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Consequences of Vopěnka's Principle over weak set theories

by

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Dedicated to the memory of Petr Vopěnka

Abstract. It is shown that Vopěnka's Principle (VP) can restore almost the entire ZF over a weak fragment of it. Namely, if EST is the theory consisting of the axioms of Extensionality, Empty Set, Pairing, Union, Cartesian Product, Δ_0-Separation and Induction along ω, then EST + VP proves the axioms of Infinity, Replacement (thus also Separation) and Powerset. The result was motivated by previous ones (2014), as well as by H. Friedman's (2015), where a distinction is made among various forms of VP. As a corollary, EST + Foundation + VP $=$ ZF + VP and EST + Foundation + AC + VP $=$ ZFC + VP. Also, it is shown that the Foundation axiom is independent of ZF $-$ {Foundation} + VP. It is open whether the Axiom of Choice is independent of ZF + VP. A very weak form of choice follows from VP, and some other similar forms of choice are introduced.

1. Introduction. Vopěnka's Principle (henceforth abbreviated VP) is mainly known as a (very) large cardinal axiom (see [11]). Also, several other implications of the principle were proved long ago, especially in category theory (see [1]). Recently there has been a revived interest in VP through new set-theoretic proofs of category-theoretic results (see Bagaria and Brooke-Taylor [2]). Furthermore, Brooke-Taylor [5] showed the relative consistency of VP with almost all usual ZFC-independent statements, like GCH and diamond principles (see the Introduction of [5]).

In all of the above results the underlying theory is ZFC. In contrast, the aim of the present paper is to reveal a still different capability of VP: the capability to restore the most basic axioms of ZF, namely Replacement (thus also Separation) and Powerset, as well as Infinity (if the latter is missing),

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when added to a suitable weak set theory. By "weak set theory" we generally mean the following.

Definition 1.1. A *weak set theory* is one that does not include the axioms of Powerset and Replacement.

Various weak systems of set theory have been considered in the literature. Perhaps the most well-known of them is Kripke–Platek's theory KP on which the theory of admissible sets is based (see [6, p. 48]). A system weaker than KP is Devlin's Basic Set Theory (BS) used in [6, p. 36]. An extensive and detailed treatment of an array of weak systems, among them Devlin's BS, can be found in [13].

A weak set theory may or may not include Infinity. Also, it may or may not be a fragment of ZF. For example the theory LZFC ("local ZFC") of [16] and [17] is weak, proves Infinity, but is not a fragment of ZFC. On the other hand, the system EST introduced in Section 2 below, as a ground theory for VP, is a fragment of ZF but does not include Infinity. Yet EST + VP proves this axiom. It is worth pointing out that EST, even augmented with Infinity, is weaker than BS because of lack of Foundation. A fortiori it is weaker than KP.

Throughout we shall refer to the well-known axioms of ZFC with their usual names, and without further explanations. These are: Extensionality, Empty Set, Pairing, Union, Powerset, Infinity, Separation, Foundation, Replacement and Choice. Sometimes it is convenient to denote them by abbreviations, especially within theorems. Specifically, we often write *Ext* for Extensionality, *Pair* for Pairing, *Pow* for Powerset, *Sep* and Δ_0-*Sep* for Separation and Δ_0-Separation, respectively, *Rep* and Δ_0-*Rep* for Replacement and Δ_0-Replacement, respectively, *Found* for Foundation, *Inf* for Infinity and AC for Choice. Another weak axiom that will be used below is Cartesian Product, abbreviated *CartProd*, which says that for any sets x, y, $x \times y$ is a set.

The capability of VP to restore the axioms of Replacement and Powerset was first noticed in [17]. In that paper we showed that if we add VP to a strengthened variant of LZFC, then Replacement and Powerset are recovered. Later on, when we came across Friedman's [8], where seven variants of VP are given in apparently decreasing strength but still equivalent over ZFC, we realized that what was used in [17] was not the full principle VP but only a weaker form, denoted VP_4 in Friedman's list. Moreover, we saw that using VP instead of VP_4 shows the result of [17] for the theory LZFC itself rather than a strengthened variant of it. That led us to focus on what VP can prove over a weak fragment of ZF rather than LZFC.

The content of the paper is as follows. In Section 2.1 we introduce the variants of VP, especially the general one VP and the weaker one VP_4, and describe their basic differences. Also, we outline the way in which VP acts as a set-existence principle. In Section 2.2 we introduce the weak theory EST. In Section 2.3 we show that VP is expressible in EST. Section 2.4 contains the main results of Section 2, namely that EST+VP proves Infinity, Replacement and Powerset.

In Section 3 we prove that Foundation is independent of ZF + VP, that is, if ZF + VP is consistent, then so is $\mathrm{ZF}_0 + \mathrm{VP} + \neg Found$, where $\mathrm{ZF}_0 = \mathrm{ZFC} - \{Found\}$.

In Section 4 the question whether AC is independent of ZF+VP is raised. The question remains open. In particular, it is open whether the question can be settled with the help of symmetric and permutation models. It is also observed that a very weak form of choice follows from VP. This gives the chance to introduce some other similar forms of choice whose relative strength over ZF, as well as over ZF + VP, is also open.

2. Vopěnka's Principle over some weak set theories

2.1. VP and its variants. Let $\mathcal{L} = \{\in\}$ be the language of set theory. Given a formula $\phi(x)$ of $\mathcal{L}$ in one free variable, let X_ϕ denote the extension $\{x : \phi(x)\}$ of $\phi(x)$. As usual, we refer to X_ϕ as "classes". Vopěnka's Principle is a statement that quantifies over classes, so cannot be formulated in ZF as a single axiom; it can be formulated however as an axiom-scheme. Clearly, for every ϕ, "X_ϕ is a proper class" is a first-order sentence. Therefore so also are the statements

VP_ϕ: *If X_ϕ is a proper class of L-structures, for some first-order language L, then there are distinct $\mathcal{M}, \mathcal{N} \in X_\phi$ such that $\mathcal{M} \precsim \mathcal{N}$.*

Here $\mathcal{M} \precsim \mathcal{N}$ means that there is an elementary embedding $f : \mathcal{M} \to \mathcal{N}$. Let

$$\mathrm{VP} = \{\mathrm{VP}_\phi : \phi(x) \text{ a formula of } \mathcal{L}\}.$$

[Notice that, throughout the paper, we refer to the arbitrary first-order languages mentioned in the statement of VP_ϕ above by the symbol "L", to avoid confusion with the fixed language $\mathcal{L} = \{\in\}$ of set theory. Of course $\mathcal{L}$ is *one* of those L, but a very specific one.]

Let us refer to the above formulation of VP_ϕ as being "direct". We also have the contrapositive formulation: "If X_ϕ is a class of L-structures, for some first-order language L, and there are no distinct $\mathcal{M}, \mathcal{N} \in X_\phi$ such that $\mathcal{M} \precsim \mathcal{N}$, then X_ϕ is not a proper class". But "X_ϕ is not a proper class" means exactly that X_ϕ is a set. This latter formulation enables VP_ϕ to act as a set-existence principle: it says that "if such and such is the case about ϕ and X_ϕ, then X_ϕ is a *set*".

The direct formulation of VP is suitable for ZF and ZFC, where it is usually easy to decide whether its premise is satisfied, that is, whether a class X_ϕ of structures is proper. In contrast, the contrapositive formulation makes VP suitable for weak set theories. The reason is that such theories have poorly defined universes, where it is often unclear which classes X_ϕ are sets and which are proper ones. So in weak set theories we are more in need of principles entailing that such and such classes X_ϕ are sets. For example, in a weak set theory where Replacement and Powerset are missing, such instances of VP might be used to establish that the classes $\{x : (\exists y \in A)(F_\psi(y) = x)\}$, where $F_\psi(y) = x$ is a functional relation, or $\{x : x \subseteq A\}$ for any set A, are actually sets, thus proving the aforementioned axioms. As already said above, we first noticed this fact when working on [17]. Motivated by results in that paper and also by Friedman's [8], we pursued the above idea more systematically and showed that in fact, by the contrapositive action of VP, the three pillar axioms of ZF, Powerset, Replacement and Infinity, can be established over a very weak fragment of ZF.

H. Friedman [8] gave seven variants of VP, namely VP above plus six weaker forms VP_i, $1 \le i \le 6$, almost in decreasing strength, each of them resulting either from a narrowing of the range of first-order languages that can be engaged in the scheme, or from replacing "elementary embedding" with "embedding", or from both. Specifically, VP_1 results from VP by restricting the term "first-order language" to "language of finite relational type". VP_2 results from VP_1 by replacing "elementary embedding" by "embedding". VP_3 results from VP_2 by replacing "language of finite relational type" by "language with a single binary operation". VP_4 results from VP_3 by replacing "language with a single binary operation" with "language with a single binary relation". VP_4 is essentially the variant used in [17], so we omit the other variants of VP. More precisely, let

$\mathrm{VP}_{4,\phi}$: *If X_ϕ is a proper class of L-structures of a language L with a single binary relation $\mathbf{R}$, then there are distinct $\mathcal{M}, \mathcal{N} \in X_\phi$ such that $\mathcal{M}$ is embeddable into $\mathcal{N}$.*

Let also

$$\mathrm{VP}_4 = \{\mathrm{VP}_{4,\phi} : \phi(x) \text{ is a formula of } \mathcal{L}\}.$$

As Friedman states in [8], all these seven variants are equivalent over the theory of classes Neumann–Gödel–Bernays with Choice (NGBC), when VP_i are formulated in the language of the theory of classes. That means that the above formulations of VP_i in ZF, as schemes, are also equivalent over ZFC. The equivalence of VP_i should have been well-known since the start, at least among category-theorists. Also, as indicated in [1, Historical Remarks, pp. 278–279], the original formulation of Vopěnka's Principle was VP_4 rather than VP. However, these variants need *not* be equivalent over weaker theories, like the one we deal with below (see Corollary 2.16 for the difference between VP and VP_4 over EST, and the comments following that).

Although in this paper we have made the shift from the weak variant VP_4 to the most general one VP, in the proofs of our main results below we still do not need the full generality of VP. To be precise: the main difference between VP_4 and VP, as used below, is that the first-order languages involved in VP may contain, for our purposes, in contrast to those involved in VP_4, an *arbitrary set of constants.* On the other hand, for our purposes, the languages in VP need not contain more than one binary and one unary predicate. Moreover, for the derivation of Replacement and Powerset it suffices that VP talks just about *embeddings* rather than elementary embeddings. (See Remark 2.17 below.)

Having made the distinction between VP and VP_4, we can now describe the result of [17] that has largely motivated this paper. (Although in [17] we refer to VP, what we actually use is VP_4.) To formulate it we need an old theorem of P. Vopěnka, A. Pultr and Z. Hedrlín [18], that we abbreviate V-P-H, the proof of which was crucial for the result of [17].

Given a set A with a binary relation $R \subseteq A \times A$, we refer to the ordered pair $\langle A, R\rangle$ as a *graph.* Given a graph $\langle A, R\rangle$, a mapping $f : A \to A$ is an *endomorphism* if for all $x, y \in A$, $\langle x, y\rangle \in R$ implies $\langle f(x), f(y)\rangle \in R$. The graph $\langle A, R\rangle$ is said to be *rigid* if the only endomorphism of $\langle A, R\rangle$ is the identity.

The V-P-H theorem is the following:

THEOREM 2.1 (V-P-H [18]). (ZFC) *For every infinite set A, there is a binary relation $R \subset A \times A$ such that the graph $\langle A, R\rangle$ is rigid.*

Obviously, the preceding theorem refers to structures with a single binary relation only. Also, rigidity is a property that refers to (lack of even) *endomorphisms* rather than embeddings (let alone elementary embeddings). This is why what is proved in [17] is in essence the following.

THEOREM 2.2 ([17, Theorem 6.3]). *If T is a theory such that* LZFC $\subseteq T$ *and* $T \vdash$ V-P-H, *then* $T + VP_4$ *proves Replacement and Powerset, that is,* ZFC $\subseteq T + VP_4$.

[This formulation differs from that of [17] only in that we mention VP_4 in place of VP ([1]).]

([1]) The precise definition of the theory LZFC is not needed here. It suffices to say that it is a weak set theory in the sense of Definition 1.1. It is not a fragment of ZF, but it satisfies Infinity (in the sense that ω is the least inductive set), AC, Cartesian Product, and others. For later use we note also that LZFC is much stronger than the theory EST introduced below.

In Theorem 2.2, the assumption that the theory T proves V-P-H guarantees the existence of a rigid binary relation R on every set A. Then applying VP_4 to structures suitably equipped with such an R, we show that $\mathcal{P}(A)$ and $F``A$ are not proper classes. The variant VP_4 suffices for this purpose since we need to employ a language with a binary relation $\mathbf{R}$ only, as well as just *endomorphisms* instead of embeddings or elementary embeddings.

The basic observation that led from Theorem 2.2 above to the results of the present article is that, in the absence of V-P-H, the rigidity property can be alternatively guaranteed by employing an infinity of constants in the language of the structure, namely a constant $\mathbf{c}_a$ for each element $a \in A$. Such languages are allowed in VP though not in

VP_4.

2.2. A weak fragment of ZF for expressing VP. Below we define the weak fragment of ZF called EST (for Elementary Set Theory). First let us set

$$\mathrm{EST}_0 = \{\mathit{Ext}, \mathit{Emptyset}, \mathit{Pair}, \mathit{Union}, \mathit{CartProd}, \Delta_0\text{-}\mathit{Sep}\}.$$

For any sets x, y, the ordered pair $\langle x, y\rangle$ is defined in EST_0 as usual, that is, as the set $\{\{x\}, \{x, y\}\}$. For any given sets M, N, $M \times N = \{\langle x, y\rangle : x \in M, y \in N\}$ is a set by *CartProd*. Binary relations between M and N are defined as sets $R \subseteq M \times N$. Functions $f : M \to N$ are defined as special binary relations $f \subseteq M \times N$. Throughout, the symbol f ranges over functions, so $(\exists f)(\cdots)$ abbreviates $(\exists f)(f \text{ is a function} \wedge \cdots)$.

Next we define the class ω of natural numbers, which also need not be a set. Let $\mathit{Tr}(x)$ denote the predicate "x is a transitive set". Let us also define the predicates

$$\begin{aligned}
\mathit{Ord}(x) &:= \mathit{Tr}(x) \wedge (\forall y, z \in x)(y \in z \vee y = z \vee z \in y),\\
\mathit{Succ}(x) &:= (\exists y \in x)(x = y \cup \{y\}),\\
\mathit{Nat}(x) &:= \mathit{Ord}(x) \wedge (\forall y)[(y \in x \vee y = x) \to (y = \emptyset \vee \mathit{Succ}(y))].
\end{aligned}$$

Finally let us set

$$\omega = \{x : \mathit{Nat}(x)\}.$$

We call ω the *class of natural numbers.* Throughout, in writing formulas of $\mathcal{L}$, it is convenient to use the notation $x \in \omega$, although ω is a class in general, as an abbreviation of the predicate $\mathit{Nat}(x)$. In particular, $(\exists x \in \omega)\phi$ and $(\forall x \in \omega)\phi$ stand for $(\exists x)(\mathit{Nat}(x) \wedge \phi)$ and $(\forall x)(\mathit{Nat}(x) \to \phi)$, respectively. Also, as usual, the letters m and n will range over elements of ω.

Further we need *induction* to hold along ω, that is, that every non-empty subclass $X \subseteq \omega$ has a least element. For that purpose an additional axiom is needed. This is the Induction scheme, $\mathit{Ind}(\omega)$, given below, which is almost identical to the Induction scheme of Peano Arithmetic. Namely, for every formula $\phi(x)$ of $\mathcal{L}$ such that $X_\phi \subseteq \omega$ (that is, $(\forall x)(\phi(x) \to Nat(x))$), let $Ind_\phi(\omega)$ denote the formula

$$Ind_\phi(\omega):\quad [\phi(\emptyset) \wedge (\forall x \in \omega)(\phi(x) \to \phi(x \cup \{x\}))] \to (\forall x \in \omega)\phi(x).$$

Let also

$$Ind(\omega) = \{Ind_\phi(\omega) : X_\phi \subseteq \omega\}.$$

[Equivalently, $Ind(\omega)$ says that every non-empty $X_\phi \subseteq \omega$ has a $\in$-least element.] Finally we set

$$\mathrm{EST} = \mathrm{EST}_0 + Ind(\omega).$$

EST is the weak theory that will be used below as a base theory for VP.

Remark 2.3. The definition of ω above is as in [4, pp. 468ff.] (see also [3]), except that in [4] the predicate $Ord(x)$ says that "x is well-ordered with respect to $\in$", while in our definition of $Ord(x)$, "x is linearly ordered with respect to $\in$". Note the following:

(i) A difference between the two properties is that "linearly ordered" is Δ_0 while "well-ordered" is not (it is Π_1). This fact will be needed in the proof of Theorem 2.12 below. On the other hand, with the help of $Ind(\omega)$, one can easily prove that for every $x \in \omega$, x is indeed well-ordered with respect to $\in$.

(ii) If Foundation were available, the properties "x is linearly ordered" and "x is well-ordered" (with respect to $\in$) would be equivalent.

(iii) However, even if Foundation were available, $\mathrm{EST}_0 + Found$ could not prove $Ind(\omega)$. That would need in addition Separation or Replacement. That is, $\mathrm{EST}_0 + Found + Sep \vdash Ind(\omega)$.

With the help of $Ind_\phi(\omega)$ one can prove all basic facts about natural numbers. Some of them needed below are the following:

Fact 2.4.

(i) *For any $x, y \in \omega$, if $x \in y$, then $x \subsetneq y$, and x is an initial segment of y.*
(ii) *If $x, y \in \omega$ and $x \subsetneq y$, then there is no injection $f : y \to x$.*
(iii) *For any $x, y \in \omega$, we have $x \in y \vee y \in x \vee x = y$.*
(iv) *If $x \in \omega$ and $x = y \cup \{y\}$, then y is the greatest element of x.*

Having defined ordered pairs and natural numbers, we can define ordered n-tuples, for $n \in \omega$, as usual by induction. Namely, for every $n > 2$, $\langle x_0, \dots, x_{n+1}\rangle = \langle\langle x_0, \dots, x_n\rangle, x_{n+1}\rangle$. Moreover, by the axiom *CartProd*, for $n \in \omega$, $n > 0$, and every n-tuple of sets $M_0, \dots, M_{n-1}$, the product $M_0 \times \cdots \times M_{n-1} = \{\langle x_0, \dots, x_{n-1}\rangle : x_i \in M_i\}$ is a set.

Below we shall also need the set of functions $f : n \to M$, denoted ${}^nM$, for any set M and any $n \in \omega$. Since Replacement is not available in EST we shall define the elements of ${}^nM$ in a slightly different way. Namely, while an $f \in {}^nM$ typically has the form

$$f = \{\langle 0, x_0\rangle, \ldots, \langle n-1, x_{n-1}\rangle\},$$

obviously we can identify the latter set with the n-tuple

$$\langle\langle 0, x_0\rangle, \ldots, \langle n-1, x_{n-1}\rangle\rangle,$$

which is an element of the Cartesian product

$$(\{0\} \times M) \times \cdots \times (\{n-1\} \times M).$$

The latter is a set as we saw above, so we can define

$${}^nM := (\{0\} \times M) \times \cdots \times (\{n-1\} \times M).$$

As usual, we shall let letters g, h range over elements of ${}^nM$ and we shall write $h(i) = x$ to denote the fact that $\langle i, x\rangle$ is the ith element of the n-tuple h.

As already mentioned in the Introduction, EST, even augmented with Infinity, is weaker than Devlin's system BS. In connection with the definition of the *Sat* predicate in EST that will be given in the next section, the referee kindly informed me that BS, though stronger than EST, is famously insufficient for the purpose Devlin introduced it, including defining the satisfaction predicate ([2]). (The flaws of BS with respect to this point are discussed and remedied in [13, §10].) The main reason that EST succeeds where BS fails appears to be the unorthodox definition of the set ${}^nM$ given above. Unexpectedly enough, this unusual yet legitimate formalization of ${}^nM$ is all we need to make things work.

2.3. Expressibility of VP in EST. From now on we work in EST, with language $\mathcal{L} = \{\in\}$. In this subsection we show how the concepts required for the formulation of VP can be defined in EST. The formulation of VP (at least for the needs of the present article) requires the following:

(a) The definition of a language L_A, for every set A, that contains at most one unary and one binary relation symbol, but contains a constant $\mathbf{c}_a$ for each $a \in A$ and infinitely many variables. Due to the lack of Replacement, L_A need not be a set.

(b) The definition of terms, formulas and sentences of L_A, in a way that these classes of objects are *inductive,* so that one can prove inductively the usual facts about these syntactic objects. Again the classes of terms, formulas and sentences need not be sets.

(c) The definition of L_A-structures for every language L_A.

(d) The definition of satisfaction relation "$\mathcal{M} \models \sigma(x_0, \ldots, x_{n-1})$" for any L_A-structure $\mathcal{M}$, any formula $\sigma(v_0, \ldots, v_{n-1})$ of L_A, with free variables

([2]) A discussion on this issue can be found at http://mathoverflow.net/questions/77734/devlins-constructibility-as-a-resource.

$v_0, \ldots, v_{n-1}$, and any n-tuple of elements $x_0, \ldots, x_{n-1}$ of M. In view of this, the relation $\mathcal{M} \precsim \mathcal{N}$ of elementary embeddability (or simple embeddability) between L_A-structures is immediately defined.

We show below how definitions (a)–(d) can be implemented in EST. Since the definability of the satisfaction relation is crucial for the expressibility of VP in EST, we shall give explicitly the necessary definitions below.

Definition 2.5. For every $n \in \omega$, let $v_n := \langle 0, n\rangle$. Then v_n is the nth *variable* (of every language). For every set a, let $\mathbf{c}_a := \langle 1, a\rangle$. The sets $\mathbf{c}_a$ are called *constants.* Also let us identify the (sufficient) logical symbols $\equiv$, $\neg$, $\wedge$ and $\exists$ with elements of ω as follows: $\equiv := 2$, $\neg := 3$, $\wedge := 4$, $\exists := 5$. Finally, let $\mathbf{U} := 6$ and $\mathbf{R} := 7$. The symbols $\equiv$, $\mathbf{U}$ and $\mathbf{R}$ are referred to as *predicates.* The predicates $\equiv$ and $\mathbf{R}$ are *binary*, while $\mathbf{U}$ is a *unary* one. For every set A, let

$$L_A = \{\equiv, \neg, \wedge, \exists\} \cup \{v_n : n \in \omega\} \cup \{\mathbf{c}_a : a \in A\} \cup \{\mathbf{U}, \mathbf{R}\}.$$

Also let

$$\begin{aligned} V(L_A) &= \{v_n : n \in \omega\}, \\ C(L_A) &= \{\mathbf{c}_a : a \in A\}, \\ \mathit{Term}(L_A) &= \{v_n : n \in \omega\} \cup \{\mathbf{c}_a : a \in A\} \end{aligned}$$

be the classes of variables, constants and *terms of* L_A, respectively.

Definition 2.6. Given any set A, define the class $\mathit{AFml}(L_A)$ of *atomic formulas* of L_A as follows:

$$\begin{aligned} \mathit{AFml}(L_A) = \{\langle \equiv, t, s\rangle : t, s \in \mathit{Term}(L_A)\} &\cup \{\langle \mathbf{U}, t\rangle : t \in \mathit{Term}(L_A)\} \\ \cup \{\langle \mathbf{R}, t, s\rangle : t, s \in \mathit{Term}(L_A)\}&. \end{aligned}$$

[The meaning of the above codings is straightforward: $\langle \equiv, t, s\rangle$, $\langle \mathbf{U}, t\rangle$ and $\langle \mathbf{R}, t, s\rangle$ represent the formulas $t \equiv s$, $\mathbf{U}(t)$ and $\mathbf{R}(t, s)$, respectively.]

Definition 2.7. The class $\mathit{Fml}(L_A)$ of *formulas of* L_A is defined as follows. First let the predicate $\mathit{Fml}_{L_A}(x, f, n)$ be given by

$$\begin{aligned} \mathit{Fml}_{L_A}(x, f, n) := \big[n \neq 0 \wedge \mathit{dom}(f) = n \wedge f(n-1) = x \\ \wedge(\forall k < n)[(f(k) \in \mathit{AFml}(L_A) \vee (\exists j < k)(f(k) = \langle \neg, f(j)\rangle) \\ \vee(\exists j, l < k)(f(k) = \langle \wedge, f(j), f(l)\rangle) \\ \vee(\exists j < k)(\exists m \in \omega)(f(k) = \langle \exists, m, f(j)\rangle)]\big]. \end{aligned}$$

Then set

$$\mathit{Fml}(L_A) = \{x : (\exists f)(\exists n \in \omega)\mathit{Fml}_{L_A}(x, f, n)\}.$$

The preceding definition of formulas is essentially the one given in [7, Chapter 3, Definition 5.2]. It goes through smoothly despite the fact that ω need not be a set. However the above definition of $\mathit{Fml}(L_A)$ is not quite precise. By writing $Fml_{L_A}(x, f, n)$ we intend f to be a function with domain n that enumerates the subformulas of x and *only them.* But if, for example, $dom(f) = 3$ and $f(0)$, $f(1)$ and $f(2)$ are atomic formulas, and $f(2) = x$, then $Fml(x, f, 3)$ holds according to 2.7, although $f(0)$ and $f(1)$ are not subformulas of x. Thus an additional constraint must be added to the definition of $Fml_{L_A}(x, f, n)$ in order to prevent f from enumerating irrelevant atomic formulas. This is simply the requirement for the domain of f to be *minimal,* specifically that $dom(f) = |Sub(x)|$ (the number of subformulas of x). In the above example the domain of a function enumerating the subformulas of an atomic formula should be 1 not 3. This requirement can be formally expressed by a simple modification to the definition of $Fml_{L_A}(x, f, n)$, and hence to that of $Fml(L_A)$ ([3]). Henceforth we assume that this requirement is implicitly satisfied whenever we write $Fml_{L_A}(x, f, n)$.

The crucial thing about 2.7 is its capability to support inductive proofs and recursive definitions. We let the letters σ, τ denote elements of $Fml(L_A)$. As a first application of 2.7, every $\sigma \in Fml(L_A)$ is assigned a *length*, which is the domain n of some enumerating function f for the subformulas of σ, or, since these functions are all minimal, the number of subformulas σ. More generally, in view of the validity of induction along ω the following holds.

Lemma 2.8. *Let $X \subseteq Fml(L_A)$ be a subclass of $Fml(L_A)$. If*

$$AFml(L_A) \subseteq X$$

and X is closed with respect to $\neg$, $\wedge$ and $\exists$, then $X = Fml(L_A)$.

Proof. Assume X is as stated and suppose $Fml(L_A) - X \neq \emptyset$. Then, by the inductive properties of ω, there is $\sigma \in Fml(L_A) - X$ of least length n. We immediately obtain a contradiction from the definition of $Fml(L_A)$. ■

Given a language L_A as above, L_A-structures are defined as follows.

Definition 2.9. For any set A, an *L_A-structure* is a quadruple

$$\mathcal{M} = \langle M, U, R, I \rangle,$$

where $U \subseteq M$, $R \subseteq M \times M$, and I is a (set) function $I : A \to M$. We refer to M as the *domain* of $\mathcal{M}$, and to I as the *constant assignment* for L_A. If either **U** or **R** is missing from L_A, the L_A-structures are triples $\mathcal{M} = \langle M, R, I \rangle$ or $\mathcal{M} = \langle M, U, I \rangle$, respectively. The *interpretation* of the extra-logical symbols

([3]) The modification is this: We set

$$Fml(L_A) = \{x : (\exists f)(\exists n \in \omega) Fml^*_{L_A}(x, f, n)\},$$

where

$$Fml^*_{L_A}(x, f, n) := Fml_{L_A}(x, f, n) \wedge (\forall g)(\forall m)[Fml_{L_A}(x, g, m) \to n \leq m].$$

of L_A in $\mathcal{M}$ is defined as follows: $\mathbf{U}^{\mathcal{M}} = U$, $\mathbf{R}^{\mathcal{M}} = R$, $\mathbf{c}_a^{\mathcal{M}} = I(a)$ for each $a \in A$. On the other hand, $\equiv^{\mathcal{M}}$ is the identity.

For every $\sigma \in Fml(L_A)$, the (finite) set of free variables of σ, denoted $FV(\sigma)$, is defined as usual by induction on the length of σ (that is, along the steps of 2.7). Also, for every σ, $|FV(\sigma)| \in \omega$. We come to the definition of the satisfaction relation $Sat(\mathcal{M}, \sigma, e)$ which formalizes the relation $\mathcal{M} \models \sigma(e(0), \ldots, e(n-1))$, for an L_A-structure $\mathcal{M} = \langle M, U, R, I \rangle$, a formula $\sigma(v_0, \ldots, v_{n-1})$ with free variables $v_0, \ldots, v_{n-1}$, and a mapping $e : n \to M$, that is, $e \in {}^nM$ (recall that by the discussion at the end of Section 2.2, ${}^nM$ is a set). The next definition is an adaptation of [7, Chapter 3, Definition 5.4]. (Recall that the letters f, g always denote functions.) We assume here that both symbols $\mathbf{U}$ and $\mathbf{R}$ occur; if some of them is missing, the definition is modified in the obvious way.

Definition 2.10. Let $Sat(\mathcal{M}, \sigma, e)$ denote the relation

$$\begin{aligned}(\exists M, U, R, I)(\exists f, g)(\exists n, m \in \omega)[\mathcal{M} = \langle M, U, R, I \rangle \\ \wedge Fml_{L_A}(\sigma, f, n) \wedge |FV(\sigma)| = m \wedge dom(g) = n \wedge e \in g(n-1) \\ \wedge (\forall k < n) S(k, f, g, m, \mathcal{M})],\end{aligned}$$

where

$$\begin{aligned}
&S(k, f, g, m, \mathcal{M}) \\
&:= (\exists i, j)[f(k) = \langle \equiv, v_i, v_j \rangle \wedge g(k) = \{h \in {}^mM : h(i) = h(j)\}] \\
&\quad \vee (\exists i)(\exists a \in A)[f(k) = \langle \equiv, v_i, \mathbf{c}_a \rangle \wedge g(k) = \{h \in {}^mM : h(i) = I(a)\}] \\
&\quad \vee (\exists j)(\exists a \in A)[f(k) = \langle \equiv, \mathbf{c}_a, v_j \rangle \wedge g(k) = \{h \in {}^mM : h(j) = I(a)\}] \\
&\quad \vee (\exists a \in A)[f(k) = \langle \equiv, \mathbf{c}_a, \mathbf{c}_a \rangle \wedge g(k) = {}^mM] \\
&\quad \vee (\exists a \neq b \in A)[f(k) = \langle \equiv, \mathbf{c}_a, \mathbf{c}_b \rangle \wedge g(k) = \emptyset] \\
&\quad \vee (\exists i)[f(k) = \langle \mathbf{U}, v_i \rangle \wedge g(k) = \{h \in {}^mM : h(i) \in U\}] \\
&\quad \vee (\exists a \in A)[f(k) = \langle \mathbf{U}, \mathbf{c}_a \rangle \wedge I(a) \in U \wedge g(k) = {}^mM] \\
&\quad \vee (\exists a \in A)[f(k) = \langle \mathbf{U}, \mathbf{c}_a \rangle \wedge I(a) \notin U \wedge g(k) = \emptyset] \\
&\quad \vee (\exists i, j)[f(k) = \langle \mathbf{R}, v_i, v_j \rangle \wedge g(k) = \{h \in {}^mM : \langle h(i), h(j) \rangle \in R\}] \\
&\quad \vee (\exists i)(\exists a \in A)[f(k) = \langle \mathbf{R}, v_i, \mathbf{c}_a \rangle \wedge g(k) = \{h \in {}^mM : \langle h(i), I(a) \rangle \in R\}] \\
&\quad \vee (\exists j)(\exists a \in A)[f(k) = \langle \mathbf{R}, \mathbf{c}_a, v_j \rangle \wedge g(k) = \{h \in {}^mM : \langle I(a), h(j) \rangle \in R\}] \\
&\quad \vee (\exists a, b \in A)[f(k) = \langle \mathbf{R}, \mathbf{c}_a, \mathbf{c}_b \rangle \wedge \langle I(a), I(b) \rangle \in R \wedge g(k) = {}^mM] \\
&\quad \vee (\exists a, b \in A)[f(k) = \langle \mathbf{R}, \mathbf{c}_a, \mathbf{c}_b \rangle \wedge \langle I(a), I(b) \rangle \notin R \wedge g(k) = \emptyset] \\
&\quad \vee (\exists i)[f(k) = \langle \neg, f(i) \rangle \wedge g(k) = {}^mM - g(i)] \\
&\quad \vee (\exists i, j)[f(k) = \langle \wedge, f(i), f(j) \rangle \wedge g(k) = g(i) \cap g(j)] \\
&\quad \vee (\exists i, j)[f(k) = \langle \exists, i, f(j) \rangle \wedge g(k) = \{h \in {}^mM : (\exists x \in M)(h(i/x) \in g(j))\}];
\end{aligned}$$

in the last clause, $h(i/x)$ is the tuple of ${}^mM$ resulting from h if we replace $\langle i, h(i)\rangle$ with $\langle i, x\rangle$.

In the preceding definition the function g, that enumerates the sets of assignments that make true the subformulas of σ, is definable in EST because on the one hand ${}^mM$ is a set, and on the other hand in each clause of the definition, $g(k)$ is a Δ_0 subclass of ${}^mM$, therefore a set.

Having defined what $\mathcal{M} \models \sigma(x_0, \ldots, x_{m-1})$ means for an L_A-structure $\mathcal{M}$, an L_A-formula $\sigma(v_0, \ldots, v_{m-1})$ with its free variables being among $v_0, \ldots, v_{m-1}$, and for $x_0, \ldots, x_{m-1} \in M$, we can then define elementary embeddings from one L_A-structure into another as usual.

Given two L_A-structures $\mathcal{M}$, $\mathcal{N}$, we say that $\mathcal{M}$ is *elementarily embeddable* in $\mathcal{N}$, notation $\mathcal{M} \precsim \mathcal{N}$, if there is a 1-1 function $f : M \to N$ such that for every formula $\sigma(v_0, \ldots, v_{m-1})$ of L_A with free variables among $v_0, \ldots, v_{m-1}$, and any $x_0, \ldots, x_{m-1} \in M$,

$$\mathcal{M} \models \sigma(x_0, \ldots, x_{m-1}) \leftrightarrow \mathcal{N} \models \sigma(f(x_0), \ldots, f(x_{m-1})).$$

Let $f : \mathcal{M} \precsim \mathcal{N}$ denote the fact that f is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Sometimes, for more precision, we need to specify the language we refer to. Then we say that $f : M \to N$ is an L_A-*elementary embedding*, and we write

$$f : \mathcal{M} \precsim_{L_A} \mathcal{N}.$$

Clearly the last relation is definable in EST. The following simple fact will be repeatedly used below.

FACT 2.11. *Let $\mathcal{M} = \langle M, U, R, I\rangle$, $\mathcal{N} = \langle N, Z, S, J\rangle$ be L_A-structures and $f : M \to N$ be an L_A-elementary embedding (or just an L_A-embedding). Then $f \circ I = J$, that is, for every $a \in A$, $f(I(a)) = J(a)$. Equivalently, for every $a \in A$, $f(\boldsymbol{c}_a^{\mathcal{M}}) = \boldsymbol{c}_a^{\mathcal{N}}$.*

Proof. For every L_A-embedding $f : \mathcal{M} \to \mathcal{N}$, by definition $f(\mathbf{c}_a^{\mathcal{M}}) = \mathbf{c}_a^{\mathcal{N}}$. Also, for every $a \in A$, $I(a) = \mathbf{c}_a^{\mathcal{M}}$ and $J(a) = \mathbf{c}_a^{\mathcal{N}}$. Therefore $f(I(a)) = J(a)$. ∎

The language $\mathcal{L} = \{\in\}$ of EST is just a particular instance of the languages L_A above, namely one with $A = \emptyset$ and with one binary relation symbol $\in$. So the classes of formulas and sentences of $\mathcal{L}$ are already definable in EST.

Given a formula $\phi(x)$ of $\mathcal{L} = \{\in\}$, let X_ϕ denote the extension of ϕ, although X_ϕ needs not be a set in EST. The expression "X_ϕ is a proper class" simply stands for the $\mathcal{L}$-sentence

$$(\forall x)(\exists y)[(\phi(y) \wedge y \notin x) \vee (\neg\phi(y) \wedge y \in x)].$$

[Note that in ZF "X_ϕ is a proper class" is formulated just as $(\forall x)(X_\phi \not\subseteq x)$ because of Separation. But in EST, where Separation is missing and $X_\phi \subseteq x$

does not imply that X_ϕ is a set, "X_ϕ is a proper class" has to be formulated as $(\forall x)(X_\phi \neq x)$.]

Given any formula $\phi(x)$ of $\mathcal{L}$ in one free variable, the instance VP_ϕ of Vopěnka's Principle is formulated as follows:

VP_ϕ: *For every set A, if X_ϕ is a proper class of L_A-structures, then there are $\mathcal{M} \neq \mathcal{N} \in X_\phi$ and f such that $f : \mathcal{M} \precsim_{L_A} \mathcal{N}$.*

Clearly VP_ϕ is an $\mathcal{L}$-sentence, so the class

$$\mathrm{VP} = \{x : (\exists \phi \in \mathit{Fml}(\mathcal{L}))(x = \mathrm{VP}_\phi)\}$$

is definable in EST. This completes the description of the scheme VP in EST.

2.4. Consequences of EST + VP. We begin with the proof of Infinity because it does not depend on any form of Replacement or Powerset. We show that the class $\omega = \{x : \mathit{Nat}(x)\}$, as defined in Section 2.2, is a set in EST + VP.

Theorem 2.12. *In* EST + VP, *the class $\omega = \{x : \mathit{Nat}(x)\}$ is a set. Therefore* EST + VP $\vdash$ *Inf.*

Proof. To avoid dealing with elementary embeddings of $\emptyset$, let $\omega^* = \omega - \{0\}$. Obviously in EST ω is a set iff ω^* is a set. So towards a contradiction assume that ω^* is a proper class. Consider the language $L_\emptyset = \{\mathbf{R}\}$ with only a binary relation symbol $\mathbf{R}$ (that is, $L_\emptyset$ contains no constants $\mathbf{c}_a$). For each $x \in \omega^*$, let $\in_x = \{\langle y, z\rangle \in x \times x : y \in z\}$ (the restriction of $\in$ to x). Since $x \times x$ is a set, by Δ_0-Separation so is $\in_x$. Let $\mathcal{M}_x = \langle x, \in_x\rangle$. Each $\mathcal{M}_x$ is an $L_\emptyset$-structure by interpreting $\mathbf{R}$ by $\in_x$. Let also

$$K = \{\mathcal{M}_x : x \in \omega^*\}.$$

We claim that K is a proper class when ω^* is so. Indeed, assume K is a set. Since

$$K = \{\langle x, \in_x\rangle : x \in \omega^*\} = \big\{\{\{x\}, \{x, \in_x\}\} : x \in \omega^*\big\},$$

clearly $\omega^* \subset \bigcup^2 K = \bigcup\bigcup K$. In particular $\omega^* = \{x \in \bigcup^2 K : \mathit{Nat}(x)\}$. Since $\bigcup^2 K$ is a set and $\mathit{Nat}(x)$ is Δ_0, ω^* is a set by Δ_0-Separation, a contradiction.

Thus K is a proper class of $L_\emptyset$-structures. By VP there exist $x_0 \neq x_1 \in \omega^*$ and a function $f : x_0 \to x_1$ such that $f : \mathcal{M}_{x_0} \precsim \mathcal{M}_{x_1}$.

But one can easily see by the clauses of Fact 2.4 that this is false. Indeed, since f is 1-1 and $x_0 \neq x_1$, by 2.4(i)–(iii), $x_0 \in x_1$, and hence x_0 is a proper initial segment of x_1. Also, by elementarity, we can see by induction on x_0 that f is the identity. [$0 = \emptyset$ is the first element of both x_0 and x_1, so $f(0) = 0$. Inductively, if $f(n) = n$ then the next element of n should be sent to the next element of $f(n)$, that is, $f(n+1) = n+1$.] The elements of ω^* are all successor ordinals, so let $x_0 = y_0 \cup \{y_0\}$ and $x_1 = y_1 \cup \{y_1\}$. Since $x_0 \neq x_1$, we have $y_0 \neq y_1$. By 2.4(iv), y_0 is the greatest element of x_0 and

$f(y_0) = y_0$, since f is the identity, while, by elementarity, $f(y_0)$ should be the greatest element y_1 of x_1. But $y_1 \neq y_0$, a contradiction. ∎

Next we come to the proof of Replacement.

THEOREM 2.13.

(i) $\mathrm{EST} + \mathrm{VP} \vdash \Delta_0$-*Rep*.
(ii) $\mathrm{EST} + \mathrm{VP} + \Delta_0$-*Rep* $\vdash$ *Rep*.
(iii) *Therefore* $\mathrm{EST} + \mathrm{VP} \vdash$ *Rep*.

Proof. (i) To prove Δ_0-*Rep*, let $\phi(x, y)$ be a Δ_0 formula such that $(\forall x)(\exists! y)\ \phi(x, y)$. This defines a class mapping $F_\phi : V \to V$ such that $F_\phi(x) = y$ iff $\phi(x, y)$. Fix a set A. It suffices to show that the class $B = F_\phi``A = \{F_\phi(a) : a \in A\}$ is a set. Let $L_A = \{\mathbf{U}\} \cup \{\mathbf{c}_a : a \in A\}$ be the language with a unary relation symbol $\mathbf{U}$ and a constant $\mathbf{c}_a$ for each $a \in A$. For every $b \in B$, $A \times \{b\}$ is a set, so for each such b consider the L_A-structure

$$\mathcal{M}_b = \langle A \times \{b\}, U_b, I_b \rangle,$$

where $U_b \subseteq A \times \{b\}$ is defined as follows: For every $a \in A$,

$$\langle a, b\rangle \in U_b \ \Leftrightarrow\ F_\phi(a) = b.$$

We have $U_b = \{\langle a, b\rangle \in A \times \{b\} : \phi(a, b)\}$. By *CartProd*, $A \times \{b\}$ is a set, and since ϕ is Δ_0, U_b is a set, by Δ_0-*Sep*, that interprets $\mathbf{U}$, that is, $\mathbf{U}^{\mathcal{M}_b} = U_b$. The constant assignment $I_b : A \to A \times \{b\}$ is defined by $I_b(a) = \langle a, b\rangle$ for each $a \in A$. Then I_b is a set too, by Δ_0-*Sep*, because $I_b = \{\langle x, \langle y, b\rangle\rangle \in A \times (A \times \{b\}) : x = y\}$ and $x = y$ is Δ_0. This means that for all $a \in A$ and $b \in B$, we have $\mathbf{c}_a^{\mathcal{M}_b} = \langle a, b\rangle$. Let

$$S = \{\mathcal{M}_b : b \in B\}.$$

It suffices to show that S is a set. For suppose that this is the case. Then clearly for some $n \in \omega$ (actually for $n = 7$), $B \subset \cup^n S$. Moreover

$$B = \{y \in \cup^n S : (\exists x \in A)(F_\phi(x) = y)\} = \{y \in \cup^n S : (\exists x \in A)\phi(x, y)\}.$$

Since $\cup^n S$ is a set and the formula $(\exists x \in A)\phi(x, y)$ is Δ_0, it follows by Δ_0-Separation that B is set.

So let us verify that S is a set. To reach a contradiction assume that S is a proper class. Then by VP there are $b, c \in B$, $b \neq c$, and a mapping $f : A \times \{b\} \to A \times \{b\}$ such that $f : \mathcal{M}_b \precsim \mathcal{M}_c$. By elementarity, for every $a \in A$,

$$f(\langle a, b\rangle) = f(\mathbf{c}_a^{\mathcal{M}_b}) = \mathbf{c}_a^{\mathcal{M}_c} = \langle a, c\rangle.$$

On the other hand, by elementarity again, for every $a \in A$,

$$F_\phi(a) = b \ \Leftrightarrow\ \langle a, b\rangle \in U_b \ \Leftrightarrow\ f(\langle a, b\rangle) \in U_c \ \Leftrightarrow \langle a, c\rangle \in U_c \ \Leftrightarrow F_\phi(a) = c,$$

which is a contradiction since $b \neq c$.

(ii) Now we work in EST+VP+Δ_0-*Rep*, and prove that full Replacement holds. The proof is for the most part similar to that of clause (i) above. Let $\phi(x, y)$ be a formula such that $(\forall x)(\exists! y)\phi(x, y)$, and let $F_\phi(x) = y$ iff $\phi(x, y)$. We fix again a set A, and show that if $B = F_\phi$"A, then B is a set. We define the structures $\mathcal{M}_b$ as before and we set $S = \{\mathcal{M}_b : b \in B\}$. As in (i), it follows by means of VP that S cannot be a proper class. Thus S is a set. The only departure from the proof of (i) is at the point of inferring that B is a set from S being a set. This now can be achieved by the help of Δ_0-Replacement: just observe that the mapping $S \ni \mathcal{M}_b \mapsto b \in B$ is clearly Δ_0-definable and onto. Therefore B is a set.

(iii) This is immediate from (i) and (ii). ∎

Now we come to the proof of Powerset, which is based on a clause of Theorem 2.13.

Theorem 2.14. *EST+VP+Δ_0-Rep $\vdash$ Pow. Hence, by Theorem 2.13(i), EST + VP $\vdash$ Pow.*

Proof. We work in EST + VP + Δ_0-*Rep*. Fix a set A, and let again $L_A = \{\mathbf{U}\} \cup \{\mathbf{c}_a : a \in A\}$, where $\mathbf{U}$ is a unary relation symbol. For each $X \in \mathcal{P}(A)$ consider the L_A-structure

$$\mathcal{M}_X = \langle A, X, id_A \rangle,$$

where for each $X \subseteq A$, $\mathbf{U}^{\mathcal{M}_X} = X$ and the constant assignment is the identity mapping $id_A : A \to A$. Note that id_A is a set in EST, by Δ_0-*Sep*, since $id_A = \{\langle x, y\rangle \in A \times A : x = y\}$. Thus $\mathbf{c}_a^{\mathcal{M}_X} = a$ for every $a \in A$. To reach a contradiction, assume that $\mathcal{P}(A)$ is a proper class. Let

$$K = \{\mathcal{M}_X : X \in \mathcal{P}(A)\}.$$

The mapping $K \ni \mathcal{M}_X \mapsto X \in \mathcal{P}(A)$ is clearly Δ_0, so by Δ_0-Replacement, the class K is proper too. By VP there are $X \neq Y \in \mathcal{P}(A)$ and an elementary embedding $f : \mathcal{M}_X \to \mathcal{M}_Y$. But then $f(a) = f(\mathbf{c}_a^{\mathcal{M}_X}) = \mathbf{c}_a^{\mathcal{M}_Y} = a$ for every $a \in A$. That is, $f = id_A$. On the other hand, by elementarity, f should map 1-1 X onto Y, hence $X = Y$, a contradiction. ∎

Now ZF = EST + $\{Inf, Pow, Rep, Found\}$. So from Theorems 2.12–2.14 we immediately obtain the following:

Corollary 2.15. *EST + Found + VP = ZF + VP, and EST + Found + AC + VP = ZFC + VP.*

As mentioned at the beginning of this section, the replacement of VP_4 (used in Theorem 2.2) by VP (used in Theorems 2.12–2.14) was necessitated by the fact that Theorem V-P-H was among the assumptions of 2.2, while this is not the case for 2.12–2.14. Inspecting the proof of V-P-H in [18], we see that it relies heavily on AC, as well as on the following two facts:

(a) For every well-ordered set $\langle x, \leq\rangle$, there is a (unique) ordinal α such that $\langle x, \leq\rangle \cong \langle \alpha, \in\rangle$.
(b) For every ordinal α, there exists the set of ordinals of countable cofinality below α, $\{\beta < \alpha : \mathrm{cf}(\beta) = \omega\}$.

Clause (a) requires Δ_1-Replacement, while (b) requires, firstly, that ω is a set, and secondly, Σ_1-Separation. These being available the proof of V-P-H goes through, so

$$\mathrm{EST} + \{\mathrm{AC}, \text{“}\omega \text{ is a set”}, \Delta_1\text{-}Rep, \Sigma_1\text{-}Sep\} \vdash \text{V-P-H}.$$

With V-P-H at hand we can work exactly as in the proof of 2.2 with VP_4 in place of VP (a rigid binary relation R on any set A does the job that the constants $\mathbf{c}_a$ do in VP). Thus we obtain the following.

Corollary 2.16. *The theory*

$$\mathrm{EST} + \{\mathrm{AC}, \text{“}\omega \textit{ is a set”}, \Delta_1\text{-}Rep, \Sigma_1\text{-}Sep\} + \mathrm{VP}_4$$

proves Replacement and Powerset.

Corollary 2.16 is in sharp contrast to the results 2.13 and 2.14 above, which together show that EST+VP *alone* proves Replacement and Powerset. This gives a measure of the difference in apparent strength between the principles VP_4 and VP over EST.

Remark 2.17. It is further worth mentioning that in the proofs of Theorems 2.13 and 2.14 we did not use the full strength of VP. A simple inspection of the proofs shows that in these results we used the fact that for a given proper class X_ϕ of L_A-structures there are distinct structures $\mathcal{M}, \mathcal{N}$ in X_ϕ and an *embedding only* $f : \mathcal{M} \to \mathcal{N}$, rather than an elementary embedding. In contrast, in Theorem 2.12 some kind of elementarity for f is required. Therefore 2.13 and 2.14 can still be established by means of the following weaker form, VP_0, of Vopěnka's Principle. Let VP_0 result from VP if “elementary embedding” is replaced by “embedding” (while the languages L_A still contain finitely many relations and an arbitrary set of constants). Then $\mathrm{EST} + \mathrm{VP}_0$ proves Replacement and Powerset.

Since $\mathrm{EST} \subset \mathrm{LZFC}$ (see footnote 1 after Theorem 2.2), as an immediate corollary to Theorems 2.13 and 2.14 we obtain the following improvement to [17, Theorem 6.3]:

Theorem 2.18. $\mathrm{LZFC} + \mathrm{VP}$ *proves Replacement and Powerset, that is,* $\mathrm{ZFC} \subseteq \mathrm{LZFC} + \mathrm{VP}$.

The improvement is of course that the requirement for T to prove V-P-H is no longer needed for LZFC.

On the other hand, $\mathrm{ZFC} + \mathrm{VP}$ implies the existence of a proper class of extendible cardinals (see [11, Lemma 20.25] and the remark after its proof).

Since for every extendible cardinal κ, $V_\kappa \models \mathrm{ZFC}$, we immediately infer that $\mathrm{ZFC} + \mathrm{VP} \vdash \mathit{Loc}(\mathrm{ZFC})$, where $\mathit{Loc}(\mathrm{ZFC})$ is the central axiom of LZFC saying that "every set belongs to a transitive model of ZFC". From this we have:

LEMMA 2.19. $\mathrm{LZFC} \subset \mathrm{ZFC} + \mathrm{VP}$.

From Theorem 2.18 and Lemma 2.19 we obtain:

THEOREM 2.20. $\mathrm{LZFC} + \mathrm{VP} = \mathrm{ZFC} + \mathrm{VP}$.

3. VP and Foundation. Let $\mathrm{ZF}_0 = \mathrm{ZF} - \{\mathit{Found}\}$. In this section we show that VP does not prove Foundation over ZF_0. Namely, the following holds:

THEOREM 3.1. *If* $\mathrm{ZF} + \mathrm{VP}$ *is consistent, then so is* $\mathrm{ZF}_0 + \mathrm{VP} + \neg\mathit{Found}$. *Similarly with* ZFC *in place of* ZF.

The proof is by the well-known method of using a non-standard membership relation $\in_\pi$ in V, produced by a definable permutation of V. Namely, it is a rather folklore result that if V is the universe of ZFC, $\pi : V \to V$ is a definable permutation, and $\in_\pi$ is the binary relation defined by $x \in_\pi y$ iff $x \in \pi(y)$, then $V_\pi \models \mathrm{ZFC}_0$ (see [12, Ch. IV, exercise 18]). For simplicity, let us abbreviate henceforth $\langle V, \in_\pi\rangle$ to V_π and $\langle V, \in\rangle$ to V. In order for Foundation to fail in V_π it suffices to take π so that $x \in \pi(x)$ for some x. In this way we shall prove the next theorem, from which Theorem 3.1 follows.

THEOREM 3.2. *If* $V \models \mathrm{ZF} + \mathrm{VP}$, *then there is a permutation* $\pi : V \to V$ *such that* $V_\pi \models \mathrm{VP} + \neg\mathit{Found}$.

We shall need first some preliminary definitions and lemmas. Given a definable permutation $\pi : V \to V$, let us denote by ϕ^π, for every formula ϕ of $\mathcal{L} = \{\in\}$, the formula resulting from ϕ if we replace every atomic subformula $x \in y$ occurring in ϕ by $x \in \pi(y)$. The following is easy to check by induction on the length of ϕ.

LEMMA 3.3.

(i) *The mapping* $\phi \mapsto \phi^\pi$ *commutes with connectives and quantifiers, that is,* $(\phi \to \psi)^\pi = (\phi^\pi \to \psi^\pi)$, $(\neg\phi)^\pi = \neg\phi^\pi$, *and* $(\forall x\phi)^\pi = (\forall x)\phi^\pi$.
(ii) *For every sentence* ϕ, $V_\pi \models \phi$ *iff* $V \models \phi^\pi$.

For every standard notion of V, like singleton, ordered pair, n-tuple, relation, function, there is a corresponding π-notion for V_π. For instance a π-*pair* is a set z such that $V_\pi \models$ "z is an ordered pair". The latter holds iff for some x, y we have $V_\pi \models z = \langle x, y\rangle$. By 3.3(ii), this is equivalent to $(z = \langle x, y\rangle)^\pi$ (we often write just ϕ instead of $V \models \phi$), more conveniently denoted by $z = \langle x, y\rangle^\pi$. Similarly the fact that Q is a π-binary relation between sets M and N means that $(Q \subseteq M \times N)^\pi$ is true. A π-notion expressed by a sentence ϕ is said to be *absolute* if $\phi^\pi \leftrightarrow \phi$. In the next two lemmas we give some simple sufficient conditions concerning the permutation π in order for some key notions to be absolute.

LEMMA 3.4. *Suppose that $\pi : V \to V$ fixes all finite sets. Then:*

(i) *π-pairs are absolute, that is, for all x, y, $\langle x, y\rangle^\pi = \langle x, y\rangle$.*
(ii) *For all M, N, Q, $(Q \subseteq M \times N)^\pi \leftrightarrow \pi(Q) \subseteq \pi(M) \times \pi(N)$.*
(iii) *$[f : M \to N$ is a function$]^\pi$ is equivalent to $\pi(f) : \pi(M) \to \pi(N)$ being a function.*

Proof. (i) Assume π fixes all finite sets. Then so also does π^{-1}. Analyzing the definition of $(z = \langle x, y\rangle)^\pi := (z = \{\{x\}, \{x, y\}\})^\pi$, it is easy to see that

$$z = \langle x, y\rangle^\pi \ \leftrightarrow\ z = \pi^{-1}\big(\{\pi^{-1}(\{x\}), \pi^{-1}(\{x, y\})\}\big). \tag{1}$$

All the arguments of π^{-1} on the right-hand side of (1) are finite, so π^{-1} fixes them. Therefore (1) implies

$$z = \langle x, y\rangle^\pi \ \leftrightarrow\ z = \{\{x\}, \{x, y\}\} = \langle x, y\rangle.$$

(ii) Analyzing the definition of $(Q \subseteq M \times N)^\pi$ we see that

$$(Q \subseteq M \times N)^\pi \ \leftrightarrow\ \pi(Q) \subseteq \{\langle x, y\rangle^\pi : x \in \pi(M),\, y \in \pi(N)\}. \tag{2}$$

By (i) above, $\langle x, y\rangle^\pi = \langle x, y\rangle$ for every pair. So (2) gives

$$(Q \subseteq M \times N)^\pi \ \leftrightarrow\ \pi(Q) \subseteq \{\langle x, y\rangle : x \in \pi(M),\, y \in \pi(N)\} = \pi(M) \times \pi(N).$$

(iii) Easy to check. ∎

Recall that a language L_A consists of the symbols $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{c}_a$ for $a \in A$, and an L_A-structure is a quadruple $\mathcal{M} = \langle M, U, R, I\rangle$, where $R \subseteq M \times M$, $U \subseteq M$ and $I : A \to M$ is a mapping.

Given an L_A-structure $\mathcal{M} = \langle M, U, R, I\rangle$ and a permutation $\pi : V \to V$, set

$$\mathcal{M}^\pi := \langle \pi(M), \pi(U), \pi(R), \pi(I)\rangle.$$

LEMMA 3.5. *Suppose that $\pi : V \to V$ fixes all finite sets and ω.*

(i) *Let L_A be a first-order language in the sense of V_π, for some $A \in V_\pi$. Then $\pi(L_A) = L_{\pi(A)}$ is a language in V.*
(ii) *If σ is a formula of L_A in the sense of V_π, then σ is a formula of $L_{\pi(A)}$ in V.*
(iii) *If $\mathcal{M}$ is an L_A-structure in V_π, then $\mathcal{M}^\pi$ is an $L_{\pi(A)}$-structure in V.*
(iv) *If $\mathcal{M}$ is an L_A-structure in V_π, then*

$$V_\pi \models [x_0, x_1 \in M \wedge (\mathcal{M} \models \mathbf{R}(x_0, x_1))] \\ \leftrightarrow\ V \models [x_0, x_1 \in \pi(M) \wedge (\mathcal{M}^\pi \models \mathbf{R}(x_0, x_1))],$$

and similarly for the predicate $\mathbf{U}$.

Proof. (i) Recall that by Definition 2.5,

$$V_\pi \models L_A = \{2,3,4,5,6,7\} \cup \{\langle 0,n\rangle : n \in \omega\} \cup \{\langle 1,a\rangle : a \in A\}.$$

In view of Lemma 3.3, this is equivalently written

$$V \models \pi(L_A) = \{2,3,4,5,6,7\}^\pi \cup \{\langle 0,n\rangle^\pi : n \in \pi(\omega)\} \cup \{\langle 1,a\rangle^\pi : a \in \pi(A)\}.$$

Since π fixes all finite sets and ω, π-pairs are absolute by Lemma 3.4, $\{2,3,4,5,6,7\}^\pi = \{2,3,4,5,6,7\}$ and $\pi(\omega) = \omega$, we have

$$V \models \pi(L_A) = \{2,3,4,5,6,7\} \cup \{\langle 0,n\rangle : n \in \pi(\omega)\} \cup \{\langle 1,a\rangle : a \in \pi(A)\}.$$

But the right-hand side of the above equation is clearly the language $L_{\pi(A)}$, so $\pi(L_A) = L_{\pi(A)}$.

(ii) Since $\pi(\omega) = \omega$ and $\pi(n) = n$ for every $n \in \omega$, ω is absolute in V_π. Thus the claim follows by a simple induction on σ along the steps of Definition 2.7.

(iii) That $\mathcal{M} = \langle M, U, R, I\rangle$ is an L_A-structure in the sense of V_π means that $(U \subseteq M)^\pi$, $(R \subseteq M \times M)^\pi$ and $(I : A \to M$ is a function$)^\pi$. Since π fixes all finite sets, these facts are translated into V, according to 3.4, as $\pi(U) \subseteq \pi(M)$, $\pi(R) \subseteq \pi(M) \times \pi(M)$ and $\pi(I) : \pi(A) \to \pi(M)$ is a function, respectively. But this means that $\langle \pi(M), \pi(U), \pi(R), \pi(I)\rangle$, that is, $\mathcal{M}^\pi$ is an $L_{\pi(A)}$-structure.

(iv) Let $\mathcal{M} = \langle M, U, R, I\rangle$ be an L_A-structure in V_π and let

$$V_\pi \models [x_0, x_1 \in M \wedge (\mathcal{M} \models \mathbf{R}(x_0, x_1))].$$

Obviously, this is equivalently written as

$$V_\pi \models [x_0, x_1 \in M \wedge \langle x_0, x_1\rangle \in R].$$

Its translation to V is

$$V \models [x_0, x_1 \in \pi(M) \wedge \langle x_0, x_1\rangle^\pi \in \pi(R)].$$

Since $\langle x_0, x_1\rangle^\pi = \langle x_0, x_1\rangle$ by the condition on π, the latter is also equivalent to

$$V \models [x_0, x_1 \in \pi(M) \wedge (\mathcal{M}^\pi \models \mathbf{R}(x_0, x_1))]. \blacksquare$$

LEMMA 3.6. *Let π be a permutation that fixes all finite sets and ω, and let $\mathcal{M} = \langle M, U, R, I\rangle$ and $\mathcal{N} = \langle N, Z, S, J\rangle$ be L_A-structures in V_π. Then for any f,*

$$V_\pi \models [f : \mathcal{M} \precsim_{L_A} \mathcal{N}] \quad \textit{iff} \quad V \models [\pi(f) : \mathcal{M}^\pi \precsim_{L_{\pi(A)}} \mathcal{N}^\pi].$$

Proof. Let us sketch the proof of "$\to$". The other direction is similar. Assume that π is as stated, $\mathcal{M}$, $\mathcal{N}$ are L_A-structures in V_π and $V_\pi \models [f : \mathcal{M} \precsim_{L_A} \mathcal{N}]$, that is, $f : M \to N$ is an L_A-elementary embedding. By Lemma 3.5, $\mathcal{M}^\pi, \mathcal{N}^\pi$ are $L_{\pi(A)}$-structures in V. We have to show that $\pi(f) : \pi(M) \to \pi(N)$ is an $L_{\pi(A)}$-elementary embedding. For this we must show that for every $L_{\pi(A)}$-formula $\sigma(v_0,\ldots,v_{n-1})$ and any $x_0,\ldots,x_{n-1}\in\pi(M)$,

$$\mathcal{M}^\pi \models \sigma(x_0,\ldots,x_{n-1}) \ \leftrightarrow\ \mathcal{N}^\pi \models \sigma\big(\pi(f)(x_0),\ldots,\pi(f)(x_{n-1})\big).$$

This is done by routine induction on the length of σ. Let us just show the above for the atomic sentences $\mathbf{R}(x_0,x_1)$ of $L_{\pi(A)}$. This amounts to showing that if $x_0,x_1\in\pi(M)$ and $y_0,y_1\in\pi(N)$, then (in V):

$$\text{(3)}\qquad \langle x_0,y_0\rangle\in\pi(f)\wedge\langle x_1,y_1\rangle\in\pi(f) \\ \qquad\qquad \to\ (\mathcal{M}^\pi\models\mathbf{R}(x_0,x_1)\leftrightarrow\mathcal{N}^\pi\models\mathbf{R}(y_0,y_1)).$$

But by our assumption $V_\pi\models[f:\mathcal{M}\precsim_{L_A}\mathcal{N}]$, for all $x_0,x_1\in M$ and $y_0,y_1\in N$ we have

$$\text{(4)}\qquad V_\pi\models[\langle x_0,y_0\rangle\in f\wedge\langle x_1,y_1\rangle\in f] \\ \qquad\qquad \to\ (\mathcal{M}\models\mathbf{R}(x_0,x_1)\leftrightarrow\mathcal{N}\models\mathbf{R}(y_0,y_1)).$$

By Lemmas 3.3 and 3.5(iv) (that hold because of our conditions about π), (3) is just the translation of (4) to V. The other steps of the induction are routine. ∎

Proof of Theorem 3.2. Let $V\models \mathrm{ZF}+\mathrm{VP}$. Pick a permutation π of V that fixes all finite sets and ω. Then Lemmas 3.4–3.6 above hold. Suppose also that for some sets X and Y such that $X\in Y$, π exchanges X and Y, that is, $\pi(X)=Y$ and $\pi(Y)=X$, so Foundation fails in V_π. It remains to show that $V_\pi\models\mathrm{VP}$.

Let L_A be a language in the sense of V_π. The set A, which essentially contains the constants of the language, can be arbitrary, so in particular we may have $A=X$ or $A=Y$. For that reason in general $\pi(A)\neq A$. Let $\phi(x)$ be a formula of $\mathcal{L}=\{\in\}$ such that

$$\text{(5)}\qquad V_\pi\models \text{“}X_\phi \text{ is a proper class of } L_A\text{-structures”}.$$

We have to show that

$$\text{(6)}\qquad V_\pi\models(\exists\mathcal{M}\neq\mathcal{N}\in X_\phi)(\mathcal{M}\precsim_{L_A}\mathcal{N}).$$

Now (5) implies that X_ϕ is a proper class in V_π, that is,

$$\text{(7)}\qquad V_\pi\models(\forall x)(\exists y)(\phi(y)\wedge y\notin x),$$

and also

$$\text{(8)}\qquad V_\pi\models(\forall x)[\phi(x)\to x \text{ is an } L_A\text{-structure}].$$

From (7) we have

$$\text{(9)}\qquad V\models(\forall x)(\exists y)(\phi^\pi(y)\wedge y\notin\pi(x)),$$

that is, X_{ϕ^π} is a proper class in V. Moreover, if $\mathcal{M}\in X_{\phi^\pi}$, then $V_\pi\models\phi(\mathcal{M})$, so by (8), $\mathcal{M}$ is an L_A-structure in V_π. By Lemma 3.5(iii), $\mathcal{M}^\pi$ is an $L_{\pi(A)}$-

structure in V. Thus

$$\mathcal{M} \in X_{\phi^\pi} \ \rightarrow\ \mathcal{M}^\pi \text{ is an } L_{\pi(A)}\text{-structure.} \tag{10}$$

Consider the formula $\psi(x)$ of $\mathcal{L}$ defined by

$$\psi(x) := (\exists \mathcal{M})(x = \mathcal{M}^\pi \wedge \phi^\pi(\mathcal{M})).$$

Then clearly for every $\mathcal{M}$,

$$V \models \phi^\pi(\mathcal{M}) \ \Leftrightarrow\ V_\pi \models \phi(\mathcal{M}) \ \Leftrightarrow\ V \models \psi(\mathcal{M}^\pi). \tag{11}$$

By (10) and (11) the elements of X_ψ are $L_{\pi(A)}$-structures. Moreover, the functional correspondence $X_{\phi^\pi} \ni \mathcal{M} \mapsto \mathcal{M}^\pi \in X_\psi$ is 1-1, so X_ψ is a proper class since X_{ϕ^π} is so. Since VP is true in V, we have

$$V \models (\exists x, y)(\psi(x) \wedge \psi(y) \wedge x \neq y \wedge x \precsim_{L_{\pi(A)}} y). \tag{12}$$

Pick two distinct structures $\mathcal{M}^\pi$, $\mathcal{N}^\pi$ of X_ψ such that

$$\mathcal{M}^\pi \precsim_{L_{\pi(A)}} \mathcal{N}^\pi.$$

Then, by Lemma 3.6, it follows that $V_\pi \models \mathcal{M} \precsim_{L_A} \mathcal{N}$ and also $\mathcal{M} \neq \mathcal{N} \in X_{\phi^\pi}$. Therefore

$$V_\pi \models \mathcal{M} \neq \mathcal{N} \in X_\phi \wedge \mathcal{M} \precsim_{L_A} \mathcal{N}.$$

But this is the required conclusion (6). ∎

4. VP and Choice. What still remains open with respect to VP and the axioms of ZFC is the relationship of VP with AC, namely the following:

QUESTION 4.1. *Assume that* ZFC+VP *is consistent. Is* AC *independent of* ZF + VP*?*

Below we make two comments, one concerning the independence of AC and one concerning the opposite direction.

4.1. VP and symmetric models. We guess that AC is independent of ZF + VP. To establish this, however, the most natural way seems to be through the technique of permutation models of ZFA or symmetric generic models of ZF, which are the standard tools for refuting AC. The technical details of the method can be found in [11]. Also, for a comprehensive list of various symmetric models and their applications one can consult [9, Part III].

These methods lead inevitably to the following steps. We start with a model V of ZFA + AC + VP (or ZFC + VP) and choose a symmetric model $HS \subset V$ for which we intend to show that $HS \models \text{VP} + \neg\text{AC}$. Assuming that already $HS \models \neg\text{AC}$, it remains to establish that $HS \models \text{VP}$. Let X_ϕ be a proper class of L_A-structures, in the sense of HS, for some $A \in HS$. We have to show that

$$HS \models (\exists x, y)(\phi(x) \wedge \phi(y) \wedge x \neq y \wedge x \precsim y). \tag{13}$$

Since HS is an inner submodel of V, $X_{\phi^{HS}}$ is a proper class of L_A-structures in the sense of V, where ϕ^{HS} is the usual relativization of ϕ to HS. Since VP holds in V, we have

$$V \models (\exists x, y)(\phi^{HS}(x) \wedge \phi^{HS}(y) \wedge x \neq y \wedge x \precsim y). \tag{14}$$

Thus the proof of the independence of AC amounts to showing that (13) *can be derived from* (14).

In fact, proving the derivation (14)$\to$(13) is a challenging problem that cannot be settled in the "easy way". The easy way would be the deduction of (14)$\to$(13) through an implication of the following form: For any language L_A and any L_A-structures $\mathcal{M}$, $\mathcal{N}$ in HS,

$$V \models \mathcal{M} \precsim \mathcal{N} \ \to \ HS \models \mathcal{M} \precsim \mathcal{N}. \tag{15}$$

Obviously, if (15) were true for every L_A, the implication (14)$\to$ (13) would be true as well. But (15) is false in general. For if we take $L_A = \emptyset$, the L_A-structures are just sets, and elementary embeddings are simple injections. So (15) would imply in particular that for all $x, y \in HS$,

$$V \models |x| \leq |y| \ \to \ HS \models |x| \leq |y|. \tag{16}$$

But since $V \models$ AC,

$$V \models (\forall x, y)(|x| \leq |y| \vee |y| \leq |x|).$$

So by (16),

$$HS \models (\forall x, y)(|x| \leq |y| \vee |y| \leq |x|).$$

The last sentence says that in HS the cardinalities of all sets are comparable, and this is well-known to be equivalent to AC (see for example [14, Theorem 3.1]). Therefore $HS \models$ AC, which is false!

Summing up: Answering Question 4.1 in the affirmative amounts to finding a symmetric model HS and a non-straightforward proof of the implication (14)$\to$(13), for any $L_A \in HS$ and any proper class X_ϕ of L_A-structures.

4.2. Weak forms of Choice related to VP. Now let us have a look at the opposite direction. Despite the fact that ZF+VP is unlikely to prove AC, weaker forms of AC might be derived.

In the previous subsection we mentioned the well-known equivalence of AC with the fact that the cardinalities of any two sets are comparable. This last formulation of AC admits natural weakenings, and one extreme such weakening is a consequence of VP.

To facilitate discussion let us say that, in ZF, two sets x, y are *comparable* if their cardinalities are, that is, if either $|x| \leq |y|$ or $|y| \leq |x|$. Otherwise they are said to be *incomparable.* For each formula $\phi(x)$ in one free variable, consider the following *comparability* axiom:

Comp$_\phi$: If X_ϕ is a proper class of sets, then it contains at least two comparable elements.

Then the following fact shows a slight dependence of VP with Choice.

FACT 4.2. *For every property* $\phi(x)$*,* $\mathrm{ZF} + \mathrm{VP} \vdash \mathit{Comp}_\phi$.

Proof. Let X_ϕ be a proper class of sets. As already mentioned in Subsection 4.1 above, X_ϕ can be thought of as a proper class of L-structures for $L = \emptyset$, for which 1-1 mappings are elementary embeddings. Thus, by VP, there are $x \neq y \in X_\phi$ such that $|x| \leq |y|$. ■

Once Comp_ϕ are defined, other similar axioms come up naturally. For every standard cardinal number (4) $\kappa \geq 2$, consider the following comparability axiom:

Comp_κ: $(\forall x)[|x| = \kappa \to (\exists y \neq z \in x)(y, z \text{ are comparable})]$.

[Comp_κ says that every set of cardinality κ contains distinct comparable elements.] Comp_κ becomes weaker and weaker as κ increases. Moreover, Comp_ϕ look like "weakest limits" of Comp_κ, although, apart from the implications $\mathit{Comp}_n \to \mathit{Comp}_\phi$, $n \in \omega$, which obviously hold in ZF, it was unknown for which $\kappa \geq \omega$ (if any) $\mathit{Comp}_\kappa \to \mathit{Comp}_\phi$ are also true.

However, quite recently Lefteris Tachtsis [15] proved that Comp_ω implies (over ZF) that finite sets coincide with Dedekind-finite sets (where X is *Dedekind-finite* if $\aleph_0 \not\leq |X|$, or equivalently, if there is no injection $f : X \to X$ such that $f``X \subsetneq X$). Let F and DF denote the classes of finite and Dedekind-finite sets, respectively. It is well-known that over ZF we have $\mathsf{F} \subseteq \mathsf{DF}$, while AC implies $\mathsf{F} = \mathsf{DF}$.

THEOREM 4.3 (Tachtsis [15]). *Over* ZF*,* Comp_ω *implies* $\mathsf{F} = \mathsf{DF}$.

By 4.3 one can show the following.

PROPOSITION 4.4. *In* ZF*, the following holds: for every* ϕ*, if* X_ϕ *is a proper class, then* $\mathit{Comp}_\omega \to \mathit{Comp}_\phi$.

Proof. Assume Comp_ω is true and X_ϕ is a proper class. Let us write X instead of X_ϕ. In view of Comp_ω, to show that Comp_ϕ is true it suffices to show that there is a set $x \subset X$ such that $|x| = \aleph_0$. Define $f : \omega \to \mathit{On}$ inductively as follows: $f(0) = 0$, $f(n+1) = \min\{\beta > f(n) : X \cap V_{f(n)} \subsetneq X \cap V_\beta\}$. Since X is a proper class, f is defined for every $n \in \omega$. (Otherwise, there exists $k \in \omega$ such that for every $\beta > f(k)$, $X \cap V_{f(k)} = X \cap V_\beta$. But then $X = X \cap V = X \cap V_{f(k)}$, so X is a set.) Let $y_n = X \cap V_{f(n)}$. Each y_n is a set and $y_n \subsetneq y_{n+1}$. So clearly $\bigcup_n y_n$ is an infinite set. By Theorem 4.3, $\bigcup_n y_n$ is Dedekind-infinite, thus there is $x \subseteq \bigcup_n y_n \subset X$ such that $|x| = \aleph_0$. ■

(4) By a standard cardinal number we mean an initial ordinal.

In fact the axioms $Comp_\kappa$ are not entirely new. $Comp_2$ is the already mentioned equivalent of AC, that any two sets are comparable. But also the axioms $Comp_n$, for $2 \le n < \omega$, are considered in [14, p. 22], under the name T3(n), and are attributed to A. Tarski (1964). Moreover, a significant result is shown in [14, Theorem 3.4]: $Comp_n$ is equivalent to AC for every $2 \le n < \omega$ ([5]).

The consistency of $\neg Comp_\kappa$, for every $\kappa \ge 2$, and $\neg Comp_\phi$, for *some* ϕ, can be shown by using the following result of [10]:

Theorem 4.5 ([10, Thm. 11.1]). *Let V be a model of* ZFA + AC, *with a set A of atoms, and let $\langle I, \preceq\rangle$ be a partially ordered set in V such that $|A| = |I| \cdot \aleph_0$. Then there is a permutation model $HS \subset V$ satisfying the following: There exists a family of sets $\{S_i : i \in I\}$ such that for all $i, j \in I$,*

$$i \preceq j \;\leftrightarrow\; |S_i| \le |S_j|. \tag{17}$$

The method of permutation models also works if the atoms form a proper class rather than a set (see [10, p. 139]). Starting with such a model it is not difficult to strengthen 4.5 as follows:

Theorem 4.6. *Let V be a model of* ZFA + AC, *where now A is a proper class of atoms. Let also $\langle I, \preceq\rangle$ be a partially ordered proper class such that for every $i \in I$, $\{j : j \preceq i\}$ is a set, and $A = \{a_{in} : i \in I,\, n \in \omega\}$. Then there is a permutation model $HS \subset V$ and a proper class $\{S_i : i \in I\} \subset HS$ such that* (17) *holds.*

Sketch of proof. By assumption, for every $i \in I$, $\hat{i} = \{j \in I : j \preceq i\}$ is a set and $i \preceq j \leftrightarrow \hat{i} \subseteq \hat{j}$. Thus $\langle I, \preceq\rangle$ is embedded in $\langle \mathcal{P}(I), \subseteq\rangle$, where $\mathcal{P}(I)$ is the class of subsets of I, so it suffices to show that there is a class $\{S_x : x \subset I\}$ such that $x \subseteq y \leftrightarrow |S_x| \le |S_y|$ for all $x, y \subset I$. By assumption again, $A = \{a_{in} : i \in I,\, n \in \omega\}$. For each $x \in \mathcal{P}(I)$, let $S_x = \{a_{in} : i \in x,\, n \in \omega\}$. Consider the class-group $\mathcal{G}$ of permutations π of A such that $\pi(S_{\{i\}}) = S_{\{i\}}$, that is, for every $i \in I$ and every $n \in \omega$ there is $m \in \omega$ such that $\pi(a_{in}) = a_{im}$ ($\mathcal{G}$ is a class of coded classes). Also let $\mathcal{F}$ be the filter of (suitably coded) subgroups of $\mathcal{G}$ generated by the ideal of finite subsets of A. Then the proof goes as for Theorem 4.5 above. ∎

Now by taking $\langle I, \preceq\rangle$ to be an antichain, that is, $i \preceq i \leftrightarrow i = j$, we obtain as a corollary to the preceding theorems the relative consistency of $\neg Comp_\kappa$, and $\neg Comp_\phi$ for a specific ϕ.

Corollary 4.7.

(i) *Let $V \models$* ZFA + AC, *with a set A of atoms, and let κ be a cardinal number such that $|A| = \kappa \cdot \aleph_0$. Then there is a permutation model $HS \subset V$ such that $HS \models \neg Comp_\kappa$.*

([5]) I am indebted to Lefteris Tachtsis for bringing this reference to my attention.

(ii) *Let* $V \models \mathrm{ZFA} + \mathrm{AC}$, *with a proper class of atoms* $A = \{a_{in} : i \in I, n \in \omega\}$. *Then there is a permutation model* $HS \subset V$ *and a class* $X_\phi = \{S_i : i \in I\}$ *in* HS *such that* $HS \models \neg Comp_\phi$.

Proof. (i) Taking I to be an antichain such that $|I| = \kappa$, we get the claim immediately from Theorem 4.5.

(ii) Similarly, this follows from Theorem 4.6 for an antichain I which is a proper class and $X_\phi = \{S_i : i \in I\}$. ∎

We close with two questions concerning the relative strength of comparability axioms and their relationship with VP.

QUESTIONS 4.8.

(1) *Is any of the implications* $Comp_\kappa \to Comp_\lambda$ *for* $\omega \le \kappa < \lambda$ *reversible over* ZF, *or over* ZF + VP*?*
(2) *Does* ZF + VP *prove* $Comp_\kappa$ *for some* $\kappa \ge 2$*?*

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