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Chen–Ricci inequalities for submanifolds of Riemannian and Kaehlerian product manifolds

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Abstract. Some examples of slant submanifolds of almost product Riemannian manifolds are presented. The existence of a useful orthonormal basis in proper slant submanifolds of a Riemannian product manifold is proved. The sectional curvature, the Ricci curvature and the scalar curvature of submanifolds of locally product manifolds of almost constant curvature are obtained. Chen–Ricci inequalities involving the Ricci tensor and the squared mean curvature for submanifolds of locally product manifolds of almost constant curvature are established. Chen–Ricci inequalities for different kinds of submanifolds of Kaehlerian product manifolds are also given.

1. Introduction. Since the celebrated theory of J. F. Nash of isometric immersion of a Riemannian manifold into a suitable Euclidean space gives very important and effective motivation to view each Riemannian manifold as a submanifold in a Euclidean space, the problem of discovering simple basic relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifold theory. The main extrinsic invariant is the squared mean curvature, and the main intrinsic invariants include the classical curvature invariants: the Ricci curvature and the scalar curvature. There are also many important modern intrinsic invariants of (sub)manifolds introduced by B.-Y. Chen (cf. [C:3], [C:4], [C:7], [C:9]). The basic relationships discovered so far are (sharp) inequalities involving intrinsic and extrinsic invariants, and the study of this topic has attracted a lot of attention during the last two decades.

In 1999, B.-Y. Chen (cf. [C:5, Theorem 4], [C:6, Theorem 1]) obtained a basic inequality involving the Ricci curvature and the squared mean curva-

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ture of submanifolds in a real space form. This inequality drew attention of several authors and they established similar inequalities for different kinds of submanifolds in ambient manifolds possessing various structures. Motivated by the result of B.-Y. Chen [C:5, Theorem 4], in [HT] and [HMT] the authors presented a general theory for submanifolds of Riemannian manifolds by proving a basic inequality (see Theorem 2.1), called the Chen–Ricci inequality [T:2], involving the Ricci curvature and the squared mean curvature of the submanifold.

Apart from Hermitian geometry, the theory of product manifolds has important physical and geometrical aspects. In physics, the spacetime of Einstein's general relativity could be considered as a product of 3-dimensional space and 1-dimensional time, both having their metrics, thus its topology is generated by the metrics of these spaces. There are also nice applications of product manifolds in Kaluza–Klein theory, brane theory and gauge theory (cf. [BEE], [CY], [H], [HS]). Also, product manifolds and their submanifolds have been studied by many mathematicians. Locally product manifolds were first introduced by S. Tachibana [Ta]. Invariant, anti-invariant and non-invariant submanifolds of a locally product manifold were studied by T. Adati [Ad]; semi-invariant submanifolds of a locally product manifold were investigated by A. Bejancu [B]; almost semi-invariant submanifolds of a locally product manifold were studied by the second author and K. D. Singh [T:1], [TS]; and skew semi-invariant submanifolds (a special class of almost semi-invariant submanifolds) of a locally product manifold were studied by X. Liu and F.-M. Shao [LS].

Since the inception of the theory of slant submanifolds in Kaehlerian manifolds created by B.-Y. Chen [C:1], this theory has shown an increasing development. Recently, many authors investigated slant submanifolds of various spaces (complex manifolds, contact manifolds etc.)—see [ACCM], [At], [CCFF], [CZ], [KUS], [S:1], [S:2], [SK], [SUK], [V]. It is known that proper slant submanifolds of Kaehlerian manifolds are always even-dimensional, while proper slant submanifolds of almost contact metric manifolds are always odd-dimensional. However, B. Sahin [S:1] showed that proper slant submanifolds in almost Riemannian product manifolds may have even or odd dimension. Hence slant submanifolds of almost product manifold are quite different from slant submanifolds of complex manifolds and contact manifolds.

Motivated by these facts, in the present paper, we initiate the study of the Chen–Ricci inequality for slant submanifolds of almost product Riemannian manifolds. The paper is organized as follows. Section 2 is concerned with some necessary preliminaries. In Section 3, we recall some basic facts about the locally product manifolds. In Section 4, some examples of slant submanifolds of almost product Riemannian manifolds are given. The existence of a useful orthonormal basis in proper slant submanifolds of a Riemannian product manifold is proved. Some basic results are also presented. In Section 5, first we find the sectional curvature, the Ricci curvature and the scalar curvature of submanifolds of locally product manifolds of almost constant curvature. Then we obtain Chen–Ricci inequalities involving the Ricci tensor and the squared mean curvature for submanifolds of locally product manifolds of almost constant curvature. In Section 6, we obtain Chen–Ricci inequalities for slant submanifolds of Kaehlerian product manifolds. In particular, we also investigate these relations for F-invariant submanifolds and F-totally real submanifolds of Kaehlerian product manifolds.

2. Preliminaries. Let (M, g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The Gauss and Weingarten formulas are

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian connection, the induced Riemannian connection and the induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively. We denote the inner product of both the metrics g and $\widetilde{g}$ by $\langle\,,\rangle$. The second fundamental form σ is related to the shape operator A_N by $\langle\sigma(X, Y), N\rangle = \langle A_N X, Y\rangle$. The Gauss equation is

$$R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) + \langle\sigma(X, W), \sigma(Y, Z)\rangle - \langle\sigma(X, Z), \sigma(Y, W)\rangle \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M, respectively. The mean curvature vector H is $H = (1/n)\,\mathrm{trace}(\sigma)$. If $\sigma = 0$, then the submanifold is called *totally geodesic* in $\widetilde{M}$. If $H = 0$, then the submanifold is called *minimal.* If $\sigma(X, Y) = \langle X, Y\rangle H$ for all $X, Y \in TM$, then the submanifold is called *totally umbilical* [C:2]. The *relative null space* of M at p is defined by

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X, Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* $\mathcal{D}(p)$ at p [C:5], [C:6].

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, and suppose e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. We set

$$\sigma_{ij}^r = \langle\sigma(e_i, e_j), e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle\sigma(e_i, e_j), \sigma(e_i, e_j)\rangle.$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. Thus, we can say that K_{ij} and $\widetilde{K}_{ij}$ are the "intrinsic" and "extrinsic" sectional curvatures of $\operatorname{span}\{e_i, e_j\}$ at p, respectively. In view of (2.1), we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \tag{2.2}$$

From (2.2) it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2\|H\|^2 - \|\sigma\|^2, \tag{2.3}$$

where $\widetilde{\tau}(T_pM)$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. Thus, we can say that $\tau(p)$ and $\widetilde{\tau}(T_pM)$ are the "intrinsic" and "extrinsic" scalar curvature of the submanifold at p, respectively. We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid \langle X, X\rangle = 1\}.$$

We recall the following result [HT].

THEOREM 2.1. *Let M be an n-dimensional submanifold of a Riemannian manifold $\widetilde{M}$. Then:*

(a) *For $X \in T_p^1M$ we have*

$$\operatorname{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_pM)}(X), \tag{2.4}$$

where $\widetilde{\operatorname{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *For a fixed $X \in T_p^1M$, equality holds in (2.4) if and only if*

$$\sigma(X,X) = \tfrac{1}{2}nH(p) \quad \text{and} \quad \sigma(X,Y) = 0 \tag{2.5}$$

for all $Y \in T_pM$ such that $\langle X, Y\rangle = 0$.

(c) *Equality holds in (2.4) for all $X \in T_p^1M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Note that (2.4) is a special case of [C:8, Theorem 3.1, inequality (3.3)]. From Theorem 2.1, we immediately have the following:

COROLLARY 2.2. *Let M be an n-dimensional submanifold of a Riemannian manifold. For $X \in T_p^1M$ any two of the following three statements imply the remaining one:*

(a) *The mean curvature vector $H(p)$ vanishes.*
(b) *The unit vector X belongs to the relative null space $\mathcal{N}_p$.*
(c) *The unit vector X satisfies the equality case of (2.4), namely*

$$\operatorname{Ric}(X) = \tfrac{1}{4}n^2\|H\|^2 + \widetilde{\operatorname{Ric}}_{(T_pM)}(X). \tag{2.6}$$

3. Locally product manifolds. Let $\widetilde{M}$ be a smooth manifold equipped with a tensor of type $(1,1)$ such that $F^2 = I$, where I denotes the identity transformation. Then $\widetilde{M}$ is called an *almost product manifold* and F is called an *almost product structure* on $\widetilde{M}$. The eigenvalues of F are $+1$ and -1. If we set

$$P = \tfrac{1}{2}(I + F), \qquad Q = \tfrac{1}{2}(I - F),$$

then

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$

If an almost product manifold $\widetilde{M}$ admits a Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX, FY) = \widetilde{g}(X, Y) \tag{3.1}$$

for all vector fields X and Y on $\widetilde{M}$, then $\widetilde{M}$ is called an *almost product Riemannian manifold* [YK:2].

Let $\widetilde{M}$ be an m-dimensional manifold. Suppose that the indices a, b, c, d run over the range $1, \dots, m_1$, the indices $\alpha, \beta, \gamma, \nu$ run over $m_1 + 1, \dots, m_1 + m_2 = m$, and the indices i, j, k, h run over $1, \dots, m$. A system of coordinate neighborhoods on $\widetilde{M}$ is said to be a *separating coordinate system* if in the intersection of any two coordinate neighborhoods (x^i) and $(x^{i'})$ we have

$$x^{a'} = x^{a'}(x^a), \qquad x^{\alpha'} = x^{\alpha'}(x^\alpha),$$
$$\det\left(\frac{\partial x^{a'}}{\partial x^a}\right) \neq 0, \quad \det\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right) \neq 0.$$

Suppose that $\widetilde{M}$ is covered by a separating coordinate system. We denote by $\widetilde{M}_1$ the system of subspaces defined by

$$x^\alpha = \text{constant}, \qquad \alpha \in \{m_1 + 1, \dots, m_1 + m_2 = m\},$$

and by $\widetilde{M}_2$ the system of subspaces defined by

$$x^a = \text{constant}, \qquad a \in \{1, \dots, m_1\}.$$

Then the manifold $\widetilde{M}$ is locally the product $\widetilde{M}_1 \times \widetilde{M}_2$ of two manifolds and is called a *locally product manifold.* A locally product manifold always admits a natural tensor field F of type $(1,1)$ given by

$$F_j^i = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_\beta^\alpha \end{pmatrix},$$

which satisfies

$$F^2 = I.$$

If each contravariant vector tangent to $\widetilde{M}_\ell$, $\ell \in \{1, 2\}$, parallel transported along $\widetilde{M}_\ell$, is still tangent to $\widetilde{M}_\ell$, then $\widetilde{M}_\ell$ is said to be *totally geodesic* in

$\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$. A locally product manifold $\widetilde{M}$ equipped with a Riemannian metric

$$ds^2 = \widetilde{g}_{ij}(x) dx^i dx^j$$

satisfying (3.1) is called a *locally product Riemannian manifold.* If the metric $\widetilde{g}$ of a locally product Riemannian manifold $\widetilde{M}$ has the form

$$ds^2 = \widetilde{g}_{ab}(x^c) dx^a dx^b + \widetilde{g}_{\alpha\beta}(x^\gamma)\, dx^\alpha\, dx^\beta,$$

then $\widetilde{M}$ is called a *locally decomposable Riemannian manifold.* A locally product Riemannian manifold is locally decomposable if and only if $\widetilde{\nabla} F = 0$, where $\widetilde{\nabla}$ is the Riemannian connection of $(\widetilde{M}, \widetilde{g})$.

Let $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ be a locally decomposable Riemannian manifold with $\dim(\widetilde{M}_\ell) = m_\ell > 2$, $\ell = 1, 2$. It is known [YK:2, Theorem 2.4, p. 421] that both $\widetilde{M}_1$ and $\widetilde{M}_2$ are Einstein, that is,

$$\widetilde{S}_{ab} = \mu_1 \widetilde{g}_{ab}, \qquad S_{\alpha\beta} = \mu_2 \widetilde{g}_{\alpha\beta}$$

if and only if

$$\widetilde{S}_{ij} = k_1 \widetilde{g}_{ij} + k_2 b F_{ij},$$

where

$$k_1 = \tfrac{1}{2}(\mu_1 + \mu_2), \qquad k_2 = \tfrac{1}{2}(\mu_1 - \mu_2).$$

Next, $\widetilde{M}_1$ and $\widetilde{M}_2$ are of constant sectional curvatures λ_1 and λ_2, respectively, that is,

$$\widetilde{R}_{abcd} = \lambda_1(\widetilde{g}_{ad}\widetilde{g}_{bc} - \widetilde{g}_{ac}\widetilde{g}_{bd}), \quad \widetilde{R}_{\alpha\beta\gamma\nu} = \lambda_2(\widetilde{g}_{\alpha\nu}\widetilde{g}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\nu})$$

if and only if

$$\begin{aligned}\widetilde{R}_{hijk} = a\{&(\widetilde{g}_{hk}\widetilde{g}_{ij} - \widetilde{g}_{hj}\widetilde{g}_{ik}) + (F_{hk}F_{ij} - F_{hj}F_{ik})\} \\ &+ b\{(F_{hk}\widetilde{g}_{ij} - F_{hj}\widetilde{g}_{ik}) + (\widetilde{g}_{hk}F_{ij} - \widetilde{g}_{hj}F_{ik})\},\end{aligned}$$

where

$$a = \tfrac{1}{4}(\lambda_1 + \lambda_2), \qquad b = \tfrac{1}{4}(\lambda_1 - \lambda_2).$$

A locally decomposable Riemannian manifold is called a *manifold of almost constant curvature*, denoted $\widetilde{M}(a, b)$, if its curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = a\{&(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) \\ &+ (\langle X,FW\rangle\langle Y,FZ\rangle - \langle X,FZ\rangle\langle Y,FW\rangle)\} \\ + b\{&(\langle X,FW\rangle\langle Y,Z\rangle - \langle X,FZ\rangle\langle Y,W\rangle) \\ &+ (\langle X,W\rangle\langle Y,FZ\rangle - \langle X,Z\rangle\langle Y,FW\rangle)\}\end{aligned} \tag{3.2}$$

for all vector fields X, Y, Z, W in $\widetilde{M}$. For more details, we refer to [Ta], [Y], [YK:1] and [YK:2].

4. Slant submanifolds of almost product Riemannian manifolds. Let (M, g) be an n-dimensional Riemannian submanifold of a Riemannian manifold $\widetilde{M}$ equipped with an almost product Riemannian structure $(F, \widetilde{g})$. For any vector field X tangent to M, we can write

$$FX = fX + \omega X, \tag{4.1}$$

where fX is the tangential part of FX and ωX is the normal part of FX. From (3.1) and (4.1), we see that

$$g(fX, Y) = g(X, fY) \tag{4.2}$$

for all vector fields in M. The submanifold M is said to be F-*invariant* (resp. F-*anti-invariant*) if $\omega = 0$ (resp. $f = 0$). The submanifold M is F-invariant if and only if (f, g) is an almost product Riemannian structure on M. The squared norm of f at $p \in M$ is given by

$$\|f\|^2 = \sum_{i,j=1}^{n} g(fe_i, e_j)^2,$$

where $\{e_1, \dots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

For each non-zero vector X to M at p, let $\theta(p)$ be the angle between FX and X. If $\theta(p)$ is independent of the choice of $p \in M$ and $X \in T_pM$, then M is called a *slant submanifold*. If $\theta = 0$, then M is an F-*invariant submanifold*, and if $\theta = \pi/2$, then M is an F-*anti-invariant submanifold*. A slant submanifold which is neither invariant nor anti-invariant (totally real) is called a *proper slant submanifold*. If M is a slant submanifold of $\widetilde{M}$, then $\cos\theta = \frac{\langle FX, fX\rangle}{\|X\|\|fX\|}$ is constant for any vector field X in M. Thus, a submanifold M of an almost product Riemannian manifold $\widetilde{M}$ is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda$. If θ is the slant angle of M, then $\lambda = \cos^2\theta$. A slant submanifold is called a *product slant submanifold* if the endomorphism f is parallel [S:1].

Example 4.1. Consider the Euclidean 6-space $\mathbb{R}^6$ with standard coordinates $(x^1, \dots, x^6)$. Let F be the almost product structure on $\mathbb{R}^6$ given by

$$F(x^1, x^2, x^3, x^4, x^5, x^6) = (x^1, -x^2, x^3, -x^4, x^5, -x^6).$$

We consider the submanifold M given by

$$x(u^1, u^2, u^3) = (u^1\cos\theta, u^1\sin\theta, u^2\cos\theta, u^2\sin\theta, u^3\cos\theta, u^3\sin\theta).$$

We have

$$\begin{aligned} e_1 &= (\cos\theta, \sin\theta, 0, 0, 0, 0),\\ e_2 &= (0, 0, \cos\theta, \sin\theta, 0, 0),\\ e_3 &= (0, 0, 0, 0, \cos\theta, \sin\theta), \end{aligned}$$

$$Fe_1 = (\cos\theta, -\sin\theta, 0, 0, 0, 0),$$
$$Fe_2 = (0, 0, \cos\theta, -\sin\theta, 0, 0),$$
$$Fe_3 = (0, 0, 0, 0, \cos\theta, -\sin\theta),$$

so that

$$\langle Fe_1, e_1\rangle = \cos 2\theta, \quad \langle Fe_2, e_2\rangle = \cos 2\theta, \quad \langle Fe_3, e_3\rangle = \cos 2\theta,$$
$$\langle Fe_i, e_j\rangle = 0 \quad \text{for } i \neq j \in \{1, 2, 3\}$$

and

$$fe_1 = (\cos 2\theta)e_1, \quad fe_2 = (\cos 2\theta)e_2, \quad fe_3 = (\cos 2\theta)e_3.$$

Thus M is a proper θ-slant submanifold with slant angle 2θ.

Example 4.2. Consider the Euclidean 8-space $\mathbb{R}^8$ with coordinates $(x^1, \ldots, x^8)$. Let F be the almost product structure on $\mathbb{R}^8$ given by

$$F(x^1, x^2, x^3, x^4) = (x^1, x^2, x^3, x^4, -x^5, -x^6, -x^7, -x^8).$$

We consider the submanifold M given by

$$x(u^1, u^2, u^3, u^4) = (u^1{+}u^2, u^1{+}u^2, u^3{+}u^4, u^3{+}u^4, \sqrt{2}\,u^1, \sqrt{2}\,u^2, \sqrt{2}\,u^3, \sqrt{2}\,u^4).$$

Then

$$e_1 = \tfrac{1}{2}(1, 1, 0, 0, \sqrt{2}, 0, 0, 0),$$
$$e_2 = \tfrac{1}{2}(1, 1, 0, 0, 0, \sqrt{2}, 0, 0),$$
$$e_3 = \tfrac{1}{2}(0, 0, 1, 1, 0, 0, \sqrt{2}, 0),$$
$$e_4 = \tfrac{1}{2}(0, 0, 1, 1, 0, 0, 0, \sqrt{2})$$

so that

$$Fe_1 = \tfrac{1}{2}(1, 1, 0, 0, -\sqrt{2}, 0, 0, 0),$$
$$Fe_2 = \tfrac{1}{2}(1, 1, 0, 0, 0, -\sqrt{2}, 0, 0),$$
$$Fe_3 = \tfrac{1}{2}(0, 0, 1, 1, 0, 0, -\sqrt{2}, 0),$$
$$Fe_4 = \tfrac{1}{2}(0, 0, 1, 1, 0, 0, 0, -\sqrt{2}),$$
$$\langle Fe_1, e_2\rangle = \langle Fe_2, e_1\rangle = \langle Fe_3, e_4\rangle = \langle Fe_4, e_3\rangle = 1/2$$

and

$$fe_1 = \tfrac{1}{2}e_2, \quad fe_2 = \tfrac{1}{2}e_1, \quad fe_3 = \tfrac{1}{2}e_4, \quad fe_4 = \tfrac{1}{2}e_3.$$

Hence M is a slant submanifold with slant angle $\theta = \pi/3$.

Lemma 4.3 ([S:1]). *Let M be a slant submanifold of an almost product Riemannian manifold $\widetilde{M}$. Then*

$$\langle fX, fY\rangle = \cos^2\theta\,\langle X, Y\rangle \quad \text{and} \quad \langle \omega X, \omega Y\rangle = \sin^2\theta\,\langle X, Y\rangle$$

for all vector fields X, Y on M.

THEOREM 4.4. *Let M be an n-dimensional proper slant submanifold of an almost product Riemannian manifold $\widetilde{M}$. Then any orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM $(p \in M)$ satisfies the following condition:*

For any vector e_a belonging to the basis $\{e_1, \dots, e_n\}$, there exists a vector e_b in this basis such that

$$\langle fe_a, e_b\rangle = \langle e_a, fe_b\rangle = \cos\theta,$$
$$\langle fe_a, e_c\rangle = \langle fe_b, e_c\rangle = 0 \quad \text{for } c \neq a \text{ and } c \neq b.$$

Proof. For $n = 2$, let $\{e_1, e_2\}$ be any orthonormal basis of T_pM. Then we can write

$$fe_1 = a_1e_1 + a_2e_2, \tag{4.3}$$

where $a_1, a_2 \in \mathbb{R}$. From Lemma 4.3, we have

$$\langle fe_1, fe_1\rangle = a_1^2 + a_2^2 = \cos^2\theta. \tag{4.4}$$

Thus, we get

$$fe_1 = \cos\theta\,(\cos\varphi_1\, e_1 + \sin\varphi_1\, e_2), \qquad \varphi_1 \in [0, 2\pi]. \tag{4.5}$$

Similarly, we also get

$$fe_2 = \cos\theta\,(\cos\varphi_2\, e_1 + \sin\varphi_2\, e_2), \qquad \varphi_2 \in [0, 2\pi]. \tag{4.6}$$

From (4.2), it follows that

$$\sin\varphi_1 = \cos\varphi_2. \tag{4.7}$$

Using the fact that $\langle fe_1, fe_2\rangle = 0$, we obtain

$$\cos^2\theta\,(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) = 0. \tag{4.8}$$

Since M is a proper slant submanifold, we have

$$\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 = 0. \tag{4.9}$$

Taking into account (4.7) and (4.9), we get either $\varphi_1 = 0$ and $\varphi_2 = \pi/2$ (or $\varphi_2 = 3\pi/2$, or $\varphi_1 = \pi/2$ (or $\varphi_2 = 3\pi/2$) and $\varphi_2 = 0$.

If $\varphi_1 = 0$ and $\varphi_2 = \pi/2$ (or $\varphi_2 = 3\pi/2$), then

$$fe_1 = \cos\theta\, e_1, \quad fe_2 = \cos\theta\, e_2 \quad \text{and} \quad \langle fe_1, e_2\rangle = \langle fe_2, e_1\rangle = 0. \tag{4.10}$$

If $\varphi_1 = \pi/2$ (or $\varphi_2 = 3\pi/2$) and $\varphi_2 = 0$, then

$$fe_1 = \cos\theta\, e_2, \quad fe_2 = \cos\theta\, e_1 \quad \text{and} \quad \langle fe_1, e_2\rangle = \langle fe_2, e_1\rangle = 0. \tag{4.11}$$

Thus the assertion of the theorem is satisfied for $n = 2$.

For $n = 3$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis of T_pM. Then we can write

$$fe_1 = a_1e_1 + a_2e_2 + a_3e_3, \tag{4.12}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. From Lemma 4.3, we have

$$\langle fe_1, fe_1\rangle = a_1^2 + a_2^2 + a_3^2 = \cos^2\theta. \tag{4.13}$$

Thus, we get

$$fe_1 = \cos\theta\,(\cos\varphi_1 \sin\alpha_1\, e_1 + \sin\varphi_1 \sin\alpha_1\, e_2 + \cos\alpha_1\, e_3) \tag{4.14}$$

for $\varphi_1 \in [0, 2\pi]$ and $\alpha_1 \in [0, \pi]$. Similarly, we can write

$$fe_2 = \cos\theta\,(\cos\varphi_2 \sin\alpha_2\, e_1 + \sin\varphi_2 \sin\alpha_2\, e_2 + \cos\alpha_2\, e_3), \tag{4.15}$$

$$fe_3 = \cos\theta\,(\cos\varphi_3 \sin\alpha_3\, e_1 + \sin\varphi_3 \sin\alpha_3\, e_2 + \cos\alpha_3\, e_3), \tag{4.16}$$

for $\varphi_2, \varphi_3 \in [0, 2\pi]$ and $\alpha_2, \alpha_3 \in [0, \pi]$. From (4.2), it follows that

$$\cos\alpha_1 = \cos\varphi_3 \sin\alpha_3, \qquad \cos\alpha_2 = \sin\varphi_3 \sin\alpha_3. \tag{4.17}$$

Using (4.17) and Lemma 4.3, we have the following equalities:

$$\sin^2\varphi_1 \sin^2\alpha_1 + \sin^2\varphi_2 \sin^2\alpha_2 + \sin^2\varphi_3 \sin^2\alpha_3 = 1, \tag{4.18}$$

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1, \tag{4.19}$$

$$\cos\varphi_1 \sin\alpha_1 = \sin\varphi_2 \sin\alpha_2, \tag{4.20}$$

$$\sin\varphi_1 \sin\alpha_1 = \cos\varphi_2 \sin\alpha_2. \tag{4.21}$$

Taking into account (4.17), (4.19), (4.20) and (4.21), we have

$$\begin{aligned} \cos\varphi_1 &= \sin\varphi_2, \qquad \sin\varphi_1 = \cos\varphi_2, \\ \sin\alpha_1 &= \sin\alpha_2, \qquad \sin\alpha_1 \neq 0. \end{aligned} \tag{4.22}$$

Using $\langle fe_1, fe_2\rangle = 0$, we obtain

$$2\sin\varphi_1 \cos\varphi_1 \sin^2\alpha_1 + \cos^2\alpha_2 = 0,$$

which implies $\sin\varphi_1 \cos\varphi_1 = 0$ and $\cos\alpha_2 = 0$. If we do the analysis as in (4.10) and (4.11) for $n = 2$, we have the assertion of the theorem for $n = 3$.

For $n = k$, let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of T_pM. Then we can write

$$fe_\ell = \sum_{r=1}^{k} a_r^\ell e_r, \tag{4.23}$$

where $a_r^\ell \in \mathbb{R}$, $r, \ell \in \{1, \ldots, k\}$. From Lemma 4.3, we have

$$\langle fe_\ell, fe_\ell\rangle = \sum_{r=1}^{k} (a_r^\ell)^2 = \cos^2\theta. \tag{4.24}$$

Thus, we get

$$\begin{aligned} fe_\ell = \cos\theta \Big(&\prod_{r=1}^{k-1} \sin\varphi_r^\ell e_1 + \Big[\prod_{r=1}^{k-2} \sin\varphi_r^\ell\Big] \cos\varphi_{r-1}^\ell e_2 + \cdots + \\ &\sin\varphi_1^\ell \sin\varphi_2^\ell \cos\varphi_3^\ell e_{k-2} + \sin\varphi_1^\ell \cos\varphi_2^\ell e_{k-1} + \cos\varphi_1^\ell e_k \Big) \end{aligned} \tag{4.25}$$

for $\varphi_r^\ell \in [0, 2\pi]$, $r, \ell \in \{1, \dots, k\}$. From (4.2), it follows that

$$\begin{aligned}
&\Big[\prod_{r=1}^{k-2} \sin\varphi_r^1\Big] \cos\varphi_{k-1}^1 = \prod_{r=1}^{k-1} \sin\varphi_r^2,\\
&\Big[\prod_{r=1}^{k-3} \sin\varphi_r^1\Big] \cos\varphi_{k-2}^1 = \prod_{r=1}^{k-1} \sin\varphi_r^3,\\
&\quad\vdots\\
&\sin\varphi_1^1 \cos\varphi_2^1 = \prod_{r=1}^{k-1} \sin\varphi_r^{k-1},\\
&\cos\varphi_1^1 = \prod_{r=1}^{k-1} \sin\varphi_r^k,\\
&\Big[\prod_{r=1}^{k-3} \sin\varphi_r^2\Big] \cos\varphi_{k-2}^2 = \Big[\prod_{r=1}^{k-2} \sin\varphi_r^3\Big] \cos\varphi_{k-1}^3,\\
&\quad\vdots\\
&\sin\varphi_1^2 \cos\varphi_2^2 = \Big[\prod_{r=1}^{k-2} \sin\varphi_r^{k-1}\Big] \cos\varphi_{k-1}^{k-1},\\
&\cos\varphi_1^2 = \Big[\prod_{r=1}^{k-2} \sin\varphi_r^k\Big] \cos\varphi_{k-1}^k,\\
&\quad\vdots\\
&\cos\varphi_1^{k-1} = \sin\varphi_1^k \cos\varphi_1^k.
\end{aligned}$$

Using the above equalities and Lemma 4.3, we also obtain

$$\sum_{\ell=1}^{k} \prod_{r=1}^{k-1} \sin^2\varphi_r^\ell = 1, \qquad \sum_{r=1}^{k-1} \cos^2\varphi_r^\ell = 1.$$

By a straightforward computation, we get $\cos\varphi_r^\ell = 1$ and $\cos\varphi_r^s = 0$ for $\ell \neq s \in \{1, \dots, k\}$ and $r \in \{1, \dots, k-1\}$. If a similar analysis is carried out as for $n = 2$ and $n = 3$, we deduce the assertion of the theorem for $n = k$. ∎

Corollary 4.5. *Let M be a proper slant surface of an almost Riemannian product manifold $\widetilde{M}$. If $\{e_1, e_2\}$ is an orthonormal basis of T_pM, $p \in M$, then one of the following two cases holds:*

(a) $\langle fe_1, e_1\rangle = \langle fe_2, e_2\rangle = \cos\theta$ *and* $\langle fe_1, e_2\rangle = 0$,
(b) $\langle fe_1, e_2\rangle = \cos\theta$ *and* $\langle fe_1, e_1\rangle = \langle fe_2, e_2\rangle = 0$.

Corollary 4.6. *Let M be an n-dimensional proper slant submanifold of an almost Riemannian product manifold $\widetilde{M}$. If $\{e_1, \dots, e_n\}$ is an or-*

thonormal basis of M, then $\{fe_1, \dots, fe_n\}$ is an orthogonal basis, that is, the members of $\{fe_1, \dots, fe_n\}$ are mutually orthogonal.

Proof. For $i, j \in \{1, \dots, n\}$ we have

$$\langle fe_i, fe_j \rangle = \cos^2\theta \, \langle e_i, e_j \rangle,$$

which entails the conclusion. ∎

THEOREM 4.7. *Let M be a proper θ-slant submanifold of an almost product Riemannian manifold $\widetilde{M}$ and $\{e_1, \dots, e_n\}$ be any orthonormal frame for M. Then $\nabla_X f = 0$ for all $X \in TM$ if and only if for each $i \in \{1, \dots, n\}$, either $\langle fe_i, e_i \rangle = \cos\theta$ or e_i is parallel.*

Proof. For fixed i, we have $\langle fe_i, e_j \rangle = \cos\theta$ for some j. Thus

$$0 = X\langle fe_i, e_j \rangle = \langle \nabla_X fe_i, e_j \rangle + \langle fe_i, \nabla_X e_j \rangle = \cos\theta \, \langle \nabla_X e_j, e_j \rangle. \tag{4.26}$$

Evidently, we have $\langle \nabla_X e_j, e_j \rangle = 0$. Moreover,

$$\begin{aligned} (\nabla_X f)e_i &= \nabla_X fe_i - f\nabla_X e_i \\ &= \cos\theta \, \nabla_X e_j \\ &\quad - f\{\langle \nabla_X e_i, e_1 \rangle e_1 + \dots + \langle \nabla_X e_i, e_i \rangle e_i \\ &\qquad + \dots + \langle \nabla_X e_i, e_{i-1} \rangle e_{i-1} + \langle \nabla_X e_i, e_{i+1} \rangle e_{i+1} \\ &\qquad + \dots + \langle \nabla_X e_i, e_n \rangle e_n\}. \end{aligned} \tag{4.27}$$

From (4.27), it follows that $\nabla_X f = 0$ if and only if either $\langle fe_i, e_i \rangle = \cos\theta$, i.e. $i = j$, or e_i is parallel. ∎

Recall the following theorem of B. Sahin [S:1].

THEOREM 4.8. *Let M be a submanifold of an almost product Riemannian manifold $\widetilde{M}$. Then $\nabla f = 0$ if and only if M is locally a product $M_1 \times \dots \times M_k$, where M_i is either an F-invariant submanifold with $\nabla^i f_i = 0$, an F-anti-invariant submanifold, or a product slant submanifold of $\widetilde{M}$, where $f_i = f|_{TM_i}$ and ∇^i is the Riemannian connection of M_i.*

COROLLARY 4.9. *Suppose that M is a proper product θ-slant submanifold of an almost product Riemannian manifold $\widetilde{M}$. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_pM and no vector field e_i is parallel for $i \in \{1, \dots, n\}$, then $\langle fe_i, e_i \rangle = \cos\theta$ for all i.*

5. Submanifolds of locally decomposable manifolds. The following two results are well known from [YK:1].

THEOREM 5.1. *An F-invariant submanifold of a Riemannian product manifold $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ is again a Riemannian product manifold $M = M_1 \times M_2$, where M_ℓ is a totally geodesic submanifold of $\widetilde{M}_\ell$, $\ell \in \{1, 2\}$.*

THEOREM 5.2. *If M is any F-anti-invariant submanifold of a Riemannian product manifold $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$, then $A_{FX}Y = 0$. Moreover, if $\dim(\widetilde{M}) = 2\dim(M)$, then M is totally geodesic.*

LEMMA 5.3. *Let M be an n-dimensional submanifold of a manifold $\widetilde{M}(a,b)$ of almost constant curvature. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space T_pM. Then*

$$\widetilde{K}_{ij} = a\{1 + \langle e_i, fe_i\rangle\langle e_j, fe_j\rangle - \langle e_i, fe_j\rangle^2\} + b\{\langle e_i, fe_i\rangle + \langle e_j, fe_j\rangle\}, \tag{5.1}$$

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = a\{(n-1) + \langle e_i, fe_i\rangle \operatorname{trace}(f) - \|fe_i\|^2\} + b\{(n-2)\langle e_i, fe_i\rangle + \operatorname{trace}(f)\}, \tag{5.2}$$

$$\widetilde{\tau}(T_pM) = \tfrac{1}{2}a\{(n-1)n + (\operatorname{trace}(f))^2 - \|f\|^2\} + b(n-1)\operatorname{trace}(f). \tag{5.3}$$

Proof. We get (5.1) from (3.2). Using

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \sum_{j=1,\, j\neq i}^{n} \widetilde{K}_{ij}$$

in (5.1), we get (5.2). Next, using

$$2\widetilde{\tau}(T_pM) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i),$$

from (5.2) we obtain (5.3). ∎

THEOREM 5.4. *If M is an n-dimensional submanifold of a manifold $\widetilde{M}(a,b)$ of almost constant curvature, then:*

(a) *For all $X \in T_p^1M$,*

$$\mathrm{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + a\{(n-1) + \langle X, fX\rangle \operatorname{trace}(f) - \|fX\|^2\} + b\{(n-2)\langle X, fX\rangle + \operatorname{trace}(f)\}. \tag{5.4}$$

(b) *For a fixed $X \in T_p^1M$, equality holds in* (5.4) *if and only if* (2.5) *is true. If $H(p) = 0$, then $X \in T_p^1M$ satisfies equality in* (5.4) *if and only if $X \in \mathcal{N}_p$.*

(c) *Equality holds in* (5.4) *for all $X \in T_p^1M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Proof. Using (5.2) in (2.4), we find (5.4). The rest is straightforward. ∎

THEOREM 5.5. *If M is an n-dimensional F-anti-invariant submanifold of a manifold $\widetilde{M}(a,b)$ of almost constant curvature, then:*

(a) *For all* $X \in T_p^1 M$,

$$\mathrm{Ric}(X) \le \tfrac{1}{4} n^2 \|H\|^2 + (n-1)a. \tag{5.5}$$

(b) *For a fixed* $X \in T_p^1 M$, *equality holds in* (5.5) *if and only if* (2.5) *is true. If* $H(p) = 0$, *then* $X \in T_p^1 M$ *satisfies the equality case of* (5.5) *if and only if* $X \in \mathcal{N}_p$.

(c) *Equality holds in* (5.5) *for all* $X \in T_p^1 M$ *if and only if either* p *is a totally geodesic point, or* $n = 2$ *and* p *is a totally umbilical point.*

Proof. Using $f = 0$ in (5.4), we find (5.5). The rest is straightforward. ∎

By polarization, from Theorem 5.5 we derive the following

THEOREM 5.6. *Let* M *be an* n*-dimensional* F*-anti-invariant submanifold of a manifold* $\widetilde{M}(a, b)$ *of almost constant curvature. Then the Ricci tensor* S *satisfies*

$$S \le \big(\tfrac{1}{4} n^2 \|H\|^2 + (n-1)a\big) g, \tag{5.6}$$

where g *is the induced Riemannian metric on* M. *Equality holds in* (5.6) *if and only if either* M *is a totally geodesic submanifold, or* $n = 2$ *and* M *is a totally umbilical submanifold.*

Using Theorem 5.2 in Theorem 5.6, we have the following

EXAMPLE 5.7. Every n-dimensional F-anti-invariant submanifold of a manifold $\widetilde{M}(a, b)$ of almost constant curvature and of dimension $2n$ satisfies the equality case of (5.6):

$$S = \big(\tfrac{1}{4} n^2 \|H\|^2 + (n-1)a\big) g. \tag{5.7}$$

6. Submanifolds of Kaehlerian product manifolds. Let M_1 and M_2 be Kaehlerian manifolds with complex dimension m_1 and m_2, respectively. We denote by J_1 and J_2 almost complex structures of M_1 and M_2, respectively. We consider the Kaehlerian product $\widetilde{M} = M_1 \times M_2$ and set

$$JX = J_1 PX + J_2 QX$$

for any vector field X on $\widetilde{M}$, where P and Q denote the projection operators. Then we see that

$$J_1 P = PJ, \quad J_2 Q = QJ, \quad FJ = JF,$$

where F is the natural almost product structure on $\widetilde{M}$. If M_1 and M_2 are complex space forms with constant holomorphic sectional curvature c_1 and c_2, respectively, then the Riemannian curvature tensor $\widetilde{R}$ of a Kaehlerian product manifold $\widetilde{M}$ is given by

$$\begin{aligned}(6.1)\quad \widetilde{R}(X,Y,Z,W) = \tfrac{1}{16}(c_1+c_2)\{&\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle \\ &+ \langle JY,Z\rangle\langle JX,W\rangle - \langle JX,Z\rangle\langle JY,W\rangle \\ &+ 2\langle X,JY\rangle\langle JZ,W\rangle + 2\langle FY,Z\rangle\langle FX,W\rangle \\ &- \langle FX,Z\rangle\langle FY,W\rangle + \langle FJY,Z\rangle\langle FJX,W\rangle \\ &- \langle FJX,Z\rangle\langle FJY,W\rangle + 2\langle FX,JY\rangle\langle FJZ,W\rangle\} \\ + \tfrac{1}{16}(c_1-c_2)\{&\langle FY,Z\rangle\langle X,W\rangle - \langle FX,Z\rangle\langle Y,W\rangle \\ &+ \langle Y,Z\rangle\langle FX,W\rangle - \langle X,Z\rangle\langle FY,W\rangle \\ &+ \langle FJY,Z\rangle\langle JX,W\rangle - \langle FJX,Z\rangle\langle JY,W\rangle \\ &+ \langle JY,Z\rangle\langle FJX,W\rangle - \langle JX,Z\rangle\langle FJX,W\rangle \\ &+ 2\langle FX,JY\rangle\langle JZ,W\rangle + 2\langle X,JY\rangle\langle JFZ,W\rangle\}\end{aligned}$$

for all vector fields X,Y,Z,W on $\widetilde{M}$. Furthermore, the sectional curvature of $\widetilde{M}$ is given by

$$\begin{aligned}(6.2)\quad \widetilde{K}(X\wedge Y) = \tfrac{1}{16}(c_1+c_2)\{&1+3\langle X,JY\rangle^2 + 2\langle FY,Y\rangle\langle FX,X\rangle \\ &- \langle FX,Y\rangle^2 + 3\langle X,JFY\rangle^2\} \\ + \tfrac{1}{16}(c_1-c_2)\{&\langle FY,Y\rangle + \langle FX,X\rangle + 6\langle FJX,Y\rangle\langle JX,Y\rangle\}\end{aligned}$$

for all X,Y on $\widetilde{M}$ [YK:2].

Theorem 6.1. *Let M be a proper θ-slant submanifold of a Kaehlerian product manifold $\widetilde{M}$. Then:*

(a) *For all $X \in T_p^1M$,*

$$\begin{aligned}(6.3)\quad \mathrm{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + \tfrac{1}{16}(c_1+c_2)\{&(n-1) + 3(1+\cos^2\theta)\|PX\|^2 \\ &+ 2\langle fX,X\rangle\,\mathrm{trace}(f) - \langle fX,X\rangle^2 - \cos^2\theta\} \\ + \tfrac{1}{16}(c_1-c_2)\{&\mathrm{trace}(f) + 6\langle fPX,PX\rangle + (n-2)\langle fX,X\rangle\}.\end{aligned}$$

(b) *For a fixed $X \in T_p^1M$, equality holds in* (6.3) *if and only if*

$$\sigma(X,X) = \tfrac{1}{2}n\,H(p) \quad \textit{and} \quad \sigma(X,Y) = 0$$

for all $Y \in T_pM$ orthogonal to X.

(c) *Equality holds in* (6.3) *for all $X \in T_p^1M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Proof. Let $\{e_1,\dots,e_n,e_{n+1},\dots,e_m\}$ be an orthonormal basis of $T_p\widetilde{M}$ and $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM. From (6.2), we have

$$\begin{aligned}(6.4)\quad \widetilde{K}_{1j} = \tfrac{1}{16}(c_1+c_2)\{&1+3\langle Pe_1,e_j\rangle^2 \\ &+ 2\langle fe_1,e_1\rangle\langle fe_j,e_j\rangle - \langle fe_1,e_j\rangle^2 + 3\langle fPe_1,e_j\rangle^2\} \\ + \tfrac{1}{16}(c_1-c_2)\{&\langle fe_1,e_1\rangle + \langle fe_j,e_j\rangle + 6\langle fPe_1,e_j\rangle\langle Pe_1,e_j\rangle\}\end{aligned}$$

for $j \in \{2, \ldots, n\}$. By using (6.4), we get

$$\widetilde{\mathrm{Ric}}_{T_pM}(e_1) = \tfrac{1}{16}(c_1+c_2)\Big\{(n-1) + 3\sum_{j=1}^{n}\langle Pe_1, e_j\rangle^2 - 3\langle Pe_1, e_1\rangle^2 + 2\sum_{j=1}^{n}\langle fe_1, e_1\rangle\langle fe_j, e_j\rangle - 2\langle fe_1, e_1\rangle^2 - \sum_{j=1}^{n}\langle fe_1, e_j\rangle^2 + \langle fe_1, e_1\rangle^2 + 3\sum_{j=1}^{n}\langle fPe_1, e_j\rangle^2 - 3\langle fPe_1, e_1\rangle^2\Big\} + \tfrac{1}{16}(c_1-c_2)\Big\{(n-1)\langle fe_1, e_1\rangle + \sum_{j=1}^{n}\langle fe_j, e_j\rangle - \langle fe_1, e_1\rangle + 6\sum_{j=1}^{n}\langle fPe_1, e_j\rangle\langle Pe_1, e_j\rangle - 6\langle fPe_1, e_1\rangle\langle Pe_1, e_1\rangle\Big\}. \tag{6.5}$$

Since

$$\sum_{j=1}^{n}\langle fPe_1, e_j\rangle^2 = \cos^2\theta\,\|Pe_1\|^2,$$

setting $e_1 = X$ and using Theorem 4.4 we get

$$\widetilde{\mathrm{Ric}}_{T_pM}(X) = \tfrac{1}{16}(c_1+c_2)\{(n-1) + 3(1+\cos^2\theta)\|PX\|^2 + 2\langle fX, X\rangle\,\mathrm{trace}(f) - \langle fX, X\rangle^2 - \cos^2\theta\} + \tfrac{1}{16}(c_1-c_2)\Big\{\mathrm{trace}(f) + 6\sum_{j=2}^{n}\langle fPX, e_j\rangle\langle PX, e_j\rangle + (n-2)\langle fX, X\rangle\Big\}. \tag{6.6}$$

From (5.4), we find (6.3). The rest is straightforward. ■

THEOREM 6.2. *Let M be an almost proper θ-slant submanifold of a Kaehlerian product manifold $\widetilde{M}$, $\{e_i\}$ be an orthonormal basis of T_pM and suppose $\{e_i\}$ are not parallel. Then:*

(a) *For all $X \in T_p^1M$,*

$$\mathrm{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + \tfrac{1}{16}(c_1+c_2)\{(n-1) + 3(1+\cos^2\theta)\|PX\|^2 + 2(n-1)\cos^2\theta\} + \tfrac{1}{16}(c_1-c_2)\{2(n-1)\cos\theta + 6\cos\theta\,\|PX\|^2\}. \tag{6.7}$$

(b) *For a fixed $X \in T_p^1M$, equality holds in (6.7) if and only if*

$$\sigma(X,X) = \tfrac{1}{2}nH(p) \quad \textit{and} \quad \sigma(X,Y) = 0$$

for all $Y \in T_pM$ such that $\langle X, Y\rangle = 0$.

(c) *Equality holds in (6.7) for all $X \in T_p^1M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Proof. If we write $\langle fX, X\rangle = \cos\theta$ in (6.5) and set $X = e_1$, then we obtain

$$\widetilde{\mathrm{Ric}}_{T_pM}(X) = \tfrac{1}{16}(c_1+c_2)\{(n-1)+3(1+\cos^2\theta)\|PX\|^2 + 2(n-1)\cos^2\theta + 2(n-1)\cos^2\theta\} + \tfrac{1}{16}(c_1-c_2)\{2(n-1)\cos\theta + 6\cos\theta\|PX\|^2\}. \tag{6.8}$$

In view of (6.8) in (5.4), we obtain (6.7). ∎

THEOREM 6.3. *Let M be an n-dimensional F-invariant submanifold of a Kaehlerian product manifold $\widetilde{M}$. Then:*

(a) *For all $X \in T_p^1M$,*

$$\mathrm{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + \tfrac{1}{16}(c_1+c_2)\{(n-2)+6\|PX\|^2 + 2n\langle fX, X\rangle - \langle fX, X\rangle^2\} + \tfrac{1}{16}(c_1-c_2)\{n+(n-2)\langle fX, X\rangle + 6\langle fPX, PX\rangle\}. \tag{6.9}$$

(b) *For a fixed $X \in T_p^1M$, equality holds in* (6.9) *if and only if*

$$\begin{cases} \sigma(X,Y) = 0 & \text{for all } Y \in T_pM \text{ orthogonal to } X,\\ \sigma(X,X) = \tfrac{1}{2}n\,H(p). \end{cases} \tag{6.10}$$

(c) *Equality holds in* (6.9) *for all $X \in T_p^1M$ if and only if either p is a totally geodesic point, or $n = 2$ and p is a totally umbilical point.*

Proof. Since M is an F-invariant submanifold of $\widetilde{M}$, we have $FX = fX$ for each $X \in T_pM$, $\mathrm{trace}(f) = n$ and $\langle fPe_1, e_j\rangle^2 = \|fPe_1\|^2 = \|Pe_1\|^2$. If we set $X = e_1$ and use (6.5), we obtain

$$\widetilde{\mathrm{Ric}}_{T_pM}(X) = \tfrac{1}{16}(c_1+c_2)\{(n-2)+3\|PX\|^2 + 2n\langle fX, X\rangle - \langle fX, X\rangle^2 + 3\|PX\|^2\} + \tfrac{1}{16}(c_1-c_2)\Big\{n+(n-2)\langle fX, X\rangle + 6\sum_{j=1}^{n}\langle fPX, e_j\rangle\langle PX, e_j\rangle\Big\}. \tag{6.11}$$

In view of (6.11) in (5.4), we obtain (6.9). ∎

THEOREM 6.4. *Let M be an n-dimensional F-totally real submanifold of a Kaehlerian product manifold $\widetilde{M}$. Then:*

(a) *For all $X \in T_p^1M$,*

$$\mathrm{Ric}(X) \le \tfrac{1}{4}n^2\|H\|^2 + \tfrac{1}{16}(c_1+c_2)\{(n-1)+3\|PX\|^2\}. \tag{6.12}$$

(b) *For a fixed $X \in T_p^1M$, equality holds in* (6.12) *if and only if*

$$\begin{cases} \sigma(X,Y) = 0 & \text{for all } Y \in T_pM \text{ orthogonal to } X,\\ \sigma(X,X) = \tfrac{1}{2}nH(p). \end{cases} \tag{6.13}$$

(c) *Equality holds in* (6.12) *for all* $X \in T_p^1 M$ *if and only if either* p *is a totally geodesic point, or* $n = 2$ *and* p *is a totally umbilical point.*

Proof. Since M is an anti-invariant submanifold on $\widetilde{M}$, we have $f = 0$. From (6.5) and (2.4), we obtain (6.12). ■

COROLLARY 6.5. *Let* M *be an* n*-dimensional submanifold of a Kaehlerian product manifold* $\widetilde{M}$ *and* $\{e_1, \dots, e_n\}$ *be an orthonormal basis of* $T_p M$*. For* $X \in T_p^1 M$ *any two of the following three statements imply the remaining one:*

(a) *The mean curvature vector* $H(p)$ *vanishes.*
(b) *The unit vector* X *belongs to the relative null space* $\mathcal{N}_p$*.*
(c) *The unit vector* X *satisfies the equality case of the inequalities given in the following table:*

	M	*Inequality*
(1)	*proper* θ*-slant*	$\mathrm{Ric}(X) \le \frac{1}{4} n^2 \|H\|^2 + \frac{1}{16}(c_1 + c_2)\{(n-1) + 3(1 + \cos^2\theta)\|PX\|^2 + 2\langle fX, X\rangle \operatorname{trace}(f) - \langle fX, X\rangle^2 - \cos^2\theta\} + \frac{1}{16}(c_1 - c_2)\{\operatorname{trace}(f) + 6\langle fPX, PX\rangle + (n-2)\langle fX, X\rangle\}$
(2)	*proper product* θ*-slant with all* e_i *non-parallel*	$\mathrm{Ric}(X) \le \frac{1}{4} n^2 \|H\|^2 + \frac{1}{16}(c_1 + c_2)\{(n-1) + 3(1 + \cos^2\theta)\|PX\|^2 + 2(n-1)\cos^2\theta\} + \frac{1}{16}(c_1 - c_2)\{2(n-1)\cos\theta + 6\cos\theta\|PX\|^2\}$
(3)	F*-invariant*	$\mathrm{Ric}(X) \le \frac{1}{4} n^2 \|H\|^2 + \frac{1}{16}(c_1 + c_2)\{(n-2) + 6\|PX\|^2 + 2n\langle fX, X\rangle - \langle fX, X\rangle^2\} + \frac{1}{16}(c_1 - c_2)\{n + (n-2)\langle fX, X\rangle + 6\langle fPX, PX\rangle\}$
(4)	F*-totally real*	$\mathrm{Ric}(X) \le \frac{1}{4} n^2 \|H\|^2 + \frac{1}{16}(c_1 + c_2)\{(n-1) + 3\|PX\|^2\}$

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