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A note on small gaps between primes in arithmetic progressions

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1. Introduction. A long standing problem concerning the distribution of prime numbers is the *prime k-tuples conjecture*. We call a set $\mathcal{H} = \{h_1, \dots, h_k\}$ an *admissible k-tuple* if the h_i do not cover all residue classes modulo p for any prime p. The prime k-tuples conjecture then states that there are infinitely many integers n such that all of the numbers $n + h_i$, $i = 1, \dots, k$, are simultaneously prime. Recently, there have been breakthrough developments towards proving this conjecture. First Zhang [8], refining a method of Goldston, Pintz and Yıldırım [3], proved that for k large enough, the sets $n + \mathcal{H}$ contain two primes infinitely often, thus settling the *bounded gaps conjecture*. Then Maynard [5] and Tao (unpublished) independently devised another modification of the Goldston–Pintz–Yıldırım method which could detect m primes in k-tuples for any m, provided k is large enough.

In this paper we present an implementation of the Maynard–Tao method to yield a corresponding result concerning primes in an arithmetic progression, with a bound that is uniform in the modulus of the progression. The proof goes along the same lines, after tweaking the set-up to pick out only the primes in the arithmetic progression under consideration. The key ingredient will be a Bombieri–Vinogradov type theorem that is tailored to the case at hand; see Section 4.

The author would like to thank Roger Baker and Liangyi Zhao for calling his attention to a result of theirs [1] which precedes the present work and is of a similar nature. The similarities and differences between the two works will be briefly discussed at the end of the paper.

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2. Notation and setup. Throughout, the letters c and C will denote constants which need not be the same at every instance. When we need to track constants, we employ subscripts or superscripts.

The method requires that we restrict ourselves to arithmetic progressions in which primes are reasonably well-distributed, i.e. progressions to moduli whose associated Dirichlet L-functions do not vanish too close to $s = 1$.

For the imaginary part γ of a zero of an L-function, we shall denote $|\gamma|+1$ by $\widetilde{\gamma}$ for the sake of brevity. We first recall some basic facts concerning zero-free regions of L-functions [2, §14]. There is a constant c_0 (the bounds cited below are known in fact for different constants, but we take c_0 to be the minimum of those to simplify notation) such that an L-function $L(s,\chi)$ to the modulus q has no zero $\beta + i\gamma$ in the region

$$\beta \geq 1 - \frac{c_0}{\log q\widetilde{\gamma}}, \tag{2.1}$$

except possibly a single real zero, which can exist for at most one real character χ (mod q). We call a modulus to which there is such a *primitive* character an *exceptional modulus*, and the corresponding zero an *exceptional zero*. Exceptional moduli are of the form $q = 2^\nu p_1 \dots p_m$, where $\nu \leq 3$ and $p_1 < \dots < p_m$ are distinct odd primes, whence, by the Prime Number Theorem, we have $p_m \gg \sum_{p \leq p_m} \log p \gg \log q$. On the other hand we have, for the real zeros, the unconditional bound

$$\beta < 1 - \frac{c_0}{q^{1/2}(\log q)^2}. \tag{2.2}$$

Also, if χ_1 and χ_2 are distinct real primitive characters to moduli q_1 and q_2 respectively and the corresponding L-functions have real zeros β_1 and β_2, then the Landau–Page theorem states that these zeros must satisfy

$$\min(\beta_1, \beta_2) < 1 - \frac{c_0}{\log q_1 q_2}. \tag{2.3}$$

We shall have to confine ourselves to L-functions which do not have a zero in the region

$$\beta \geq 1 - \frac{c^* \log\log X}{\log X}, \qquad \widetilde{\gamma} \leq \exp(c^\sharp \sqrt{\log X}) \tag{2.4}$$

for a parameter X and given constants c^* and $c^\sharp$. This is a consequence of (2.2) when $q \ll (\log X/(\log\log X)^2)^2$. We suppose that X is large enough in terms of $c^\sharp$ and c^* such that

$$c^\sharp \leq \frac{c_0}{4c^*} \frac{\sqrt{\log X}}{\log\log X}, \tag{2.5}$$

and argue that there is at most one modulus $\leq \exp(2c^\sharp \sqrt{\log X})$ to which there is a primitive character whose L-function vanishes in the region (2.4). By (2.1), no non-exceptional zeros exist in the region stated, so we only need to consider real zeros. Suppose there are two such moduli q_1 and q_2, with corresponding real zeros β_1 and β_2. Then using (2.3) we have

$$1-\frac{c^*\log\log X}{\log X}<1-\frac{c_0}{\log q_1q_2}\le 1-\frac{c_0}{4c^\sharp\sqrt{\log X}}, \tag{2.6}$$

which is impossible by (2.5). We denote this possibly existing unique modulus by q_0 and the greatest prime dividing q_0 by p_0, or set $p_0 = 1$ in case q_0 does not exist. We note that $q_0 \gg (\log X/(\log\log X)^2)^2$, whence $p_0 \gg \log\log X$.

Our main parameter X is large enough and $f(X)$ is a given increasing function of X with $f(X) \ll X^{5/12-5\vartheta/6}$ for some positive number $\vartheta < 1/2$. The modulus M of the arithmetic progression does not exceed $f(X)$ and is not a multiple of any number in a set $\mathcal{Z}$ of exceptions whose size Z_f satisfies

$$Z_f=\begin{cases}0 & \text{if } f(X)\ll(\log X)^C,\\ 1 & \text{if } f(X)\ll \exp(c\sqrt{\log X}),\\ O((\log\log X)^C) & \text{otherwise.}\end{cases} \tag{2.7}$$

We denote characters modulo q, M, and qM by ψ, ξ, and χ respectively. A summation $\sum_\chi^*$ over characters with an asterisk in the superscript denotes that the summation is over primitive characters only.

Set $x = X/M$ and let $W = \prod_{p\le D_0} p$ be the product of primes not exceeding $D_0 = \log\log\log X$, and in turn set $W' = W/(W, P_fM)$ and $V = W'M$, where

$$P_f=\begin{cases}1 & \text{if } f(X)\ll(\log X)^C,\\ p_0 & \text{otherwise.}\end{cases} \tag{2.8}$$

Furthermore set $R = N^{\vartheta/2-\delta}$ for some small positive δ. Let $\mathcal{H} = \{h_1,\dots,h_k\}$ be an admissible k-tuple with $\operatorname{diam}(\mathcal{H}) < D_0M$ such that $h_i \equiv a \pmod M$, $i = 1,\dots,k$, for a given residue class $a \pmod M$ coprime to M. The weights $\lambda_{d_1,\dots,d_k}$ are supported on $(\prod_i d_i, VP_f) = 1$, $\prod_i d_i < R$, and $\mu(\prod_i d_i)^2 = 1$ (the last condition implies, of course, that $(d_i,d_j) = 1$). We also choose ν_0 such that $(M\nu_0 + h_i, W') = 1$ for $i = 1,\dots,k$ (this is possible because $\mathcal{H}$ is admissible).

With these, we will consider the sum

$$S^{(\rho)}=\sum_{\substack{x\le n<2x\\ n\equiv\nu_0\ (\mathrm{mod}\ W')}}\Big(\sum_{i=1}^k\chi_{\mathbb{P}}(nM+h_i)-\rho\Big)\Big(\sum_{d_i\mid nM+h_i}\lambda_{d_1,\dots,d_k}\Big)^2, \tag{2.9}$$

where $\chi_{\mathbb{P}}$ is the characteristic function of the primes. Clearly, the positivity of $S^{(\rho)}$ implies that for at least one $n \in [x, 2x)$ the inner sum is positive, and this establishes the existence of at least $\lfloor\rho + 1\rfloor$ primes among the numbers

$nM+h_i$, $i=1,\dots,k$, but nM lies in $[X,2X)$ and each $nM+h_i$ is congruent to $a \pmod M$ by the condition on $\mathcal{H}$.

3. Results. Our main theorem is the following.

THEOREM 1. *Let k be a given integer, $\vartheta < 1/2$, and $f(X) \ll X^{5/12-5\vartheta/6}$ an increasing function of X. Further, let $\mathcal{S}_k$ be the set of all piecewise differentiable functions $\mathbb{R}^k \to \mathbb{R}$ supported on $\mathcal{R}_k = \{(x_1,\dots,x_k) \in [0,1]^k : \sum_{i=1}^k x_i \le 1\}$, and define*

$$M_k = \sup_{F\in\mathcal{S}_k} \frac{\sum_{m=1}^k J_k^{(m)}(F)}{I_k(F)}, \tag{3.1}$$

where

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1,\dots,t_k)^2\, dt_1 \dots dt_k, \tag{3.2}$$

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \Big(\int_0^1 F(t_1,\dots,t_k)\, dt_m\Big)^2 dt_1 \dots dt_{m-1}\, dt_{m+1}\, dt_k. \tag{3.3}$$

If X is large enough, then for all $M \le f(X)$ except multiples of numbers in a set of size Z_f, all residue classes $a \pmod M$ coprime to M, and all admissible k-tuples $\mathcal{H} = \{h_1,\dots,h_k\}$ such that $h_i \equiv a \pmod M$, $i=1,\dots,k$, there is a multiple nM of M with $nM \in [X,2X]$ such that at least $r_k = \lceil \vartheta M_k/2\rceil$ of the numbers $nM+h_i$, $i=1,\dots,k$, are primes.

We can instantiate this to some concrete cases to deduce certain facts. We denote by p'_n the nth prime that is congruent to $a \pmod M$. First note that if $f(X) \le \exp(c\sqrt{\log X})$ for some c, we can apply the theorem with ϑ as close to $1/2$ as we like, and the set of exceptions will be empty or a singleton according as $f(X) \ll (\log X)^C$ for some C or not. In either case, taking $k=105$ suffices to produce two primes, by Proposition 4.3 of Maynard [5], and if we use the refinement $M_{54} > 4.002$ from the Polymath Project [7, Theorem 23], then an admissible 54-tuple, which exists with diameter 270 [7, Theorem 17], is sufficient. To produce r primes with arbitrary r, we use the bound $M_k > \log k + O(1)$ [7, Theorem 23] to see that a $\lceil e^{4r+C}\rceil$-tuple suffices. From any admissible tuple $\{h_i\}_i$ we can obtain a tuple $\{Mh_i+a\}_i$ whose members are all congruent to $a \pmod M$, with diameter dilated by M. Using the admissible tuple $\{Mp_{\pi(k)+1}+a,\dots,Mp_{\pi(k)+k}+a\}$ of diameter $\ll Mk\log k$ when r is large, we have the following theorems.

THEOREM 2. *Let C be a given positive constant. If X is sufficiently large, then for all $M \ll (\log X)^C$ and all a with $(a,M)=1$, there is a*

$p'_n \in [X, 2X]$ *such that*

$$p'_{n+1} - p'_n \le 270M. \tag{3.4}$$

THEOREM 3. *Let c be a given positive constant. If X is sufficiently large, then for all $M \ll \exp(c\sqrt{\log X})$ except multiples of a single number, and all a with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+1} - p'_n \le 270M. \tag{3.5}$$

THEOREM 4. *Let r be a positive integer and C be a given positive constant. If X is sufficiently large, then for all $M \ll (\log X)^C$ and all a with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll re^{4r}M. \tag{3.6}$$

THEOREM 5. *Let r be a positive integer and c be a given positive constant. If X is sufficiently large, then for all $M \ll \exp(c\sqrt{\log X})$ except multiples of a single number, and all a with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll re^{4r}M. \tag{3.7}$$

When M is allowed to grow as large as a power of X, our tuple lengths have to grow and our bounds get much weaker. Suppose $M \ll X^{5/12-\eta}$ for some positive η. In that case Theorem 1 applies with $\vartheta = 6\eta/5$, so that to find $r + 1$ primes we need k such that

$$\frac{3\eta M_k}{5} > r. \tag{3.8}$$

We again use the fact that

$$M_k > \log k + O(1) \tag{3.9}$$

when k is sufficiently large to see that if $k \ge Ce^{5r/(3\eta)}$ for some absolute constant C, (3.8) is satisfied. We take $k = \lceil Ce^{5r/(3\eta)} \rceil$, take the admissible tuple $\{Mp_{\pi(k)+1} + a, \ldots, Mp_{\pi(k)+k} + a\}$ of diameter $Mk \log k$, and obtain

THEOREM 6. *Let η be given with $0 < \eta < 5/12$, and let r be a positive integer. If X is sufficiently large, then for all $M \ll X^{5/12-\eta}$ except multiples of numbers in a set of size $\ll (\log X)^C$, and all a with $(a, M) = 1$, there is a $p'_n \in [X, 2X]$ such that*

$$p'_{n+r} - p'_n \ll \frac{r}{\eta} e^{5r/(3\eta)} M. \tag{3.10}$$

In order to prove Theorem 1, we write

$$S^{(\rho)} = S_2 - \rho S_1, \tag{3.11}$$

where

$$S_1 = \sum_{\substack{x\le n<2x\\ n\equiv \nu_0 \,(\mathrm{mod}\, W')}} \Big(\sum_{d_i\mid nM+h_i} \lambda_{d_1,\dots,d_k}\Big)^2, \tag{3.12}$$

$$\begin{aligned} S_2 &= \sum_{m=1}^{k} S_2^{(m)} \\ &= \sum_{m=1}^{k} \sum_{\substack{x\le n<2x\\ n\equiv \nu_0 \,(\mathrm{mod}\, W')}} \chi_{\mathbb{P}}(nM+h_m)\Big(\sum_{d_i\mid nM+h_i} \lambda_{d_1,\dots,d_k}\Big)^2, \end{aligned} \tag{3.13}$$

so that we can estimate $S^{(\rho)}$ by using the following proposition.

Proposition 1. *Let k be a given integer and let X be a parameter that is large enough. Let $\lambda_{d_1,\dots,d_k}$ be defined in terms of a fixed piecewise differentiable function F by*

$$\lambda_{d_1,\dots,d_k} = \Big(\prod_{i=1}^{k}\mu(d_i)d_i\Big) \sum_{\substack{r_1,\dots,r_k\\ d_i\mid r_i\,\forall i\\ (r_i,V)=1\,\forall i}} \frac{\mu\big(\prod_{i=1}^{k} r_i\big)^2}{\prod_{i=1}^{k}\varphi(r_i)} F\Big(\frac{\log r_1}{\log R},\dots,\frac{\log r_k}{\log R}\Big) \tag{3.14}$$

whenever $(\prod_{i=1}^{k} d_i, VP_f) = 1$, and let $\lambda_{d_1,\dots,d_k} = 0$ otherwise. Moreover, let F be supported on $\mathcal{R}_k = \{(x_1,\dots,x_k)\in[0,1]^k : \sum_{i=1}^{k} x_i \le 1\}$. Then

$$S_1 = (1+o(1))\frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k} I_k(F), \tag{3.15}$$

$$S_2 = (1+o(1))\frac{\varphi(VP_f)^k X(\log R)^{k+1}}{V(VP_f)^k \log X} \sum_{m=1}^{k} J_k^{(m)}(F), \tag{3.16}$$

provided $I_k(F)\ne 0$ and $J_k^{(m)}(F)\ne 0$ for each m, where $I_k(F)$ and $J_k^{(m)}(F)$ are given by (3.2) *and* (3.3) *respectively.*

Proof of Theorem 1. Let $\mathcal{S}_k$ and M_k be as in Theorem 1. Then for any $\delta > 0$, we can find $F_0 \in \mathcal{S}_k$ such that $\sum_{m=1}^{k} J_k^{(m)}(F_0) > (M_k-\delta)I_k(F_0)$. With this F_0, by (3.11) and Proposition 1 we have

$$\begin{aligned} S^{(\rho)} &= \frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k}\Big(\frac{\log R}{\log N}\sum_{m=1}^{k} J_k^{(m)}(F_0) - \rho I_k(F_0) + o(1)\Big) \\ &\ge \frac{\varphi(VP_f)^k X(\log R)^k}{V(VP_f)^k} I_k(F)\Big(\Big(\frac{\vartheta}{2}-\delta\Big)(M_k-\delta) - \rho + o(1)\Big). \end{aligned}$$

If $\rho = \vartheta M_k/2 - \delta'$, then with δ sufficiently small, we have $S^{(\rho)} > 0$ for all large enough X, implying that at least $\lfloor \rho+1\rfloor$ of the $nM+h_i$ are prime. Since $\lfloor\rho+1\rfloor = \lceil \vartheta M_k/2\rceil$ for δ' small enough, we obtain our result. ∎

4. A Bombieri–Vinogradov type theorem. Recall that a summation $\sum_{\chi}^{*}$ over characters with an asterisk in the superscript denotes that the sum runs over primitive characters only. We quote here a zero-density result [4, Theorem 10.4 and the following Remark] which we will need in our proof.

THEOREM 7. *Let m be given and $N(1-\delta, T, \chi)$ be the number of zeros $\beta + i\gamma$ of $L(s,\chi)$ in the region $1-\delta \le \beta$, $|\gamma| \le T$. Set*

$$N(1-\delta, m, Q, T) = \sum_{\substack{q\le Q\\ (q,m)=1}} \sideset{}{^*}\sum_{\psi \,(\mathrm{mod}\, q)} \sum_{\xi\,(\mathrm{mod}\, m)} N(1-\delta, T, \psi\xi). \tag{4.1}$$

Then for $\delta < 1/2$ and any $\varepsilon > 0$ we have

$$N(1-\delta, m, Q, T) \ll \big((mQT)^{2\delta} + (mQ^2T)^{c(\delta)\delta}\big)(\log mQT)^A \tag{4.2}$$

for some constant A, where

$$c(\delta) = \min\bigg(\frac{3}{1+\delta}, \frac{3}{2-3\delta}\bigg). \tag{4.3}$$

We estimate the number of the moduli we will have to exclude in the following proposition.

PROPOSITION 2. *Let c^* and $c^\sharp$ be given constants. There is a set $\mathcal{Z}$ of exceptions with $|\mathcal{Z}| \ll (\log X)^C$ such that if X is large enough, then for any $M \le f(X)$ that is not a multiple of any number in $\mathcal{Z}$ and all $q \le \exp(c^\sharp\sqrt{\log X})$ with $(q, Mp_0) = 1$, the L-functions $L(s,\psi\xi)$, where ψ $(\mathrm{mod}\ q)$ is primitive and ξ is any character modulo M, have no zeros in the region $1 - c^* \log\log X/\log X \le \beta \le 1$, $|\gamma| \le \exp(c^\sharp\sqrt{\log X})$. The set $\mathcal{Z}$ can have elements $\le \exp(c^\sharp\sqrt{\log X})$ only if q_0 exists, in which case those elements are all multiples of p_0.*

Proof. Suppose M is a modulus such that for some character ξ $(\mathrm{mod}\ M)$ and a primitive character ψ modulo q, $L(s,\psi\xi)$ has a zero in the region indicated. Then $\psi\xi$ must be induced by a character of the form $\psi\xi^*$, where ξ^* $(\mathrm{mod}\ m)$ is a primitive character modulo $m \mid M$. We estimate the number of such m. Let

$$\mathcal{Z} = \Big\{ m \le f(X) : \text{there exist } q \le \exp(c^\sharp\sqrt{\log X}) \text{ and } \chi \ (\mathrm{mod}\ mq) \text{ with } (q, mp_0) = 1 \text{ and } \chi \text{ primitive such that } L(\beta+i\gamma,\chi) = 0 \text{ for some } \beta > 1 - \frac{c^*\log\log X}{\log X}\Big\} \tag{4.4}$$

be the set of exceptions whose size we wish to bound. We divide the ranges $1 \le m \le f(X)$, $1 \le q \le \exp(c^\sharp\sqrt{\log X})$, and $\widetilde{\gamma} \le \exp(c^\sharp\sqrt{\log X})$ into dyadic segments $[M_\lambda/2, M_\lambda)$ $[Q_\mu/2, Q_\mu)$ and $[T_\nu/2, T_\nu)$ respectively. Then

$$\#\mathcal{Z} \le \sum_{\lambda,\mu,\nu} \sum_{m\le M_\lambda} \sum_{\substack{q\le Q_\mu \\ (q,mp_0)=1}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, qm)} N\biggl(1-\frac{c^*\log\log X}{\log X}, T_\nu, \chi\biggr). \tag{4.5}$$

Using Theorem 7 with $m=1$ and $M_\lambda Q_\mu$ in place of Q shows the above is

$$\ll \sum_{\lambda,\mu,\nu} (M_\lambda^2 Q_\mu^2 T_\nu)^{12c^*\log\log X/(5\log X)} (\log Q_\mu T_\nu)^C \ll (\log X)^C, \tag{4.6}$$

where C and the implicit constant depend on c^* and $c^\sharp$. Now suppose that X is large enough to satisfy (2.5). If $m\in\mathcal{Z}$ with $m\le \exp(c^\sharp\sqrt{\log X})$, so that $mq\le \exp(2c^\sharp\sqrt{\log X})$, then by the discussion in Section 2, $L(s,\chi)=0$ with primitive χ (mod mq) implies $mq=q_0$, and since $p_0\nmid q$ we have $p_0\,|\,m$. ∎

Remark 1. If $f(X)\le\exp(c\sqrt{\log X})$ for some constant c, then choosing $c^\sharp$ to be such a constant one sees that $\mathcal{Z}$ can be taken to be at most a singleton.

Remark 2. We see that asymptotically almost all moduli remain after exceptions, because the excluded moduli number at most $\ll f(X)/\log\log X$, since $p_0\ge\log\log X$.

Using Proposition 2, we prove the following

Theorem 8. *Let A be a given positive number. Then there exists a positive number B such that for all $M\le f(X)$ except multiples of numbers in a set of size Z_f, we have*

$$\sum_{\substack{q\le \frac{X^{1/2}}{M^{6/5}}(\log X)^{-B} \\ (q,Mp_0)=1}} \max_{(a,qM)=1} \biggl|\psi(X;qM,a)-\frac{\psi(X)}{\varphi(qM)}\biggr| \ll \frac{X}{\varphi(M)}(\log X)^{-A}, \tag{4.7}$$

where the implicit constants depend on A.

Proof. Let c^* be a constant to be specified later in terms of A, and pick $c^\sharp$ arbitrarily (or, in case $f(X)\le\exp(c\sqrt{\log X})$ for some c, pick $c^\sharp$ according to Remark 1), so that Proposition 2 furnishes us with a set $\mathcal{Z}$ of size Z_f. Then if M is not a multiple of any number in $\mathcal{Z}$, $q\le\exp(c^\sharp\sqrt{\log X})$ and $(q,Mp_0)=1$, then no $L(s,\psi\xi)$ with ψ primitive has a zero in the region $\beta\ge 1-c^*\log\log X/\log X$, $\widetilde{\gamma}\le\exp(c^\sharp\sqrt{\log X})$. We write $\Omega = X^{1/2}M^{-6/5}(\log X)^{-B}$ for brevity. We have

$$\psi(X;qM,a)=\frac{1}{\varphi(qM)}\sum_{\chi\,(\mathrm{mod}\,qM)}\overline{\chi}(a)\psi(X,\chi) \tag{4.8}$$

and

$$|\psi(X,\chi_0)-\psi(X)| \le \sum_{\substack{n\le X\\ (n,qM)>1}} \Lambda(n) \ll (\log qM)(\log X), \tag{4.9}$$

so it suffices to consider, within acceptable error,

$$\sum_{\substack{q\le \Omega\\ (q,Mp_0)=1}} \max_{(a,qM)=1} \left|\frac{1}{\varphi(qM)} \sum_{\substack{\chi \,(\mathrm{mod}\, qM)\\ \chi\ne\chi_0}} \overline{\chi}(a)\psi(X,\chi)\right|. \tag{4.10}$$

Since $(M,q)=1$, we can factorize χ as $\psi\xi$, where ψ and ξ are characters to the moduli q and M respectively (there is no danger of confusing $\psi(n)$ with $\psi(X;q,a)$, nor with $\psi(X,\chi)$), so that (4.10) is

$$\sum_{\substack{q\le \Omega\\ (q,Mp_0)=1}} \max_{(a,qM)=1} \left|\frac{1}{\varphi(qM)} \sum_{\substack{\psi \,(\mathrm{mod}\, q)\\ \xi\,(\mathrm{mod}\, M)\\ \psi\xi\ne\chi_0}} \overline{\psi\xi}(a)\psi(X,\psi\xi)\right|. \tag{4.11}$$

We replace each character ψ with the primitive character ψ^* inducing it. This leads to an error of

$$\sum_{\substack{q\le \Omega\\ (q,Mp_0)=1}} \frac{1}{\varphi(qM)} \sum_{\substack{\psi \,(\mathrm{mod}\, q)\\ \xi\,(\mathrm{mod}\, M)}} \sum_{\substack{n\le X\\ (n,q)>1}} \Lambda(n) \ll \frac{X^{1/2}}{M^{6/5}} \exp(-c^\sharp\sqrt{\log X})(\log X)^2, \tag{4.12}$$

and this is acceptable. Using the explicit formula for $\psi(X,\chi)$ in the form

$$\psi(X,\chi) = -\sum_{\substack{|\gamma_\chi|\le X^{1/2}\\ \beta_\chi>1/2}} \frac{X^{\rho_\chi}}{\rho_\chi} + O(X^{1/2}(\log X)^2), \tag{4.13}$$

we are left to bound

$$\frac{1}{\varphi(M)} \sum_{\substack{q\le \Omega\\ (q,Mp_0)=1}} \frac{1}{\varphi(q)} \sum_{\xi\,(\mathrm{mod}\, M)} \sum_{\psi\,(\mathrm{mod}\, q)} \sum_{\substack{|\gamma_{\psi^*\xi}|\le X^{1/2}\\ \beta_{\psi^*\xi}>1/2}} \frac{X^{\beta_{\psi^*\xi}}}{|\rho_{\psi^*\xi}|}. \tag{4.14}$$

We rearrange the sum according to the moduli of the primitive characters ψ^* that occur, hence after relabelling the dummy variables so that q is now the modulus of ψ^*, we have

$$(4.15)\quad \frac{X}{\varphi(M)} \sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \sum_{\xi\,(\mathrm{mod}\,M)} \sideset{}{^*}\sum_{\psi\,(\mathrm{mod}\,q)} \sum_{\substack{|\gamma_{\psi\xi}|\le X^{1/2}\\ \beta_{\psi\xi}>1/2}} \frac{X^{-(1-\beta_{\psi\xi})}}{|\rho_{\psi\xi}|} \sum_{\substack{k\le\Omega/q\\(k,Mp_0)=1}} \frac{1}{\varphi(kq)}$$
$$\ll \frac{X(\log X)^2}{\varphi(M)} \sum_{\substack{q\le\Omega\\(q,Mp_0)=1}} \sum_{\xi\,(\mathrm{mod}\,M)} \sideset{}{^*}\sum_{\psi\,(\mathrm{mod}\,q)} \sum_{\substack{|\gamma_{\psi\xi}|\le X^{1/2}\\ \beta_{\psi\xi}>1/2}} \frac{X^{-(1-\beta_{\psi\xi})}}{q|\rho_{\psi\xi}|}.$$

We divide the ranges for q and $\widetilde{\gamma}$ into dyadic segments, and the range for β into segments of length $(\log X)^{-1}$ as follows:

$$(4.16)\quad q\in[Q_\mu/2,Q_\mu),\qquad \widetilde{\gamma}\in[T_\nu/2,T_\nu),\qquad 1-\beta\in[\delta_\lambda-(\log X)^{-1},\delta_\lambda),$$

where $2\le Q_\mu=2^\mu<2\Omega$, $2\le T_\nu=2^\nu<2X^{1/2}$ and $(\log X)^{-1}\le\delta_\lambda=\lambda(\log X)^{-1}\le 1/2$. So our expression is

$$(4.17)\qquad \ll \frac{X(\log X)^5}{\varphi(M)} \sup_{(\lambda,\mu,\nu)} \frac{N^*(1-\delta_\lambda,M,Q_\mu,T_\nu)}{Q_\mu T_\nu} X^{-\delta_\lambda},$$

where

$$(4.18)\quad N^*(1-\delta_\lambda,M,Q_\mu,T_\nu)=\sum_{\substack{Q_\mu/2<q\le Q_\mu\\(q,Mp_0)=1}} \sideset{}{^*}\sum_{\psi\,(\mathrm{mod}\,q)} \sum_{\xi\,(\mathrm{mod}\,M)} N(1-\delta_\lambda,T_\nu,\psi\xi).$$

Thus we need to show, for all triples (λ,μ,ν), dropping the subscripts for economy of notation, the upper bound

$$(4.19)\qquad N^*(1-\delta,M,Q,T)\ll QTX^\delta(\log X)^{-A-5}.$$

To this end we use Theorem 7, which for our ranges of Q and T yields

$$(4.20)\qquad N^*(1-\delta,M,R,T)\ll((MQT)^{2\delta}+(MQ^2T)^{c(\delta)\delta})(\log X)^{C'},$$

where C' is an absolute constant.

Since for $0\le\delta\le1/2$ we have

$$(4.21)\qquad \frac{(MQT)^{2\delta}}{QT}(\log X)^{C'}\ll M^{2\delta}(\log X)^{C'},$$

the contribution of the first term on the right hand side of (4.20) is acceptable if $\delta\ge 2/15$, say. So we only need to show

$$(4.22)\qquad \frac{(MQ^2T)^{c(\delta)\delta}}{QT}\ll X^\delta(\log X)^{-(A+C'+5)}$$

for $0\le\delta\le1/2$, and

$$(4.23)\qquad \frac{(MQT)^{2\delta}}{QT}\ll X^\delta(\log X)^{-(A+C'+5)}$$

for $0\le\delta\le 2/15$.

If $1/4 \le \delta \le 1/2$, we have $c(\delta) = 3/(1+\delta)$. Here $6\delta/(1+\delta) - 1 \le 2\delta$, $3\delta/(1+\delta) - 1 \le 0$ and $3\delta/(1+\delta) \le 12\delta/5$, so

$$\text{(4.24)} \quad \frac{(MQ^2T)^{3\delta/(1+\delta)}}{QT} \ll \frac{(MQ^2)^{3\delta/(1+\delta)}}{Q} \ll M^{12\delta/5}\bigg(\frac{X^{1/2}}{M^{6/5}(\log X)^B}\bigg)^{2\delta} \ll X^{\delta}(\log X)^{-2\delta B} \ll X^{\delta}(\log X)^{-(A+C'+5)}$$

for $B \ge 2(A+C'+5)$.

If $2/15 \le \delta \le 1/4$, we have $c(\delta) = 3/(2-3\delta)$. Here also $3\delta/(2-3\delta) \le 12/5\delta$, $0 \le 6\delta/(2-3\delta) - 1 \le 4\delta/5$ and $3\delta/(2-3\delta) - 1 \le 0$, so

$$\text{(4.25)} \quad \frac{(MQ^2T)^{3\delta/(2-3\delta)}}{QT} \ll \frac{(MQ^2)^{3\delta/(2-3\delta)}}{Q} \ll M^{12\delta/5}\bigg(\frac{X^{1/2}}{M^{6/5}(\log X)^B}\bigg)^{4\delta/5} \ll M^{36\delta/25}X^{2\delta/5}(\log X)^{-4\delta B/5},$$

and this is $\ll X^{\delta}(\log X)^{-(A+C'+5)}$ if $M \ll X^{5/12}$ and $B \ge \frac{75}{8}(A+C'+5)$.

Now suppose $\delta \le 2/15$. Then $6\delta/(2-3\delta)-1 \le -1/2$ and $3\delta/(2-3\delta) \le 2\delta$, hence

$$\text{(4.26)} \quad \frac{(MQ^2T)^{3\delta/(2-3\delta)}}{QT} \ll M^{2\delta}(QT)^{-1/2},$$

as well as

$$\text{(4.27)} \quad \frac{(MQT)^{2\delta}}{QT} \ll M^{2\delta}(QT)^{-1/2} =: (*).$$

Now if $M \ll X^{5/12}$ and $QT \ge \exp(c^\sharp\sqrt{\log X})$, then

$$\text{(4.28)} \quad (*) \le M^{2\delta}\exp\bigg(-\frac{c^\sharp}{2}\sqrt{\log X}\bigg) \ll X^{\delta}(\log X)^{-(A+C'+5)}.$$

Otherwise, if $QT \le \exp(c^\sharp\sqrt{\log X})$, we use the fact that $\delta \ge c^*\log\log X/\log X$ by our assumption on M, and so

$$\text{(4.29)} \quad (*) \le \bigg(\frac{M^2}{X}\bigg)^{\delta} \le \exp\bigg(-\frac{c^*}{5}\log\log X\bigg) \le (\log X)^{-(A+C'+5)},$$

provided $c^* \ge 5(A+C'+5)$. ■

REMARK. Note that when M indeed reaches $X^{5/12}$, the sum is vacuous and the theorem is trivial. We will apply it with $M \ll X^{5/12-5\vartheta/6}$ for some positive ϑ to get the "level of distribution" ϑ.

For the shorter range $M \le (\log X)^C$, we can simply use the classical Bombieri–Vinogradov theorem (see, for instance, [2, §28]) with $A+C$ in place of A, and gain a factor of $\phi(M)$ without any further modifications.

THEOREM 9. *Let A be a given positive number and let $M \ll (\log X)^C$ be an integer. Then there is a positive number B such that*

$$\sum_{\substack{q \le X^{1/2}(\log X)^{-B} \\ (q,M)=1}} \max_{(a,qM)=1} \left| \psi(X; qM, a) - \frac{\psi(X)}{\varphi(qM)} \right| \ll \frac{X}{\varphi(M)} (\log X)^{-A}, \tag{4.30}$$

where the implicit constant depends on A and C.

We would like to express these results in a unified fashion. To that end, given an increasing function $f(X)$ of X such that $f(X) \ll X^{5/12-5\vartheta/6}$ with $\vartheta > 0$, we introduce the following notation:

$$e_f = \begin{cases} 1/2 & \text{if } f(X) \le \exp(C\sqrt{\log X}), \\ \vartheta & \text{otherwise.} \end{cases} \tag{4.31}$$

With this we have

THEOREM 10. *Let A be a given positive number and $f(X)$ an increasing function of X satisfying $f(X) \ll X^C$ with $C < 5/12$. Then for all $M \le f(X)$ except multiples of numbers in a set of size at most Z_f, and all $\delta > 0$, we have*

$$\sum_{\substack{q \le X^{e_f - \delta} \\ (q, MP_f)=1}} \max_{(a,qM)=1} \left| \psi(X; qM, a) - \frac{\psi(X)}{\varphi(qM)} \right| \ll \frac{X}{\varphi(M)} (\log X)^{-A}. \tag{4.32}$$

Now we are in a position to prove our main proposition.

5. Proof of Proposition 1. This section consists of lemmata that establish Proposition 1. They follow the corresponding results in [5] *mutatis mutandis.* In [5], the parameter W features in a dual role: first in that the weights $\lambda_{d_1,\ldots,d_k}$ are supported for $(\prod d_i, W) = 1$, and second in the "W-trick", i.e. in restricting n to $n \equiv \nu_0 \pmod W$. In our case we have VP_f in the first role and W' in the second.

LEMMA 1. *Let*

$$y_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i)\varphi(r_i)\Big) \sum_{\substack{d_1,\ldots,d_k \\ r_i \mid d_i \,\forall i}} \frac{\lambda_{d_1,\ldots,d_k}}{\prod_{i=1}^{k} d_i}, \tag{5.1}$$

and let $y_{\max} = \sup_{r_1,\ldots,r_k} |y_{r_1,\ldots,r_k}|$. Then

$$S_1 = \frac{X}{V} \sum_{u_1,\ldots,u_k} \frac{y_{u_1,\ldots,u_k}^2}{\prod_{i=1}^{k} \varphi(u_i)} + O\bigg(\frac{y_{\max}^2 \varphi(VP_f)^k X (\log X)^k}{V (VP_f)^k D_0} \bigg). \tag{5.2}$$

Proof. We start by rearranging the sum on the right hand side of (3.12) to obtain

$$S_1 = \sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} \lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \sum_{\substack{x\le n<2x\\ n\equiv \nu_0\,(\mathrm{mod}\,W')\\ [d_i,e_i]\mid nM+h_i}} 1. \tag{5.3}$$

Now when $W', [d_1,e_1],\dots,[d_k,e_k]$ are pairwise coprime, the inner sum is over a single residue class modulo $q = W'\prod_i [d_i,e_i]$ by the Chinese Remainder Theorem; otherwise it is empty, in the case $p\mid (W',[d_i,e_i])$ because of the condition $(W', M\nu_0 + h_i) = 1$, and in the case $p\mid ([d_i,e_i],[d_j,e_j])$ because it being non-empty would imply $p\mid h_i - h_j$, but $h_i - h_j = fM$ for some $f < D_0$ since h_i and h_j lie in the same residue class modulo M, and $p \nmid M$ and p cannot be a prime less than D_0 by the support of λ. Since $f < D_0$ by the diameter of $\mathcal{H}$, we deduce that there is no contribution when $([d_i,e_i],[d_j,e_j]) > 1$.

Thus the inner sum is $x/q + O(1)$, and we have

$$S_1 = \frac{X}{V}\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k [d_i,e_i]} + O\Big(\sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k}} |\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}|\Big), \tag{5.4}$$

where $\sum'$ denotes the coprimality restrictions. The error term is plainly

$$\ll \lambda_{\max}^2\Big(\sum_{d<R}\tau_k(d)\Big)^2 \ll \lambda_{\max}^2 R^2(\log X)^{2k}, \tag{5.5}$$

where $\lambda_{\max} = \sup_{d_1,\dots,d_k}\lambda_{d_1,\dots,d_k}$. To deal with the main term, we use the identity

$$\frac{1}{[d_i,e_i]} = \frac{1}{d_ie_i}\sum_{u_i\mid d_i,e_i}\varphi(u_i) \tag{5.6}$$

and rewrite it as

$$\frac{X}{V}\sum_{u_1,\dots,u_k}\Big(\prod_{i=1}^k \varphi(u_i)\Big) \sideset{}{'}\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ u_i\mid d_i,e_i\,\forall i}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k d_ie_i}. \tag{5.7}$$

By the support of λ, we may drop the requirement that W' is coprime to $[d_i,e_i]$. Also by the support of λ, terms with $(d_i,d_j) > 1$ for $i\ne j$ have no contribution. Thus our restrictions boil down to $(d_i,e_j) = 1$ for $i \ne j$. We may remove this requirement by multiplying our expression with

$\sum_{s_{i,j}|d_i,e_j} \mu(s_{i,j})$ for all i,j. Then our main term becomes

(5.8)

$$\frac{X}{V}\sum_{u_1,\dots,u_k}\Big(\prod_{i=1}^{k}\varphi(u_i)\Big)\sum_{s_{1,2},\dots,s_{k-1,k}}\Big(\prod_{\substack{1\le i,j\le k\\ i\neq j}}\mu(s_{i,j})\Big)\sum_{\substack{d_1,\dots,d_k\\ e_1,\dots,e_k\\ u_i|d_i,e_i\,\forall i\\ s_{i,j}|d_i,e_j\,\forall i\neq j}}\frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^{k}d_ie_i}.$$

We may restrict $s_{i,j}$ to be coprime to u_i, u_j, $s_{i,a}$ and $s_{b,j}$ for all $a\neq i$ and $b\neq j$ since these have no contribution by the support of λ. We denote the summation with these restrictions by $\sum^{*}$. We introduce the change of variable

$$y_{r_1,\dots,r_k}=\Big(\prod_{i=1}^{k}\mu(r_i)\varphi(r_i)\Big)\sum_{\substack{d_1,\dots,d_k\\ r_i|d_i\,\forall i}}\frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k}d_i}. \tag{5.9}$$

Thus $y_{r_1,\dots,r_k}$ is supported on $r=\prod_i r_i<R$, $(r,VP_f)=1$ and $\mu(r)^2=1$. This change is invertible and we have

$$\sum_{\substack{r_1,\dots,r_k\\ d_i|r_i\,\forall i}}\frac{y_{r_1,\dots,r_k}}{\prod_{i=1}^{k}\varphi(r_i)}=\frac{\lambda_{d_1,\dots,d_k}}{\prod_{i=1}^{k}\mu(d_i)d_i}. \tag{5.10}$$

Hence any choice of $y_{r_1,\dots,r_k}$ with the above-mentioned support will yield a choice of $\lambda_{d_1,\dots,d_k}$. We note here that Maynard's estimate of $\lambda_{\max}$ in terms of $y_{\max}=\sup_{r_1,\dots,r_k}y_{r_1,\dots,r_k}$ holds verbatim and we have

$$\lambda_{\max}\ll y_{\max}(\log X)^k. \tag{5.11}$$

So our error term (5.5) is $O(y_{\max}^2R^2(\log X)^{4k})$. Using our change of variables we obtain

$$\begin{aligned}S_1=\frac{X}{V}\sum_{u_1,\dots,u_k}\Big(\prod_{i=1}^{k}\varphi(u_i)\Big)&\sideset{}{^*}\sum_{s_{1,2},\dots,s_{k-1,k}}\Big(\prod_{\substack{1\le i,j\le k\\ i\neq j}}\mu(s_{i,j})\Big)\\ &\times\Big(\prod_{i=1}^{k}\frac{\mu(a_i)\mu(b_i)}{\varphi(a_i)\varphi(b_i)}\Big)y_{a_1,\dots,a_k}y_{b_1,\dots,b_k}+O(y_{\max}^2R^2(\log X)^{4k}),\end{aligned} \tag{5.12}$$

where $a_j=u_j\prod_{i\neq j}s_{j,i}$ and $b_j=u_j\prod_{i\neq j}s_{i,j}$. Since there is no contribution when a_j or b_j are not squarefree, we may rewrite $\mu(a_j)$ as $\mu(u_j)\prod_{i\neq j}\mu(s_{j,i})$, and similarly for $\varphi(a_j)$, $\mu(b_j)$ and $\varphi(b_j)$. This gives us

$$(5.13)\quad S_1 = \frac{X}{V}\sum_{u_1,\ldots,u_k}\Big(\prod_{i=1}^{k}\frac{\mu(u_i)^2}{\varphi(u_i)}\Big)\sideset{}{^*}\sum_{s_{1,2},\ldots,s_{k,k-1}}\Big(\prod_{\substack{1\le i,j\le k\\ i\ne j}}\frac{\mu(s_{i,j})}{\varphi(s_{i,j})^2}\Big)y_{a_1,\ldots,a_k}y_{b_1,\ldots,b_k} + O(y_{\max}^2 R^2(\log X)^{4k}).$$

There is no contribution from $s_{i,j}$ with $1 < s_{i,j} < D_0$ because of the restricted support of y. The contribution when $s_{i,j} > D_0$ is

$$(5.14)\quad \ll \frac{y_{\max}^2 X}{V}\Big(\sum_{\substack{u<R\\ (u,VP_f)=1}}\frac{\mu(u_i)^2}{\varphi(u_i)}\Big)^k\Big(\sum_{s_{i,j}>D_0}\frac{\mu(s_{i,j})^2}{\varphi(s_{i,j})^2}\Big)\Big(\sum_{s>1}\frac{\mu(s)^2}{\varphi(s)^2}\Big)^{k^2-k-1} \ll \frac{y_{\max}^2\varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}.$$

Our previous error of $y_{\max}^2 R^2(\log X)^{4k}$ can be absorbed into this error, and the terms with $s_{i,j} = 1$ give us our desired main term. ∎

Lemma 2. *Let $S_2^{(m)}$ be as defined in (3.13), and let*

$$(5.15)\quad y^{(m)}_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^{k}\mu(r_i)g(r_i)\Big)\sum_{\substack{d_1,\ldots,d_k\\ r_i\mid d_i\,\forall i\\ d_m=1}}\frac{\lambda_{d_1,\ldots,d_k}}{\prod_i\varphi(d_i)},$$

where g is the totally multiplicative function defined on primes by $g(p) = p-2$. Let $y^{(m)}_{\max} = \sup_{r_1,\ldots,r_k}|y^{(m)}_{r_1,\ldots,r_k}|$. Then for any fixed $A > 0$,

$$(5.16)\quad S_2^{(m)} = \frac{X}{\varphi(V)\log X}\sum_{u_1,\ldots,u_k}\frac{(y^{(m)}_{u_1,\ldots,u_k})^2}{\prod_{i=1}^{k}g(u_i)} + O\Big(\frac{(y^{(m)}_{\max})^2\varphi(VP_f)^{k-1}X(\log X)^{k-2}}{\varphi(V)(VP_f)^{k-1}D_0}\Big) + O\Big(\frac{y_{\max}^2 X}{\varphi(M)(\log X)^A}\Big).$$

Proof. We first rearrange the sum to obtain

$$(5.17)\quad S_2^{(m)} = \sum_{\substack{d_1,\ldots,d_k\\ e_1,\ldots,e_k}}\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}\sum_{\substack{x\le n<2x\\ n\equiv\nu_0\,(\mathrm{mod}\,W')\\ [d_i,e_i]\mid nM+h_i}}\chi_{\mathbb{P}}(nM+h_m).$$

In the inner sum, if $W', [d_1,e_1],\ldots,[d_k,e_k]$ are pairwise relatively prime, the conditions determine n modulo $q = W'\prod_i[d_i,e_i]$, since $(M,[d_i,e_i]) = 1$ by the support of λ. In turn, $nM+h_m$ is determined modulo $qM = V\prod_i[d_i,e_i]$. Note that here $(q,P_f) = 1$. Also, if $([d_i,e_i], nM+h_m) > 1$ with $i \ne m$, then $p\,|\,|h_i-h_m| = fM$ for some $p\,|\,[d_i,e_i]$ and $f < D_0$ by the diameter of $\mathcal{H}$;

since d_i and e_i are relatively prime to both M and W by the support of λ, this is not possible. So $nM + h_m$ is relatively prime to the modulus if and only if $d_m = e_m = 1$. Thus we can write

$$\sum_{\substack{x \le n < 2x \\ n \equiv \nu_0 \,(\mathrm{mod}\, W') \\ [d_i,e_i] \mid nM+h_i}} \chi_{\mathbb{P}}(nM + h_m) = \sum_{\substack{X+h_m \le n < 2X+h_m \\ n \equiv b \,(\mathrm{mod}\, qM)}} \chi_{\mathbb{P}}(n) = \frac{\mathcal{P}_X}{\varphi(V) \prod_i \varphi([d_i, e_i])} + E(X, qM) + O(1), \tag{5.18}$$

where

$$E(X, qM) = \Bigg| \sum_{\substack{X \le n < 2X \\ n \equiv b \,(\mathrm{mod}\, qM)}} \chi_{\mathbb{P}}(n) - \frac{\mathcal{P}_X}{\varphi(qM)} \Bigg|, \tag{5.19}$$

$\mathcal{P}_X$ is the number of primes in $[X, 2X]$, and the $O(1)$ term arises from ignoring the shift by h_m in the sum. Thus the main term becomes

$$\frac{\mathcal{P}_X}{\varphi(V)} \sideset{}{'}\sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} \frac{\lambda_{d_1,\dots,d_k} \lambda_{e_1,\dots,e_k}}{\prod_i \varphi([d_i, e_i])} \tag{5.20}$$

where $\sum'$ denotes the constraint that $W', [d_1, e_1], \dots, [d_k, e_k]$ are pairwise relatively prime. As before, there is no contribution when $(W', [d_i, e_i]) > 1$ or $(d_i, d_j) > 1$, and we remove the conditions $(d_i, e_j) = 1$ by multiplying our expression by $\sum_{s_{i,j} \mid d_i, e_j} \mu(s_{i,j})$. We also use the identity (valid for squarefree d_i and e_i)

$$\frac{1}{\varphi([d_i, e_i])} = \frac{1}{\varphi(d_i)\varphi(e_i)} \sum_{u_i \mid d_i, e_i} g(u_i), \tag{5.21}$$

where g is the totally multiplicative function defined on primes by $g(p) = p - 2$. The main term then becomes

(5.22)
$$\frac{\mathcal{P}_X}{\varphi(V)} \sum_{u_1,\dots,u_k} \Big(\prod_{i=1}^{k} g(u_i) \Big) \sum_{s_{1,2},\dots,s_{k-1,k}} \Big(\prod_{\substack{1 \le i,j \le k \\ i \ne j}} \mu(s_{i,j}) \Big) \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ u_i \mid d_i, e_i \,\forall i \\ s_{i,j} \mid d_i, e_j \,\forall i \ne j \\ d_m = e_m = 1}} \frac{\lambda_{d_1,\dots,d_k} \lambda_{e_1,\dots,e_k}}{\prod_i \varphi(d_i)\varphi(e_i)}.$$

We again restrict $s_{i,j}$ to be coprime to u_i, u_j, $s_{i,a}$ and $s_{b,j}$ for all $a \ne i$ and $b \ne j$, and make the change of variable

$$y^{(m)}_{r_1,\dots,r_k} = \Big(\prod_{i=1}^{k} \mu(r_i) g(r_i) \Big) \sum_{\substack{d_1,\dots,d_k \\ r_i \mid d_i \,\forall i \\ d_m = 1}} \frac{\lambda_{d_1,\dots,d_k}}{\prod_i \varphi(d_i)}. \tag{5.23}$$

This is invertible, and $y^{(m)}_{r_1,\dots,r_k}$ is supported on $(\prod_i r_i, VP_f)=1$, $\prod_i r_i<R$, $\mu(\prod_i r_i)^2=1$ and $r_m=1$. Then the main term becomes

$$\frac{\mathcal{P}_X}{\varphi(V)}\sum_{u_1,\dots,u_k}\left(\prod_{i=1}^k\frac{\mu(u_i)^2}{g(u_i)}\right)\sideset{}{^*}\sum_{s_{1,2},\dots,s_{k-1,k}}\left(\prod_{\substack{1\le i,j\le k\\ i\ne j}}\frac{\mu(s_{i,j})}{g(s_{i,j})^2}\right)y^{(m)}_{a_1,\dots,a_k}y^{(m)}_{b_1,\dots,b_k}, \tag{5.24}$$

where $a_j=u_j\prod_{i\ne j}s_{j,i}$ and $b_j=u_j\prod_{i\ne j}s_{i,j}$ for each $1\le j\le k$. Because of the restricted support of y, there is no contribution from terms with $(s_{i,j},VP_f)>1$. So we only need to consider $s_{i,j}=1$ or $s_{i,j}>D_0$. The contribution when $s_{i,j}>D_0$ is

$$\begin{aligned}\tag{5.25} &\ll\frac{(y^{(m)}_{\max})^2X}{\varphi(V)\log X}\Bigg(\sum_{\substack{u<R\\(u,VP_f)=1}}\frac{\mu(u)^2}{g(u)}\Bigg)^{k-1}\Bigg(\sum_s\frac{\mu(s)^2}{g(s)^2}\Bigg)^{k(k-1)-1}\sum_{s_{i,j}>D_0}\frac{\mu(s_{i,j})^2}{g(s_{i,j})^2}\\ &\ll\frac{(y^{(m)}_{\max})^2\varphi(VP_f)^{k-1}X(\log X)^{k-2}}{\varphi(V)(VP_f)^{k-1}D_0}.\end{aligned}$$

The contribution from $s_{i,j}=1$ gives us the main term which is

$$\frac{\mathcal{P}_X}{\varphi(V)}\sum_{u_1,\dots,u_k}\frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^k g(u_i)}. \tag{5.26}$$

By the prime number theorem, $\mathcal{P}_X=X/\log X+O(X/(\log X)^2)$, and the error here contributes

$$\frac{(y^{(m)}_{\max})^2X}{\varphi(V)(\log X)^2}\Bigg(\sum_{\substack{u<R\\(u,VP_f)=1}}\frac{\mu(u)^2}{\varphi(u)}\Bigg)^{k-1}\ll\frac{(y^{(m)}_{\max})^2\varphi(VP_f)^{k-1}X(\log X)^{k-3}}{\varphi(V)(VP_f)^{k-1}}, \tag{5.27}$$

which can be absorbed in the error term from (5.25).

Now we turn to the contribution of the error terms in (5.18), which is

$$\ll\sum_{\substack{d_1,\dots,d_k\\e_1,\dots,e_k}}|\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}|(E(X,qM)+1). \tag{5.28}$$

From the support of λ, we see that we only need to consider squarefree q with $q<W'R^2$ and $(q,MP_f)=1$. Since for a squarefree integer q there are at most $\tau_{3k}(q)$ choices of $d_1,\dots,d_k,e_1,\dots,e_k$ for which $q=W'\prod_i[d_i,e_i]$, we see that the error is

$$\ll\lambda_{\max}^2\sum_{\substack{q<W'R^2\\(q,MP_f)=1}}\mu(q)^2\tau_{3k}(q)E(X,qM)+\lambda_{\max}^2\sum_{\substack{q<W'R^2\\(q,MP_f)=1}}\mu(q)^2\tau_{3k}(q). \tag{5.29}$$

Now the second term is $\ll\lambda_{\max}^2W'R^2\log(W'R^2)^{3k-1}$. We use the Cauchy–Schwarz inequality and the trivial bound $E(X,qM)\ll X/\varphi(qM)$ to see that the first term is

(5.30)

$$\ll \frac{\lambda_{\max}^2}{\varphi(M)^{1/2}}\Bigg(\sum_{\substack{q<W'R^2\\(q,MP_f)=1}} \mu(q)^2\tau_{3k}(q)^2\frac{X}{\varphi(q)}\Bigg)^{1/2}\Bigg(\sum_{\substack{q<W'R^2\\(q,MP_f)=1}} \mu(q)^2E(X,qM)\Bigg)^{1/2}.$$

The first sum is $\ll X\log(W'R^2)^{3k}$. Now for X large enough, $W'R^2 \le X^{e_f-\delta}$, so that Theorem 10 applies to show that the second sum is $\ll \frac{X}{\varphi(M)}(\log X)^{-A}$ for A arbitrarily large. Thus the total contribution is

$$\ll \frac{y_{\max}^2 X}{\varphi(M)(\log X)^A}, \tag{5.31}$$

and this completes the proof. ∎

Lemma 3. *If $r_m = 1$, then*

$$y^{(m)}_{r_1,\ldots,r_k} = \sum_{a_m} \frac{y_{r_1,\ldots,r_{m-1},a_m,r_{m+1},\ldots,r_k}}{\varphi(a_m)} + O\bigg(\frac{y_{\max}\varphi(VP_f)\log X}{VP_fD_0}\bigg). \tag{5.32}$$

Proof. We assume that $r_m = 1$. We substitute (5.10) into (5.23) and obtain

$$y^{(m)}_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^k \mu(r_i)g(r_i)\Big)\sum_{\substack{d_1,\ldots,d_k\\ r_i|d_i\,\forall i\\ d_m=1}}\Big(\prod_{i=1}^k\frac{\mu(d_i)d_i}{\varphi(d_i)}\Big)\sum_{\substack{a_1,\ldots,a_k\\ d_i|a_i\,\forall i}}\frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k\varphi(a_i)}. \tag{5.33}$$

Swapping summations over d and a, we have

$$y^{(m)}_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^k \mu(r_i)g(r_i)\Big)\sum_{\substack{a_1,\ldots,a_k\\ r_i|a_i\,\forall i}}\frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k\varphi(a_i)}\sum_{\substack{d_1,\ldots,d_k\\ d_i|a_i,\, r_i|d_i\,\forall i\\ d_m=1}}\prod_{i=1}^k\frac{\mu(d_i)d_i}{\varphi(d_i)}. \tag{5.34}$$

The inner sum can be directly computed when a_i is squarefree, which is the only case that matters by the support of y. We have

$$\begin{aligned}\sum_{d_i|a_i,\, r_i|d_i}\frac{\mu(d_i)d_i}{\varphi(d_i)} &= \frac{\mu(r_i)r_i}{\varphi(r_i)}\sum_{d_i|a_i/r_i}\frac{\mu(d_i)d_i}{\varphi(d_i)} = \frac{\mu(r_i)r_i}{\varphi(r_i)}\prod_{p|a_i/r_i}\frac{-1}{p-1}\\ &= \frac{\mu(r_i)r_i}{\varphi(r_i)}\,\frac{\mu(a_i/r_i)}{\varphi(a_i/r_i)} = \frac{\mu(a_i)r_i}{\varphi(a_i)}.\end{aligned} \tag{5.35}$$

Hence

$$y^{(m)}_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^k \mu(r_i)g(r_i)\Big)\sum_{\substack{a_1,\ldots,a_k\\ r_i|a_i\,\forall i}}\frac{y_{a_1,\ldots,a_k}}{\prod_{i=1}^k\varphi(a_i)}\prod_{i\neq m}\frac{\mu(a_i)r_i}{\varphi(a_i)}. \tag{5.36}$$

By the support of y, we need only consider a_j with $(a_j, VP_f) = 1$. This implies $a_j = r_j$ or $a_j > D_0 r_j$. The total contribution from $a_j \neq r_j$ when $j \neq m$ is

$$(5.37)\quad \ll y_{\max}\Big(\prod_{i=1}^{k} g(r_i)r_i\Big)\Big(\sum_{a_j > D_0 r_j} \frac{\mu(a_j)^2}{\varphi(a_j)^2}\Big) \times \Big(\sum_{\substack{a_m < R\\ (a_m, VP_f)=1}} \frac{\mu(a_j)^2}{\varphi(a_j)}\Big) \prod_{\substack{1\le i\le k\\ i\neq j,m}} \Big(\sum_{r_i \mid a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2}\Big) \ll \Big(\prod_{i=1}^{k} \frac{g(r_i)r_i}{\varphi(r_i)^2}\Big) \frac{y_{\max}\varphi(VP_f)\log R}{VP_f D_0} \ll \frac{y_{\max}\varphi(VP_f)\log X}{VP_f D_0}.$$

Thus we find that

$$(5.38)\quad y^{(m)}_{r_1,\ldots,r_k} = \Big(\prod_{i=1}^{k} \frac{g(r_i)r_i}{\varphi(r_i)^2}\Big) \sum_{a_m} \frac{y_{r_1,\ldots,r_{m-1},a_m,r_{m+1},\ldots,r_k}}{\varphi(a_m)} + O\Big(\frac{y_{\max}\varphi(VP_f)\log X}{VP_f D_0}\Big).$$

Since the product is $1 + O(D_0^{-1})$, we have the result. ∎

Lemma 4. *Let $y_{r_1,\ldots,r_k}$ be given in terms of a piecewise differentiable function F supported on $\mathcal{R}_k = \{(x_1,\ldots,x_k) \in [0,1]^k : \sum_{i=1}^k x_i \le 1\}$ by*

$$(5.39)\quad y_{r_1,\ldots,r_k} = F\Big(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_k}{\log R}\Big)$$

whenever $r = \prod_i r_i$ is squarefree and satisfies $(r, VP_f) = 1$. Set

$$(5.40)\quad F_{\max} = \sup_{(t_1,\ldots,t_k)\in[0,1]^k} |F(t_1,\ldots,t_k)| + \sum_{i=1}^{k} \Big|\frac{\partial F}{\partial t_i}(t_1,\ldots,t_k)\Big|.$$

Then

$$(5.41)\quad S_1 = \frac{\varphi(VP_f)^k X (\log R)^k}{V(VP_f)^k} I_k(F) + O\Big(\frac{F_{\max}^2 \varphi(VP_f)^k X (\log X)^{k-1} \log\log X}{V(VP_f)^k D_0}\Big),$$

where

$$(5.42)\quad I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1,\ldots,t_k)^2\, dt_1 \ldots dt_k.$$

Proof. We substitute (5.39) into our expression for S_1 from Lemma 1 and obtain

$$S_1 = \frac{X}{V} \sum_{\substack{u_1,\dots,u_k \\ (u_i,u_j)=1\,\forall i\neq j \\ (u_i,VP_f)=1\,\forall i}} \left(\prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)}\right) F\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_k}{\log R}\right)^2 + O\left(\frac{F_{\max}^2 \varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}\right). \tag{5.43}$$

Now if $(u_i,u_j)>1$ for some $i\neq j$ and $(u_i,VP_f)=(u_j,VP_f)=1$, then there is a prime $p\,|\,(u_i,u_j)$ with $p\nmid VP_f$, so *a fortiori* $p\nmid W$ and $p>D_0$. Thus the cost of dropping the condition $(u_i,u_j)=1$ is an error of size

$$\begin{aligned} &\ll \frac{F_{\max}^2 X}{V} \sum_{p>D_0} \sum_{\substack{u_1,\dots,u_k<R \\ p|u_i,u_j \\ (u_i,VP_f)=1\,\forall i}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \\ &\ll \frac{F_{\max}^2 X}{V} \sum_{p>D_0} \frac{1}{(p-1)^2} \left(\sum_{\substack{u<R \\ (u,VP_f)=1}} \frac{\mu(u)^2}{\varphi(u)}\right)^k \\ &\ll \frac{F_{\max}^2 \varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}. \end{aligned} \tag{5.44}$$

Thus we are left to evaluate

$$\sum_{\substack{u_1,\dots,u_k \\ (u_i,VP_f)=1\,\forall i}} \left(\prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)}\right) F\left(\frac{\log u_1}{\log R},\dots,\frac{\log u_k}{\log R}\right)^2. \tag{5.45}$$

This differs from the corresponding sum in Maynard's work only in that we have a VP_f, which does not have as small prime factors, in place of W. Let

$$\gamma(p) = \begin{cases} 1 & \text{if } p\nmid VP_f, \\ 0 & \text{otherwise.} \end{cases} \tag{5.46}$$

Then we can use Lemma 6.1 of [5] with $\kappa = 1$,

$$\begin{aligned} L &\ll 1 + \sum_{p|VP_f} \frac{\log p}{p} \ll \Big(\sum_{p\le \log R} + \sum_{\substack{p|MP_f \\ p>\log R}}\Big) \frac{\log p}{p} \\ &\ll \log\log R + \frac{\log MP_f}{\log R} \ll \log\log X, \end{aligned} \tag{5.47}$$

and A_1 and A_2 suitable constants. The lemma then yields

$$(5.48)\qquad \sum_{\substack{u_1,\dots,u_k\\ (u_i,VP_f)=1\,\forall i}} \Big(\prod_{i=1}^{k} \frac{\mu(u_i)^2}{\varphi(u_i)}\Big) F\Big(\frac{\log u_1}{\log R},\dots,\frac{\log u_k}{\log R}\Big)^2 = \frac{\varphi(VP_f)^k(\log R)^k}{(VP_f)^k} I_k(F) + O\Big(\frac{F_{\max}^2\varphi(VP_f)^k(\log X)^{k-1}\log\log X}{(VP_f)^k D_0}\Big),$$

and the proof is complete. ∎

Lemma 5. *Let $y_{r_1,\dots,r_k}$, F, and $F_{\max}$ be as in Lemma 4. Then*

$$(5.49)\qquad S_2^{(m)} = \frac{\varphi(VP_f)^k X(\log R)^{k+1}}{V(VP_f)^k\log X} J_k^{(m)}(F) + O\Big(\frac{F_{\max}^2\varphi(VP_f)^k X(\log X)^k}{V(VP_f)^k D_0}\Big),$$

where

$$(5.50)\qquad J_k^{(m)}(F) = \int_0^1\cdots\int_0^1\Big(\int_0^1 F(t_1,\dots,t_k)\,dt_m\Big)^2 dt_1\dots dt_{m-1}\,dt_{m+1}\dots dt_k.$$

Proof. From Lemma 2, we want to evaluate the sum

$$(5.51)\qquad \sum_{u_1,\dots,u_k}\frac{(y^{(m)}_{u_1,\dots,u_k})^2}{\prod_{i=1}^k g(u_i)}.$$

First we estimate $y^{(m)}_{r_1,\dots,r_k}$. Recall $y^{(m)}_{r_1,\dots,r_k}$ is supported on $(\prod_i r_i, VP_f)=1$, $\mu(\prod_i r_i)^2=1$, $(r_i,r_j)=1$ when $i\neq j$ and $r_m=1$. Then substituting (5.39) into our expression for $y^{(m)}_{r_1,\dots,r_k}$ from Lemma 3, we obtain

$$(5.52)\qquad y^{(m)}_{r_1,\dots,r_k} = \sum_{(u,VP_f\prod_i r_i)=1}\frac{\mu(u)^2}{\varphi(u)} \times F\Big(\frac{\log r_1}{\log R},\dots,\frac{\log r_{m-1}}{\log R},\frac{\log u}{\log R},\frac{\log r_{m+1}}{\log R},\dots,\frac{\log r_k}{\log R}\Big) + O\Big(\frac{F_{\max}\varphi(VP_f)\log X}{VP_f D_0}\Big).$$

From this it is plain that

$$(5.53)\qquad y^{(m)}_{\max} \ll \frac{\varphi(VP_f)}{VP_f}F_{\max}\log X.$$

Now we use [5, Lemma 6.1] again, with $\kappa=1$,

$$(5.54)\qquad \gamma(p) = \begin{cases} 1 & \text{if } p\nmid VP_f\prod_{i=1}^k r_i,\\ 0 & \text{otherwise},\end{cases}$$

$$L \ll 1 + \sum_{p \mid V \prod_i r_i} \frac{\log p}{p} \ll \Big(\sum_{p \le \log R} + \sum_{\substack{p \mid M P_f \prod_i r_i \\ p > \log R}} \Big) \frac{\log p}{p} \ll \log\log X, \tag{5.55}$$

and A_1, A_2 suitable constants to obtain

$$\begin{aligned} y^{(m)}_{r_1,\ldots,r_k} = (\log R) \frac{\varphi(VP_f)}{VP_f} \Big(\prod_{i=1}^{k} \frac{\varphi(r_i)}{r_i} \Big) F^{(m)}_{r_1,\ldots,r_k} \\ + O\Big(\frac{F_{\max} \varphi(VP_f) \log X}{VP_f D_0} \Big), \end{aligned} \tag{5.56}$$

where

$$F^{(m)}_{r_1,\ldots,r_k} = \int_0^1 F\Big(\frac{\log r_1}{\log R}, \ldots, \frac{\log r_{m-1}}{\log R}, t_m, \frac{\log r_{m+1}}{\log R}, \ldots, \frac{\log r_k}{\log R} \Big)\, dt_m. \tag{5.57}$$

This is valid if $r_m = 1$, and $r = \prod_{i=1}^{k} r_i$ satisfies $(r, VP_f) = 1$ and $\mu(r)^2 = 1$, otherwise $y^{(m)}_{r_1,\ldots,r_k} = 0$. Squared, (5.56) gives

$$\begin{aligned} (y^{(m)}_{r_1,\ldots,r_k})^2 = (\log R)^2 \frac{\varphi(VP_f)^2}{(VP_f)^2} \Big(\prod_{i=1}^{k} \frac{\varphi(r_i)^2}{r_i^2} \Big) (F^{(m)}_{r_1,\ldots,r_k})^2 \\ + O\Big(\frac{(F_{\max})^2 \varphi(VP_f)^2 (\log X)^2}{(VP_f)^2 D_0} \Big). \end{aligned} \tag{5.58}$$

Using this in the expression for $S_2^{(m)}$ from Lemma 2, we have

$$\begin{aligned} S_2^{(m)} = \frac{\varphi(VP_f)^2 X (\log R)^2}{\varphi(V)(VP_f)^2 \log X} \sum_{\substack{r_1,\ldots,r_k \\ (r_i, VP_f)=1 \\ (r_i, r_j)=1\, \forall i \ne j \\ r_m = 1}} \Big(\prod_{i=1}^{k} \frac{\mu(r_i)^2 \varphi(r_i)^2}{g(r_i) r_i^2} \Big) (F^{(m)}_{r_1,\ldots,r_k})^2 \\ + O\Big(\frac{F_{\max}^2 \varphi(VP_f)^k X (\log X)^k}{V (VP_f)^k D_0} \Big). \end{aligned} \tag{5.59}$$

We drop the condition $(r_i, r_j) = 1$ as before, this time introducing an error of size

$$\begin{aligned} &\ll \frac{F_{\max}^2 \varphi(VP_f)^2 X (\log R)^2}{\varphi(V)(VP_f)^2 \log X} \Big(\sum_{p > D_0} \frac{\varphi(p)^4}{g(p)^2 p^4} \Big) \Big(\sum_{\substack{r < R \\ (r, VP_f)=1}} \frac{\varphi(r)^2}{g(r) r^2} \Big)^{k-1} \\ &\ll \frac{F_{\max}^2 \varphi(VP_f)^{k+1} X (\log X)^k}{\varphi(V)(VP_f)^{k+1} D_0}. \end{aligned} \tag{5.60}$$

Thus we are left to evaluate

$$\sum_{\substack{r_1,\dots,r_{m-1},r_{m+1},\dots,r_k\\(r_i,VP_f)=1}} \Big(\prod_{i=1}^k \frac{\mu(r_i)^2\varphi(r_i)^2}{g(r_i)r_i^2}\Big)(F^{(m)}_{r_1,\dots,r_k})^2. \tag{5.61}$$

Again we apply [5, Lemma 6.1] with $\kappa = 1$ and with

$$\gamma(p) = \begin{cases} 1 - \dfrac{p^2-3p+1}{p^3-p^2-2p+1} & \text{if } p \nmid VP_f,\\ 0 & \text{otherwise,}\end{cases} \tag{5.62}$$

$$L \ll 1 + \sum_{p\mid VP_f} \frac{\log p}{p} \ll \log\log X, \tag{5.63}$$

and A_1, A_2 suitable constants. The singular series in this case is

$$\mathfrak{S} = \frac{\varphi(VP_f)}{VP_f}\Big(1 + O\Big(\frac{1}{D_0}\Big)\Big), \tag{5.64}$$

and we obtain

$$S_2^{(m)} = \frac{\varphi(VP_f)^{k+1}X(\log R)^{k+1}}{\varphi(V)(VP_f)^{k+1}\log X} J_k^{(m)}(F) + O\Big(\frac{F_{\max}^2\varphi(VP_f)^{k+1}X(\log X)^k}{\varphi(V)(VP_f)^{k+1}D_0}\Big). \tag{5.65}$$

Now in the main term we have

$$\frac{\varphi(VP_f)}{\varphi(V)(VP_f)} = \frac{1}{V}\cdot\frac{V}{\varphi(V)}\cdot\frac{\varphi(VP_f)}{(VP_f)} = \frac{1}{V}\prod_{p\mid V}\frac{p}{p-1}\prod_{p\mid VP_f}\frac{p-1}{p} = \frac{1}{V}\prod_{\substack{p\mid P_f\\ p\nmid V}}\frac{p-1}{p}. \tag{5.66}$$

This last product is either vacuous, or consists of a single factor $1 - p_0^{-1}$, which is $1+O((\log\log X)^{-1})$. Thus we may replace (5.65), within acceptable error, with

$$S_2^{(m)} = \frac{\varphi(VP_f)^{k}X(\log R)^{k+1}}{V(VP_f)^{k}\log X} J_k^{(m)}(F) + O\Big(\frac{F_{\max}^2\varphi(VP_f)^{k}X(\log X)^k}{V(VP_f)^{k}D_0}\Big), \tag{5.67}$$

where we have replaced $\frac{\varphi(VP_f)}{\varphi(V)(VP_f)}$ with $1/V$ in the error term as well. ∎

6. Discussion. Baker and Zhao also consider primes in arithmetic progressions, except they prove their result for certain smooth moduli (recall that a number is called *y-smooth* if it has no prime factor exceeding y). The techniques they employ involve estimating Dirichlet polynomials and appealing to a zero-free region described in terms of the largest prime and the squarefree kernel of M to obtain the required Bombieri–Vinogradov type theorem. Their result [1, Theorem 1] reads as follows (with the notation adapted where applicable to avoid confusion).

THEOREM (Baker–Zhao). *Let $\eta > 0$, $r \geq 1$, and let $M = X^\theta$ with $0 < \theta \leq 5/12 - \eta$, $(a, M) = 1$. Let*

$$K(\theta) = \begin{cases} 4/(1-2\theta) & \text{if } \theta < 2/5 - \varepsilon, \\ 40/(9-20\theta) & \text{if } \theta \geq 2/5 - \varepsilon. \end{cases}$$

Suppose that M satisfies

$$\max\{p : p \mid M\} < \exp\left(\frac{\log X}{B \log\log X}\right), \qquad \prod_{p \mid M} p < X^\delta, \qquad w \nmid M$$

with

$$B = \frac{C_1}{\eta} \exp\left(\frac{4(r+1)}{K(\theta)}\right), \qquad \delta = \frac{C_3 \eta}{r + \log(1/\eta)} \exp\left(-\frac{4(r+1)}{K(\theta)}\right)$$

for suitable absolute positive constants C_1 and C_3, and w denotes the possibly existing unique exceptional modulus to which there is a Dirichlet L-function with a zero in the region $\beta > c_1/\log X$. There are primes $p_n < \cdots < p_{n+r}$ in $(X/2, X]$ with $p_i \equiv a \pmod{M}$ such that

$$p_{n+r} - p_n < C_2 M r \exp(K(\theta) r).$$

Here C_2 is a positive absolute constant.

Recalling our Theorem 6, i.e.

$$p_{n+r} - p_n \ll \left(\frac{r}{\eta}\right) \exp\left(\frac{5r}{3\eta}\right) M,$$

one immediately sees that the Baker–Zhao bound is stronger as r grows, and also has the advantage of describing the moduli for which it holds (apart from the possibility of being a multiple of the exceptional modulus if it exists). On the other hand, as per Remark 2 following Proposition 2, the result of the present work holds for $X^{5/12-\eta}(1 - c/\log\log X)$ moduli up to $X^{5/12-\eta}$, while by Dickman's theorem (see, for instance, [6, Theorem 7.2]), there are $o(X^{5/12-\eta})$ integers with no prime divisors exceeding $\exp\left(\frac{\log X}{B \log\log X}\right)$ for which the Baker–Zhao result holds. Hence the present result is valid for a much larger class of arithmetic progressions. With these considerations the two can be regarded as complementary results concerning uniform small gaps between primes in arithmetic progressions over a range of moduli.

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