell^1(G,\omega)$ for two important non-commutative locally compact groups G: the free group $\mathbb{F}_2$, which is non-amenable, and the amenable $(\boldsymbol{ax}+\boldsymbol{b})$-group. We show that the condition that characterizes weak amenability of $\ell^1(G,\omega)$ for commutative groups G remains necessary for the non-commutative case, but it is sufficient neither for $\ell^1(\mathbb{F}_2,\omega)$ nor for $\ell^1((\boldsymbol{ax}+\boldsymbol{b}),\omega)$ to be weakly amenable. We prove that for several important classes of weights ω the algebra $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if the weight ω is diagonally bounded. In particular, the polynomial weight $\omega_\alpha(x)=(1+|x|)^\alpha$, where $|x|$ denotes the length of the element $x\in\mathbb{F}_2$ and $\alpha>0$, never makes $\ell^1(\mathbb{F}_2,\omega_\alpha)$ weakly amenable.

We also study weak amenability of an Abelian algebra $\ell^1(\mathbb{Z}^2,\omega)$. We give an example showing that weak amenability of $\ell^1(\mathbb{Z}^2,\omega)$ does not necessarily imply weak amenability of $\ell^1(\mathbb{Z},\omega_i)$, where ω_i denotes the restriction of ω to the ith coordinate $(i=1,2)$. We also provide a simple procedure for verification whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable.

1. Introduction. Let A be a Banach algebra and let X be a Banach A-bimodule. A linear map $D: A\to X$ is called a *derivation* if it satisfies $D(ab)=a\cdot D(b)+D(a)\cdot b$ for all $a,b\in A$. For every $x\in X$ the map $a\mapsto a\cdot x-x\cdot a$ is a continuous derivation, called an *inner derivation*. The Banach algebra A is called *amenable* if every continuous derivation from A to any dual Banach A-bimodule X^* is inner; and A is called *weakly amenable* if every continuous derivation from A into its Banach space dual A^* is inner. For the background and history of these notions see the monograph [3].

Since the group algebra $L^1(G)$ and the weighted group algebra $L^1(G,\omega)$ are fundamental examples of Banach algebras, a number of studies were conducted investigating amenability properties of these algebras (see [5]–[7]

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and [14]). In particular, it was proved that every group algebra $L^1(G)$, and hence every discrete group algebra $\ell^1(G)$, is weakly amenable. In this paper, we investigate weak amenability of the weighted discrete group algebra $\ell^1(G,\omega)$ for several important groups G.

Recall that a weight on a discrete group G is a function $\omega : G \to \mathbb{R}^+$ satisfying the weight inequality $\omega(xy) \le \omega(x)\omega(y)$ for $x, y \in G$. Given a weight ω on G, consider

$$\ell^1(G,\omega) = \Big\{ f = \sum_{x\in G} f(x)\delta_x \in \mathbb{C}^G : \sum_{x\in G} |f(x)|\omega(x) < \infty \Big\}.$$

Equipped with the norm

$$\|f\|_{\ell^1(G,\omega)} = \sum_{x\in G} |f(x)|\omega(x)$$

and the convolution product

$$(f * g)(x) = \sum_{y\in G} f(y^{-1}x)g(y), \qquad x \in G,$$

$\ell^1(G,\omega)$ becomes a Banach algebra, called a *discrete Beurling algebra*.

N. Gronbaek has characterized the weights ω making $\ell^1(G,\omega)$ weakly amenable in the case when the group G is Abelian.

PROPOSITION 1.1 ([6, Corollary 4.8]). *Let G be an Abelian discrete group, and ω be a weight function on G. The Beurling algebra $\ell^1(G,\omega)$ is weakly amenable if and only if*

$$(*) \qquad \sup_{g\in G} \frac{|\Phi(g)|}{\omega(g)\omega(-g)} = \infty$$

for every non-zero group homomorphism $\Phi : G \to \mathbb{C}$.

Note that in the above, $\mathbb{C}$ is considered as an additive group.

It was later proved that the condition that every non-zero homomorphism $\Phi : G \to \mathbb{C}$ satisfies $(*)$ remains necessary for weak amenability of $\ell^1(G,\omega)$ for arbitrary group G (see the PhD thesis of A. Pourabbas [9, Proposition 3.2.3] or the PhD thesis of C. R. Borwick [2, Theorem 2.8]). C. R. Borwick also shows that this condition is no longer sufficient for non-Abelian groups G. He does this by constructing a family of polynomial type weights ω_α on $SL_2(\mathbb{R})$ for which $\ell^1(SL_2(\mathbb{R}),\omega_\alpha)$ is not weakly amenable [2, Theorem 2.24] and showing that there are no non-zero homomorphisms $\Phi : SL_2(\mathbb{R}) \to \mathbb{C}$ [2, Theorem 2.16], so that the condition of Proposition 1.1 is satisfied for any weight ω on $SL_2(\mathbb{R})$.

In this paper, we study weak amenability of $\ell^1(G,\omega)$ for two important non-commutative discrete groups: the free group $\mathbb{F}_2$ on two generators and the $(\boldsymbol{ax+b})$-group. We focus on these groups because $\mathbb{F}_2$ is the simplest non-commutative and non-amenable group, and $(\boldsymbol{ax+b})$-group is one of the simplest non-commutative non-compact but yet amenable groups.

In Section 3 we study polynomial weights on $\mathbb{F}_2$ and $(\boldsymbol{ax+b})$. We show that the corresponding discrete Beurling algebras are weakly amenable only when the weight is constant. In particular, $\ell^1(\mathbb{F}_2,\omega_\alpha)$ is not weakly amenable for any weight of the form $\omega_\alpha(x)=(1+|x|)^\alpha$, $\alpha>0$, where $|x|$ denotes the length of the word x in $\mathbb{F}_2$. This contrasts with the following combination of results proved in [1], [6], and [4] (see also [11] and [14]).

Proposition 1.2. *Let G be either $\mathbb{Z}$ or $\mathbb{R}$, α be a non-negative number, and $\omega_\alpha(x)=(1+|x|)^\alpha$ $(x\in G)$. Then $L^1(G,\omega_\alpha)$ is weakly amenable if and only if $0\le\alpha<1/2$.*

In particular, our results give rise to another family of examples showing that the necessary and sufficient condition on ω for weak amenability of $\ell^1(G,\omega)$ given in Proposition 1.1 for Abelian groups G is no longer sufficient for non-commutative groups. Moreover, some of our examples will even deal with the weights that factor through the abelianization map $q:G\to G_{\mathrm{ab}}$, in which case the expression in $(*)$ only depends on G_{ab}, and so the expectations for the criterion from Proposition 1.1 to work are even higher.

Section 4 is devoted to the study of weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ for more general weights. The free group $\mathbb{F}_2$ is of special interest since it is the source of many counterintuitive results. Some study concerning Beurling algebras on $\mathbb{F}_2$ was conducted by H. G. Dales and A. T.-M. Lau [4], but the questions regarding weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ remained open. We characterize weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ for two important classes of weights. For these weights ω we show that $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is equivalent to a multiplicative weight, in which case $\ell^1(\mathbb{F}_2,\omega)\cong\ell^1(\mathbb{F}_2)$. The results obtained prompt us to conjecture that $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is diagonally bounded.

Finally, in Section 5 we study weak amenability of $\ell^1(\mathbb{Z}^2,\omega)$ on the Abelian group $\mathbb{Z}^2$. It was proved in [14] that if ω is a weight on $\mathbb{Z}^2$ and ω_i denotes the restriction of ω to the ith coordinate, $i=1,2$, then weak amenability of both $\ell^1(\mathbb{Z},\omega_1)$ and $\ell^1(\mathbb{Z},\omega_2)$ implies that of $\ell^1(\mathbb{Z}^2,\omega)$. We give an example showing that the converse is not true. We also provide a simple procedure for verification whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable.

Most of the results of this paper were obtained by the author during her PhD program in the Department of Mathematics of the University of Manitoba (see the thesis [12]).

2. Preliminaries. We start with an easy technical observation.

LEMMA 2.1. *Let G be a discrete group, and ω be a weight on G. Suppose a map D from $\{\delta_x\}_{x\in G}$ to $\ell^\infty(G,1/\omega)$ has the following properties:*

$$D(\delta_{xy}) = D(\delta_x)\cdot\delta_y + \delta_x\cdot D(\delta_y), \qquad x,y\in G, \tag{2.1}$$

$$\|D(\delta_x)\|_{\ell^\infty(G,1/\omega)} \le C\omega(x), \qquad x\in G, \tag{2.2}$$

where $C>0$ is a constant. Then D can be extended to a bounded derivation from $\ell^1(G,\omega)$ to $\ell^\infty(G,1/\omega)$.

The next result shows that the necessity part of Proposition 1.1 still holds for a general discrete group.

LEMMA 2.2 (A. Pourabbas [9, Proposition 3.2.3]; see also [2, Theorem 2.8]). *Let G be a discrete group, and ω be a weight on G. If there exists a non-zero group homomorphism $\Phi: G\to\mathbb{R}$ such that*

$$\sup_{x\in G}\frac{|\Phi(x)|}{\omega(x)\omega(x^{-1})}<\infty,$$

then $\ell^1(G,\omega)$ is not weakly amenable.

Proof. It suffices to construct a non-inner bounded derivation $D:\ell^1(G,\omega)\to\ell^\infty(G,1/\omega)$. We first define D on $\{\delta_x\}_{x\in G}$:

$$D(\delta_x)=\Phi(x)\delta_{x^{-1}}, \qquad x\in G.$$

Since Φ is a group homomorphism and

$$\delta_{(xy)^{-1}}(z)=\delta_{y^{-1}x^{-1}}(z)=\delta_{x^{-1}}(yz)=\delta_{y^{-1}}(zx), \qquad x,y,z\in G,$$

we have

$$\begin{aligned} D(\delta_{xy})(z) &= \Phi(xy)\delta_{y^{-1}x^{-1}}(z)=\Phi(x)\delta_{x^{-1}}(yz)+\Phi(y)\delta_{y^{-1}}(zx)\\ &= D(\delta_x)(yz)+D(\delta_y)(zx)=(D(\delta_x)\cdot\delta_y)(z)+(\delta_x\cdot D(\delta_y))(z), \qquad x,y,z\in G,\end{aligned}$$

and so (2.1) holds. If we set

$$C=\sup_{x\in G}\frac{|\Phi(x)|}{\omega(x)\omega(x^{-1})},$$

then for every $x\in G$ we have

$$\|D(\delta_x)\|_{\ell^\infty(G,1/\omega)}=\frac{|\Phi(x)|}{\omega(x^{-1})}=\omega(x)\frac{|\Phi(x)|}{\omega(x)\omega(x^{-1})}\le C\omega(x),$$

and (2.2) also holds. Therefore, due to Lemma 2.1, D can be extended to a bounded derivation from $\ell^1(G,\omega)$ to $\ell^\infty(G,1/\omega)$. We now show that the extended derivation is not inner. Assume, to the contrary, that there exists $\varphi\in\ell^\infty(G,1/\omega)$ such that $D(h)=h\cdot\varphi-\varphi\cdot h$ for all $h\in\ell^1(G,\omega)$. Then

$$D(\delta_x)(x^{-1})=(\delta_x\cdot\varphi)(x^{-1})-(\varphi\cdot\delta_x)(x^{-1})=\varphi(e)-\varphi(e)=0, \qquad x\in G, \tag{2.3}$$

where e denotes the identity of G. On the other hand, according to the definition of D, we have $D(\delta_x)(x^{-1}) = \Phi(x)$. Combined with (2.3), this yields $\Phi \equiv 0$, which contradicts the assumption that Φ is a non-zero group homomorphism. So, D is not inner, and hence $\ell^1(G,\omega)$ is not weakly amenable. ∎

The following is another necessary condition for weak amenability of $\ell^1(G,\omega)$. It was first obtained by C. R. Borwick in his PhD thesis [2]. Since this thesis is not easily accessible, we include a proof here for the sake of completeness.

Lemma 2.3. *Let G be a discrete group, and ω be a weight on G. If there is a function $\psi : G \to \mathbb{R}$, $x_0 \in G$, and a constant $C > 0$ such that ω is bounded away from zero on the conjugacy class $\{yx_0y^{-1}\}_{y\in G}$,*

$$|\psi(xy) - \psi(yx)| \le C\omega(x)\omega(y), \qquad x, y \in G, \tag{2.4}$$

$$\sup_{y\in G} \frac{|\psi(yx_0y^{-1})|}{\omega(yx_0y^{-1})} = \infty, \tag{2.5}$$

then $\ell^1(G,\omega)$ is not weakly amenable.

Proof. As usual, to show that $\ell^1(G,\omega)$ is not weakly amenable we construct a bounded derivation $D : \ell^1(G,\omega) \to \ell^\infty(G, 1/\omega)$ which is not inner. We first define an operator $D : \{\delta_x\}_{x\in G} \to \ell^\infty(G, 1/\omega)$ in the following way:

$$D(\delta_x)(y) = \psi(xy) - \psi(yx) \ (= (\psi \cdot \delta_x)(y) - (\delta_x \cdot \psi)(y)), \qquad x, y \in G.$$

It is easy to see that D really ranges in $\ell^\infty(G, 1/\omega)$ because of (2.4):

$$\begin{aligned} \|D(\delta_x)\|_{\ell^\infty(G,1/\omega)} &= \sup_{y\in G} \frac{|D(\delta_x)(y)|}{\omega(y)} = \sup_{y\in G} \frac{|\psi(xy) - \psi(yx)|}{\omega(y)} \\ &\underset{(2.4)}{\le} \sup_{y\in G} \frac{C\omega(x)\omega(y)}{\omega(y)} = C\omega(x) < \infty. \end{aligned}$$

At the same time, we have just proved that the operator D satisfies condition (2.2) of Lemma 2.1. So, if we show that D also satisfies (2.1), we will be able to apply Lemma 2.1 to extend D by linearity and continuity to a bounded derivation from $\ell^1(G,\omega)$ to $\ell^\infty(G, 1/\omega)$. Indeed, we have

$$\begin{aligned} D(\delta_{xy})(t) &= \psi(xyt) - \psi(txy) = (\psi(xyt) - \psi(ytx)) + (\psi(ytx) - \psi(txy)) \\ &= D(\delta_x)(yt) + D(\delta_y)(tx) = (D(\delta_x) \cdot \delta_y)(t) + (\delta_x \cdot D(\delta_y))(t), \qquad x, y, t \in G, \end{aligned}$$

and so D can be extended in the desired way. To finish the proof, we only need to show that the extended derivation D is not inner. Suppose, to the contrary, that D is inner. Then there exists a function $\varphi \in \ell^\infty(G, 1/\omega)$ such that $D(h) = \varphi \cdot h - h \cdot \varphi$ for all $h \in \ell^1(G,\omega)$. In particular,

$$D(\delta_x)(y) = (\varphi \cdot \delta_x)(y) - (\delta_x \cdot \varphi)(y) = \varphi(xy) - \varphi(yx), \qquad x, y \in G. \tag{2.6}$$

On the other hand, by definition of D we have $D(\delta_x)(y) = \psi(xy) - \psi(yx)$ $(x, y \in G)$. Taking $x = x_0 y^{-1}$, we obtain

$$\psi(x_0) - \psi(yx_0y^{-1}) = D(\delta_{x_0y^{-1}})(y) = \varphi(x_0) - \varphi(yx_0y^{-1}), \quad y \in G,$$

which implies

$$\varphi(yx_0y^{-1}) = \psi(yx_0y^{-1}) + \varphi(x_0) - \psi(x_0), \quad y \in G.$$

Then using (2.5) and the fact that $\inf_{y\in G} \omega(yx_0y^{-1}) > 0$ we have

$$\begin{aligned}\|\varphi\|_{\ell^\infty(G,1/\omega)} &= \sup_{x\in G} \frac{|\varphi(x)|}{\omega(x)} \ge \sup_{y\in G} \frac{|\varphi(yx_0y^{-1})|}{\omega(yx_0y^{-1})} \\ &= \sup_{y\in G} \frac{\big|\psi(yx_0y^{-1}) + \big(\varphi(x_0) - \psi(x_0)\big)\big|}{\omega(yx_0y^{-1})} \\ &\ge \sup_{y\in G} \frac{|\psi(yx_0y^{-1})|}{\omega(yx_0y^{-1})} - \big|\varphi(x_0) - \psi(x_0)\big| \cdot \sup_{y\in G} \frac{1}{\omega(yx_0y^{-1})} = \infty,\end{aligned}$$

contradicting $\varphi \in \ell^\infty(G, 1/\omega)$. This proves that D is not inner, and hence $\ell^1(G, \omega)$ is not weakly amenable. ∎

3. Polynomial weights on $\mathbb{F}_2$ and $(ax + b)$. In this section we consider polynomial weights on two basic non-commutative groups: the free group $\mathbb{F}_2$ and the $(ax + b)$-group. We show that contrary to the expectations based on the theory of weak amenability for Abelian discrete Beurling algebras, the corresponding discrete weighted group algebras are not weakly amenable.

We will make use of the following technical result.

Lemma 3.1. *Let* $0 < \gamma \le 1$. *Then* $\big||x|^\gamma - |y|^\gamma\big| \le |x-y|^\gamma$ *for all* $x, y \in \mathbb{R}$.

Proof. It is enough to show that

$$f(x, y) = x^\gamma + y^\gamma - (x + y)^\gamma \ge 0, \quad x, y \ge 0. \tag{3.1}$$

Fix $y \ge 0$. When $x = 0$, we have $f(0, y) = 0$, and (3.1) holds. We also have

$$\frac{\partial f}{\partial x}(x, y) = \gamma(x^{\gamma-1} - (x + y)^{\gamma-1}) \ge 0,$$

since $0 < \gamma \le 1$ and $y \ge 0$. This immediately implies (3.1) for arbitrary $x \ge 0$. ∎

3.1. Polynomial weights on $\mathbb{F}_2$. First, let us define several notions.

Definition 3.2. Let a and b denote the two generators of the free group $\mathbb{F}_2$. Then every $x \in \mathbb{F}_2$ can be written in a non-cancelable form $x = a^{k_1}b^{l_1} \dots a^{k_n}b^{l_n}$, i.e., $k_i, l_i \in \mathbb{Z}$, and all k_i, l_i are non-zero except possibly k_1 and l_n, $1 \le i \le n$, $n \in \mathbb{N}$. We denote $|x| = \sum_{i=1}^n (|k_i| + |l_i|)$ and call it the *length* of x. The number $\sum_{i=1}^n k_i$ (resp. $\sum_{i=1}^n l_i$) will be called the *total power of* a (resp. the *total power of* b) in x, and we denote it by $A(x)$ (resp. $B(x)$). Note that both A and B are group homomorphisms from $\mathbb{F}_2$ to $\mathbb{Z}$.

PROPOSITION 3.3. *Let* $\alpha > 0$ *and* ω_α *be a function on* $\mathbb{F}_2$ *defined by* $\omega_\alpha(x) = (1 + |x|)^\alpha$, $x \in \mathbb{F}_2$. *Then* ω_α *is a weight on* $\mathbb{F}_2$ (*called a* polynomial weight), *and* $\ell^1(\mathbb{F}_2, \omega_\alpha)$ *is not weakly amenable.*

Proof. Since the length function $|\cdot|$ on $\mathbb{F}_2$ obviously satisfies the triangle inequality $|xy| \le |x| + |y|$ $(x, y \in \mathbb{F}_2)$, it follows that ω_α is a weight on $\mathbb{F}_2$:

$$\begin{aligned}\omega_\alpha(xy) &= (1 + |xy|)^\alpha \le (1 + |x| + |y|)^\alpha \\ &\le (1 + |x|)^\alpha (1 + |y|)^\alpha = \omega_\alpha(x)\omega_\alpha(y), \qquad x, y \in \mathbb{F}_2.\end{aligned}$$

To prove that $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable, we first consider the case when $\alpha \ge 1/2$. Since $|A(t)| \le |t|$ for every $t \in \mathbb{F}_2$, we have

$$\sup_{t \in \mathbb{F}_2} \frac{|A(t)|}{\omega_\alpha(t)\omega_\alpha(t^{-1})} = \sup_{t \in \mathbb{F}_2} \frac{|A(t)|}{(1 + |t|)^{2\alpha}} \le \sup_{t \in \mathbb{F}_2} \frac{|t|}{(1 + |t|)^{2\alpha}} < \infty,$$

and because $A : \mathbb{F}_2 \to \mathbb{Z}$ is a group homomorphism, Lemma 2.2 implies that $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable.

Now let $0 < \alpha < 1/2$. In this case we will use Lemma 2.3. Take an arbitrary $\beta \in (\alpha, 2\alpha)$, and consider the function $\psi : \mathbb{F}_2 \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} |t|^\beta & \text{if } x = tat^{-1},\ t \in \mathbb{F}_2, \text{ and the word } tat^{-1} \text{ is non-cancelable}, \\ 0 & \text{otherwise}. \end{cases}$$

We claim that ψ satisfies the conditions of Lemma 2.3 for $x_0 = a$. The weight ω_α is obviously bounded away from zero on the whole $\mathbb{F}_2$, and in particular on $\{yay^{-1}\}_{y \in \mathbb{F}_2}$. Now we prove that ψ satisfies (2.4) for $C = 1$, i.e.,

$$|\psi(xy) - \psi(yx)| \le \omega_\alpha(x)\omega_\alpha(y), \qquad x, y \in \mathbb{F}_2. \tag{3.2}$$

By the definition of ψ, it vanishes off the conjugacy class $E = \{tat^{-1}\}_{t \in \mathbb{F}_2}$. Since $yx = y(xy)y^{-1}$, the elements xy and yx always belong to the same conjugacy class, and so we only need to prove (3.2) in the case when both xy and yx are in E. Let $xy = uau^{-1}$ and $yx = vav^{-1}$, both representations being non-cancelable. Assume without loss of generality that $|u| \le |v|$. Because

$$vav^{-1} = yx = y(xy)y^{-1} = yuau^{-1}y^{-1},$$

we have $(u^{-1}y^{-1}v)a = a(u^{-1}y^{-1}v)$. So, the elements a and $u^{-1}y^{-1}v$ commute, which can happen in a free group only if both are powers of a third element (see, for example, [8, Proposition 2.17]). Since a is one of the generators of $\mathbb{F}_2$, it is only a power of itself, which implies that $u^{-1}y^{-1}v = a^k$ for some $k \in \mathbb{Z}$. In other words, $yu = va^{-k}$. We consider two cases: $k = 0$ and $k \ne 0$.

If $k=0$, then $y=vu^{-1}$ and $x=(xy)y^{-1}=(uau^{-1})(uv^{-1})=uav^{-1}$. In this case, the inequality (3.2) that we want to prove becomes

$$\big||u|^\beta-|v|^\beta\big|\le(1+|vu^{-1}|)^\alpha(1+|uav^{-1}|)^\alpha.$$

Since $0<\alpha\le 1/2$ and $\alpha<\beta<2\alpha$, we see that $0<\beta\le 1$, and so $\big||u|^\beta-|v|^\beta\big|\le\big||u|-|v|\big|^\beta$ by Lemma 3.1. We also have $|vu^{-1}|\ge\big||u|-|v|\big|$ and $|uav^{-1}|\ge\big||u|-|v|\big|-1$. Therefore,

$$\begin{aligned}(1+|vu^{-1}|)^\alpha(1+|uav^{-1}|)^\alpha &\ge \big(1+\big||u|-|v|\big|\big)^\alpha\big||u|-|v|\big|^\alpha \ge \big||u|-|v|\big|^{2\alpha}\\ &\ge \big||u|-|v|\big|^\beta \ge \big||u|^\beta-|v|^\beta\big|,\end{aligned}$$

because $\beta\le 2\alpha$ and $\big||u|-|v|\big|\in\mathbb{N}\cup\{0\}$. Hence, (3.2) is verified for the case $k=0$.

Now let $k\ne 0$. Then $yu=va^{-k}$. Recall that both expressions uau^{-1} and vav^{-1} were non-cancelable. This means that both u and v end with a non-zero power of the second generator b of $\mathbb{F}_2$. Hence, the equality $yu=va^{-k}$ is only possible for $k\ne 0$ if $y=tu^{-1}$, and the expression tu^{-1} is non-cancelable. In this case, $t=va^{-k}$ and $|t|=|v|+|k|$, implying that $|v|=|t|-|k|$. We also have $x=(xy)y^{-1}=(uau^{-1})(ut^{-1})=uat^{-1}$. Thus, the inequality (3.2) that we want to prove becomes

$$\big||u|^\beta-(|t|-|k|)^\beta\big|\le(1+|tu^{-1}|)^\alpha(1+|uat^{-1}|)^\alpha.$$

Recall that we assumed from the very beginning that $|u|\le|v|=|t|-|k|$, and so, using the same arguments as in the previous case, we obtain

$$\begin{aligned}\big||u|^\beta-(|t|-|k|)^\beta\big| &= (|t|-|k|)^\beta-|u|^\beta \le \big||t|-|k|-|u|\big|^\beta \le \big||t|-|u|\big|^\beta\\ &\le (1+|tu^{-1}|)^\alpha(1+|uat^{-1}|)^\alpha,\end{aligned}$$

and (3.2) is verified for $k\ne 0$ as well.

Finally, we check that ψ satisfies (2.5) for $x_0=a$:

$$\sup_{y\in\mathbb{F}_2}\frac{|\psi(yay^{-1})|}{\omega(yay^{-1})}\ge\sup_{n\in\mathbb{N}}\frac{|\psi(b^nab^{-n})|}{\omega(b^nab^{-n})}=\sup_{n\in\mathbb{N}}\frac{n^\beta}{(2n+2)^\alpha}=\infty,$$

since $\beta>\alpha$. So the function ψ satisfies all the conditions of Lemma 2.3, and hence $\ell^1(\mathbb{F}_2,\omega_\alpha)$ is not weakly amenable. ∎

We will see later that Proposition 3.3 can also be obtained as a corollary of a more general Theorem 4.2, but we believe that the direct proof given above is of independent interest.

As was mentioned in the introduction, C. R. Borwick showed in [2, Chapter 2] that the necessary and sufficient condition from Proposition 1.1 for weak amenability of the Abelian discrete Beurling algebra $\ell^1(G,\omega)$ is no longer sufficient if the group G is not Abelian. Proposition 3.3 gives rise to another family of examples illustrating this point.

EXAMPLE 3.4. If $0 < \alpha < 1/2$, then the weight ω_α satisfies the condition from Proposition 1.1, but, according to Proposition 3.3, $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is not weakly amenable. Indeed, if $\Phi : \mathbb{F}_2 \to \mathbb{C}$ is a non-trivial group homomorphism, then $\Phi(x) = c_1 A(x) + c_2 B(x)$ ($x \in \mathbb{F}_2$), where $c_1, c_2 \in \mathbb{C}$ and $|c_1|^2 + |c_2|^2 \neq 0$. Assuming without loss of generality that $c_1 \neq 0$, we have

$$\sup_{x\in\mathbb{F}_2} \frac{|\Phi(x)|}{\omega_\alpha(x)\omega_\alpha(x^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{|\Phi(a^n)|}{\omega_\alpha(a^n)\omega_\alpha(a^{-n})} = \sup_{n\in\mathbb{N}} \frac{|c_1|n}{(1+n)^{2\alpha}} = \infty,$$

since $\alpha < 1/2$.

Note that the group $\mathbb{F}_2$ is not amenable, which still leaves a possibility for Proposition 1.1 to hold at least for amenable groups G. However, in the next subsection we give an example of a weight ω on an amenable group G that satisfies the conditions of Proposition 1.1 but still makes $\ell^1(G,\omega)$ not weakly amenable.

3.2. Polynomial weights on the $(\boldsymbol{ax+b})$-group. We consider the non-commutative amenable group $(\boldsymbol{ax+b})$ of all orientation-preserving affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$, where the map $x \mapsto ax + b$ is identified with the pair (a, b). Multiplication in this group is given by the composition of the corresponding transformations of $\mathbb{R}$, which can be expressed as

$$(a,b)(c,d) = (ac, ad+b), \qquad a,c > 0,\ b,d \in \mathbb{R}.$$

The identity of $(\boldsymbol{ax+b})$ is the pair $(1,0)$ corresponding to the identity map on $\mathbb{R}$. Therefore,

$$(a,b)^{-1} = \left(\frac{1}{a}, \frac{-b}{a}\right), \qquad a > 0,\ b \in \mathbb{R}.$$

Throughout the remainder of this subsection for the sake of notational convenience we denote the group $(\boldsymbol{ax+b})$ by G.

THEOREM 3.5. *Let α be a positive number, and ω_α be the function on G defined by $\omega_\alpha(a,b) = (1 + |\log a|)^\alpha$, $(a,b) \in G$. Then ω_α is a weight on G, and $\ell^1(G,\omega_\alpha)$ is not weakly amenable.*

Proof. To verify the weight inequality for ω_α, let $(a,b),(c,d) \in G$. Then

$$\begin{aligned}\omega_\alpha((a,b)(c,d)) = \omega_\alpha(ac, ad+b) = (1 + |\log(ac)|)^\alpha &\leq (1 + |\log a| + |\log c|)^\alpha \\ &\leq (1 + |\log a|)^\alpha (1 + |\log c|)^\alpha = \omega_\alpha(a,b)\omega_\alpha(c,d).\end{aligned}$$

The proof of the fact that $\ell^1(G,\omega_\alpha)$ is not weakly amenable will be very similar to the corresponding part of the proof of Proposition 3.3. Again, we consider two possibilities: $\alpha \geq 1/2$ and $0 < \alpha < 1/2$. Suppose first that

$\alpha \geq 1/2$. Then

$$\sup_{(a,b)\in G} \frac{|\log a|}{\omega_\alpha(a,b)\omega_\alpha((a,b)^{-1})} = \sup_{a>0} \frac{|\log a|}{(1+|\log a|)^\alpha(1+|\log(1/a)|)^\alpha} = \sup_{a>0} \frac{|\log a|}{(1+|\log a|)^{2\alpha}} < \infty,$$

and since $(a,b) \mapsto \log a$ is a group homomorphism from G to $\mathbb{C}$, we conclude that $\ell^1(G,\omega_\alpha)$ is not weakly amenable by Lemma 2.2.

Now suppose that $0 < \alpha < 1/2$. In this case we use Lemma 2.3. We define the function $\psi : G \to \mathbb{R}$ as follows:

$$\psi(a,b) = \begin{cases} |\log b|^{2\alpha} & \text{if } a = 1,\ b > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that ψ satisfies all conditions of Lemma 2.3 for $x_0 = (1,1)$. Again, as in Proposition 3.3, ω_α is bounded away from zero on the whole G, and in particular on $\{yx_0y^{-1}\}_{y\in G}$. Next we show that ψ satisfies (2.4) for $C = 1$, i.e.,

$$|\psi(uv) - \psi(vu)| \leq \omega_\alpha(u)\omega_\alpha(v), \qquad v,u \in G. \tag{3.3}$$

Let $u = (a,b)$ and $v = (c,d)$, where $a,c > 0$, $b,d \in \mathbb{R}$. Then $uv = (ac, ad+b)$ and $vu = (ac, bc+d)$. If $ac \neq 1$, then $\psi(uv) = \psi(vu) = 0$, and (3.3) holds automatically. Suppose now that $ac = 1$. Then

$$c = 1/a, \qquad bc + d = b/a + d = (ad+b)/a.$$

Since $a > 0$, either both $ad+b$ and $bc+d$ are negative, in which case (3.3) again holds automatically, or both $ad+b$ and $bc+d$ are positive. In the latter case, applying Lemma 3.1 to $\gamma = 2\alpha \in (0,1)$, we obtain

$$\begin{aligned} |\psi(uv) - \psi(vu)| &= \left|\psi(1, ad+b) - \psi\left(1, \frac{ad+b}{a}\right)\right| \\ &= \left| |\log(ad+b)|^{2\alpha} - \left|\log\frac{ad+b}{a}\right|^{2\alpha} \right| \\ &= \big| |\log(ad+b)|^{2\alpha} - |\log(ad+b) - \log a|^{2\alpha} \big| \leq (1+|\log a|)^{2\alpha} \\ &= \omega_\alpha(a,b)\omega_\alpha(1/a,d) = \omega_\alpha(a,b)\omega_\alpha(c,d) = \omega_\alpha(u)\omega_\alpha(v), \end{aligned}$$

and (3.3) is verified.

Finally, we check that ψ satisfies (2.5) for $x_0 = (1,1)$. For this we note that $\{(a,b)(1,1)(a,b)^{-1}\}_{(a,b)\in G} = \{(1,a)\}_{a>0}$, implying

$$\sup_{y\in G} \frac{|\psi(yx_0y^{-1})|}{\omega_\alpha(yx_0y^{-1})} = \sup_{a>0} \frac{|\psi(1,a)|}{\omega_\alpha(1,a)} = \sup_{a>0} |\log a|^{2\alpha} = \infty,$$

and (2.5) is also verified. Hence, ψ satisfies all the conditions of Lemma 2.3, which means that $\ell^1(G,\omega_\alpha)$ is not weakly amenable. ∎

EXAMPLE 3.6. It is natural to call the weight ω_α defined in Theorem 3.5 a *polynomial weight* on $(\boldsymbol{ax+b})$. Note that, unlike $\mathbb{F}_2$, the group $(\boldsymbol{ax+b})$ is amenable. Theorem 3.5 demonstrates that even a "nice" weight on an amenable group may still make the corresponding weighted group algebra not weakly amenable. It also shows that Proposition 1.1 does not hold for general amenable non-commutative groups. Indeed, since it is easy to see that a group homomorphism $\Phi : (\boldsymbol{ax+b}) \to \mathbb{C}$ must have the form $\Phi(a,b) = c\log a$, $c \in \mathbb{C}$, the algebra $\ell^1((\boldsymbol{ax+b}),\omega_{1/3})$ is not weakly amenable, although $\omega_{1/3}$ satisfies the conditions of Proposition 1.1.

REMARK 3.7. In fact, the proof of Theorem 3.5 can be adopted to produce an example of a finitely generated (and hence separable) non-commutative amenable group $\widetilde{G}$ such that Proposition 1.1 does not hold for $\widetilde{G}$. Indeed, all our arguments will work for the subgroup

$$\widetilde{G} = \{(2^n, b) : n \in \mathbb{Z},\ b \in \mathbb{Z}[1/2]\} = \langle (2,0),(1,1)\rangle$$

of the $(\boldsymbol{ax+b})$-group and the weight $\omega_{1/3}$ restricted to $\widetilde{G}$. This shows that the pathology of the example is really the result of non-commutativity rather than of non-separability of the group. The author would like to thank N. Spronk for this observation.

4. Weak amenability of $\ell^1(\mathbb{F}_2,\omega)$ for more general weights. The only known sufficient condition for weak amenability of $L^1(G,\omega)$, and hence for weak amenability of $\ell^1(G,\omega)$, when G is not necessarily Abelian is the following direct extension of weak amenability of non-weighted group algebras. Recall that a weight ω on a group G is called *diagonally bounded* if the function $x \mapsto \omega(x)\omega(x^{-1})$ is bounded on G.

PROPOSITION 4.1 ([10, Theorem 3.14]). *Let G be a locally compact group and ω be a diagonally bounded weight on G. Then the Beurling algebra $L^1(G,\omega)$ is weakly amenable.*

In fact, one only needs to slightly modify the proof by M. Despić and F. Ghahramani [5] for weak amenability of the group algebra $L^1(G)$ to obtain Proposition 4.1.

Using Proposition 4.1, Lemma 2.3, and Lemma 2.2 we can show that for two natural classes of weights ω the algebra $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is diagonally bounded.

Again, the generators of $\mathbb{F}_2$ are denoted by a and b. We continue to use the notation from Definition 3.2: $|x|$ stands for the length of x, and $A(x)$ and $B(x)$ for the total powers of a and b in x respectively, $x \in \mathbb{F}_2$.

THEOREM 4.2. *Let ω be a weight on $\mathbb{F}_2$ such that there exists an increasing function W from $\mathbb{N} \cup \{0\}$ to $[1,\infty)$ and constants $c_1, c_2 > 0$ such* *that*

$$c_1 W(|x|) \le \omega(x) \le c_2 W(|x|), \qquad x \in \mathbb{F}_2,$$

where $|x|$ *is the length of* $x \in \mathbb{F}_2$. *Then* $\ell^1(\mathbb{F}_2, \omega)$ *is weakly amenable if and only if* ω *is bounded.*

Before proving Theorem 4.2, we establish the following auxiliary result.

LEMMA 4.3. *Let* $W : \mathbb{N} \cup \{0\} \to [1, \infty)$ *be an increasing function such that*

$$\sup_{n\in\mathbb{N}} W(n) = \sup_{n\in\mathbb{N}} \frac{n}{W(n)} = \infty.$$

Then there exists a function $f : \mathbb{N} \cup \{0\} \to [1, \infty)$ *with the following properties:*

$$f \text{ is increasing;} \tag{4.1}$$

$$f(m+n) - f(m-n) \le W(m)W(n), \qquad m, n \in \mathbb{N},\ m \ge n; \tag{4.2}$$

$$\sup_{n\in\mathbb{N}} \frac{f(n)}{W(n)} = \infty. \tag{4.3}$$

Proof. The idea of the proof is to define f inductively to be as large as possible subject to (4.2), and then show that such a function will also satisfy (4.1) and (4.3).

For technical convenience, we extend W to an increasing function from $[0, \infty)$ to $[1, \infty)$ using piecewise-linear interpolation. Then we inductively define $f : \mathbb{N} \cup \{0\} \to [1, \infty)$ and $F : \{(j, k) \in (\mathbb{N} \cup \{0\})^2 : 0 \le j \le k - 1\} \to [1, \infty)$ by the following formulas:

$$\begin{aligned} f(0) &= 1, \\ F(j,k) &= W\left(\frac{k+j}{2}\right) W\left(\frac{k-j}{2}\right) + f(j), \qquad 0 \le j \le k-1, \\ f(k) &= \min_{0\le j\le k-1} F(j,k). \end{aligned}$$

We now verify (4.2). Let $m \ge n$ be natural numbers. Then $0 \le m - n \le m + n - 1$, and so

$$f(m+n) = \min_{0\le j\le m+n-1} F(j, m+n) \le F(m-n, m+n) = W(m)W(n) + f(m-n),$$

which immediately implies (4.2).

Next we show that f is increasing. Obviously, it is enough to show that $f(k+1) \ge f(k)$ for every $k \in \mathbb{N} \cup \{0\}$. For $k = 0$ we have

$$f(1) = F(0,1) = \left(W\left(\frac{1}{2}\right)\right)^2 + f(0) \ge f(0).$$

Now let $k \in \mathbb{N}$. Then

$$f(k+1) = \min_{0\le j\le k} F(j, k+1), \qquad f(k) = \min_{0\le j\le k-1} F(j,k).$$

Since W is a positive increasing function, for each $j \in [0, k-1]$ we have

$$\begin{aligned} F(j,k+1) &= W\left(\frac{(k+1)+j}{2}\right)W\left(\frac{(k+1)-j}{2}\right) + f(j) \\ &\geq W\left(\frac{k+j}{2}\right)W\left(\frac{k-j}{2}\right) + f(j) = F(j,k). \end{aligned}$$

Hence, in order to prove that $f(k+1) \geq f(k)$, it is enough to show that $F(k,k+1) \geq f(k)$. But

$$F(k,k+1) = W\left(\frac{2k+1}{2}\right)W\left(\frac{1}{2}\right) + f(k) \geq f(k),$$

since W is positive, and so f is indeed an increasing function.

Finally, we prove (4.3). Suppose, to the contrary, that there exists a positive integer N such that

$$\sup_{n\in\mathbb{N}} \frac{f(n)}{W(n)} \leq N. \tag{4.4}$$

Our goal is to show that in this case for sufficiently large m we will have $f(3m) \geq 3NW(m)$. Then, using (4.4) and the fact that W is increasing, we will conclude that $\sup_{n\in\mathbb{N}} n/W(n) < \infty$, contradicting the growth constraint on W.

By definition,

$$f(k) = \min_{0\leq j\leq k-1} W\left(\frac{k+j}{2}\right)W\left(\frac{k-j}{2}\right) + f(j), \qquad k \in \mathbb{N}.$$

Hence, for each $k \in \mathbb{N}$ there exists $J(k) \in [0, k-1]$ such that

$$f(k) = W\left(\frac{k+J(k)}{2}\right)W\left(\frac{k-J(k)}{2}\right) + f(J(k)). \tag{4.5}$$

Since W is unbounded, there exists $n_0 \in \mathbb{N}$ such that $W(n_0) \geq 3N$. We are going to prove that

$$f(3m) \geq 3NW(m) \qquad \text{for } m \geq 6n_0N. \tag{4.6}$$

Fix some $m \geq 6n_0N$. Applying (4.5) to $k = 3m$ and using the monotonicity of W, we obtain

$$\begin{aligned} f(3m) &= W\left(\frac{3m+J(3m)}{2}\right)W\left(\frac{3m-J(3m)}{2}\right) + f(J(3m)) \\ &\geq W(m)W\left(\frac{3m-J(3m)}{2}\right) + f(J(3m)). \end{aligned} \tag{4.7}$$

If $J(3m) < 3m - 2n_0$, then $(3m - J(3m))/2 > n_0$, and using the facts that f is positive, W is increasing, and $W(n_0) \geq 3N$, we deduce from (4.7) that

$$f(3m) \geq W(m)W(n_0) \geq 3NW(m).$$

Thus, (4.6) is verified in this case. Now assume that $J(3m) \geq 3m - 2n_0$ and set $j_1 = J(3m)$. Since $W \geq 1$, it follows from (4.7) that

$$f(3m) \geq W(m) + f(j_1). \tag{4.8}$$

By our assumption $m \geq 6n_0N$, and so

$$2n_0 \leq \frac{m}{3N}. \tag{4.9}$$

Then since $j_1 = J(3m) \geq 3m - 2n_0$, we have

$$\frac{j_1}{2} \geq \frac{3m - \frac{m}{3N}}{2} \geq m.$$

Applying (4.5) to $k = j_1$ and using the monotonicity of W we then obtain

$$\begin{aligned} f(j_1) &= W\left(\frac{j_1 + J(j_1)}{2}\right) W\left(\frac{j_1 - J(j_1)}{2}\right) + f(J(j_1)) \\ &\geq W(m) W\left(\frac{j_1 - J(j_1)}{2}\right) + f(J(j_1)). \end{aligned}$$

Combined with (4.8) this yields

$$f(3m) \geq W(m) + W(m) W\left(\frac{j_1 - J(j_1)}{2}\right) + f(J(j_1)). \tag{4.10}$$

If $J(j_1) < j_1 - 2n_0$, it follows that

$$f(3m) \geq W(m) W(n_0) \geq N W(m),$$

and (4.6) is verified. If $J(j_1) \geq j_1 - 2n_0$, we set $j_2 = J(j_1)$ and repeat for j_2 the steps we performed for j_1. Namely, from (4.10) we obtain

$$f(3m) \geq 2W(m) + f(j_2);$$

then applying (4.9) we get

$$\frac{j_2}{2} \geq \frac{j_1 - 2n_0}{2} \geq \frac{3m - 2 \cdot 2n_0}{2} \geq \frac{3m - \frac{2m}{3N}}{2} \geq m,$$

which combined with (4.5) for $k = j_2$ implies

$$f(3m) \geq 2W(m) + W(m) W\left(\frac{j_2 - J(j_2)}{2}\right) + f(J(j_2)). \tag{4.11}$$

And again, if $J(j_2) < j_2 - 2n_0$ then (4.6) is verified, while if $J(j_2) > j_2 - 2n_0$ we set $j_3 = J(j_2)$ and repeat the whole procedure. Continuing, we either end up with some j_q ($q < 3N$) such that $J(j_q) < j_q - 2n_0$ and then (4.6) will be verified since

$$\frac{j_q}{2} \geq \frac{3m - q \cdot 2n_0}{2} \geq \frac{3m - \frac{qm}{3N}}{2} \geq m,$$

or we get to j_{3N} for which, analogously to (4.11), we will have

$$f(3m) \geq 3NW(m) + W(m)W\left(\frac{j_{3N} - J(j_{3N})}{2}\right) + f(J(j_{3N})),$$

and (4.6) will follow just from the positivity of W and f. So, (4.6) is proved.

We will now use (4.6) to obtain a contradiction with $\sup_{n\in\mathbb{N}} n/W(n) = \infty$. It follows from (4.4) and (4.6) that

$$W(3m) \geq \frac{f(3m)}{N} \geq 3W(m) \quad \text{for } m \geq 6n_0N.$$

If we set $m_0 = 6n_0N$, then it is easy to show by induction that $W(3^p m_0) \geq 3^p W(m_0)$ for all $p \in \mathbb{N} \cup \{0\}$. For each integer $n \geq m_0$ there exists a unique $p \in \mathbb{N} \cup \{0\}$ such that $3^p m_0 \leq n < 3^{p+1} m_0$. Using the monotonicity of W we obtain

$$\frac{n}{W(n)} \leq \frac{3^{p+1} m_0}{W(3^p m_0)} \leq \frac{3^{p+1} m_0}{3^p W(m_0)} = \frac{3m_0}{W(m_0)},$$

which immediately implies that $\sup_{n\in\mathbb{N}} n/W(n) < \infty$ and gives the desired contradiction. Thus, $\sup_{k\in\mathbb{N}} f(k)/W(k) = \infty$, and the proof is complete. ■

Proof of Theorem 4.2. If ω is a bounded weight, then $\ell^1(\mathbb{F}_2,\omega)$ is isomorphic to $\ell^1(\mathbb{F}_2)$, and $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable due to weak amenability of $\ell^1(\mathbb{F}_2)$. So, the non-trivial part is to prove that if ω is not bounded and satisfies the conditions of Theorem 4.2, then $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable.

As noted above, the function A of total power of the generator a is a continuous homomorphism from $\mathbb{F}_2$ to $\mathbb{Z} \subset \mathbb{C}$. By Lemma 2.2, $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable if

$$\sup_{x\in\mathbb{F}_2} \frac{|A(x)|}{\omega(x)\omega(x^{-1})} < \infty.$$

Assume now that

$$\sup_{x\in\mathbb{F}_2} \frac{|A(x)|}{\omega(x)\omega(x^{-1})} = \infty.$$

Since obviously $|A(x)| \leq |x|$, $|x^{-1}| = |x|$, and $\omega(x) \geq c_1 W(|x|)$, it follows that

$$\sup_{x\in\mathbb{F}_2} \frac{|x|}{(W(|x|))^2} = \infty,$$

and hence

$$\sup_{n\in\mathbb{N}} \frac{n}{W(n)} \geq \sup_{n\in\mathbb{N}} \frac{\sqrt{n}}{W(n)} = \infty.$$

Therefore, we can apply Lemma 4.3 to the function W to get an increasing function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that

$$f(m+n) - f(m-n) \leq W(m)W(n), \qquad m, n \in \mathbb{N},\ m \geq n, \tag{4.12}$$

$$\sup_{n\in\mathbb{N}} \frac{f(n)}{W(n)} = \infty. \tag{4.13}$$

We will show that $\psi(x) = f(|x|)$ satisfies the conditions of Lemma 2.3 either for $x_0 = a$, or for $x_0 = a^2$, implying that $\ell^1(\mathbb{F}_2, \omega)$ is not weakly amenable. Note that because $\omega(x) \geq c_1 W(|x|)$ and $W : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ is an increasing function, the weight ω is bounded away from zero on the whole group $\mathbb{F}_2$, and in particular on any conjugacy class $\{yx_0y^{-1}\}_{y\in\mathbb{F}_2}$.

We now aim to find a constant $C > 0$ such that

$$|\psi(xy) - \psi(yx)| \leq C\omega(x)\omega(y), \qquad x, y \in \mathbb{F}_2. \tag{4.14}$$

Let $x, y \in \mathbb{F}_2$ be given. According to the definition of ψ, we have

$$|\psi(xy) - \psi(yx)| = \big|f(|xy|) - f(|yx|)\big|.$$

Let $|x| = m$, $|y| = n$, and assume without loss of generality that $m \geq n$. By the triangle inequality,

$$m - n = \big||x| - |y|\big| \leq |xy|, \qquad |yx| \leq |x| + |y| = m + n.$$

Since f is an increasing function, it follows that

$$\big|f(|xy|) - f(|yx|)\big| \leq f(m+n) - f(m-n).$$

Together with (4.12) and the inequality $\omega(t) \geq c_1 W(|t|)$ $(t \in \mathbb{F}_2)$, this implies the desired inequality (4.14) with $C = 1/c_1^2$:

$$\begin{aligned}|\psi(xy) - \psi(yx)| = \big|f(|xy|) - f(|yx|)\big| &\leq f(m+n) - f(m-n)\\ &\leq W(m)W(n) = W(|x|)W(|y|) \leq \frac{1}{c_1^2}\omega(x)\omega(y).\end{aligned}$$

Finally, we check condition (2.5) of Lemma 2.3 for the function ψ. We take x_0 to be either a or a^2, and consider the conjugacy classes $\{xax^{-1}\}_{x\in\mathbb{F}_2}$ and $\{xa^2x^{-1}\}_{x\in\mathbb{F}_2}$:

$$\sup_{y\in G} \frac{\psi(yay^{-1})}{\omega(yay^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{\psi(b^n a b^{-n})}{\omega(b^n a b^{-n})} \geq \sup_{n\in\mathbb{N}} \frac{f(2n+1)}{c_2 W(2n+1)} = \frac{1}{c_2} \sup_{n\in\mathbb{N}} \frac{f(2n+1)}{W(2n+1)},$$

$$\sup_{y\in G} \frac{\psi(ya^2y^{-1})}{\omega(ya^2y^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{\psi(b^n a^2 b^{-n})}{\omega(b^n a^2 b^{-n})} \geq \frac{1}{c_2} \sup_{n\in\mathbb{N}} \frac{f(2n+2)}{W(2n+2)}.$$

Therefore, it is enough to show that either

$$\sup_{n\in\mathbb{N}} \frac{f(2n+1)}{W(2n+1)} = \infty \quad \text{or} \quad \sup_{n\in\mathbb{N}} \frac{f(2n+2)}{W(2n+2)} = \infty.$$

But this is a direct consequence of (4.13), and the proof is complete. ∎

One can now obtain Proposition 3.3 as an easy corollary of Theorem 4.2.

REMARK 4.4. In fact, Theorem 4.2 can be extended to a more general class of groups including all finitely generated free groups (see [12, Proposition 4.11]).

Since weak amenability of $\ell^1(G,\omega)$ is completely characterized in the case when the group G is Abelian, it is natural to look for some commutativity when studying weak amenability of $\ell^1(\mathbb{F}_2,\omega)$. This motivates the study of a special class of weights ω on $\mathbb{F}_2$ that depend only on the abelianization of $\mathbb{F}_2$. Because the abelianization of the free group $\mathbb{F}_2$ is a free Abelian group on two generators, we consider the weights ω of the form $\omega(x) = W(A(x), B(x))$ $(x \in \mathbb{F}_2)$. We characterize the weights of this type that make $\ell^1(\mathbb{F}_2,\omega)$ weakly amenable.

THEOREM 4.5. *Let ω be a weight on $\mathbb{F}_2$ of the form $\omega(x) = W(A(x), B(x))$, $x \in \mathbb{F}_2$, for some function $W : \mathbb{Z}^2 \to \mathbb{R}^+$. Then the Beurling algebra $\ell^1(\mathbb{F}_2,\omega)$ is weakly amenable if and only if ω is diagonally bounded.*

Proof. The sufficiency part is a direct consequence of Proposition 4.1. So, we only need to show that if ω is not diagonally bounded, then $\ell^1(\mathbb{F}_2,\omega)$ is not weakly amenable. Let $x = aba^{-1}b^{-1} \in \mathbb{F}_2$. We show that the function

$$\psi(t) = \begin{cases} \log(\omega(y)\omega(y^{-1})) & \text{if } t = yxy^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies the conditions of Lemma 2.3 for the conjugacy class $\{yxy^{-1}\}_{y\in\mathbb{F}_2}$. First, we note that since $\omega(x) = W(A(x), B(x))$ and both A and B are group homomorphisms, it follows that ω is constant, and hence bounded away from zero, on each conjugacy class, in particular on $\{yxy^{-1}\}_{y\in\mathbb{F}_2}$. Next we check that ψ is well-defined, i.e., if $t \in \mathbb{F}_2$ has two different representations $t = y_1xy_1^{-1} = y_2xy_2^{-1}$, then $\psi(y_1xy_1^{-1}) = \psi(y_2xy_2^{-1})$. To this end, it is enough to show that $A(y_1) = A(y_2)$ and $B(y_1) = B(y_2)$. We only show that $A(y_1) = A(y_2)$, since the proof of the fact that $B(y_1) = B(y_2)$ is completely analogous. Note that

$$(y_1xy_2^{-1})(y_2y_1^{-1}) = y_1xy_1^{-1} = y_2xy_2^{-1} = (y_2y_1^{-1})(y_1xy_2^{-1}), \tag{4.15}$$

which means that the elements $y_1xy_2^{-1}$ and $y_2y_1^{-1}$ commute. In a free group two elements commute if and only if both of them are powers of a third element (see, for example, [8, Proposition 2.17]). So, since $\mathbb{F}_2$ is a free group, (4.15) implies the existence of $u \in \mathbb{F}_2$ and integers k, l such that $y_1xy_2^{-1} = u^k$ and $y_2y_1^{-1} = u^l$. Because A is a group homomorphism, we have

$$A(y_2) - A(y_1) = A(y_2y_1^{-1}) = lA(u).$$

Hence, to prove that $A(y_1) = A(y_2)$, it suffices to show that $A(u) = 0$. Since

$x = aba^{-1}b^{-1}$, we have $A(x) = 0$, and so

$$0 = A(x) = A(y_1 x y_1^{-1}) = A((y_1 x y_2^{-1})(y_2 y_1^{-1})) = A(u^{k+l}) = (k+l)A(u).$$

When $k + l \neq 0$, it immediately follows that $A(u) = 0$, and our claim is proved. If $k + l = 0$, then $y_1 x y_1^{-1} = u^{k+l} = e$, which implies that $x = e$, contradicting the choice of x. This proves that the function ψ is well-defined.

Our next goal is to show that ψ satisfies the conditions of Lemma 2.3. First, we prove that there exists a constant $C > 0$ such that

$$|\psi(uv) - \psi(vu)| \leq C\omega(u)\omega(v), \qquad u, v \in \mathbb{F}_2.$$

Since ψ is non-zero only on the conjugacy class $\{yxy^{-1}\}_{y\in\mathbb{F}_2}$, and the elements uv and vu always belong to the same conjugacy class ($vu = v(uv)v^{-1}$), we only need to consider the case when $uv, vu \in \{yxy^{-1}\}_{y\in\mathbb{F}_2}$. Let $uv = yxy^{-1}$. Then $vu = (vy)x(vy)^{-1}$, and we have

$$\begin{aligned} |\psi(uv) - \psi(vu)| &= |\psi(yxy^{-1}) - \psi((vy)x(vy)^{-1})| \\ &= \big|\log(\omega(y)\omega(y^{-1})) - \log\big(\omega(vy)\omega((vy)^{-1})\big)\big| \\ &= \left|\log \frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})}\right|. \end{aligned}$$

Using the weight inequality for ω, we obtain

$$\omega(y) \leq \omega(v^{-1})\omega(vy), \qquad \omega(y^{-1}) \leq \omega(y^{-1}v^{-1})\omega(v),$$

which implies

$$\frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})} \leq \omega(v^{-1})\omega(v). \tag{4.16}$$

We also have

$$\omega(vy) \leq \omega(v)\omega(y), \qquad \omega(y^{-1}v^{-1}) \leq \omega(y^{-1})\omega(v^{-1}),$$

yielding

$$\frac{\omega(vy)\omega(y^{-1}v^{-1})}{\omega(y)\omega(y^{-1})} \leq \omega(v)\omega(v^{-1}). \tag{4.17}$$

From the inequalities (4.16) and (4.17) it follows that

$$\left|\log \frac{\omega(y)\omega(y^{-1})}{\omega(vy)\omega(y^{-1}v^{-1})}\right| \leq \log(\omega(v)\omega(v^{-1})).$$

Since $\omega(v)\omega(v^{-1}) \geq \omega(e) = \text{const} > 0$, by elementary calculus there exists a constant $C > 0$ such that

$$\log(\omega(v)\omega(v^{-1})) \leq C\omega(v)\omega(v^{-1}). \tag{4.18}$$

Combining all of the above, we get

$$|\psi(uv) - \psi(vu)| \leq C\omega(v)\omega(v^{-1}).$$

Recalling that $uv = yxy^{-1}$, we obtain $v^{-1} = yx^{-1}y^{-1}u$, and so $A(v^{-1}) = A(u)$ and $B(v^{-1}) = B(u)$, since $A(x) = B(x) = 0$ and both A and B are group homomorphisms. Hence,

$$\omega(v^{-1}) = W(A(v^{-1}), B(v^{-1})) = W(A(u), B(u)) = \omega(u),$$

which implies the desired inequality

$$|\psi(uv) - \psi(vu)| \le C\omega(u)\omega(v).$$

Finally, we show that $\sup_{y\in \mathbb{F}_2} \psi(yxy^{-1})/\omega(yxy^{-1}) = \infty$. Since ω factors through the abelianization map from $\mathbb{F}_2$ to $\mathbb{Z}^2$, it is constant on conjugacy classes. Therefore,

$$\begin{aligned}\sup_{y\in \mathbb{F}_2} \frac{\psi(yxy^{-1})}{\omega(yxy^{-1})} &= \sup_{y\in \mathbb{F}_2} \frac{\log(\omega(y)\omega(y^{-1}))}{\omega(x)}\\ &= \frac{1}{\omega(x)} \log\Big(\sup_{y\in\mathbb{F}_2} \omega(y)\omega(y^{-1})\Big) = \infty,\end{aligned}$$

since ω is not diagonally bounded. Applying Lemma 2.3, we conclude that $\ell^1(\mathbb{F}_2, \omega)$ is not weakly amenable, and the proposition is proved. ■

Remark 4.6. The diagonal boundedness of ω is obviously equivalent to the diagonal boundedness of W, which is a weight on an amenable group $\mathbb{Z}^2$. As follows from [13, Lemma 1], W is diagonally bounded if and only if it is equivalent to a multiplicative weight. Hence, under conditions of Theorem 4.5, $\ell^1(\mathbb{F}_2, \omega)$ is weakly amenable if and only if $\ell^1(\mathbb{F}_2, \omega) \cong \ell^1(\mathbb{F}_2)$.

Remark 4.7. Theorems 3.5 and 4.5 give rise to examples of weights ω that factor through the abelianization map $q : G \to G_{\mathrm{ab}}$, but for which the criterion from Proposition 1.1 still fails.

The results of Proposition 4.1, Theorem 4.2, and Theorem 4.5 naturally lead us to the following conjecture.

Conjecture 4.8. *Let ω be a weight on $\mathbb{F}_2$. Then $\ell^1(\mathbb{F}_2, \omega)$ is weakly amenable if and only if ω is diagonally bounded.*

Remark 4.9. Based on ideas of B. E. Johnson and M. C. White, it was shown by several authors that a diagonally bounded weight on $\mathbb{F}_2$ does not have to be equivalent to a multiplicative weight (see, for instance, [4, Example 10.1] or [10, Example 3.15]). At the same time we note that every diagonally bounded weight on an amenable group is equivalent to a multiplicative weight. Indeed, if the group G is amenable and ω is a weight on G, then according to [13, Lemma 1] there exists a continuous character function $\phi : G \to \mathbb{R}^+$ (i.e., $\phi(xy) = \phi(x)\phi(y)$, $x, y \in G$) such that $\phi \le \omega$ on G. So, if

ω is diagonally bounded, we have

$$\phi(x) \le \omega(x) = \frac{\omega(x)\omega(x^{-1})}{\omega(x^{-1})} \le \frac{\omega(x)\omega(x^{-1})}{\phi(x^{-1})} = \phi(x)\omega(x)\omega(x^{-1})$$
$$\le \phi(x) \sup_{y\in G} \omega(y)\omega(y^{-1}) = c\phi(x), \qquad c = \text{const}.$$

This precisely means that ω is equivalent to a multiplicative weight ϕ on G.

5. Weak amenability of $\ell^1(\mathbb{Z}^2,\omega)$. It was proved in [14] that if ω is a weight on $\mathbb{Z}^2$ such that both $\ell^1(\mathbb{Z},\omega_1)$ and $\ell^1(\mathbb{Z},\omega_2)$ are weakly amenable, where ω_i denotes the restriction of ω to the ith coordinate, $i = 1, 2$, i.e., $\omega_1(k) = \omega(k, 0)$, $\omega_2(k) = \omega(0, k)$, $k \in \mathbb{Z}$, then $\ell^1(\mathbb{Z}^2,\omega)$ is also weakly amenable. We present an example showing that the converse is not true.

Consider the function ω on $\mathbb{Z}^2$ defined by

$$\omega(k,m) = (1+|k|)^{1/3}(1+|k+m|)^{1/3}, \qquad k,m \in \mathbb{Z}. \tag{5.1}$$

It is easy to see that ω is a weight on $\mathbb{Z}^2$. This follows from the fact that both mappings $(k,m) \mapsto k$ and $(k,m) \mapsto k+m$ from $\mathbb{Z}^2$ to $\mathbb{Z}$ are linear, and from the obvious inequality

$$(1+|a+b|) \le (1+|a|)(1+|b|), \qquad a,b \in \mathbb{Z}.$$

Example 5.1. For the weight ω defined by (5.1), the algebra $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable, but the algebra $\ell^1(\mathbb{Z},\omega_1)$ is not weakly amenable.

Proof. The weight ω_1 is precisely given by $\omega_1(k) = \omega(k,0) = (1+|k|)^{2/3}$, $k \in \mathbb{Z}$, and so $\ell^1(\mathbb{Z},\omega_1)$ is not weakly amenable by Proposition 1.2. We now prove that $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable. According to Proposition 1.1, it is enough to show that

$$\sup_{t\in\mathbb{Z}^2} \frac{|\Phi(t)|}{\omega(t)\omega(-t)} = \infty$$

for every non-trivial group homomorphism $\Phi : \mathbb{Z}^2 \to \mathbb{C}$. Since every such homomorphism is of the form $\Phi(k,m) = ck+dm$, $k,m \in \mathbb{Z}$, for some complex numbers c, d with $|c|^2 + |d|^2 \ne 0$, we only need to show that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} = \sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} = \infty$$

for all $c, d \in \mathbb{C}$ with $|c|^2 + |d|^2 \ne 0$. If $d \ne 0$, then

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} \underset{\text{set } k=0}{\ge} \sup_{m\in\mathbb{Z}} \frac{|d|\cdot|m|}{(1+|m|)^{2/3}} = \infty.$$

Now, if $d = 0$, then $c \ne 0$ since $|c|^2 + |d|^2 \ne 0$, and we have

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} \underset{\text{set } m=-k}{\ge} \sup_{k\in\mathbb{Z}} \frac{|c|\cdot|k|}{(1+|k|)^{2/3}} = \infty.$$

So, we conclude that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{(1+|k|)^{2/3}(1+|k+m|)^{2/3}} = \infty$$

for all non-trivial pairs $(c,d) \in \mathbb{C}^2$. Hence, $\ell^1(\mathbb{Z}^2,\omega)$ is indeed weakly amenable. ■

Remark 5.2. In fact, in the forthcoming paper of Y. Zhang and the author, the idea of the construction of the weight ω in Example 5.1 has been generalized to show that, for any Abelian locally compact groups G_1 and G_2 admitting non-trivial continuous group homomorphisms into $\mathbb{C}$, weak amenability of $L^1(G_1 \times G_2, \omega)$ does not imply weak amenability of $L^1(G_i, \omega_i)$ $(i = 1, 2)$ (see also [12, Proposition 5.11]).

The rest of this section is devoted to developing an easy procedure for verification whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable.

First we note that the complex-valued homomorphisms Φ in the characterization of weak amenability of $\ell^1(G,\omega)$ for Abelian groups G from Proposition 1.1 can be replaced with real-valued homomorphisms, see [14, Theorem 3.5]. It follows that $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable if and only if for every non-trivial group homomorphism $\Phi : \mathbb{Z}^2 \to \mathbb{R}$ we have

$$\sup_{t\in\mathbb{Z}^2} \frac{|\Phi(t)|}{\omega(t)\omega(-t)} = \infty.$$

Because every such homomorphism has the form $\Phi(k,m) = ck+dm$ for some $c,d \in \mathbb{R}$ with $c^2+d^2 \neq 0$, the group algebra $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable if and only if

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} = \infty$$

for every pair $(c,d) \in \mathbb{R}^2$ such that $c^2+d^2 \neq 0$. We aim to find a procedure that allows us to determine weak amenability of $\ell^1(\mathbb{Z}^2,\omega)$ by checking the supremums for only two pairs (c,d), instead of all non-trivial pairs (c,d). This will significantly simplify the verification process in most cases. We start by proving the following simple technical lemma.

Lemma 5.3. *Suppose that ω is a weight on $\mathbb{Z}^2$. Let c_1, d_1, c_2, d_2 be real numbers satisfying the relation $c_1d_2 - c_2d_1 \neq 0$ and such that*

$$\sup_{k,m\in\mathbb{Z}} \frac{|c_ik+d_im|}{\omega(k,m)\omega(-k,-m)} < \infty, \qquad i = 1, 2.$$

Then for all $c,d \in \mathbb{R}$ we have

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} < \infty.$$

Proof. Denote

$$M_i = \sup_{k,m\in\mathbb{Z}} \frac{|c_i k + d_i m|}{\omega(k,m)\omega(-k,-m)}, \qquad i = 1,2.$$

Then for every $k, m \in \mathbb{Z}$ we have

$$|c_i k + d_i m| \le M_i \omega(k,m)\omega(-k,-m), \qquad i = 1,2.$$

Since $c_1 d_2 - c_2 d_1 \neq 0$, the vectors (c_1, d_1) and (c_2, d_2) are linearly independent in $\mathbb{R}^2$. Fix arbitrary $(c,d) \in \mathbb{R}^2$. Then there exist real coefficients α, β such that

$$(c,d) = \alpha(c_1,d_1) + \beta(c_2,d_2),$$

and we obtain

$$\begin{aligned}|ck + dm| &= |\alpha(c_1 k + d_1 m) + \beta(c_2 k + d_2 m)| \\ &\le |\alpha| \cdot |c_1 k + d_1 m| + |\beta| \cdot |c_2 k + d_2 m| \\ &\le (|\alpha| M_1 + |\beta| M_2)\omega(k,m)\omega(-k,-m), \qquad k,m \in \mathbb{Z}.\end{aligned}$$

This immediately implies that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} \le |\alpha| M_1 + |\beta| M_2 < \infty. \ \blacksquare$$

It follows from Lemma 5.3 that for any weight ω on $\mathbb{Z}^2$ there are three possible situations:

S1. For every non-trivial pair $(c,d) \in \mathbb{R}^2$,

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} = \infty,$$

and $\ell^1(\mathbb{Z}^2, \omega)$ is weakly amenable. (For example, this is the case for the weight $\omega(k,m) = (1 + |k| + |m|)^{1/3}$.)

S2. For every pair $(c,d) \in \mathbb{R}^2$,

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} < \infty,$$

and $\ell^1(\mathbb{Z}^2, \omega)$ is not weakly amenable. (For example, this holds for the weight $\omega(k,m) = (1 + |k| + |m|)^{2/3}$.)

S3. There is a unique, up to a non-zero multiple, non-trivial pair $(c,d) \in \mathbb{R}^2$ such that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck + dm|}{\omega(k,m)\omega(-k,-m)} < \infty,$$

and $\ell^1(\mathbb{Z}^2, \omega)$ is not weakly amenable. (For example, this is the situation with the weight $\omega(k,m) = (1 + |k + m|)^{2/3}$.)

Employing this observation, we can prove the following result.

PROPOSITION 5.4. *Let ω be a weight on $\mathbb{Z}^2$ which is symmetric and even, i.e.,*

$$\omega(k,m)=\omega(k,-m)=\omega(m,k), \qquad k,m\in\mathbb{Z}. \tag{5.2}$$

Then $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable if and only if there exist $c,d\in\mathbb{R}$ such that

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)}=\infty. \tag{5.3}$$

REMARK 5.5. The conclusion of Proposition 5.4 means that if (5.3) holds for one pair (c,d), then it holds for all pairs (c,d) of real numbers. So in practice, if ω is symmetric and even, then one simply computes the supremum from (5.3) for any single non-trivial pair $(c,d)\in\mathbb{R}^2$ to determine whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable. If the supremum is infinite, then $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable; if the supremum is finite, then $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable.

Proof of Proposition 5.4. We only need to prove that S3 is not possible for any weight ω satisfying (5.2). According to Lemma 5.3, it is enough to show that if for some non-trivial pair $(c_0,d_0)\in\mathbb{R}^2$ the corresponding supremum is finite, then there exists another pair $(c,d)\in\mathbb{R}^2$, not proportional to (c_0,d_0), for which the supremum is also finite.

First we consider the case when $c_0\neq\pm d_0$. Then the pair (d_0,c_0) is not proportional to (c_0,d_0), and for this pair we still have

$$\sup_{k,m\in\mathbb{Z}} \frac{|d_0k+c_0m|}{\omega(k,m)\omega(-k,-m)} \underset{\omega(k,m)=\omega(m,k)}{=} \sup_{k,m\in\mathbb{Z}} \frac{|d_0k+c_0m|}{\omega(m,k)\omega(-m,-k)} \underset{k\leftrightarrow m}{=} \sup_{k,m\in\mathbb{Z}} \frac{|c_0k+d_0m|}{\omega(k,m)\omega(-k,-m)}<\infty.$$

Now, if $c_0=d_0$ or $c_0=-d_0$, then $d_0\neq 0$ (since the pair (c_0,d_0) is non-trivial), and so the pair $(c_0,-d_0)$ is not proportional to (c_0,d_0). For this pair we still have

$$\sup_{k,m\in\mathbb{Z}} \frac{|c_0k-d_0m|}{\omega(k,m)\omega(-k,-m)} \underset{\omega(k,m)=\omega(k,-m)}{=} \sup_{k,m\in\mathbb{Z}} \frac{|c_0k-d_0m|}{\omega(k,-m)\omega(-k,m)} \underset{m\leftrightarrow -m}{=} \sup_{k,m\in\mathbb{Z}} \frac{|c_0k+d_0m|}{\omega(k,m)\omega(-k,-m)}<\infty.$$

The proof is complete. ∎

REMARK 5.6. In particular, Proposition 5.4 holds for any weight of the form $\omega(k,m)=W(\|(k,m)\|)$, i.e., any weight depending only on the norm $\|(k,m)\|=\sqrt{k^2+m^2}$, $k,m\in\mathbb{Z}$.

Now let us consider situation S3 in more detail. Let ω be a weight for which we have this situation. Without loss of generality, we can assume that the corresponding supremum is finite for a pair (c,d) with $c=1$, i.e., that there exists $d\in\mathbb{R}$ such that

$$\sup_{k,m\in\mathbb{Z}}\frac{|k+dm|}{\omega(k,m)\omega(-k,-m)}=M<\infty.$$

This implies

$$\frac{1}{M}|k+dm|\le\omega(k,m)\omega(-k,-m),\qquad k,m\in\mathbb{Z}.\tag{5.4}$$

Since we are in situation S3, the supremum is infinite for every pair (c',d') that is not proportional to $(1,d)$, in particular for the pair $(0,1)$. So, we have

$$\sup_{k,m\in\mathbb{Z}}\frac{|m|}{\omega(k,m)\omega(-k,-m)}=\infty.$$

This means that there exists a sequence $\{(k_n,m_n)\}_{n=1}^{\infty}\subset\mathbb{Z}^2$ such that

$$\frac{|m_n|}{\omega(k_n,m_n)\omega(-k_n,-m_n)}>n,$$

and hence

$$\frac{|m_n|}{n}>\omega(k_n,m_n)\omega(-k_n,-m_n),\qquad n\in\mathbb{N}.$$

Combining the last inequality with (5.4), we obtain

$$\frac{1}{M}|k_n+dm_n|\le\omega(k_n,m_n)\omega(-k_n,-m_n)<\frac{|m_n|}{n},\qquad n\in\mathbb{N}.$$

Dividing by (non-zero) $|m_n|$ and multiplying by M, we finally get

$$\left|\frac{k_n}{m_n}+d\right|<\frac{M}{n},\qquad n\in\mathbb{N}.$$

It follows that $d=-\lim_{n\to\infty}k_n/m_n$.

Now we are ready to formulate the aforementioned procedure involving calculation of at most two supremums.

Procedure for verification of whether $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable:

Step 1. We calculate

$$\sup_{k,m\in\mathbb{Z}}\frac{|m|}{\omega(k,m)\omega(-k,-m)}.$$

If it is finite, then $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable. If it is infinite, then we are either in situation S1 or in situation S3, and we proceed to the second step.

Step 2. We choose $\{(k_n,m_n)\}_{n=1}^{\infty}\subset\mathbb{Z}^2$ such that

$$\frac{|m_n|}{\omega(k_n,m_n)\omega(-k_n,-m_n)}>n,\qquad n\in\mathbb{N},$$

and consider $\lim_{n\to\infty} k_n/m_n$. If the limit does not exist or is infinite, then, according to what we have discussed above, we cannot be in situation S3. This means that we are in situation S1, and so $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable. Now, if $\lim_{n\to\infty} k_n/m_n$ exists and is finite, we denote it by $-d$ and proceed to the last step.

Step 3. We calculate

$$\sup_{k,m\in\mathbb{Z}} \frac{|k+dm|}{\omega(k,m)\omega(-k,-m)}.$$

If it is finite, then $\ell^1(\mathbb{Z}^2,\omega)$ is not weakly amenable. On the other hand, if it is infinite, we cannot be in situation S3, so we must be in situation S1, which means that $\ell^1(\mathbb{Z}^2,\omega)$ is weakly amenable.

Remark 5.7. The procedure above will also work if in the first step we start from any other

$$\sup_{k,m\in\mathbb{Z}} \frac{|ck+dm|}{\omega(k,m)\omega(-k,-m)} \quad \text{instead of} \quad \sup_{k,m\in\mathbb{Z}} \frac{|m|}{\omega(k,m)\omega(-k,-m)},$$

with minor adjustments in the subsequent steps.

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