ACTA ARITHMETICA
172.3 (2016)

Coppersmith–Rivlin type inequalities and the order of vanishing of polynomials at 1

by

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1. Introduction. In [B-99] and [B-13] we examined a number of problems concerning polynomials with coefficients restricted in various ways. We are particularly interested in how small such polynomials can be on the interval $[0,1]$. For example, we proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\bigl(-c_1\sqrt{n}\bigr) \le \min_{0 \not\equiv Q \in \mathcal{F}_n} \Bigl\{ \max_{x \in [0,1]} |Q(x)| \Bigr\} \le \exp\bigl(-c_2\sqrt{n}\bigr)$$

for every $n \ge 2$, where $\mathcal{F}_n$ denotes the set of all polynomials of degree at most n with coefficients from $\{-1,0,1\}$.

Littlewood considered minimization problems of this variety on the unit disk. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view. See [A-90], [B-98], [B-95], [F-80], [O-93].

One key to the analysis is a study of the related problem of giving an upper bound for the multiplicity of the zero these restricted polynomials can have at 1. In [B-99] and [B-13] we answer this latter question precisely for the class of polynomials of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1,\ a_j \in \mathbb{C},\ j = 1, \ldots, n,$$

with fixed $|a_0| \neq 0$.

2010 *Mathematics Subject Classification*: 11C08, 41A17, 26C10, 30C15.
Key words and phrases: polynomials, restricted coefficients, order of vanishing at 1, Coppersmith–Rivlin type inequalities.
Received 8 April 2015; revised 30 September 2015.
Published online 4 February 2016.

Variants of these questions have attracted considerable study, though rarely have precise answers been possible to give. See in particular [A-79], [B-32], [B-87], [E-50], [Sch-33], [Sz-34]. Indeed, the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n.) See [H-82], [B-94b], or [B-02].

For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ we define the following numbers. Let $\kappa_p(n,L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| \geq L\Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}, \qquad a_j \in \mathbb{C},$$

such that $(x-1)^k$ divides $Q(x)$. For $n \in \mathbb{N}$ and $L > 0$ let $\kappa_\infty(n,L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| \geq L \max_{1 \leq j \leq n} |a_j|, \qquad a_j \in \mathbb{C},$$

such that $(x-1)^k$ divides $Q(x)$. In [B-99] we proved that there is an absolute constant $c_3 > 0$ such that

$$\min\left\{\frac{1}{6}\sqrt{(n(1-\log L)} - 1, n\right\} \leq \kappa_\infty(n,L) \leq \min\left\{c_3\sqrt{n(1-\log L)}, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0,1]$. However, we were far from being able to establish the right result in the case of $L \geq 1$. In [B-13] we proved the right order of magnitude of $\kappa_\infty(n,L)$ and $\kappa_2(n,L)$ in the case of $L \geq 1$. Our results in [B-99] and [B-13] sharpen and generalize results of Schur [Sch-33], Amoroso [A-90], Bombieri and Vaaler [B-87], and Hua [H-82] who gave versions for polynomials with integer coefficients. Our results in [B-99] have turned out to be related to a number of recent papers from a wide range of research areas. See [A-02], [B-98], [B-95], [B-96], [B-97a], [B-97b], [B-97c], [B-00], [B-07], [B-08a], [B-13], [B-08b], [B-94a], [B-94b], [Bu-99], [C-02], [C-13], [C-10], [D-99], [D-01], [D-14], [D-03], [E-08a], [E-08b], [E-15], [F-00], [G-05], [K-04], [M-69], [M-03], [N-94], [O-93], [P-99], [P-13], [R-04], [R-07], [S-99], [T-07], [T-84], for example. More results on the zeros of polynomials with Littlewood type coefficient constraints may be found in [E-02b]. Markov and Bernstein type inequalities under Erdős type coefficient constraints are surveyed in [E-02a].

For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$ we define the following numbers. Let $\mu_q(n,L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L}\Big(\sum_{j=1}^{n} |Q(j)|^q\Big)^{1/q}.$$

For $n \in \mathbb{N}$ and $L > 0$ let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{1\le j\le n} |Q(j)|.$$

It is a simple consequence of Hölder's inequality (see Lemma 3.6) that

$$\kappa_p(n, L) \le \mu_q(n, L)$$

whenever $n \in \mathbb{N}$, $L > 0$, $1 \le p, q \le \infty$, and $1/p + 1/q = 1$.

In this paper we find the the size of $\kappa_p(n, L)$ and $\mu_q(n, L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \le p, q \le \infty$. The result about $\mu_\infty(n, L)$ is due to Coppersmith and Rivlin [C-92a], but our proof presented here is completely different and much shorter even in that special case. Our results in [B-99] may be viewed as finding the size of $\kappa_\infty(n, L)$ and $\mu_1(n, L)$ for all $n \in \mathbb{N}$ and $L \in (0, 1]$. Our results in [B-13] in turn may be viewed as finding the size of $\kappa_\infty(n, L)$, $\mu_1(n, L)$, $\kappa_2(n, L)$, and $\mu_2(n, L)$ for all $n \in \mathbb{N}$ and $L > 0$.

2. New results. We extend some of our main results in [B-13] to the case $L \ge 1$.

Theorem 2.1. *Let $p \in (1, \infty]$ and $q \in [1, \infty)$ satisfy $1/p + 1/q = 1$. There are absolute constants $c_1, c_2 > 0$ such that*

$$\sqrt{n}\,(c_1L)^{-q/2} - 1 \le \kappa_p(n, L) \le \mu_q(n, L) \le \sqrt{n}\,(c_2L)^{-q/2} + 2$$

for every $n \in \mathbb{N}$ and $L > 1/2$, and

$$\begin{aligned} c_3 \min\big\{\sqrt{n(-\log L)}, n\big\} &\le \kappa_p(n, L) \le \mu_q(n, L) \\ &\le c_4 \min\big\{\sqrt{n(-\log L)}, n\big\} + 4 \end{aligned}$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Here $c_1 := 1/53$, $c_2 := 40$, $c_3 := 2/7$, and $c_4 := 13$ are appropriate choices.

Theorem 2.2. *There are absolute constants $c_1, c_2 > 0$ such that*

$$c_1\sqrt{n(1-L)} - 1 \le \kappa_1(n, L) \le \mu_\infty(n, L) \le c_2\sqrt{n(1-L)} + 1$$

for every $n \in \mathbb{N}$ and $L \in (1/2, 1]$, and

$$\begin{aligned} c_3 \min\big\{\sqrt{n(-\log L)}, n\big\} &\le \kappa_1(n, L) \le \mu_\infty(n, L) \\ &\le c_4 \min\big\{\sqrt{n(-\log L)}, n\big\} + 4 \end{aligned}$$

for every $n \in \mathbb{N}$ *and* $L \in (0, 1/2]$. *Note that* $\kappa_1(n, L) = \mu_\infty(n, L) = 0$ *for every* $n \in \mathbb{N}$ *and* $L > 1$. *Here* $c_1 := 1/5$, $c_2 := 1$, $c_3 := 2/7$, *and* $c_4 := 13$ *are appropriate choices.*

3. Lemmas. In this section we list our lemmas needed in the proofs of Theorems 2.1 and 2.2. These lemmas are proved in Section 4. Let $\mathcal{P}_n$ be the set of all polynomials of degree at most n with real coefficients. Let $\mathcal{P}_n^c$ be the set of all polynomials of degree at most n with complex coefficients.

LEMMA 3.1. *Let* $p \in (1, \infty)$. *For any* $1 \le M$ *there are polynomials* P_n *of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ a_{0,n} \ge \frac{3M}{\pi^2} + o(M),$$
$$\Big(\sum_{j=1}^{n} |a_{j,n}|^p\Big)^{1/p} \le 16 M^{1/p},$$

such that P_n *has at least* $\lfloor \sqrt{n/M} \rfloor$ *zeros at* 1.

LEMMA 3.2. *Let* $p, q \in (1, \infty)$ *satisfy* $1/p + 1/q = 1$. *For any* $L \ge 1/48$ *there are polynomials* P_n *of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ a_{0,n} \ge L + o(L),\ \sum_{j=1}^{n} |a_{j,n}|^p \le 1,$$

such that P_n *has at least* $\lfloor \sqrt{n} (cL)^{-q/2} \rfloor$ *zeros at* 1 *with* $c := \frac{3}{16\pi^2}$.

LEMMA 3.3. *Let* $p \in [1, \infty)$. *For any* $L \in (0, 1/17)$ *there are polynomials* P_n *of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ a_{0,n} = L,\ \sum_{j=1}^{n} |a_{j,n}|^p \le 1,$$

such that P_n *has at least* $\frac{2}{7} \min\big\{\sqrt{n(1 - \log L)}, n\big\}$ *zeros at* 1.

LEMMA 3.4. *For any* $L \in (0, 1)$ *there are polynomials* $P_n \not\equiv 0$ *of the form*

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ a_{0,n} \ge L \sum_{j=1}^{n} |a_{j,n}|,$$

such that P_n *has at least* $\frac{1}{5}\sqrt{(n-1)(1-L)}$ *zeros at* 1.

The observation below is well known, easy to prove, and recorded in several papers. See [B-99], for example.

LEMMA 3.5. *Let* $P \not\equiv 0$ *be a polynomial of the form* $P(x) = \sum_{j=0}^{n} a_j x^j$. *Then* $(x-1)^k$ *divides* P *if and only if* $\sum_{j=0}^{n} a_j Q(j) = 0$ *for all polynomials* $Q \in \mathcal{P}_{k-1}^c$.

Our next lemma is a simple consequence of Hölder's inequality.

LEMMA 3.6. *Let $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. Then for every $n \in \mathbb{N}$ and $L > 0$, we have*

$$\kappa_p(n, L) \le \mu_q(n, L).$$

The next lemma is [K-09, Lemma 3.4].

LEMMA 3.7. *For $A, M > 0$, there exists a polynomial G such that $F = G^2 \in \mathcal{P}_m$ with*

$$m < \sqrt{\pi}\,\sqrt{A}\,\sqrt[4]{M} + 2$$

such that $F(0) = M$ and

$$|F(x)| \le \min\{M, x^{-2}\}, \qquad x \in (0, A].$$

We also need Lemma 5.7 from [B-99], which may be stated as follows.

LEMMA 3.8. *Let n and R be positive integers with $1 \le R \le \sqrt{n}$. Then there exists a polynomial $F \in \mathcal{P}_m$ with*

$$m \le 4\sqrt{n} + \frac{9}{7}R\sqrt{n} + R + 4 \le \frac{44}{7}R\sqrt{n} + 4$$

such that

$$F(1) = F(2) = \cdots = F(R^2) = 0$$

and

$$\begin{aligned} |F(0)| &> \exp(R^2)\big(|F(R^2+1)| + |F(R^2+2)| + \cdots + |F(n)|\big) \\ &\ge \exp(R^2)\Big(\sum_{j=1}^{n} |F(j)|^2\Big)^{1/2}. \end{aligned}$$

We will use Lemmas 3.6 and 3.7 to prove the following two:

LEMMA 3.9. *Let $q \in [1, \infty)$. For every $n \in \mathbb{N}$, $q \in [1, \infty)$, and $K > 0$, there are polynomials $F \in \mathcal{P}_m$ satisfying*

$$|F(0)| > K\Big(\sum_{j=1}^{n} |F(j)|^q\Big)^{1/q}$$

and

$$m \le \begin{cases} \sqrt{n}\,(40K)^{q/2} + 2, & 0 < K < 2, \\ 13\min\big\{\sqrt{n\log K}, n\big\} + 4, & K \ge 2. \end{cases}$$

LEMMA 3.10. *For every $n \in \mathbb{N}$ and $K > 1$, there are polynomials $F \in \mathcal{P}_m$ satisfying*

$$|F(0)| > K \max_{j \in \{1,\dots,n\}} |F(j)|$$

and

$$m \le \begin{cases} \sqrt{n(K-1)/2} + 1, & 1 < K < 2, \\ 13\min\big\{\sqrt{n\log K}, n\big\} + 4, & K \ge 2. \end{cases}$$

4. Proofs of the lemmas

Proof of Lemma 3.1. Modifying the construction on p. 138 of [B-95] for $x \in (0, \infty)$ we define $H_1(x) := 1$ and

$$H_m(x) := \frac{(-1)^{m+1} 2(m!)^2}{2\pi i} \int_\Gamma \frac{x^t \, dt}{(t-2) \prod_{j=0}^m (t-j^2)}, \qquad m = 2, 3, \dots,$$

where the simple closed contour Γ surrounds the zeros of the denominator of the integrand. Observe that we have added the factor $t-2$ to the denominator and the factor $(-1)^{m+1}2$ to achieve our goals. (It is left to the reader to see what role this modification plays in our proof.) Then H_m is a polynomial of degree m^2 with a zero at 1 with multiplicity at least $m+1$. (This can be seen easily by repeated differentiation and then evaluation of the above contour integral by expanding the contour to infinity.)

Also, by the residue theorem,

$$H_m(x) = 1 + d_m x^2 + \sum_{k=1}^m c_{k,m} x^{k^2}, \qquad m = 2, 3, \dots, \tag{4.1}$$

where

$$c_{k,m} = \frac{(-1)^{m+1} 2(m!)^2}{(k^2-2) \prod_{j=0,\, j \neq k}^m (k^2 - j^2)} = \frac{4}{k^2-2} \, \frac{(-1)^{k+1}(m!)^2}{(m-k)!(m+k)!},$$

$$d_m = \frac{(-1)^{m+1} 2(m!)^2}{\prod_{j=0}^m (2-j^2)}.$$

It follows that each $c_{k,m}$ is real and

$$|c_{k,m}| \le \frac{4}{|k^2-2|}, \qquad k = 1, \dots, m, \tag{4.2}$$

and a simple calculation shows that

$$|d_m| \le 8, \qquad m = 2, 3, \dots. \tag{4.3}$$

(No effort has been made to optimize the bound in (4.3).)

Let S_M be the collection of all odd square free integers in $[1, M]$. Let $m := \lfloor \sqrt{n/M} \rfloor$. If $m = 0$ then there is nothing to prove. So we may assume that $m \ge 1$. It is well known that

$$|S_M| \ge \frac{3M}{\pi^2} + o(M),$$

where $|A|$ denotes the number of elements in a finite set A. This follows from the fact that if S_M^* is the collection of all square free integers in $[1, M]$, then

$$|S_M^*| = \frac{6M}{\pi^2} + o(M)$$

(see [H-38, pp. 267–268], for example), by observing that the number of odd square free integers in $[1, M]$ is not less than the number of even square free integers in $[1, M]$ (if a is an even square free integer then $a/2$ is an odd square free integer). We define

$$P_n(x) := \sum_{j \in S_M} H_m(x^j).$$

Then P_n is of the form

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ j = 0, 1, \ldots, n.$$

We have

$$a_{0,n} = |S_M^*| \geq \frac{3M}{\pi^2} + o(M).$$

First assume that $m = 1$. Then

$$\sum_{j=1}^{n} |a_{j,n}|^p = 2|S_M| \leq 2M,$$

and as P_n has one zero at 1, the lemma follows. Now assume that $m \geq 2$. We have $ju \neq lv$ whenever $j, l \in S_M$, $j \neq l$, and $u, v \in \{1^2, 2^2, \ldots, m^2\} \cup \{2\}$. Combining this with (4.1)–(4.3), we obtain

$$\begin{aligned}\sum_{j=1}^{n} |a_{j,n}|^p &\leq |S_M| \left(8^p + \sum_{k=1}^{m} \left(\frac{4}{|k^2 - 2|} \right)^p \right) \leq |S_M| \left(8^p + \sum_{k=1}^{m} \frac{4^p}{|k^2 - 2|} \right) \\ &= M(8^p + 8^p) \leq 16^p M.\end{aligned}$$

Observe that each term in P_n has a zero at 1 with multiplicity at least $m + 1 > \lfloor \sqrt{n/M} \rfloor$, and hence so does P_n. ∎

Proof of Lemma 3.2. The statement follows from Lemma 3.1 by choosing $1 \leq M$ so that

$$L := \frac{3}{16\pi^2} M^{1-1/p} = \frac{3}{16\pi^2} M^{1/q}.$$

This can be done when $\frac{3}{16\pi^2} \leq L$. ∎

Proof of Lemma 3.3. Let $L \in (0, 1/17]$. We define

$$k := \min\left\{ \left\lfloor \frac{-\log L}{\log 17} \right\rfloor, n \right\} \quad \text{and} \quad m := \lfloor \sqrt{n/k} \rfloor.$$

Observe that $k, m \geq 1$. Let $P_n := LH_m^k \in \mathcal{P}_n$, where $H_m \in \mathcal{P}_{m^2}$ is defined by (4.1). Then

$$P_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ j = 0, 1, \ldots, n,$$

has at least

$$km \geq k \frac{1}{2} \sqrt{n/k} = \frac{1}{2} \sqrt{nk} = \frac{1}{2\sqrt{\log 17}} \min\{ \sqrt{n(-\log L)}, n \}$$

zeros at 1, where $2\sqrt{\log 17} < 7/2$. Clearly, $a_{0,n} = P_n(0) = L$, and using the notation in (4.1) again, we can deduce that

$$\sum_{j=1}^{n} |a_{j,n}|^p \le L^p \Big(\sum_{j=1}^{n} |a_{j,n}|\Big)^p \le L^p \Big(1 + |d_m| + \sum_{k=1}^{m} |c_{k,m}|\Big)^{kp}$$
$$\le L^p (1+8+8)^{kp} = L^p \, 17^{kp} \le L^p L^{-p} = 1$$

if $m \ge 2$, and

$$\sum_{j=1}^{n} |a_{j,n}|^p \le L^p \Big(\sum_{j=1}^{n} |a_{j,n}|\Big)^p \le L^p 2^{kp} \le L^p L^{-p} = 1$$

if $m = 1$. ∎

Proof of Lemma 3.4. Let

$$r := \left\lfloor 12\,\frac{1+L}{1-L} \right\rfloor + 1 \quad \text{and} \quad m := \left\lfloor \sqrt{\frac{n-1}{r}} \right\rfloor.$$

When $m \le 1$ we have $\lfloor (1/9)\sqrt{n(1-L)} \rfloor = 0$, so there is nothing to prove. Now assume that $m \ge 2$. Let $P_n \in \mathcal{P}_n$ be defined by $P_n(x) := H_m(x^r)$, where $H_m \in \mathcal{P}_{m^2}$ is as in (4.1). Let $Q_n \in \mathcal{P}_n$ be defined by

$$Q_n(x) = -\int_0^1 P_n(t)\,dt + \int_0^x P_n(t)\,dt.$$

Then, using the notation in (4.1) again, we have

$$Q_n(x) = -1 - \frac{d_m}{2r+1} - \sum_{k=1}^{m} \frac{c_{k,m}}{rk^2+1} + x + \frac{d_m x^{2r+1}}{2r+1} + \sum_{k=1}^{m} \frac{c_{k,m} x^{rk^2+1}}{rk^2+1}.$$

Writing

$$Q_n(x) = \sum_{j=0}^{n} a_{j,n} x^j, \qquad a_{j,n} \in \mathbb{R},\ j = 0, 1, \ldots, n,$$

and recalling (4.2) and (4.3), we have

$$|a_{0,n}| \ge 1 - \frac{8}{2r+1} - \sum_{k=1}^{m} \frac{4}{|k^2-4|(rk^2+1)} \ge 1 - \frac{8}{2r+1} - \frac{8}{r} > 1 - \frac{12}{r}$$

and

$$\sum_{j=1}^{n} |a_{j,n}| \le 1 + \frac{8}{2r+1} + \sum_{k=1}^{m} \frac{4}{(k^2-2)(rk^2+1)} < 1 + \frac{12}{r}.$$

Combining these two formulas, we obtain

$$\frac{|a_{0,n}|}{\sum_{j=1}^{n} |a_{j,n}|} > \frac{1 - 12/r}{1 + 12/r} \ge \frac{1 - (1-L)/(1+L)}{1 + (1-L)/(1+L)} = L.$$

Also Q_n has at least $m+1 \ge \lfloor \sqrt{(n-1)/r} \rfloor + 1 \ge \frac{1}{5}\sqrt{(n-1)(1-L)}$ zeros at 1. ∎

Proof of Lemma 3.6. We assume that $p, q \in (1,\infty)$; in the cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$ the result can be proved similarly with straightforward modifications. Let $m := \mu_q(n, L)$. Let Q be a polynomial of degree m with complex coefficients such that

$$|Q(0)| > \frac{1}{L}\Big(\sum_{j=1}^{n} |Q(j)|^q\Big)^{1/q}.$$

Now let P be a polynomial of the form

$$P(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_0| \ge L\Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}, \quad a_j \in \mathbb{C}.$$

It follows from Hölder's inequality that

$$\Big|\sum_{j=1}^{n} a_j Q(j)\Big| \le \Big(\sum_{j=1}^{n} |a_j|^p\Big)^{1/p}\Big(\sum_{j=1}^{n} |Q(j)|^q\Big)^{1/q} < \frac{|a_0|}{L} L|Q(0)| = |a_0 Q(0)|.$$

Then $\sum_{j=0}^{n} a_j Q(j) \ne 0$, and hence Lemma 3.5 implies that $(x-1)^{m+1}$ does not divide P. We conclude that $\kappa_p(n, L) \le m = \mu_q(n, L)$. ∎

Proof of Lemma 3.9. Note that $\mu_q(n, K) \le n$ for all $n \in \mathbb{N}$ and $L > 0$, as is justified by $H \in \mathcal{P}_n$ defined by $H(x) := \prod_{j=1}^{n} (x - j)$.

CASE 1: $0 < K < n^{-1/q}$. The choice $F \equiv 1$ gives the lemma.

CASE 2: $n^{-1/q} \le K < 2$. Let F be the polynomial given in Lemma 3.7 with $A := n$ and $M := (4K)^{2q}$. Then

$$\begin{aligned}\sum_{j=1}^{n} |F(j)|^q &\le \sum_{j \le M^{-1/2}} M^q + \sum_{j > M^{1/2}} \frac{1}{j^{2q}} < M^{q-1/2} + \frac{1}{2q-1}\lfloor M^{-1/2} \rfloor^{-2q+1} \\ &\le (1 + 2^{2q-1}) M^{q-1/2},\end{aligned}$$

so

$$\Big(\sum_{j=1}^{n} |F(j)|^q\Big)^{1/q} < 4M^{1-1/(2q)} = K^{-1}F(0),$$

and the degree m of F satisfies

$$m < \pi\sqrt{n}\,\sqrt[4]{M} + 2 < \pi\sqrt{n}\,(4K)^{q/2} + 2 \le \sqrt{n}\,(40K)^{q/2} + 2.$$

CASE 3: $2 \le K \le \exp(n - 2\sqrt{n})$. Let $R := \lfloor \sqrt{\log K} \rfloor + 1$, and let F be the polynomial given in Lemma 3.8 with this R. Then

$$|F(0)| > K\sum_{j=1}^{n} |F(j)| \ge K\Big(\sum_{j=1}^{n} |F(j)|^q\Big)^{1/q},$$

and the degree m of F satisfies

$$m \le \frac{44}{7} R\sqrt{n} + 4 \le 13\sqrt{n \log K} + 4.$$

Case 4: $K > \exp(n - 2\sqrt{n})$, $n \ge 9$. Then $\log K > n - 2\sqrt{n} \ge n/3$ for all $n \ge 9$. Hence the polynomial $F \in \mathcal{P}_n$ defined by

$$F(x) := \prod_{j=1}^{n} (x - j) \tag{4.4}$$

shows that

$$\mu_q(n, K) \le n \le \sqrt{3} \min\{\sqrt{n \log K}, n\}.$$

Case 5: $K \ge 2$ and $n < 9$. Now the polynomial (4.4) shows

$$\mu_q(n, K) \le n \le 4 \min\{\sqrt{n \log K}, n\}. \ \blacksquare$$

Proof of Lemma 3.10. First let $1 < K < 2$. Let $m = \lfloor \sqrt{n(K-1)/2} \rfloor + 1$. Let T_m be the Chebyshev polynomial of degree m defined by

$$T_m(\cos t) = \cos(mt), \qquad t \in \mathbb{R}.$$

It is well known that $|T_m'(1)| = m^2$ and $T_m'(x)$ is increasing on $[1, \infty)$, hence $T_m(1 + x) \ge 1 + m^2 x$ for all $x > 0$. Now we define $F \in \mathcal{P}_m$ by

$$F(x) := T_m\left(\frac{-2x}{n-1} + \frac{n+1}{n-1}\right).$$

Then $|F(x)| \le 1$ for all $x \in [1, n]$, and

$$F(0) \ge T_m\left(1 + \frac{2}{n-1}\right) > 1 + \frac{m^2}{n-1} > 1 + \frac{m^2}{n} \ge K,$$

which finishes the proof in the case of $1 < K < 2$. Now let $k \ge 2$. Then the polynomial $F \in \mathcal{P}_m$ chosen for $q = 1$, $n \in \mathbb{N}$, and $K \ge 2$ by Lemma 3.9 gives

$$|F(0)| > K\left(\sum_{j=1}^{n} |F(j)|^q\right)^{1/q} \ge K \max_{j \in \{1, \ldots, n\}} |F(j)|,$$

with

$$m \le 13 \min\{\sqrt{n \log K}, n\} + 4. \ \blacksquare$$

5. Proofs of the theorems

Proof of Theorem 2.1. Without loss of generality we may assume that $p \in (1, \infty)$, as the case $p = \infty$ follows by a simple limiting argument (or we may as well refer to the main result in [B-13]). By Lemma 3.6 we have

$$\kappa_p(n, L) \le \mu_q(n, L)$$

for every $n \in \mathbb{N}$ and $L > 0$. The lower bounds for $\kappa_p(n, L)$ follow from Lemmas 3.2 and 3.3. The upper bounds for $\mu_q(n, L)$ follow from Lemma 3.9 with $K = L^{-1}$. ∎

Proof of Theorem 2.2. By Lemma 3.6 we have

$$\kappa_1(n, L) \le \mu_\infty(n, L)$$

for every $n \in \mathbb{N}$ and $L > 0$. The lower bounds for $\kappa_1(n, L)$ follow from Lemmas 3.3 and 3.4. The upper bounds for $\mu_\infty(n, L)$ follow from Lemma 3.10 with $K = L^{-1}$. ∎

6. Remarks and problems. A question that we have not really considered in this paper is the following: Are there examples of n, L, and p for which the values of $\kappa_p(n, L)$ are significantly smaller if the coefficients are required to be rational (perhaps together with other restrictions)? The same question may be raised about $\mu_q(n, L)$. As the conditions on the coefficients of the polynomials in Theorems 2.1 and 2.2 are homogeneous, assuming rational coefficients and integer coefficients lead to the same results. Three special classes of interest are

$$\mathcal{F}_n := \Big\{Q : Q(z) = \sum_{j=0}^{n} a_j z^j,\ a_j \in \{-1, 0, 1\}\Big\},$$

$$\mathcal{L}_n := \Big\{Q : Q(z) = \sum_{j=0}^{n} a_j z^j,\ a_j \in \{-1, 1\}\Big\},$$

$$\mathcal{K}_n := \Big\{Q : Q(z) = \sum_{j=0}^{n} a_j z^j,\ a_j \in \mathbb{C},\ |a_j| = 1\Big\}.$$

The following three problems arise naturally.

Problem 6.1. *How many zeros can a polynomial $0 \not\equiv Q \in \mathcal{F}_n$ have at 1?*

Problem 6.2. *How many zeros can a polynomial $Q \in \mathcal{L}_n$ have at 1?*

Problem 6.3. *How many zeros can a polynomial $Q \in \mathcal{K}_n$ have at 1?*

The case $p = \infty$ and $L = 1$ in our Theorem 2.1 shows that every $0 \not\equiv Q \in \mathcal{F}_n$, every $Q \in \mathcal{L}_n$, and every $Q \in \mathcal{K}_n$ can have at most $cn^{1/2}$ zeros at 1, where $c > 0$ is an absolute constant. However, one may expect better results by utilizing the additional pieces of information on their coefficients.

It was observed in [B-99] that for every integer $n \ge 2$ there is a $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This can be shown by a simple pigeon-hole argument. However, as far as we know, closing the gap between $cn^{1/2}$ and $c(n/\log n)^{1/2}$ in Problem 6.1 is an open and most likely a very difficult problem.

As far as Problem 6.2 is concerned, Boyd [B-97c] showed that for $n \geq 3$ every $Q \in \mathcal{L}_n$ has at most

$$\frac{c(\log n)^2}{\log\log n} \tag{6.1}$$

zeros at 1. This is the best known upper bound in Problem 6.2 even today. Boyd's proof is very clever and, up to an application of the prime number theorem, completely elementary. It is reasonable to conjecture that there is an absolute constant $c > 0$ such that every $Q \in \mathcal{L}_n$, $n \geq 2$, has at most $c\log n$ zeros at 1. It is easy to see that for every integer $n \geq 2$ there are $Q_n \in \mathcal{L}_n$ with at least $c\log n$ zeros at 1 with an absolute constant $c > 0$.

As to Problem 6.3, one may suspect that every $Q \in \mathcal{K}_n$, $n \geq 2$, has at most $c\log n$ zeros at 1. However, just to see if Boyd's bound (6.1) holds for every $Q \in \mathcal{K}_n$ seems quite challenging and beyond reach at the moment.

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