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On universal enveloping algebras in a topological setting

by

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Abstract. We study some embeddings of suitably topologized spaces of vector-valued smooth functions on topological groups, where smoothness is defined via differentiability along continuous one-parameter subgroups. As an application, we investigate the canonical correspondences between the universal enveloping algebra, the invariant local operators, and the convolution algebra of distributions supported at the unit element of any finite-dimensional Lie group, when one passes from finite-dimensional Lie groups to pre-Lie groups. The latter class includes for instance all locally compact groups, Banach–Lie groups, additive groups underlying locally convex vector spaces, and also mapping groups consisting of rapidly decreasing Lie group-valued functions.

Contents

1. Introduction. It is well known that Lie theory and the related representation theory have been successfully extended much beyond the classical setting of finite-dimensional real Lie groups, and this research area now includes locally compact groups [HM07], [HM13], Lie groups modeled on Banach spaces or even on locally convex spaces [KM97], [Bel06], [Ne06], and some other classes of topological groups which may not be locally compact [BCR81], [Gl02b], [HM05]. The differential calculus on topological groups, involving functions which are smooth along one-parameter subgroups (Definition 2.3), plays an important role for these extensions of Lie theory and

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has recently found remarkable applications also to supergroups and their representation theory [NS13a], [NS13b]. We have merely mentioned here a very few references that are closely related to the topics of our paper.

On the other hand, as one can see for instance in [War72] or [Ped94], a key fact in harmonic analysis and representation theory is that the universal enveloping algebras of finite-dimensional Lie algebras can be realized by linear functionals or operators on spaces of smooth functions on the corresponding Lie groups, for instance as convolution algebras of distributions supported at the unit element or as invariant linear differential operators. (See also [Gl12] for situations of infinite-dimensional Lie algebras $\mathfrak{g}$ whose universal enveloping algebra can be made into a topological algebra as a quotient of the tensor algebra of $\mathfrak{g}$; in particular, this is the case for any Banach–Lie algebra $\mathfrak{g}$.) It is then natural to seek for such realizations beyond the classical setting of finite-dimensional Lie groups, with motivation coming from the representation theory of groups of the aforementioned types. In the present paper we begin an investigation of that question, oriented towards a pretty large class of topological groups which have sufficiently many one-parameter subgroups, namely the pre-Lie groups; see Definition 5.4 and Examples 5.6–5.9 below.

To this end, one needs a suitable notion of distributions with compact support, that is, continuous linear functionals on the space of smooth functions on the group under consideration. While spaces of smooth functions on any topological group have already been studied in the literature, one still needs to give these function spaces a topology adequate for the purposes of turning their topological duals into associative algebras which act on function spaces by the natural operation of convolution. It should be pointed out here that although convolution of functions on a topological group requires some Haar measure on that group, this is not necessary for convolution of functions with distributions or measures (see Definition 3.5).

One of the main technical novelties of our paper is the construction of a suitable topology on the space of smooth functions on any topological group and with values in any locally convex space $\mathcal{Y}$, for which for arbitrary topological groups G and H one has the canonical topological embedding of spaces of smooth functions

$$\mathcal{C}^\infty(H\times G,\mathcal{Y})\hookrightarrow\mathcal{C}^\infty(H,\mathcal{C}^\infty(G,\mathcal{Y}))$$

(Theorem 4.16 and Remark 4.17). By using that fact, we then prove that for any topological group G, convolution with distributions with compact support (that is, linear functionals which are continuous for the aforementioned topology) does preserve the space $\mathcal{C}^\infty(G)$ of smooth functions (Proposition 5.1). By focusing on distributions supported at $\mathbf{1}\in G$, we can thus identify them with continuous linear operators on $\mathcal{C}^\infty(G)$ which commute with left translations and are local, in the sense that they do not increase the support of functions (Theorem 5.2). Recall that Peetre's theorem [Pee60] ensures that the local operators on smooth manifolds are precisely the differential operators, not necessarily of finite order. If G is any finite-dimensional Lie group, then we recover the natural correspondence between the distributions supported at $\mathbf{1} \in G$ and the left invariant differential operators on G.

The topology that we introduce on any function space $\mathcal{C}^\infty(G, \mathcal{Y})$ agrees with the topology of uniform convergence of functions and their derivatives on compact sets if G is any finite-dimensional real Lie group. However, unlike most constructions of similar topologies on spaces of test functions from the literature, our construction (Definition 3.1) does not need the group G to be locally compact. In fact, spaces of test functions, distributions, and universal enveloping algebras have already been investigated on locally compact groups which are not necessarily Lie groups, for instance:

- Basic distribution theory on abelian locally compact groups by using differentiability along one-parameter subgroups was developed in [Ri53].
- Let G be any topological group which is a projective limit of Lie groups. Under the additional hypotheses that G is simply connected, locally compact, and separable, in [Ka59], [Ka61], [Ma61], [Br61], and [BC75, Sect. 2] one endowed the space $\mathcal{C}^\infty(G)$ with the topology of a locally convex space, which is nuclear if and only if every quotient group of G whose Lie algebra is finite-dimensional is necessarily a Lie group, as proved in [BC75, Sätze 3.3, 3.5].
- Some nuclear function spaces on locally compact groups that do not use approximations by Lie groups were constructed in [Py74].
- Universal enveloping algebras of separable locally compact groups which are projective limits of Lie groups were studied in [Br61], [MM64], and [MM65].
- Differential operators in connection with distributions and convolutions on locally compact groups were also studied in [Ed88] and [Ak95].
- Significant implications in representation theory and Lie theory were recently investigated in [BB11], [BN15], [Nic14], and [Nic15].
- Also recently, Sobolev spaces on abelian locally compact groups, defined via Fourier transforms, were studied with motivation coming from some problems in mathematical physics; see [GPR13], [GoRe15], and the references therein.

Our article is organized as follows: In Section 2 we provide some basic definitions and auxiliary results from the differential calculus on topological groups. Section 3 introduces the convolution of smooth functions with compactly supported distributions and states one of the main problems which motivated the present investigation (Problem 3.15). Section 4 is devoted to proving the aforementioned embedding of spaces of smooth functions on topological groups (Theorem 4.16), which is our main technical result. Finally, in Section 5 we use that result to establish the structure of invariant local operators (Theorem 5.2).

General notation. Throughout the present paper we denote by G, H arbitrary topological groups, unless otherwise mentioned. We assume that the topology of any topological group is separated. For any topological spaces T and S we denote by $\mathcal{C}(T,S)$ the set of all continuous maps $f\colon T\to S$.

2. Preliminaries. This section presents some ideas and notions of Lie theory that play a key role in the present paper. Our basic references for Lie theory of topological groups are [BCR81], [HM05], and [HM07]. See also [BB11], [BN15], [Nic14], and [Nic15] for more recent developments.

The adjoint action of a topological group. Let G be any topological group with the set of neighborhoods of $\mathbf{1}\in G$ denoted by $\mathcal{V}_G(\mathbf{1})$. Define

$$\mathfrak{L}(G)=\{\gamma\in\mathcal{C}(\mathbb{R},G)\mid(\forall t,s\in\mathbb{R})\ \gamma(t+s)=\gamma(t)\gamma(s)\}.$$

We endow $\mathfrak{L}(G)$ with the topology of uniform convergence on compact subsets of $\mathbb{R}$. It can be described by neighborhood bases as follows. For $n\in\mathbb{N}$ and $U\in\mathcal{V}_G(\mathbf{1})$ denote

$$W_{n,U}=\{(\gamma_1,\gamma_2)\in\mathfrak{L}(G)\times\mathfrak{L}(G)\mid(\forall t\in[-n,n])\ \gamma_2(t)\gamma_1(t)^{-1}\in U\}.$$

For $\gamma_1\in\mathfrak{L}(G)$ define $W_{n,U}(\gamma_1)=\{\gamma_2\in\mathfrak{L}(G)\mid(\gamma_1,\gamma_2)\in W_{n,U}\}$. Then there exists a unique topology on $\mathfrak{L}(G)$ such that for each $\gamma\in\mathfrak{L}(G)$ the family $\{W_{n,U}(\gamma)\mid n\in\mathbb{N},\ U\in\mathcal{V}_G(\mathbf{1})\}$ is a fundamental system of neighborhoods of γ.

Definition 2.1. The *adjoint action* of the topological group G is the map

$$\mathrm{Ad}_G\colon G\times\mathfrak{L}(G)\to\mathfrak{L}(G),\qquad(g,\gamma)\mapsto\mathrm{Ad}_G(g)\gamma:=g\gamma(\cdot)g^{-1}.$$

The map Ad_G indeed takes values in $\mathfrak{L}(G)$ and is a group action, since the action of G on itself by inner automorphisms, $G\times G\to G$, $(g,h)\mapsto ghg^{-1}$, is continuous.

We now recall the following result for later use:

Lemma 2.2. *The adjoint action of every topological group is a continuous mapping.*

Proof. See [BCR81, Lemma 0.1.4.1]. ∎

Differentiability along one-parameter subgroups

Definition 2.3. Let G be a topological group, $V\subseteq G$ an open subset, and $\mathcal{Y}$ a real locally convex space. If $\varphi\colon V\to\mathcal{Y}$, $\gamma\in\mathfrak{L}(G)$, and $g\in V$, then we denote

$$(D^\lambda_\gamma\varphi)(g)=\lim_{t\to 0}\frac{\varphi(g\gamma(t))-\varphi(g)}{t} \tag{2.1}$$

if the limit exists.

We define $\mathcal{C}^1(V,\mathcal{Y})$ as the set of all $\varphi\in\mathcal{C}(V,\mathcal{Y})$ for which the function

$$D^\lambda\varphi\colon V\times\mathfrak{L}(G)\to\mathcal{Y},\qquad (D^\lambda\varphi)(g;\gamma):=(D^\lambda_\gamma\varphi)(g),$$

is well defined and continuous. We also denote $D^\lambda\varphi=(D^\lambda)^1\varphi$.

Now let $n\ge 2$ and assume the space $\mathcal{C}^{n-1}(V,\mathcal{Y})$ and the mapping $(D^\lambda)^{n-1}$ have been defined. Then we define $\mathcal{C}^n(V,\mathcal{Y})$ as the set of all functions $\varphi\in\mathcal{C}^{n-1}(V,\mathcal{Y})$ for which the function

$$\begin{aligned}(D^\lambda)^n\varphi\colon V\times\mathfrak{L}(G)\times\cdots\times\mathfrak{L}(G)\to\mathcal{Y},&\\ (g;\gamma_1,\dots,\gamma_n)\mapsto(D^\lambda_{\gamma_n}(D^\lambda_{\gamma_{n-1}}\cdots(D^\lambda_{\gamma_1}\varphi)\cdots))(g),&\end{aligned}$$

is well defined and continuous.

Moreover we define $\mathcal{C}^\infty(V,\mathcal{Y}):=\bigcap_{n\ge1}\mathcal{C}^n(V,\mathcal{Y})$. If $\mathcal{Y}=\mathbb{C}$, then we write simply $\mathcal{C}^n(G):=\mathcal{C}^n(V,\mathbb{C})$ etc. for $n=1,2,\dots,\infty$.

Notation 2.4. It will be convenient to use the notation

$$D^\lambda_\gamma\varphi:=D^\lambda_{\gamma_n}(D^\lambda_{\gamma_{n-1}}\cdots(D^\lambda_{\gamma_1}\varphi)\cdots)\colon G\to\mathcal{Y}$$

whenever $\gamma:=(\gamma_1,\dots,\gamma_n)\in\mathfrak{L}(G)\times\cdots\times\mathfrak{L}(G)$ and $\varphi\in\mathcal{C}^n(G,\mathcal{Y})$.

Some auxiliary facts. For later use we record the following well known facts.

Lemma 2.5. *Let X and T be topological spaces, $\mathcal{Y}$ be a locally convex space, and $f:X\times T\to\mathcal{Y}$ be a continuous function. Pick $x_0\in X$ and a compact set $K\subseteq T$. Then for any continuous seminorm $|\cdot|$ on $\mathcal{Y}$ we have*

$$\lim_{x\to x_0}\sup_{t\in K}|f(x,t)-f(x_0,t)|=0. \tag{2.2}$$

Proof. This result is well known and is related to the exponential law for continuous functions, $\mathcal{C}(X\times K,\mathcal{Y})\simeq\mathcal{C}(X,\mathcal{C}(K,\mathcal{Y}))$; see for instance [AD51, Th. 4.21]. Alternatively, the assertion can be derived from the Wallace theorem [Ke75, Ch. 5, Th. 12]. ∎

In the following lemma we record the continuity with respect to parameters for weak integrals in locally convex spaces which may not be complete; see [Gl02a] for a thorough discussion of that integral, related differential calculus, and applications to Lie theory.

Lemma 2.6. *Let X be a topological space, $\mathcal{Y}$ be a locally convex space, $a,b\in\mathbb{R}$, $a<b$, and $f\colon X\times[a,b]\to\mathcal{Y}$ be a continuous function such that the weak integral $h(x)=\int_a^b f(x,t)\,dt$ exists for every $x\in X$. Then the resulting function $h\colon X\to\mathcal{Y}$ is continuous.*

Proof. To prove that h is continuous, let $|\cdot|$ be any continuous seminorm on $\mathcal{Y}$. It follows by [Gl02a, Lemma 1.7] that

$$(\forall x, y \in X) \quad |h(x) - h(y)| \le (b-a) \sup_{t \in [a,b]} |f(x,t) - f(y,t)|,$$

and now Lemma 2.5 shows that $h\colon X \to \mathcal{Y}$ is continuous. ∎

LEMMA 2.7. *Let H be a topological group, $\mathcal{Y}$ be a locally convex space, and $h \in \mathcal{C}(H, \mathcal{Y})$. If $X \in \mathfrak{L}(H)$ and the derivative $D^\lambda_X h\colon H \to \mathcal{Y}$ exists and is continuous, then there exists a continuous function $\chi\colon \mathbb{R} \times H \to \mathcal{Y}$ such that for all $g \in H$,*

$$(\forall t \in \mathbb{R}) \quad h(gX(t)) = h(g) + t(D^\lambda_X h)(g) + t\chi(t, g)$$

and $\chi(0, g) = 0$.

Proof. This follows from [NS13a, Lemma 2.5]; see also [BB11, Prop. 2.3]. ∎

3. Distributions with compact support, convolutions, and local operators. In this section we give a precise statement of the problem that motivated the present paper (see Problem 3.15 below).

Topologies on spaces of smooth functions. Spaces of smooth functions and their topologies play an important role in the theory of infinite-dimensional Lie groups modeled on locally convex spaces; see for instance [Ne06, Def. I.5.1]. We will now introduce a suitable topology on spaces of smooth functions on any topological group G, by using compact subsets of the space $\mathfrak{L}(G)$ of one-parameter subgroups and its Cartesian powers. This topology turns out to be suitable for establishing some embeddings of spaces of smooth functions (Theorem 4.16 and Remark 4.17).

DEFINITION 3.1. Let G be any topological group and denote

$$(\forall k \ge 1) \quad \mathfrak{L}^k(G) := \underbrace{\mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G)}_{k \text{ times}}.$$

Pick any open set $V \subseteq G$. If $\mathcal{Y}$ is any locally convex space, then for every $k \ge 1$, any compact subsets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq V$, and any continuous seminorm $|\cdot|$ on $\mathcal{Y}$ we define

$$p^{|\cdot|}_{K_1,K_2}\colon \mathcal{C}^\infty(V, \mathcal{Y}) \to [0, \infty),\ p^{|\cdot|}_{K_1,K_2}(f) = \sup\{|(D^\lambda_\gamma f)(x)| \mid \gamma \in K_1,\, x \in K_2\},$$

and

$$p^{|\cdot|}_{K_2}\colon \mathcal{C}^\infty(V, \mathcal{Y}) \to [0, \infty), \quad p^{|\cdot|}_{K_2}(f) = \sup\{|f(x)| \mid x \in K_2\}.$$

For simplicity we will always omit the seminorm $|\cdot|$ on $\mathcal{Y}$ from the above notation and write simply p_{K_1,K_2} instead of $p^{|\cdot|}_{K_1,K_2}$ and p_{K_2} instead of $p^{|\cdot|}_{K_2}$.

We endow the function space $\mathcal{C}^\infty(V, \mathcal{Y})$ with the locally convex topology defined by the family of seminorms p_{K_1,K_2} and p_{K_2}, and the resulting locally convex space will be denoted by $\mathcal{E}(V,\mathcal{Y})$. If $\mathcal{Y} = \mathbb{C}$ then we write simply $\mathcal{E}(V) := \mathcal{E}(V,\mathcal{Y})$.

We also denote by $\mathcal{E}'(G)$ the topological dual of $\mathcal{E}(G)$ endowed with the weak dual topology. This means that

$$\mathcal{E}'(G) = \{u\colon \mathcal{E}(G) \to \mathbb{C} \mid u \text{ is linear and continuous}\}$$

as a linear space. This space is endowed with the locally convex topology defined by the family of seminorms $\{q_B \mid B \text{ finite } \subseteq \mathcal{E}(G)\}$, where for every finite $B \subseteq \mathcal{E}(G)$ we define

$$q_B\colon \mathcal{E}'(G) \to \mathbb{C}, \qquad q_B(u) := \max_{f\in B} |u(f)|.$$

The elements of $\mathcal{E}'(G)$ will be called *distributions with compact support.*

Before we go further, we state an interesting problem related to the above definition.

PROBLEM 3.2. *Find conditions on the topological group G ensuring that every closed bounded subset of the locally convex space $\mathcal{E}(G)$ is compact.*

The above problem will not be addressed in the present paper. Let us just mention that an answer is known if G is any finite-dimensional Lie group (see [Eh56]).

DEFINITION 3.3. Assume the setting of Definition 3.1. The *support* of any $u \in \mathcal{E}'(G)$ is denoted by $\operatorname{supp} u$ and is defined as the set of all points $x \in G$ with the property that for every neighborhood U of x there exists $f \in \mathcal{E}(G)$ such that $\operatorname{supp} f \subseteq U$ and $u(f) \neq 0$.

REMARK 3.4. For every $u \in \mathcal{E}'(G)$, from its continuity with respect to the topology of $\mathcal{E}(G)$ introduced in Definition 3.1, it follows that there exist a positive constant $C > 0$, an integer $k \geq 1$, and some compact subsets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq G$ for which

$$(\forall f \in \mathcal{E}(G)) \qquad |u(f)| \leq C p_{K_1,K_2}(f).$$

This implies $\operatorname{supp} u \subseteq K_2$, hence $\operatorname{supp} u$ is compact in G, and this motivates the terminology introduced in Definition 3.1. For every compact subset $K \subseteq G$ we denote

$$\mathcal{E}'_K(G) := \{u \in \mathcal{E}'(G) \mid \operatorname{supp} u \subseteq K\}.$$

In the case $K = \{\mathbf{1}\}$ we will denote simply $\mathcal{E}'_{\mathbf{1}}(G) := \mathcal{E}'_{\{\mathbf{1}\}}(G)$.

Convolutions. We now introduce convolution of a smooth function with a distribution with compact support.

DEFINITION 3.5. Let G be any topological group. For all $\varphi \in \mathcal{E}(G)$ define $\check{\varphi} \in \mathcal{E}(G)$ by

$$(\forall x \in G) \qquad \check{\varphi}(x) := \varphi(x^{-1}).$$

Then for every $u \in \mathcal{E}'(G)$ we define $\check{u} \in \mathcal{E}'(G)$ by

$$(\forall \varphi \in \mathcal{E}(G)) \quad \check{u}(\varphi) := u(\check{\varphi}).$$

Finally, for all $\varphi \in \mathcal{E}(G)$ and $u \in \mathcal{E}'(G)$ we define their *convolution* as the function

$$\varphi * u \colon G \to \mathbb{C}, \quad (\varphi * u)(x) := \check{u}(\varphi \circ L_x),$$

where for all $x \in G$ we define $L_x \colon G \to G$, $L_x(y) := xy$. We will show in Propositions 3.8 and 3.10 that these definitions are correct, in the sense that $\check{\varphi}, \varphi \circ L_x \in \mathcal{E}(G)$.

Remark 3.6. For later use we note that in Definition 3.5 for all $\varphi \in \mathcal{E}(G)$ and $u \in \mathcal{E}'(G)$ one has

$$\mathrm{supp}(\varphi * u) \subseteq (\mathrm{supp}\,\varphi) \cdot (\mathrm{supp}\,u) := \{xy \mid x \in \mathrm{supp}\,\varphi,\ y \in \mathrm{supp}\,u\}.$$

To see this, we will prove that if $w \in W := G \setminus ((\mathrm{supp}\,\varphi) \cdot (\mathrm{supp}\,u))$ then $w \notin \mathrm{supp}(\varphi * u)$.

The set $\mathrm{supp}\,u$ is compact (by Remark 3.4) and $\mathrm{supp}\,\varphi$ is closed, hence the product $(\mathrm{supp}\,\varphi) \cdot (\mathrm{supp}\,u)$ is closed (see for instance [HeRo63, Ch. II, Th. 4.4]), and then its complement W is an open neighborhood of w.

We will show that $(\varphi * u)(x) = 0$ for all $x \in W$. To this end it suffices to check that $\mathrm{supp}(\varphi \circ L_x) \cap \mathrm{supp}\,\check{u} = \emptyset$, because $(\varphi * u)(x) = \check{u}(\varphi \circ L_x)$. It is easily seen that $\mathrm{supp}(\varphi \circ L_x) = x^{-1} \cdot \mathrm{supp}\,\varphi$ and $\mathrm{supp}\,\check{u} = (\mathrm{supp}\,u)^{-1}$, hence we must prove that $x \notin (\mathrm{supp}\,\varphi) \cdot (\mathrm{supp}\,u)$. But this holds true because $x \in W = G \setminus ((\mathrm{supp}\,\varphi) \cdot (\mathrm{supp}\,u))$.

In Definition 3.5, if G is a Lie group (see also [Eh56]), then $\check{\varphi}, \varphi \circ L_x \in \mathcal{E}(G)$ for all $x \in G$ and $\varphi \in \mathcal{E}(G)$. We will show in Propositions 3.8 and 3.10 that this property is shared by arbitrary topological groups. We begin by the following simple computation.

Remark 3.7. If $\varphi \in \mathcal{C}^1(G, \mathcal{Y})$, $x \in G$, and $\gamma \in \mathfrak{L}(G)$, then

$$\begin{aligned}
(D^\lambda_\gamma \check{\varphi})(x) &= \lim_{t \to 0} \frac{\varphi(\gamma(-t) \cdot x^{-1}) - \varphi(x^{-1})}{t} \\
&= -\lim_{t \to 0} \frac{\varphi(\gamma(t) \cdot x^{-1}) - \varphi(x^{-1})}{t} \\
&= -\lim_{t \to 0} \frac{\varphi(x^{-1} \cdot (\mathrm{Ad}_G(x)\gamma)(t)) - \varphi(x^{-1})}{t} \\
&= -(D^\lambda_{\mathrm{Ad}_G(x)\gamma} \varphi)(x^{-1}).
\end{aligned}$$

Similarly, if $n \geq 1$, $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(G)$, then for all $x \in G$,

$$(D^\lambda_{\gamma_1} \cdots D^\lambda_{\gamma_n} \varphi)(x) = (-1)^n (D^\lambda_{\mathrm{Ad}_G(x)\gamma_n} \cdots D^\lambda_{\mathrm{Ad}_G(x)\gamma_n} \varphi)(x^{-1})$$

(see [BCR81, p. 45]).

PROPOSITION 3.8. *If G is any topological group, then for all $\varphi \in \mathcal{E}(G)$ we have $\check{\varphi} \in \mathcal{E}(G)$. Moreover, the mapping $\mathcal{E}(G) \to \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is an isomorphism of locally convex spaces.*

Proof. The linear map $\varphi \mapsto \check{\varphi}$ is equal to its inverse, hence it suffices to prove that it is continuous. To this end define $\Psi_0 \colon G \to G$, $\Psi_0(x) = x^{-1}$, and for $n \geq 1$,

$$\Psi_n \colon \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G \to \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G,$$
$$\Psi_n(\gamma_1, \dots, \gamma_n, x) = (\mathrm{Ad}_G(x)\gamma_1, \dots, \mathrm{Ad}_G(x)\gamma_n, x^{-1}).$$

It follows directly from Lemma 2.2 that Ψ_n is a homeomorphism for all $n \geq 0$, hence for every $\varphi \in \mathcal{C}(G)$ we have $\check{\varphi} = \varphi \circ \Psi_0 \in \mathcal{C}(G)$. Moreover, by Remark 3.7, for all $n \geq 1$ and $\varphi \in \mathcal{C}^n(G)$ we have

$$(D^\lambda)^n \check{\varphi} = (-1)^n ((D^\lambda)^n \varphi) \circ \Psi_n, \tag{3.1}$$

hence $\check{\varphi} \in \mathcal{C}^n(G)$. This shows that if $\varphi \in \mathcal{E}(G)$, then $\check{\varphi} \in \mathcal{E}(G)$.

To check that the map $\mathcal{E}(G) \to \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is also continuous, let $k \geq 1$ be any integer and pick compact sets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq G$. Define

$$K_1' := \{(\mathrm{Ad}_G(x)\gamma_1, \dots, \mathrm{Ad}_G(x)\gamma_k) \mid x \in K_2,\ (\gamma_1, \dots, \gamma_k) \in K_1\}$$

and $K_2' := \{x^{-1} \mid x \in K_2\}$. Since both the inversion mapping and the adjoint action of G are continuous (Lemma 2.2), the sets K_1' and K_2' are compact. Moreover, by (3.1) along with Definition 3.1,

$$(\forall \varphi \in \mathcal{E}(G)) \quad p_{K_1,K_2}(\check{\varphi}) \leq p_{K_1',K_2'}(\varphi) \quad \text{and} \quad p_{K_2}(\check{\varphi}) = p_{K_2'}(\varphi),$$

hence the map $\mathcal{E}(G) \to \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is indeed continuous. ∎

REMARK 3.9. If $\varphi \in \mathcal{C}^1(G, \mathcal{Y})$, $x, g \in G$, and $\gamma \in \mathfrak{L}(G)$, then

$$(D^\lambda_\gamma(\varphi \circ L_x))(g) = \lim_{t \to 0} \frac{\varphi(xg\gamma(t)) - \varphi(xg)}{t} = (D^\lambda_\gamma \varphi)(xg).$$

Therefore $D^\lambda_\gamma(\varphi \circ L_x) = (D^\lambda_\gamma \varphi) \circ L_x$.

PROPOSITION 3.10. *If G is a topological group and $\mathcal{Y}$ is a locally convex space, then for every $x \in G$ the map*

$$\Lambda_x \colon \mathcal{C}^\infty(G, \mathcal{Y}) \to \mathcal{C}^\infty(G, \mathcal{Y}), \qquad \Lambda_x(\varphi) = \varphi \circ L_x,$$

is well defined and is an isomorphism of locally convex spaces.

Proof. For every $n \geq 1$ we have the homeomorphism

$$F^x_n \colon \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G \to \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G,$$
$$F^x_n(\gamma_1, \dots, \gamma_n, g) = (\gamma_1, \dots, \gamma_n, xg).$$

On the other hand, iterating Remark 3.9, we see that for every $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$ we have $(D^\lambda)^n(\varphi \circ L_x) = ((D^\lambda)^n \varphi) \circ F^x_n$, hence $(D^\lambda)^n(\varphi \circ L_x)$ is a continuous function, so $\varphi \circ L_x \in \mathcal{C}^\infty(G, \mathcal{Y})$.

Since F_n^x is a homeomorphism, it easily follows by the above formula for derivatives (similarly to the proof of Proposition 3.8) that the map Λ_x is continuous. Replacing x by x^{-1}, we see that these properties are shared by $\Lambda_{x^{-1}} = \Lambda_x^{-1}$, and this completes the proof. ∎

As already mentioned, Propositions 3.8 and 3.10 imply in particular that Definition 3.5 is correct. For later use we now record the version of these results for the multiplication map (see also Remark 5.5 below).

Proposition 3.11. *If G is a topological group with multiplication $m\colon G \times G \to G$, $(x, y) \mapsto xy$, then for any locally convex space $\mathcal{Y}$ the linear mapping*

$$\mathcal{E}(G, \mathcal{Y}) \to \mathcal{E}(G \times G, \mathcal{Y}), \qquad \varphi \mapsto \varphi \circ m,$$

is well defined and continuous.

Proof. Recall from [BCR81, p. 46] (see also [Nic14]) that for all $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$, $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathfrak{L}(G)$, $x, y \in G$, $k \geq 1$, we have

$$((D^\lambda)^k(\varphi \circ m))(x, y; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$$
$$= \sum_{\ell=0}^{k} \sum_{\substack{i_1 < \cdots < i_\ell \\ i_{\ell+1} < \cdots < i_k}} ((D^\lambda)^\ell \varphi)(xy; \beta_{i_1}, \dots, \beta_{i_\ell}, \mathrm{Ad}_G(y^{-1})\alpha_{i_{\ell+1}}, \dots, \mathrm{Ad}_G(y^{-1})\alpha_{i_k}),$$

where we assume $\{i_1, \dots, i_\ell, i_{\ell+1}, \dots, i_k\} = \{1, \dots, k\}$. With this formula at hand, the continuity of $\varphi \mapsto \varphi \circ m$ can be checked just as the continuity of $\varphi \mapsto \check{\varphi}$ in the proof of Proposition 3.8. ∎

Algebras of local operators

Definition 3.12. Let G be a topological group. A *local operator* on G is any continuous linear operator $D\colon \mathcal{E}(G) \to \mathcal{E}(G)$ with the property

$$(\forall f \in \mathcal{E}(G)) \qquad \mathrm{supp}(Df) \subseteq \mathrm{supp}\, f.$$

We denote by $\mathrm{Loc}(G)$ the set of all local operators on G. It is easily seen that $\mathrm{Loc}(G)$ is a unital associative algebra of continuous linear operators on the function space $\mathcal{E}(G)$.

Remark 3.13. It follows from [Pee60] that if G is any finite-dimensional Lie group, then $\mathrm{Loc}(G)$ is precisely the set of linear differential operators (possibly of infinite order) on G. Some generalizations of that statement for locally compact groups were obtained in [Ak95, Th. 2.3] and [Ed88, Th. 2.3]. See also [WD73] and [LW11] for generalizations to $G = (\mathcal{X}, +)$ for Banach spaces $\mathcal{X}$ that admit suitable bump functions (in particular for Hilbert spaces). Infinite-dimensional versions of the results of [Pee60] on local operators were also discussed in [Da15].

Definition 3.14. Let G be any topological group and recall the notation

$$(\forall x\in G)\qquad L_x\colon G\to G,\ L_x(y)=xy.$$

The *left-invariant local operators* on G are the elements of the set

$$\mathcal{U}(G):=\{D\in\mathrm{Loc}(G)\mid(\forall x\in G)(\forall f\in\mathcal{C}^\infty(G))\ D(f\circ L_x)=(Df)\circ L_x\}.$$

We note that $\mathcal{U}(G)$ is a unital associative subalgebra of $\mathrm{Loc}(G)$.

For every $\gamma\in\mathfrak{L}(G)$ we have $D^\lambda_\gamma\in\mathcal{U}(G)$ by Remark 3.9. We denote by $\mathcal{U}_0(G)$ the unital associative subalgebra of $\mathcal{U}(G)$ generated by $\{D^\lambda_\gamma\mid\gamma\in\mathfrak{L}(G)\}$.

We can now state one of the main problems that have motivated the present paper. We will address this problem in Theorem 5.2 and Corollary 5.3 below.

Problem 3.15. *For any topological group G we have the following inclusions of unital associative algebras:*

$$\mathcal{U}_0(G)\subseteq\mathcal{U}(G)\subseteq\mathrm{Loc}(G).$$

Investigate the gap between $\mathcal{U}_0(G)$ and $\mathcal{U}(G)$, and in particular find necessary or sufficient conditions on G for the equality $\mathcal{U}_0(G)=\mathcal{U}(G)$ to hold.

Remark 3.16. If G is any finite-dimensional Lie group, then it follows from the Poincaré–Birkhoff–Witt theorem (see also Remark 3.13) that $\mathcal{U}_0(G)=\mathcal{U}(G)$, and this is precisely the complexified universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the Lie algebra $\mathfrak{g}$ of G. Hence the difficulty of Problem 3.15 lies in extending the Poincaré–Birkhoff–Witt theorem from Lie groups to topological groups.

If G is a pre-Lie group (Definition 5.4) with $\mathfrak{L}(G)=\mathfrak{g}$ or a locally convex Lie group, then [BCR81, Sect. 1.3.1, $(\widetilde{4})$] shows that the mapping $\mathfrak{L}(G)\to\mathcal{U}_0(G)$, $\gamma\mapsto D^\lambda_\gamma$, extends to a natural surjective homomorphism of unital associative algebras $U(\mathfrak{g}_{\mathbb{C}})\to\mathcal{U}_0(G)$, whose injectivity can be established under the additional assumption that $\mathcal{C}^\infty(G)$ contains sufficiently many functions, in some sense. (See the method of proof of the Poincaré–Birkhoff–Witt theorem in [CW99] and also [BCR81, Cor. 4.1.1.7].)

For instance, assume that there exists a smooth function on $\mathfrak{g}$ which is equal to 1 on some neighborhood of $0\in\mathfrak{g}$ and has bounded support. Then the above natural homomorphism is an isomorphism $U(\mathfrak{g}_{\mathbb{C}})\xrightarrow{\sim}\mathcal{U}_0(G)$ for any locally exponential Lie group G (in the sense of [Ne06, Sect. IV]) whose Lie algebra is $\mathfrak{g}$, which includes in particular all Banach–Lie groups. Note however that this method is not always applicable since there exist Banach spaces that do not admit any nontrivial smooth function with bounded support, for instance $\ell^1(\mathbb{N})$; see [BF66, Sect. 2, Ex. (i)].

On the other hand, a result of the type $U(\mathfrak{g}_{\mathbb{C}})\xrightarrow{\sim}\mathcal{U}_0(G)=\mathcal{U}(G)$ was obtained in [Ak95, Cor. 2.5] in the case when G is any locally compact group, by using the Lie algebra $\mathfrak{g}=\mathfrak{L}(G)$ discovered in [Glu57] and [La57].

4. Some embeddings of spaces of smooth functions on topological groups. The main result of this section is Theorem 4.16, which provides a kind of weak exponential law for smooth functions on topological groups, which suffices for our purposes in Section 5. See for instance [KM97, Ch. I, §3] for a broad discussion of the exponential law for smooth functions on open subsets of locally convex spaces. Further information and references on this topic can be found in [AlS15], [KMR15], [Gl13], and [Al13].

NOTATION 4.1. Let G and H be topological groups. For any locally convex space $\mathcal{Y}$ and $\varphi \in \mathcal{C}^\infty(G \times H, \mathcal{Y})$ we define

$$\widetilde{\varphi}\colon G \to \mathcal{C}^\infty(H, \mathcal{Y}), \qquad \widetilde{\varphi}(x)(y) = \varphi(x, y).$$

This notation will be preserved throughout the present section.

Some basic formulas on partial derivatives. We now give a definition whose correctness is established in Lemma 4.4 below.

DEFINITION 4.2. Let $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. For $n \geq 1$ we define the partial derivatives $(D_1^\lambda)^n\varphi\colon H \times G \times \mathfrak{L}^n(H) \to \mathcal{Y}$ and $(D_2^\lambda)^n\varphi\colon H \times G \times \mathfrak{L}^n(G) \to \mathcal{Y}$ as follows.

For $n = 1$, $\beta \in \mathfrak{L}(H)$, and $\alpha \in \mathfrak{L}(G)$,

$$(D_1^\lambda\varphi)(x, g; \beta) = \frac{d}{dt}\Big|_{t=0}\varphi(x\beta(t), g),$$
$$(D_2^\lambda\varphi)(x, g; \alpha) = \frac{d}{dt}\Big|_{t=0}\varphi(x, g\alpha(t)).$$

Furthermore, we define inductively

$$((D_1^\lambda)^{n+1}\varphi)(x, g; \beta_1, \dots, \beta_n, \beta_{n+1}) = \frac{d}{dt}\Big|_{t=0}((D_1^\lambda)^n\varphi)(x\beta_{n+1}(t), g; \beta_1, \dots, \beta_n)$$

and

$$((D_2^\lambda)^{n+1}\varphi)(x, g; \alpha_1, \dots, \alpha_n, \alpha_{n+1}) = \frac{d}{dt}\Big|_{t=0}((D_2^\lambda)^n\varphi)(x, g\alpha_{n+1}(t); \alpha_1, \dots, \alpha_n).$$

NOTATION 4.3. By $\mathbf{1} \in \mathfrak{L}(G)$ we denote the constant function from $\mathbb{R}$ to G given by $\mathbf{1}(t) = \mathbf{1} \in G$ for all $t \in \mathbb{R}$.

The following lemma ensures the existence and continuity of the maps $(D_1^\lambda)^n\varphi$ and $(D_2^\lambda)^n\varphi$ from Definition 4.2.

LEMMA 4.4. *Let $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, $x \in H$, $g \in G$, and $n \geq 1$.*

(a) *For all $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$ we have*

$$((D_1^\lambda)^n\varphi)(x, g; \beta_1, \dots, \beta_n) = ((D^\lambda)^n\varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1})).$$

(b) *For all $\alpha_1, \dots, \alpha_n \in \mathfrak{L}(G)$ we have*

$$((D_2^\lambda)^n\varphi)(x, g; \alpha_1, \dots, \alpha_n) = ((D^\lambda)^n\varphi)(x, g; (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_n)).$$

(c) *The maps*

$$(D_1^\lambda)^n\varphi\colon H\times G\times\mathfrak{L}^n(H)\to\mathcal{Y}$$

and

$$(D_2^\lambda)^n\varphi\colon H\times G\times\mathfrak{L}^n(G)\to\mathcal{Y}$$

are continuous.

Proof. Assertions (a) and (b) are straightforward, and (c) follows from (a) and (b), by using the hypothesis $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$. ∎

Proposition 4.5. *For all $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$, $n\ge 1$, and $\beta_1,\dots,\beta_n\in\mathfrak{L}(H)$,*

$$\begin{aligned}((D^\lambda)^n\widetilde{\varphi})(x;\beta_1,\dots,\beta_n)(g)&=((D_1^\lambda)^n\varphi)(x,g;\beta_1,\dots,\beta_n)\\&=((D^\lambda)^n\varphi)(x,g;(\beta_1,\mathbf{1}),\dots,(\beta_n,\mathbf{1})).\end{aligned}$$

Proof. The last equality follows from Lemma 4.4(a). For the first equality we use induction on n.

The case $n=1$: For $x\in H$, $g\in G$, and $\beta\in\mathfrak{L}(H)$ we have

$$(D^\lambda\widetilde{\varphi})(x;\beta)(g)=\frac{d}{dt}\Big|_{t=0}\widetilde{\varphi}(x\beta(t))(g)=\frac{d}{dt}\Big|_{t=0}\varphi(x\beta(t),g)=(D_1^\lambda\varphi)(x,g;\beta).$$

Now suppose that the assertion is already proved for n. Then

$$\begin{aligned}&((D^\lambda)^{n+1}\widetilde{\varphi})(x;\beta_1,\dots,\beta_n,\beta_{n+1})(g)\\&\qquad=\frac{d}{dt}\Big|_{t=0}((D^\lambda)^n\widetilde{\varphi})(x\beta_{n+1}(t);\beta_1,\dots,\beta_n)(g)\\&\qquad=\frac{d}{dt}\Big|_{t=0}((D_1^\lambda)^n\varphi)(x\beta_{n+1}(t),g;\beta_1,\dots,\beta_n)\\&\qquad=((D_1^\lambda)^{n+1}\varphi)(x,g;\beta_1,\dots,\beta_n,\beta_{n+1}).\end{aligned}$$

The proof is complete. ∎

Remark 4.6. The formula from Proposition 4.5 gives us the point values of the derivatives of the function $\widetilde{\varphi}$ introduced in Notation 4.1. We still have to show that the convergence of the differential quotients to these values holds in the topology of $\mathcal{E}(G,\mathcal{Y})$. This will be done in Proposition 4.14.

Lemma 4.7. *If $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$, then for all $x\in H$, $g\in G$, $n\ge 1$, and $\alpha_1,\dots,\alpha_n\in\mathfrak{L}(G)$ we have*

$$\begin{aligned}((D^\lambda)^n(\widetilde{\varphi}(x)))(g;\alpha_1,\dots,\alpha_n)&=((D_2^\lambda)^n\varphi)(x,g;\alpha_1,\dots,\alpha_n)\\&=((D^\lambda)^n\varphi)(x,g;(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_n)).\end{aligned}$$

Proof. The proof is similar to the one of Proposition 4.5. ∎

Remark 4.8. It follows from Lemmas 4.7 and 4.4(c) that if $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$, then $\widetilde{\varphi}(x):=\varphi(x,\cdot)\in\mathcal{C}^\infty(G,\mathcal{Y})$ for all $x\in G$, hence the function $\widetilde{\varphi}\colon H\to\mathcal{E}(G,\mathcal{Y})$ (see Notation 4.1) is well defined.

Continuity of $\widetilde{\varphi}$

Proposition 4.9. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then $\widetilde{\varphi}\colon H \to \mathcal{E}(G, \mathcal{Y})$ (see Notation 4.1) is continuous.*

Proof. We will show that for any seminorm p on $\mathcal{E}(G, \mathcal{Y})$ of the form p_{K_2} or p_{K_1,K_2} as in Definition 3.1 we have $\lim_{x\to x_0} p(\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)) = 0$, which is tantamount to the following condition:

$$(\forall x_0 \in H)(\forall \varepsilon > 0)(\exists U \in \mathcal{V}(x_0))(\forall x \in U) \quad p(\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)) \le \varepsilon.$$

We will analyze the two types of seminorms separately. Let $x_0 \in H$ and $\varepsilon > 0$ be fixed throughout the proof.

Case (a): $p = p_{K_2}$, where $K_2 \subseteq G$ is compact. Set $E(x) := p(\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0))$. Then

$$E(x) = \sup_{g\in K_2} |\widetilde{\varphi}(x)(g) - \widetilde{\varphi}(x_0)(g)| = \sup_{g\in K_2} |\varphi(x, g) - \varphi(x_0, g)|,$$

hence the conclusion follows directly by applying Lemma 2.5 with $x_0 \in X = H$, $T = G$, $K = K_2$, and $f = \varphi\colon H \times G \to \mathcal{Y}$, which is a continuous function since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$.

Case (b): $p = p_{K_1,K_2}$ for compact $K_2 \subseteq G$ and $K_1 \subseteq \mathfrak{L}^n(G)$, $n \ge 1$. Denote again $E(x) := p(\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0))$. In this case we have

$$\begin{aligned} E(x) &= \sup\{|((D^\lambda)^n(\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)))(g;\gamma)| \mid g \in K_2,\ \gamma \in K_1\} \\ &= \sup\{|((D^\lambda)^n(\widetilde{\varphi}(x)))(g;\gamma) - ((D^\lambda)^n(\widetilde{\varphi}(x_0)))(g;\gamma)| \mid g \in K_2,\ \gamma \in K_1\}. \end{aligned}$$

Using Lemma 4.7 we obtain

$$E(x) = \sup\{|((D_2^\lambda)^n\varphi)(x, g;\gamma) - ((D_2^\lambda)^n\varphi)(x_0, g;\gamma)| \mid g \in K_2,\ \gamma \in K_1\},$$

hence the conclusion follows by applying Lemma 2.5 with $x_0 \in X = H$, $T = G \times \mathfrak{L}^n(G)$, and the compact set $K = K_2 \times K_1 \subseteq T$, since $f = (D_2^\lambda)^n\varphi\colon H \times G \times \mathfrak{L}^n(G) \to \mathcal{Y}$ is continuous by Lemma 4.4(c). ∎

Smoothness of $\widetilde{\varphi}$

Definition 4.10. Let G be a topological group. We say that $\alpha, \beta \in \mathfrak{L}(G)$ *commute* if

$$(\forall s, t \in \mathbb{R}) \quad \alpha(t)\beta(s) = \beta(s)\alpha(t).$$

Remark 4.11. If G and H are topological groups, then every $\alpha \in \mathfrak{L}(G)$ commutes with $\mathbf{1} \in \mathfrak{L}(G)$, and for every $\alpha \in \mathfrak{L}(G)$ and $\beta \in \mathfrak{L}(H)$ the elements $(\mathbf{1}, \alpha)$ and $(\beta, \mathbf{1})$ from $\mathfrak{L}(H \times G)$ commute.

Lemma 4.12. *Let H be a topological group, $n \ge 2$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(H)$. Assume that γ_i commutes with γ_{i+1} for some $i \in \{1, \dots, n-1\}$. Then for* *any* $f \in \mathcal{C}^n(H,\mathcal{Y})$ *and* $x \in H$ *we have*

$$
\begin{aligned}
&((D^\lambda)^n f)(x;\gamma_n,\dots,\gamma_{i+2},\gamma_{i+1},\gamma_i,\gamma_{i-1},\dots,\gamma_1)\\
&\qquad\qquad = ((D^\lambda)^n f)(x;\gamma_n,\dots,\gamma_{i+2},\gamma_i,\gamma_{i+1},\gamma_{i-1},\dots,\gamma_1).
\end{aligned}
$$

Proof. The function

$$(t_1,\dots,t_n)\mapsto f(x\gamma_1(t_1)\gamma_2(t_2)\cdots\gamma_n(t_n))$$

belongs to $\mathcal{C}^n(\mathbb{R}^n,\mathcal{Y})$ by [BCR81, Prop. 1.2.2.1]. Therefore

$$
\begin{aligned}
&((D^\lambda)^n f)(x;\gamma_n,\dots,\gamma_{i+2},\gamma_{i+1},\gamma_i,\gamma_{i-1},\dots,\gamma_1)\\
&\quad = \frac{\partial^n}{\partial t_1\cdots\partial t_{i-1}\partial t_i\partial t_{i+1}\partial t_{i+2}\dots\partial t_n}\bigg|_{t_1=\dots=t_n=0} f(x\gamma_1(t_1)\cdots\gamma_n(t_n))\\
&\quad = \frac{\partial^n}{\partial t_1\cdots\partial t_{i-1}\partial t_{i+1}\partial t_i\partial t_{i+2}\cdots\partial t_n}\bigg|_{t_1=\dots=t_n=0} f(x\gamma_1(t_1)\cdots\gamma_n(t_n))\\
&\quad = \frac{\partial^n}{\partial t_1\cdots\partial t_{i-1}\partial t_{i+1}\partial t_i\partial t_{i+2}\cdots\partial t_n}\bigg|_{t_1=\dots=t_n=0}\\
&\qquad\qquad f(x\gamma_1(t_1)\cdots\gamma_{i-1}(t_{i-1})\gamma_{i+1}(t_{i+1})\gamma_i(t_i)\gamma_{i+2}(t_{i+2})\cdots\gamma_n(t_n))\\
&\quad = ((D^\lambda)^n f)(x;\gamma_n,\dots,\gamma_{i+2},\gamma_i,\gamma_{i+1},\gamma_{i-1},\dots,\gamma_1). \ \blacksquare
\end{aligned}
$$

LEMMA 4.13. *Let* G *and* H *be topological groups and* $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$. *For some* $s\geq 1$ *let* $\alpha_1,\dots,\alpha_s\in\mathfrak{L}(G)$. *Then the following assertions hold for all* $x\in H$ *and* $g\in G$:

(a) *For every* $\beta\in\mathfrak{L}(H)$ *we have*

$$
\begin{aligned}
&((D^\lambda)^{s+1}\varphi)(x,g;(\beta,\mathbf{1}),(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s))\\
&\qquad\qquad = ((D^\lambda)^{s+1}\varphi)(x,g;(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s),(\beta,\mathbf{1})).
\end{aligned}
$$

(b) *For every* $n\geq 1$ *and* $\beta_1,\dots,\beta_{n+1}\in\mathfrak{L}(H)$ *we have*

$$
\begin{aligned}
&((D^\lambda)^{n+s+1}\varphi)(x,g;(\beta_1,\mathbf{1}),\dots,(\beta_n,\mathbf{1}),(\beta_{n+1},\mathbf{1}),(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s))=\\
&((D^\lambda)^{n+s+1}\varphi)(x,g;(\beta_1,\mathbf{1}),\dots,(\beta_n,\mathbf{1}),(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s),(\beta_{n+1},\mathbf{1})).
\end{aligned}
$$

Proof. In both assertions one starts from the right-hand side of the equality to be proved, and uses Remark 4.11 and Lemma 4.12 for H replaced by $H\times G$ for the pairs $(\mathbf{1},\alpha_i)$ and $(\beta_{n+1},\mathbf{1})$. $\blacksquare$

We are now in a position to solve the problem mentioned in Remark 4.6.

PROPOSITION 4.14. *Let* G *and* H *be topological groups,* $\mathcal{Y}$ *be a locally convex space, and for any* $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$ *define* $\widetilde{\varphi}\colon H\to\mathcal{E}(G,\mathcal{Y})$,

$\widetilde{\varphi}(x)(g) = \varphi(x,g)$. *Then for all* $x_0 \in G$ *and* $\beta_1^0,\dots,\beta_{n+1}^0 \in \mathfrak{L}(H)$ *we have*

$$\lim_{t\to 0} \frac{((D^\lambda)^n\widetilde{\varphi})(x_0\beta_{n+1}^0(t);\beta_1^0,\dots,\beta_n^0) - ((D^\lambda)^n\widetilde{\varphi})(x_0;\beta_1^0,\dots,\beta_n^0)}{t} = ((D_1^\lambda)^{n+1}\varphi)(x_0,\bullet\,;\beta_1^0,\dots,\beta_n^0,\beta_{n+1}^0)$$

in the topology of $\mathcal{E}(G,\mathcal{Y})$ *from Definition* 3.1.

Proof. Define $h\colon \mathbb{R}\to\mathcal{E}(G,\mathcal{Y})$ by

$$h(t) = \begin{cases} \dfrac{((D^\lambda)^n\widetilde{\varphi})(x_0\beta_{n+1}^0(t);\beta_1^0,\dots,\beta_n^0) - ((D^\lambda)^n\widetilde{\varphi})(x_0;\beta_1^0,\dots,\beta_n^0)}{t} & \text{if } t\neq 0,\\ ((D_1^\lambda)^{n+1}\varphi)(x_0,\bullet\,;\beta_1^0,\dots,\beta_n^0,\beta_{n+1}^0) & \text{if } t=0.\end{cases}$$

We must prove that $\lim_{t\to 0} h(t) = h(0)$ in $\mathcal{E}(G,\mathcal{Y})$, that is, for every seminorm p (see Definition 3.1) we have $\lim_{t\to 0} p(h(t)-h(0)) = 0$.

Again we distinguish two cases.

Case 1: $p = p_{K_2}$ for a compact set $K_2\subseteq G$. Set $E(t) = p(h(t)-h(0))$; then

$$\begin{aligned} E(t) &= \sup\{|h(t)(g)-h(0)(g)| \mid g\in K_2\}\\ &= \sup\Big\{\Big|\frac{((D_1^\lambda)^n\varphi)(x_0\beta_{n+1}^0(t),g;\beta_1^0,\dots,\beta_n^0) - ((D_1^\lambda)^n\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0)}{t}\\ &\qquad - ((D_1^\lambda)^{n+1}\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0,\beta_{n+1}^0)\Big|\ \Big|\ g\in K_2\Big\}\\ &= \sup\{|F(t,g)-F(0,g)| \mid g\in K_2\}\end{aligned}$$

where $F\colon \mathbb{R}\times G\to\mathcal{Y}$ is defined by

$$F(t,g) = \begin{cases} \dfrac{((D_1^\lambda)^n\varphi)(x_0\beta_{n+1}^0(t),g;\beta_1^0,\dots,\beta_n^0) - ((D_1^\lambda)^n\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0)}{t} & \text{if } t\neq 0,\\ ((D_1^\lambda)^{n+1}\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0,\beta_{n+1}^0) & \text{if } t=0.\end{cases}$$

The desired property $\lim_{t\to 0} E(t) = 0$ will follow by an application of Lemma 2.5 for $X=\mathbb{R}$, $T=G$, $x_0 = 0\in\mathbb{R}$, $K=K_2\subseteq G$, and $f=F\colon \mathbb{R}\times G\to\mathcal{Y}$, as soon as we check that F is continuous.

To this end, first note that for $g\in G$ we have

$$\begin{aligned}\lim_{t\to 0} F(t,g) &= \frac{d}{dt}\Big|_{t=0} ((D_1^\lambda)^n\varphi)(x_0\beta_{n+1}^0(t),g;\beta_1^0,\dots,\beta_n^0)\\ &= ((D_1^\lambda)^{n+1}\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0,\beta_{n+1}^0) = F(0,g).\end{aligned}$$

Next, we will show that Lemma 2.7 applies with H replaced by $H \times G$, $(x_0, g) \in H \times G$, $X = (\beta^0_{n+1}, \mathbf{1}) \in \mathfrak{L}(H \times G)$, and $f \colon H \times G \to \mathcal{Y}$, $f(x, y) = ((D^\lambda_1)^n \varphi)(x, y; \beta^0_1, \dots, \beta^0_n)$, which is continuous since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. Note that the derivative $D^\lambda_X f \colon H \times G \to \mathcal{Y}$ is given by $(D^\lambda_X f)(x, y) = ((D^\lambda_1)^{n+1} \varphi)(x, y; \beta^0_1, \dots, \beta^0_n, \beta^0_{n+1})$, and this is a continuous function since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$.

Therefore Lemma 2.7 provides a continuous function $\chi \colon \mathbb{R} \times G \to \mathcal{Y}$ such that $\chi(0, g) = 0$ and $f(x_0 \beta^0_{n+1}(t), g) = f(x_0, g) + t(D^\lambda_X f)(x_0, g) + t\chi(t, g)$ for all $g \in G$. We have

$$\begin{aligned}
(D^\lambda_X f)(x_0, g) = (D^\lambda f)(x_0, g; \beta^0_{n+1}, 1) &= \frac{d}{dt}\Big|_{t=0} f(x_0 \beta^0_{n+1}(t), g) \\
&= \frac{d}{dt}\Big|_{t=0} ((D^\lambda_1)^n \varphi)(x_0 \beta^0_{n+1}(t), g; \beta^0_1, \dots, \beta^0_n) \\
&= ((D^\lambda_1)^{n+1} \varphi)(x_0, g; \beta^0_1, \dots, \beta^0_n, \beta^0_{n+1}) = F(0, g)
\end{aligned}$$

and $F(t, g) = F(0, g) + \chi(t, g)$, hence F is the sum of two continuous functions, since χ is continuous by Lemma 2.7 and

$$g \mapsto F(0, g) = ((D^\lambda_1)^{n+1} \varphi)(x_0, g; \beta^0_1, \dots, \beta^0_n, \beta^0_{n+1})$$

is continuous since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. This settles Case 1.

Case 2: $p = p_{K_1, K_2}$ for compact sets $K_2 \subseteq G$ and $K_1 \subseteq \mathfrak{L}^s(G)$, where $s \geq 1$. Denote again $E(t) = p(h(t) - h(0))$ for $t \in \mathbb{R}$. Then

$$E(t) = \sup_{g, \alpha} |((D^\lambda)^s (h(t))(g; \alpha_1, \dots, \alpha_s) - ((D^\lambda)^s (h(0))(g; \alpha_1, \dots, \alpha_s)|$$

where $g \in K_2$ and $\alpha = (\alpha_1, \dots, \alpha_s) \in K_1$. Until the end of this proof it is convenient to denote $k := n + s$. Then $E(t)$ is the supremum of the values of the seminorm $|\cdot|$ involved in the definition of $p = p_{K_1, K_2}$ (see Definition 3.1) on the vectors in $\mathcal{Y}$ of the form

$$\begin{aligned}
&\frac{1}{t}\big(((D^\lambda)^k \varphi)(x_0 \beta^0_{n+1}(t), g; (\beta^0_1, \mathbf{1}), \dots, (\beta^0_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\
&\qquad - ((D^\lambda)^k \varphi)(x_0, g; (\beta^0_1, \mathbf{1}), \dots, (\beta^0_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s))\big) \\
&\qquad - ((D^\lambda)^{k+1} \varphi)(x_0, g; (\beta^0_1, \mathbf{1}), \dots, (\beta^0_n, \mathbf{1}), (\beta^0_{n+1}, \mathbf{1}), (1, \alpha_1), \dots, (\mathbf{1}, \alpha_s))
\end{aligned}$$

where again $g \in K_2$ and $\alpha = (\alpha_1, \dots, \alpha_s) \in K_1$.

Therefore $E(t) = \sup\{|F(t, g, \alpha) - F(0, g, \alpha)| \mid g \in K_2,\ \alpha \in K_1\}$ where $F \colon \mathbb{R} \times G \times \mathfrak{L}^s(G) \to \mathcal{Y}$ is given by

$$\begin{aligned}
&F(t, g, \alpha) \\
&\quad = \frac{1}{t}\big(((D^\lambda)^k \varphi)(x_0 \beta^0_{n+1}(t), g; (\beta^0_1, 1), \dots, (\beta^0_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\
&\qquad\qquad - ((D^\lambda)^k \varphi)(x_0, g; (\beta^0_1, \mathbf{1}), \dots, (\beta^0_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s))\big)
\end{aligned}$$

if $t \neq 0$, and

$$F(t,g,\alpha) = ((D^\lambda)^{k+1}\varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\beta_{n+1}^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s))$$

if $t = 0$.

The desired property $\lim_{t\to 0} E(t) = 0$ then follows by an application of Lemma 2.5 for $X = \mathbb{R}$, $T = G \times \mathfrak{L}^s(G)$, $x_0 = 0 \in \mathbb{R}$, the compact $K = K_2 \times K_1 \subseteq G \times \mathfrak{L}^s(G)$, and the function $f = F$, as soon as we prove that F is continuous.

Just as in Case 1, we first note that

$$\begin{aligned}
\lim_{t\to 0} F(t,g,\alpha) &= \frac{d}{dt}\Big|_{t=0} ((D^\lambda)^k\varphi)(x_0\beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\
&= ((D^\lambda)^{k+1}\varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})) \\
&= F(0,g,\alpha)
\end{aligned}$$

by using Lemma 4.13(b).

Now define $B\colon \mathbb{R} \to \mathcal{Y}$ by

$$B(t) = ((D^\lambda)^k\varphi)(x_0\beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)).$$

Since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, we have $B \in \mathcal{C}^1(\mathbb{R}, \mathcal{Y})$, $B'(0) = F(0,g,\alpha)$ by Lemma 4.13(b) and

$$B'(t) = ((D^\lambda)^k\varphi)(x_0\beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})).$$

The fundamental theorem of calculus for functions with values in the space $\mathcal{Y}$ which may not be complete (see [Gl02a, Th. 1.5]) yields

$$B(t) = B(0) + t\int_0^1 B'(tz)\,dz = B(0) + tB'(0) + t\int_0^1 B'(tz)\,dz - tB'(0)$$

and therefore

$$F(t,g,\alpha) = F(0,g,\alpha) + \chi(g,t,\alpha) \tag{4.1}$$

where $\chi\colon G \times \mathbb{R} \times \mathfrak{L}^s(G) \to \mathcal{Y}$ is given by

$$\begin{aligned}
\chi(g,t,\alpha) &= \int_0^1 ((D^\lambda)^{k+1}\varphi)(x_0\beta_{n+1}^0(tz), g; (\beta_j^0, \mathbf{1})_{j=1,\dots,n}, (\mathbf{1}, \alpha_i)_{i=1,\dots,s}, (\beta_{n+1}^0, \mathbf{1}))\,dz \\
&\quad - ((D^\lambda)^{k+1}\varphi)(x_0, g; (\beta_j^0, \mathbf{1})_{j=1,\dots,n}, (\mathbf{1}, \alpha_i)_{i=1,\dots,s}, (\beta_{n+1}^0, \mathbf{1}))
\end{aligned}$$

with

$$(\beta_j^0, \mathbf{1})_{j=1,\dots,n} := ((\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1})),$$
$$(\mathbf{1}, \alpha_i)_{i=1,\dots,s} := ((\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)).$$

We have $\chi(g, 0, \alpha) = 0$ and χ is continuous by Lemma 2.6 applied for $X = G \times \mathbb{R} \times \mathfrak{L}^s(G)$ and $f\colon G \times \mathbb{R} \times \mathfrak{L}^s(G) \times [0,1] \to \mathcal{Y}$ given by

$$\begin{aligned} &f(g, t, \alpha, z) \\ &\quad = ((D^\lambda)^{k+1}\varphi)(x_0\beta_{n+1}^0(tz), g; (\beta_j^0, \mathbf{1})_{j=1,\dots,n}, (\mathbf{1}, \alpha_i)_{i=1,\dots,s}, (\beta_{n+1}^0, \mathbf{1})), \end{aligned}$$

which is continuous since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$.

Finally, by (4.1), F is the sum of two continuous functions, which completes the proof. ■

LEMMA 4.15. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then the following assertions hold.*

(a) *Let $x \in H$ and $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$. Then the function*

$$h := ((D_1^\lambda)^n \varphi)(x, \bullet\,; \beta_1, \dots, \beta_n)\colon G \to \mathcal{Y}$$

belongs to $\mathcal{C}^\infty(G, \mathcal{Y})$ and for all $s \geq 1$ and $\alpha_1, \dots, \alpha_s \in \mathfrak{L}(G)$ we have

$$\begin{aligned} &((D^\lambda)^s h)(g; \alpha_1, \dots, \alpha_s) \\ &\qquad = ((D^\lambda)^{n+s}\varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)). \end{aligned}$$

(b) *Let $x \in H$, $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(G)$. Then the function*

$$h := ((D^\lambda)^n \varphi)(x, \bullet\,; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n))\colon G \to \mathcal{Y}$$

is in $\mathcal{C}^\infty(G, \mathcal{Y})$ and for every $s \geq 1$ and $\alpha_1, \dots, \alpha_s \in \mathfrak{L}(G)$ we have

$$\begin{aligned} &((D^\lambda)^s h)(g; \alpha_1, \dots, \alpha_s) \\ &\qquad = ((D^\lambda)^{n+s}\varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)). \end{aligned}$$

Proof. Assertion (a) follows from (b) for $\gamma_1 = \dots = \gamma_n = \mathbf{1} \in \mathfrak{L}(G)$, by using Lemma 4.4(a).

We prove (b) by induction on $s \geq 1$. Since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ and $h(g) = ((D^\lambda)^n\varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n))$ it follows that h is continuous.

The case $s = 1$: We have

$$\begin{aligned} (D^\lambda h)(g; \alpha) &= \frac{d}{dt}\Big|_{t=0} h(g\alpha(t)) \\ &= \frac{d}{dt}\Big|_{t=0} ((D^\lambda)^n\varphi)(x, g\alpha(t); (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n)) \\ &= ((D^\lambda)^{n+1}\varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha)), \end{aligned}$$

and since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ we obtain $h \in \mathcal{C}^1(G, \mathcal{Y})$.

Now suppose the assertion has been proved for s. Then

$$\begin{aligned}
&((D^\lambda)^{s+1}h)(g;\alpha_1,\dots,\alpha_s,\alpha_{s+1}) \\
&\quad = \frac{d}{dt}\Big|_{t=0}((D^\lambda)^s h)(g\alpha_{s+1}(t);\alpha_1,\dots,\alpha_s) \\
&\quad = \frac{d}{dt}\Big|_{t=0}((D^\lambda)^{n+s}\varphi)(x,g\alpha_{s+1}(t);(\beta_1,\gamma_1),\dots,(\beta_n,\gamma_n),(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s)) \\
&\quad = ((D^\lambda)^{n+s+1}\varphi)(x,g;(\beta_1,\gamma_1),\dots,(\beta_n,\gamma_n),(\mathbf{1},\alpha_1),\dots,(\mathbf{1},\alpha_s),(\mathbf{1},\alpha_{s+1}))
\end{aligned}$$

as desired.

Moreover, since $\varphi \in \mathcal{C}^\infty(H\times G,\mathcal{Y})$, we obtain $h\in\mathcal{C}^s(G,\mathcal{Y})$ for every $s\ge 1$. This shows that $h\in\mathcal{C}^\infty(G,\mathcal{Y})$, and the proof is complete. ∎

Theorem 4.16. *Let G and H be topological groups and $\mathcal{Y}$ be a locally convex space. Then the following assertions hold:*

(i) *For every $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$, the function*

$$\widetilde{\varphi}\colon H\to\mathcal{C}^\infty(G,\mathcal{Y}),\qquad \widetilde{\varphi}(x)(g):=\varphi(x,g),$$

belongs to $\mathcal{C}^\infty(H,\mathcal{E}(G,\mathcal{Y}))$.

(ii) *The map*

$$\Phi\colon\mathcal{E}(H\times G,\mathcal{Y})\to\mathcal{E}(H,\mathcal{E}(G,\mathcal{Y})),\qquad \varphi\mapsto\widetilde{\varphi},$$

is a linear topological embedding of locally convex spaces.

Proof. To prove (i), let $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$. The fact that $\widetilde{\varphi}$ is continuous follows by Proposition 4.9. We will show that for every $n\ge 1$ the derivative $(D^\lambda)^n\widetilde{\varphi}\colon H\times\mathfrak{L}^n(H)\to\mathcal{E}(G,\mathcal{Y})$ exists and is continuous. The existence follows from Proposition 4.14. The fact that the derivative takes values in $\mathcal{E}(G,\mathcal{Y})$ is a consequence of Lemma 4.15(a).

For the continuity we will prove that for every seminorm p on $\mathcal{E}(G,\mathcal{Y})$ as in Definition 3.1 and every $x_0\in H$, $\beta_1^0,\dots,\beta_n^0\in\mathfrak{L}(H)$ and $\varepsilon>0$ there exists a neighborhood U of $(x;\beta_1^0,\dots,\beta_n^0)\in H\times\mathfrak{L}^n(H)$ such that for every $(x;\beta_1,\dots,\beta_n)\in U$ we have

$$p(((D^\lambda)^n\widetilde{\varphi})(x;\beta_1,\dots,\beta_n)-((D^\lambda)^n\widetilde{\varphi})(x_0;\beta_1^0,\dots,\beta_n^0))\le\varepsilon.$$

Case (a): $p=p_{K_2}$ with $K_2\subseteq G$ compact. As in the proof of Proposition 4.9, we denote

$$\begin{aligned}
&E(x;\beta_1,\dots,\beta_n) \\
&\quad := \sup_{g\in K_2}|((D^\lambda)^n\widetilde{\varphi})(x;\beta_1,\dots,\beta_n)(g)-((D^\lambda)^n\widetilde{\varphi})(x_0;\beta_1^0,\dots,\beta_n^0)(g)|.
\end{aligned}$$

Applying Proposition 4.5 we obtain

$$E(x;\beta_1,\dots,\beta_n) \\ = \sup_{g\in K_2} |((D_1^\lambda)^n\varphi)(x,g;\beta_1,\dots,\beta_n) - ((D_1^\lambda)^n\varphi)(x_0,g;\beta_1^0,\dots,\beta_n^0)|.$$

Now the conclusion follows by applying Lemma 2.5 for $(x_0;\beta_1^0,\dots,\beta_n^0)\in H\times\mathfrak{L}^n(H)=X$, $K=K_2$ compact in $T=G$, and

$$f\colon H\times\mathfrak{L}^n(H)\times G\to\mathcal{Y},\qquad f(x;\beta_1,\dots,\beta_n,g)=((D_1^\lambda)^n\varphi)(x,g;\beta_1,\dots,\beta_n),$$

which is continuous since $(D_1^\lambda)^n\varphi\colon H\times G\times\mathfrak{L}^n(H)\to\mathcal{Y}$ is continuous by Lemma 4.4(c).

Case (b): $p=p_{K_1,K_2}$ with compact sets $K_2\subseteq G$ and $K_1\subseteq\mathfrak{L}^s(G)$, where $s\ge 1$. We denote

$$\begin{aligned}E(x;\beta_1,\dots,\beta_n)=\sup\{|((D^\lambda)^s((D^\lambda)^n\widetilde{\varphi})(x;\beta_1,\dots,\beta_n))(g;\gamma_1,\dots,\gamma_s)\\ -((D^\lambda)^s((D^\lambda)^n\widetilde{\varphi})(x_0;\beta_1^0,\dots,\beta_n^0))(g;\gamma_1,\dots,\gamma_s)|\mid\\ g\in K_2,\ \gamma=(\gamma_1,\dots,\gamma_s)\in K_1\}.\end{aligned}$$

By Proposition 4.5 we obtain

$$\begin{aligned}E(x;\beta_1,\dots,\beta_n)=\sup\{|((D^\lambda)^s((D_1^\lambda)^n\varphi)(x,\bullet;\beta_1,\dots,\beta_n))(g;\gamma_1,\dots,\gamma_s)\\ -((D^\lambda)^s((D_1^\lambda)^n\varphi)(x_0,\bullet;\beta_1^0,\dots,\beta_n^0))(g;\gamma_1,\dots,\gamma_s)|\mid\\ g\in K_2,\ \gamma=(\gamma_1,\dots,\gamma_s)\in K_1\}.\end{aligned}$$

Furthermore, by Lemma 4.15(a),

$$\begin{aligned}&E(x;\beta_1,\dots,\beta_n)\\ &=\sup\{|((D^\lambda)^{n+s}\varphi)(x,g;(\beta_1,\mathbf{1}),\dots,(\beta_n,\mathbf{1}),(\mathbf{1},\gamma_1),\dots,(\mathbf{1},\gamma_s))\\ &\quad-((D^\lambda)^{n+s}\varphi)(x_0,g;(\beta_1^0,\mathbf{1}),\dots,(\beta_n^0,\mathbf{1}),(\mathbf{1},\gamma_1),\dots,(\mathbf{1},\gamma_s))|\mid\\ &\quad g\in K_2,\ \gamma=(\gamma_1,\dots,\gamma_s)\in K_1\}.\end{aligned}$$

The conclusion now follows by Lemma 2.5 for $T=G\times\mathfrak{L}^s(G)$, $K=K_2\times K_1$, $(x_0;\beta_1^0,\dots,\beta_n^0)\in H\times\mathfrak{L}^n(H)=X$, and $f\colon H\times\mathfrak{L}^n(H)\times G\times\mathfrak{L}^s(G)\to\mathcal{Y}$ given by

$$\begin{aligned}&f(x,\beta_1,\dots,\beta_n,g,\gamma_1,\dots,\gamma_s)\\ &=((D^\lambda)^{n+s}\varphi)(x,g;(\beta_1,\mathbf{1}),\dots,(\beta_n,\mathbf{1}),(\mathbf{1},\gamma_1),\dots,(\mathbf{1},\gamma_s)).\end{aligned}$$

The function f is continuous since $(D^\lambda)^{n+s}\varphi$ is continuous as a consequence of the hypothesis $\varphi\in\mathcal{C}^\infty(H\times G,\mathcal{Y})$, and this concludes the proof of the fact that $\widetilde{\varphi}\in\mathcal{E}(H,\mathcal{E}(G,\mathcal{Y}))$.

For (ii), note that the inverse map

$$\operatorname{Ran}\Phi\to\mathcal{E}(H\times G,\mathcal{Y}),\qquad\widetilde{\varphi}\mapsto\varphi,$$

is well defined. Moreover, the continuity of $\varphi \mapsto \widetilde{\varphi}$ and $\widetilde{\varphi} \mapsto \varphi$ follows easily by taking into account the relations between the derivatives of φ and $\widetilde{\varphi}$ provided by Proposition 4.5 and Lemma 4.4 (see also [BCR81, Prop. 1.2.1.5]). ∎

Remark 4.17. It is easily seen that the proof of Theorem 4.16 has a local character, in the sense that it actually leads to a more general result:

Let G and H be any topological groups and $\mathcal{Y}$ be any locally convex space. Pick any open sets $V \subseteq G$ and $W \subseteq H$. Then for any $\varphi \in \mathcal{C}^\infty(W \times V, \mathcal{Y})$, the function $\widetilde{\varphi}\colon W \to \mathcal{C}^\infty(V, \mathcal{Y})$, $\widetilde{\varphi}(x)(g) := \varphi(x, g)$, belongs to $\mathcal{C}^\infty(W, \mathcal{E}(V, \mathcal{Y}))$. Moreover, the map

$$\mathcal{E}(W \times V, \mathcal{Y}) \to \mathcal{E}(W, \mathcal{E}(V, \mathcal{Y})), \quad \varphi \mapsto \widetilde{\varphi},$$

is a linear topological embedding of locally convex spaces.

We also note that the map Φ of Theorem 4.16(ii) may not be surjective; see [AlS15] for details.

5. Structure of invariant local operators. In this final section, we establish the structure of invariant local operators on an arbitrary topological group G (Theorem 5.2) and we use that result to compare to some extent the two candidates $\mathcal{U}_0(G) \subseteq \mathcal{U}(G)$ for the role of universal enveloping algebra of G; cf. Problem 3.15. Our main result in this connection is Corollary 5.3 below.

General results

Proposition 5.1. *If G is a topological group, then for every $f \in \mathcal{E}(G)$ and $u \in \mathcal{E}'(G)$ we have $f * u \in \mathcal{E}(G)$.*

Proof. Let $m\colon G \times G \to G$, $m(x, y) = xy$. By denoting $\check{u} = v \in \mathcal{E}'(G)$ we have $\check{v} = u$ and $(f * u)(x) = \check{u}(f \circ L_x) = v(f \circ L_x)$. Now define $\varphi\colon G \times G \to \mathbb{C}$, $\varphi(x, y) = f(xy)$. Since $\varphi = f \circ m$, it follows by Proposition 3.11 that $\varphi \in \mathcal{C}^\infty(G \times G)$. If we define $\widetilde{\varphi}\colon G \to \mathcal{C}^\infty(G)$, $\widetilde{\varphi}(x)(y) = \varphi(x, y)$, as in Notation 4.1, then by using Theorem 4.16(i) we obtain $\widetilde{\varphi} \in \mathcal{C}^\infty(G, \mathcal{E}(G))$.

Since $\widetilde{\varphi}(x)(y) = f(xy) = (f \circ L_x)(y)$, we have $\widetilde{\varphi}(x) = f \circ L_x$ for all $x \in G$, and therefore $f * u = v \circ \widetilde{\varphi}$. As $\widetilde{\varphi} \in \mathcal{C}^\infty(G, \mathcal{E}(G))$ by Theorem 4.16(i), we obtain $f * u \in \mathcal{E}(G)$. ∎

We can now prove the following theorem, which extends a well known property of finite-dimensional Lie groups.

Theorem 5.2. *Let G be a topological group and for every $u \in \mathcal{E}'(G)$ define the linear operator $D_u\colon \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G)$, $D_u f = f * u$. Then the operator $\Psi\colon \mathcal{E}'_{\mathbf{1}}(G) \to \mathcal{U}(G)$, $\Psi(u) = D_u$, is well defined, invertible, and its inverse is*

$$\Psi^{-1}\colon \mathcal{U}(G) \to \mathcal{E}'_{\mathbf{1}}(G),\ (\Psi^{-1}(D))(f) = (D\check{f})(\mathbf{1}) \text{ for } f \in \mathcal{C}^\infty(G),\ D \in \mathcal{U}(G).$$

Proof. We organize the proof in three steps.

STEP 1: We show that Ψ is well defined, that is, $D_u \in \mathcal{U}(G)$ for all $u \in \mathcal{E}'_1(G)$. In fact, $D_u(f\circ L_x)(y) = ((f\circ L_x)*u)(y) = \check{u}(f\circ L_x\circ L_y)$ and on the other hand $(D_u(f)\circ L_x)(y) = (f*u)(xy) = \check{u}(f\circ L_{xy}) = \check{u}(f\circ L_x\circ L_y) = D_u(f\circ L_x)(y)$, hence $D_u(f\circ L_x) = D_u(f)\circ L_x$.

From $u\in\mathcal{E}'_{\mathbf{1}}(G)$ it follows that $\operatorname{supp} u\subseteq\{\mathbf{1}\}\subseteq G$, hence

$$\operatorname{supp}(D_u f) = \operatorname{supp}(f*u) \subseteq (\operatorname{supp} f)(\operatorname{supp} u)$$

by Remark 3.6, and therefore $\operatorname{supp}(D_u f)\subseteq(\operatorname{supp} f)\{\mathbf{1}\} = \operatorname{supp} f$. Moreover, D_u is continuous as a direct consequence of Propositions 3.8 and 3.10. Thus $D_u\in\mathcal{U}(G)$.

STEP 2: We show that the mapping

$$\Phi\colon \mathcal{U}(G)\to\mathcal{E}'_{\mathbf{1}}(G),\ (\Phi(D))(f) = (D\check{f})(\mathbf{1}) \text{ for } f\in\mathcal{C}^\infty(G) \text{ and } D\in\mathcal{U}(G),$$

is well defined, that is, for every $D\in\mathcal{U}(G)$ the functional $u\colon\mathcal{E}(G)\to\mathbb{C}$, $u(f) = (D\check{f})(\mathbf{1})$, is in $\mathcal{E}'_{\mathbf{1}}(G)$.

To this end note that if $\operatorname{supp} f\subseteq U$ then $G\setminus U\subseteq\{x\in G\mid f(x)=0\}$. Now let $x\in G$ with $x\neq\mathbf{1}$. Since the topology of G is assumed to be separated, there exists an open neighborhood U of x with $\mathbf{1}\notin U$. For every $f\in\mathcal{C}^\infty(G)$ with $\operatorname{supp} f\subseteq U$ we have $\operatorname{supp}\check{f} = (\operatorname{supp} f)^{-1}\subseteq U^{-1}$, and so $\operatorname{supp}(D\check{f})\subseteq\operatorname{supp}\check{f}\subseteq U^{-1}$. Thus $G\setminus U^{-1}\subseteq\{y\in G\mid (D\check{f})(y)=0\}$.

Since $\mathbf{1}\notin U$ and $\mathbf{1}\notin U^{-1}$, we have $(D\check{f})(\mathbf{1}) = 0$, hence $x\notin\operatorname{supp} u$ for all $x\in G\setminus\{\mathbf{1}\}$, and thus $\operatorname{supp} u\subseteq\{\mathbf{1}\}$. That is, $u\in\mathcal{E}'_{\mathbf{1}}(G)$.

STEP 3: We show that $\Psi\circ\Phi = \mathrm{id}_{\mathcal{U}(G)}$ and $\Phi\circ\Psi = \mathrm{id}_{\mathcal{E}'_{\mathbf{1}}(G)}$.

To this end, let $D\in\mathcal{U}(G)$ and denote $\Phi(D) = u$. We have $u(f) = (D\check{f})(\mathbf{1})$ and $\Psi(u) = D_u$, where

$$\begin{aligned}(D_u f)(x) &= (f*u)(x) = \check{u}(f\circ L_x) = D(f\circ L_x)(\mathbf{1}) = ((Df)\circ L_x)(\mathbf{1})\\ &= (Df)(x).\end{aligned}$$

Hence $D_u f = Df$ and $D_u = D$ and we obtain $\Psi\circ\Phi = \mathrm{id}_{\mathcal{U}(G)}$.

Now let $u\in\mathcal{E}'_{\mathbf{1}}(G)$, hence $\Psi(u) = D_u$. For $v := \Phi(D_u)\in\mathcal{E}'_{\mathbf{1}}(G)$, we have $v(f) = (D_u\check{f})(\mathbf{1}) = (\check{f}*u)(\mathbf{1}) = \check{u}(\check{f}\circ L_{\mathbf{1}}) = \check{u}(\check{f}) = u(f)$. Hence $v = u$ and $\Phi\circ\Psi = \mathrm{id}_{\mathcal{E}'_{\mathbf{1}}(G)}$. ∎

If G is any pre-Lie group, then one can use Theorem 5.2 to endow $\mathcal{U}(G)$ with a natural topology for which Ψ is a homeomorphism if $\mathcal{E}'_{\mathbf{1}}(G)$ carries the weak dual topology which it inherits as a closed linear subspace of $\mathcal{E}'(G)$ (see Definition 3.1). The topology of $\mathcal{U}(G)$ can be equivalently described as the locally convex topology determined by the family of seminorms $\{D\mapsto|(Df)(\mathbf{1})|\}_{f\in\mathcal{E}(G)}$.

In the statement of the following corollary, we say that a Banach space $\mathcal{X}$ *admits bump functions* if there exists $\varphi\in\mathcal{C}^\infty(\mathcal{X})$ which is equal to 1 on some neighborhood of $0 \in \mathcal{X}$, has the support contained in some ball, and $\sup_{x\in\mathcal{X}} \| d_x^k \phi\| < \infty$ for every $k \geq 1$. Every Hilbert space *admits bump functions*; see [WD73] and [LW11] for more details and examples. In this setting, we will provide the following partial answer to Problem 3.15.

COROLLARY 5.3. *Let G be a Banach–Lie group whose Lie algebra admits bump functions. Then $\mathcal{U}_0(G)$ is a dense subalgebra of $\mathcal{U}(G)$.*

Proof. By the Hahn–Banach theorem, it suffices to show that if a continuous linear functional $\theta\colon \mathcal{U}(G) \to \mathbb{C}$ vanishes on $\mathcal{U}_0(G)$, then $\theta = 0$. To this end note that, using the above family of seminorms describing the topology of $\mathcal{U}(G)$, one can find some functions $f_1, \dots, f_n \in \mathcal{E}(G)$ with

$$|\theta(D)| \leq |(Df_1)(\mathbf{1})| + \dots + |(Df_n)(\mathbf{1})| \quad \text{for all } D \in \mathcal{U}(G).$$

Then $\theta = \theta_1 + \dots + \theta_n$ for some linear functionals $\theta_1, \dots, \theta_n \colon \mathcal{U}(G) \to \mathbb{C}$ with $|\theta_j(D)| \leq |(Df_j)(\mathbf{1})|$ for $j = 1, \dots, n$ (see for instance [SZ79, Lemma 1.1]). Then the kernel of the linear functional $D \mapsto (Df_j)(\mathbf{1})$ is contained in $\operatorname{Ker}\theta_j$, and since both these kernels are closed 1-codimensional subspaces of $\mathcal{U}(G)$, it follows that, after replacing f_j by cf_j for a suitable $c \in \mathbb{C}$, we have $\theta_j(D) = (Df_j)(\mathbf{1})$ for all $D \in \mathcal{U}(G)$. Hence for $f = f_1 + \dots + f_n$ one has $\theta(D) = (Df)(\mathbf{1})$ for all $D \in \mathcal{U}(G)$.

The assumption $\theta(D) = 0$ for all $D \in \mathcal{U}_0(G)$ is then equivalent to the fact that for all $k \geq 1$ we have $(d^k(f \circ \exp_G))(0) = 0$, where $\exp_G \colon \mathfrak{g} \to G$ is the exponential map of G, which is a local diffeomorphism at $0 \in \mathfrak{g}$. Now the hypothesis that the Lie algebra $\mathfrak{g}$ admits bump functions allows us to use [LW11, Prop. 3], which ensures that for every local operator T on $\mathfrak{g}$ we have $(T(f \circ \exp_G))(0) = 0$.

We will check that $(Df)(\mathbf{1}) = 0$ for every local operator D on G. To this end pick open sets U and V for which $\exp_G \colon V \to U$ is a diffeomorphism with inverse denoted by $\log_G$, where $\mathbf{1} \in U \subseteq G$ and $0 \in V \subseteq \mathfrak{g}$. Then use the hypothesis on $\mathfrak{g}$ to find $\psi \in \mathcal{C}^\infty(\mathfrak{g})$ with $\operatorname{supp}\psi \subseteq V$ and $\psi = 1$ on some neighborhood of $0 \in \mathfrak{g}$. Denote $\phi := \psi \circ \log_G \in \mathcal{C}^\infty(U)$ and extend it by 0 on $G \setminus U$. Then $\phi \in \mathcal{C}^\infty(G)$, $\operatorname{supp}\phi \subseteq U$, $\phi = 1$ on some neighborhood of $\mathbf{1} \in U \subseteq G$, and $\psi = \phi \circ \exp_G$. Now define

$$T \colon \mathcal{C}^\infty(\mathfrak{g}) \to \mathcal{C}^\infty(\mathfrak{g}), \qquad Th = D(((\psi h) \circ \log_G)\phi),$$

where the function $(\psi h) \circ \log_G \in \mathcal{C}^\infty(V)$ is extended by 0 on $\mathfrak{g} \setminus V$. Since D is a local operator, so is T, and hence by the above observation $(T(f \circ \exp_G))(0) = 0$, which is equivalent to $(Df)(\mathbf{1}) = 0$. That is, $\theta(D) = 0$ for all $D \in \mathcal{U}(G)$, which concludes the proof. ■

Pre-Lie groups. In order to illustrate the above general results and relate them to the earlier literature, we conclude by some specific examples of topological groups which can be studied from the perspective of Lie theory (see also Remark 3.16). This is the case of the class of pre-Lie groups introduced in [BR80] and [BCR81], closed with respect to several natural operations that may not preserve the locally compact or Lie groups, as for instance taking closed subgroups, infinite direct products, or projective limits [BCR81, Prop. 1.3.1]. Some specific pre-Lie groups are briefly mentioned in Examples 5.6–5.9 below. See also [HM07] and [Gl02b] for further information on Lie theory for topological groups which may not be Lie groups.

DEFINITION 5.4. A *pre-Lie group* is any topological group G such that:

(1) The topological space $\mathfrak{L}(G)$ has the structure of a locally convex Lie algebra over $\mathbb{R}$, whose scalar multiplication, vector addition, and bracket satisfy the following conditions for all $t, s \in \mathbb{R}$ and all γ_1, γ_2 in $\mathfrak{L}(G)$:

$$\begin{aligned}(t \cdot \gamma_1)(s) &= \gamma_1(ts);\\ (\gamma_1 + \gamma_2)(t) &= \lim_{n\to\infty} (\gamma_1(t/n)\gamma_2(t/n))^n;\\ [\gamma_1, \gamma_2](t^2) &= \lim_{n\to\infty} (\gamma_1(t/n)\gamma_2(t/n)\gamma_1(-t/n)\gamma_2(-t/n))^{n^2},\end{aligned}$$

where the convergence is assumed to be uniform on compact subsets of $\mathbb{R}$.

(2) For every nontrivial $\gamma \in \mathfrak{L}(G)$ there exists a function φ of class $\mathcal{C}^\infty$ on some neighborhood of $\mathbf{1} \in G$ such that $(D^\lambda_\gamma \varphi)(\mathbf{1}) \neq 0$.

REMARK 5.5. If G is a pre-Lie group, then the multiplication mapping $m\colon G \times G \to G$, $(x, y) \mapsto xy$, is smooth by [BCR81, Th. 1.3.2.2 and Subsect. 1.1.2] (or alternatively [BR80, Th. and Sect. 1]). In particular, by using the chain rule contained in condition (dcm) of [BCR81, Subsect. 1.3.2] (or alternatively the proof of (v) in [BR80, Th.]), we easily recover in this special case the result of Proposition 3.11, to the effect that for any locally convex space $\mathcal{Y}$ the linear mapping $\mathcal{E}(G, \mathcal{Y}) \to \mathcal{E}(G \times G, \mathcal{Y})$, $\varphi \mapsto \varphi \circ m$, is well defined and continuous.

In connection with the above discussion, it is useful to recall here from [BCR81] (see also [Nic14]) the definition of differentiability of maps between open subsets of pre-Lie groups. Let G_1, G_2 be pre-Lie groups, $X_1 \subseteq G_1$ and $X_2 \subseteq G_2$ open sets, and $f : X_1 \to X_2$ be any continuous function. We say that f *is of class* C^k if there exist maps $D^\ell f : X_1 \times \Lambda^\ell(G_1) \to \Lambda(G_2)$, $\ell = 1, \dots, k$, such that for every locally convex space $\mathcal{Y}$ and every $\varphi \in C^\ell(X_2, \mathcal{Y}), 0 \leq \ell \leq k$, we have $\varphi \circ f \in C^\ell(X_1 \cap f^{-1}(X_2), \mathcal{Y})$ and for every $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \Lambda^\ell(G_1)$ one has the chain rule

$$D^\ell(\varphi \circ f)(x; \gamma) = \sum_{k=1}^{\ell} \sum_{(A_1,\dots,A_k)} D^{D^{A_1(\gamma)} f(x)} \cdots D^{D^{A_j(\gamma)} f(x)} \varphi(f(x)). \tag{5.1}$$

The second summation in the above formula is over all partitions $\{1,\dots,\ell\} = A_1 \sqcup \cdots \sqcup A_k$ into nonempty subsets with $\min A_1 > \cdots > \min A_k$. For any fixed $k \in \{1,\dots,\ell\}$ and every $j = 1,\dots,k$, we have denoted $A_j = \{i_1^j,\dots,i_{m_j}^j\} \subseteq \{1,\dots,\ell\}$, with $i_1^j < \cdots < i_{m_j}^j$, and moreover $A_j(\gamma) := (\gamma_{i_1^j},\dots,\gamma_{i_{m_j}^j}) \in \mathfrak{L}^{m_j}(G_1)$ and

$$D^{A_j(\gamma)} f(x) := D^{m_j} f(x; A_j(\gamma)) \in \mathfrak{L}(G_2).$$

Note that $m_j = |A_j|$ for $j = 1,\dots,k$, hence $1 \le m_1,\dots,m_k \le \ell$ with $m_1 + \cdots + m_k = \ell$.

Example 5.6. Every locally compact group (in particular, every finite-dimensional Lie group) is a pre-Lie group (see [BCR81, p. 41]). In this special case our Theorem 5.2 agrees with [Ed88, Th. 1.4] and [Ak95, Cor. 2.6].

Example 5.7. Every Banach–Lie group is a pre-Lie group (see [BCR81, p. 41]). In this special case, we are unable to provide any earlier reference for the result of our Theorem 5.2.

Example 5.8. If $\mathcal{X}$ is any locally convex space, then the abelian locally convex Lie group $(\mathcal{X},+)$ is a pre-Lie group (see [BCR81, p. 41]). In this case, we are again unable to provide any earlier reference for Theorem 5.2. See however [Du73, Th. 3.4] for a related result on real Hilbert spaces.

Example 5.9. Other examples of locally convex Lie groups which are pre-Lie groups are the so-called groups of Γ-rapidly decreasing mappings with values in some Lie groups (see [BCR81, Subsect. 4.2.2]).

For completeness, we briefly recall the construction of such groups in a very special situation. Let $n \ge 1$ be any integer and $\Gamma = \{\gamma_k \mid k \ge 0\}$, where

$$(\forall k \ge 0) \qquad \gamma_k(\cdot) = (1 + |\cdot|)^k$$

and $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^n$. Let $(\mathcal{A}, \|\cdot\|)$ be any unital associative Banach algebra with some fixed norm that defines its topology, let $\mathcal{A}^\times$ denote the group of invertible elements in $\mathcal{A}$, and set

$$G := \{f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathcal{A}^\times) \mid (\forall \alpha \in \mathbb{N}^n) \ \sup_{x \in \mathbb{R}^n} \gamma_k(x) \|\partial^\alpha (f-1)(x)\| < \infty\},$$

endowed with pointwise multiplication and inversion, where ∂^α denote partial derivatives. Then the group G of *Γ-rapidly decreasing $\mathcal{A}^\times$-valued mappings* has the natural structure of a pre-Lie group. This is a very special case of [BCR81, Cor. 4.1.1.7 and Th. 4.2.2.3]. Weighted mapping groups and groups of rapidly decreasing mappings were also studied in more general form in [Wal12].

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References

[Ak95] S. S. Akbarov, *Smooth structure and differential operators on a locally compact group*, Izv. Ross. Akad. Nauk Ser. Mat. 59 (1995), 3–48 (in Russian); English transl.: Izv. Math. 59 (1995), no. 1, 1–44.

[Al13] H. Alzaareer, *Lie groups of mappings on non-compact spaces and manifolds*, PhD thesis, Univ. Paderborn, 2013.

[AlS15] H. Alzaareer and A. Schmeding, *Differentiable mappings on products with different degrees of differentiability in the two factors*, Expo. Math. 33 (2015), 184–222.

[AD51] R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. 1 (1951), 5–31.

[Bel06] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory*, Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 137, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[BN15] D. Beltiţă and M. Nicolae, *Moment convexity of solvable locally compact groups*, J. Lie Theory 25 (2015), 733–751.

[BB11] I. Beltiţă and D. Beltiţă, *On differentiability of vectors in Lie group representations*, J. Lie Theory 21 (2011), 771–785.

[BF66] R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. 15 (1966), 877–898.

[BC75] H. Boseck und G. Czichowski, *Grundfunktionen und verallgemeinerte Funktionen auf topologischen Gruppen. II*, Math. Nachr. 66 (1975), 319–332.

[BCR81] H. Boseck, G. Czichowski and K.-P. Rudolph, *Analysis on Topological Groups —General Lie Theory*, Teubner-Texte Math. 37, Teubner, Leipzig, 1981.

[BR80] H. Boseck and K.-P. Rudolph, *An axiomatic approach to differentiability on topological groups*, Math. Nachr. 98 (1980), 27–36.

[Br61] F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\mathfrak{p}$-adiques*, Bull. Soc. Math. France 89 (1961), 43–75.

[CW99] A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Math. Lecture Notes 10, Amer. Math. Soc., Providence, RI, and Berkeley Center Pure Appl. Math., Berkeley, CA, 1999.

[Da15] R. Dahmen, *Peetre's theorem in the locally convex setting*, Forum Math. 27 (2015), 2957–2979.

[Du73] D. N. Dudin, *Generalized measures, or distributions, on Hilbert space*, Trudy Moskov. Mat. Obshch. 28 (1973), 135–158 (in Russian).

[Ed88] T. Edamatsu, *Differential operators on locally compact groups*, J. Math. Kyoto Univ. 28 (1988), 445–503.

[Eh56] L. Ehrenpreis, *Some properties of distributions on Lie groups*, Pacific J. Math. 6 (1956), 591–605.

[Gl02a] H. Glöckner, *Infinite-dimensional Lie groups without completeness restrictions*, in: Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups (Będlewo, 2000), A. Strasburger et al. (eds.), Banach Center Publ. 55, Inst. Math., Polish Acad. Sci., Warszawa, 2002, 43–59.

[Gl02b] H. Glöckner, *Real and p-adic Lie algebra functors on the category of topological groups*, Pacific J. Math. 203 (2002), 321–368.

[Gl12] H. Glöckner, *Continuity of bilinear maps on direct sums of topological vector spaces*, J. Funct. Anal. 262 (2012), 2013–2030.

[Gl13] H. Glöckner, *Exponential laws for ultrametric partially differentiable functions and applications*, p-Adic Numbers Ultrametric Anal. Appl. 5 (2013), 122–159.

[Glu57] V. M. Glushkov, *Lie algebras of locally bicompact groups*, Uspekhi Mat. Nauk (N.S.) 12 (1957), no. 2 (74), 137–142 (in Russian).

[GPR13] P. Górka, H. Prado and E. G. Reyes, *On a general class of nonlocal equations*, Ann. Henri Poincaré 14 (2013), 947–966.

[GoRe15] P. Górka and E. G. Reyes, *Sobolev spaces on locally compact abelian groups and the bosonic string equation*, J. Austral. Math. Soc. 98 (2015), 39–53.

[HeRo63] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. I*, Grundlehren Math. Wiss. 115, Springer, Berlin, 1963.

[HM05] K. H. Hofmann and S. A. Morris, *Sophus Lie's third fundamental theorem and the adjoint functor theorem*, J. Group Theory 8 (2005), 115–133.

[HM07] K. H. Hofmann and S. A. Morris, *The Lie Theory of Connected Pro-Lie Groups*, EMS Tracts in Math. 2, Eur. Math. Soc., Zürich, 2007.

[HM13] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, 3rd ed., de Gruyter Stud. Math. 25, de Gruyter, Berlin, 2013.

[Ka59] G. I. Kats, *Generalized functions on locally compact groups and the decomposition of a regular representation*, Dokl. Akad. Nauk SSSR 125 (1959), 27–30 (in Russian).

[Ka61] G. I. Kats, *Generalized functions on a locally compact group and decompositions of unitary representations*, Trudy Moskov. Mat. Obshch. 10 (1961), 3–40 (in Russian).

[Ke75] J. L. Kelley, *General Topology*, Grad. Texts in Math. 27, Springer, New York, 1975.

[KM97] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, Math. Surveys Monogr. 53, Amer. Math. Soc., Providence, RI, 1997.

[KMR15] A. Kriegl, P. W. Michor and A. Rainer, *The convenient setting for Denjoy–Carleman differentiable mappings of Beurling and Roumieu type*, Rev. Mat. Complut. 28 (2015), 549–597.

[La57] R. K. Lashof, *Lie algebras of locally compact groups*, Pacific J. Math. 7 (1957), 1145–1162.

[LW11] D. H. Leung and Y.-S. Wang, *Local operators on C^p*, J. Math. Anal. Appl. 381 (2011), 308–314.

[Ma61] K. Maurin, *Distributionen auf Yamabe-Gruppen. Harmonische Analyse auf einer Abelschen l. k. Gruppe*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 (1961), 845–850.

[MM64] K. Maurin und L. Maurin, *Universelle umhüllende Algebra einer lokal kompakten Gruppe und ihre selbstadjungierte Darstellungen. Anwendungen*, Studia Math. 24 (1964), 227–243.

[MM65] K. Maurin and L. Maurin, *Automorphic forms, spectral theory and group representations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 199–203.

[Ne06] K.-H. Neeb, *Towards a Lie theory of locally convex groups*, Jpn. J. Math. 1 (2006), 291–468.

[NS13a] K.-H. Neeb and H. Salmasian, *Differentiable vectors and unitary representations of Fréchet–Lie supergroups*, Math. Z. 275 (2013), 419–451.

[NS13b] K.-H. Neeb and H. Salmasian, *Positive definite superfunctions and unitary representations of Lie supergroups*, Transform. Groups 18 (2013), 803–844.

[Nic14] M. Nicolae, *On differential calculus on pre-Lie groups*, preprint IMAR no. 6/2014, http://www.imar.ro/library.php.

[Nic15] M. Nicolae, *Tangent groups of 2-step nilpotent pre-Lie groups*, preprint IMAR no. 4/2015, http://www.imar.ro/library.php.

[Ped94] N. V. Pedersen, *Matrix coefficients and a Weyl correspondence for nilpotent Lie groups*, Invent. Math. 118 (1994), 1–36.

[Pee60] J. Peetre, *Réctification à l'article "Une caractérisation abstraite des opérateurs différentiels"*, Math. Scand. 8 (1960), 116–120.

[Py74] T. Pytlik, *Nuclear spaces on a locally compact group*, Studia Math. 50 (1974), 225–243.

[Ri53] J. Riss, *Éléments de calcul différentiel et théorie des distributions sur les groupes abéliens localement compacts*, Acta Math. 89 (1953), 45–105.

[SZ79] Ş. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Ed. Academiei, Bucureşti, and Abacus Press, Tunbridge Wells, 1979.

[Wal12] B. Walter, *Weighted diffeomorphism groups of Banach spaces and weighted mapping groups*, Dissertationes Math. 484 (2012), 128 pp.

[War72] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups. I*, Grundlehren Math. Wiss. 188, Springer, New York, 1972.

[WD73] J. C. Wells and C. R. DePrima, *Local automorphisms are differential operators on some Banach spaces*, Proc. Amer. Math. Soc. 40 (1973), 453–457.

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