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MULTILINEAR FOURIER MULTIPLIERS WITH MINIMAL SOBOLEV REGULARITY, I

BY

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Abstract. We find optimal conditions on m-linear Fourier multipliers that give rise to bounded operators from products of Hardy spaces H^{p_k}, $0 < p_k \le 1$, to Lebesgue spaces L^p. These conditions are expressed in terms of L^2-based Sobolev spaces with sharp indices within the classes of multipliers we consider. Our results extend those obtained in the linear case ($m = 1$) by Calderón and Torchinsky (1977) and in the bilinear case ($m = 2$) by Miyachi and Tomita (2013). We also prove a coordinate-type Hörmander integral condition which we use to obtain certain endpoint cases.

1. Introduction. Let σ be a bounded function on $\mathbb{R}^n$. We denote by T_σ the linear Fourier multiplier operator, whose action on Schwartz functions is given by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(\xi)\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi. \tag{1.1}$$

Mikhlin's [14] classical result states that T_σ admits an L^p-bounded extension for $1 < p < \infty$, whenever

$$|\partial_\xi^\alpha \sigma(\xi)| \le C_\alpha |\xi|^{-|\alpha|}, \quad \xi \ne 0, \tag{1.2}$$

for all multi-indices α with $|\alpha| \le [n/2] + 1$. This result was refined by Hörmander [12] who proved that (1.2) can be replaced by the Sobolev-norm condition

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j(\cdot))\widehat{\psi}\|_{W^s} < \infty \tag{1.3}$$

for some $s > n/2$, where $\widehat{\psi}$ is a smooth function supported in $1/2 \le |\xi| \le 2$ that satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$$

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for all $\xi \neq 0$. Here $\|g\|_{W^s} = \|(I-\Delta)^{s/2} g\|_{L^2}$, where I is the identity operator and $\Delta = \sum_{j=1}^n \partial_j^2$ is the Laplacian on $\mathbb{R}^n$.

Calderón and Torchinsky [1] showed that the Fourier multiplier operator in (1.1) admits a bounded extension from the Hardy space H^p to H^p with $0 < p \leq 1$ if

$$\sup_{t>0} \|\sigma(t\cdot)\widehat{\psi}\|_{W^s} < \infty$$

and $s > n/p - n/2$. Here the index $s = n/p - n/2$ is critical in the sense that the boundedness of T_σ on H^p does not hold if $s \leq n/p - n/2$. This was pointed out later by Miyachi and Tomita [15].

The bilinear counterpart of the Fourier multiplier theory has been rather similar in the formulation of results, but substantially more complicated in their proofs. The theory of multilinear operators, and in particular that of multilinear multiplier operators, originated in the work of Coifman and Meyer [2], [3], [13] and resurfaced in the work of Grafakos and Torres [11]. Multilinear Fourier multipliers are bounded functions σ on $\mathbb{R}^{mn} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ associated with the m-linear Fourier multiplier operator in the following way:

$$T_\sigma(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)} \sigma(\xi_1, \ldots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m)\, d\vec{\xi}, \tag{1.4}$$

where f_j are in the Schwartz space of $\mathbb{R}^n$ and $d\vec{\xi} = d\xi_1 \cdots d\xi_m$.

Tomita [17] obtained $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ boundedness ($1 < p_1, \ldots, p_m$, $p < \infty$) for multilinear multiplier operators under a condition analogous to (1.3). Grafakos and Si [10] extended Tomita's results to the case $p \leq 1$ by using L^r-based Sobolev norms for σ with $1 < r \leq 2$. Fujita and Tomita [4] provided weighted extensions of these results, but also noticed that the Sobolev space W^s in (1.3) can be replaced by a product-type Sobolev space $W^{(s_1,\ldots,s_m)}$ when $p > 2$. Grafakos, Miyachi and Tomita [8] extended the range of p in [4] to $p > 1$ and obtained boundedness even in the endpoint case where all but one indices p_j are equal to infinity. Miyachi and Tomita [15] provided extensions of the Calderón and Torchinsky results [1] for Hardy spaces in the bilinear case; note that in [15] it was pointed out that the conditions on the indices are sharp, even in the linear case, i.e., in the Calderón and Torchinsky theorem.

Following this stream of work, we are interested in finding conditions analogous to those in [15] in the multilinear setting, i.e., when $m \geq 3$. Our work is inspired by that of Calderón and Torchinsky [1], Grafakos and Kalton [7], and certainly of Miyachi and Tomita [15]. As in [15], we find necessary and sufficient conditions, which coincide with those in [15] when

$m = 2$, that imply boundedness for multilinear multiplier operators on products of Hardy spaces. One important aspect of this work is an appropriate regularization of the multilinear multiplier operator which allows the interchange of its action with infinite sums of H^{p_j}-atoms (see Section 3). In this article we restrict attention to the case where the domain is a product of Hardy spaces. In a subsequent article we study the case where the domain is a mix of Lebesgue and Hardy spaces.

We introduce the Sobolev spaces that will be used throughout this paper. First, for $x \in \mathbb{R}^n$ we set $\langle x\rangle = \sqrt{1+|x|^2}$. For $s_1, \dots, s_m > 0$, we denote by $W^{(s_1,\dots,s_m)}$ the Sobolev space (of product type) consisting all functions f on $\mathbb{R}^{mn}$ such that

$$\|f\|_{W^{(s_1,\dots,s_m)}} := \Big(\int\limits_{\mathbb{R}^{mn}} |\widehat{f}(y_1,\dots,y_m)\langle y_1\rangle^{s_1}\cdots\langle y_m\rangle^{s_m}|^2\, dy_1\cdots dy_m\Big)^{1/2} < \infty.$$

Notice that $W^{(s_1,\dots,s_m)}$ is a subspace of L^2.

Let ψ be a smooth function on $\mathbb{R}^{mn}$ whose Fourier transform $\widehat{\psi}$ is supported in $1/2 \le |\xi| \le 2$ and satisfies

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \xi \neq 0.$$

For $0 < p < \infty$ we denote by H^p the Lebesgue space L^p if $p > 1$ and the Hardy space H^p if $p \le 1$. The following is the main result of this paper.

THEOREM 1.1. *Let $n/2 < s_1, \dots, s_m < \infty$, $0 < p_1, \dots, p_m \le 1$, $0 < p \le 1$ be such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

and that

$$\sum_{k\in J}\left(\frac{s_k}{n} - \frac{1}{p_k}\right) > -\frac{1}{2} \tag{1.5}$$

for every subset $J \subset \{1,\dots,m\}$. If a function σ defined on $\mathbb{R}^{mn}$ satisfies

$$A := \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} < \infty, \tag{1.6}$$

then T_σ is bounded from $H^{p_1} \times \cdots \times H^{p_m}$ to L^p with constant at most a multiple of A. Moreover, the set of $2^m - 1$ conditions (1.5) is optimal.

REMARK 1.2. Conditions (1.5) imply that $s_k > n/2$ whenever $0 < p_k \le 1$ for all $1 \le k \le m$. Moreover, the condition in (1.6) is sufficient to guarantee that σ lies in $L^\infty(\mathbb{R}^{mn})$. Indeed, suppose that σ is a function on $\mathbb{R}^{mn}$ that satisfies (1.6). It is easy to see that $\widehat{\psi}\big(\frac{1}{2}x\big) + \widehat{\psi}(x) + \widehat{\psi}(2x) = 1$ for all $1 \le x \le 2$. Now we want to verify that $|\sigma(2^{j_0}x)|$ is uniformly bounded in $j_0 \in \mathbb{Z}$ for a.e. $1 \le |x| \le 2$. Applying the Cauchy–Schwarz inequality and

using the conditions $s_k > n/2$, we write

$$\begin{aligned}
|\sigma(2^{j_0}x)| &= \Big|\sum_{|l|\le 1}\sigma(2^{j_0}x)\widehat{\psi}(2^l x)\Big| \le \sum_{|l|\le 1}\Big|\int_{\mathbb{R}^{mn}}(\sigma(2^{j_0-l}\cdot)\widehat{\psi})^\vee(\xi)e^{2^{l+1}\pi i x\xi}\,d\xi\Big| \\
&\le \sum_{|l|\le 1}\int_{\mathbb{R}^{mn}}\prod_{k=1}^m(1+|\xi_k|^2)^{-s_k/2} \\
&\quad\times\Big|\prod_{k=1}^m(1+|\xi_k|^2)^{s_k/2}(\sigma(2^{j_0-l}\cdot)\widehat{\psi})^\vee(\xi_1,\dots,\xi_m)\Big|\,d\xi_1\cdots d\xi_m \\
&\le \sum_{|l|\le 1}C(s_1,\dots,s_m,n)\|\sigma_{j_0-l}\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \\
&\le 3C(s_1,\dots,s_m,n)\sup_{j\in\mathbb{Z}}\|\sigma_j\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}
\end{aligned}$$

for almost all x satisfying $1\le|x|\le 2$. Here we set $\sigma_j(\vec{\xi}) = \sigma(2^j\vec{\xi})$. Thus

$$\|\sigma\|_{L^\infty(\mathbb{R}^{mn})} \le 3C(s_1,\dots,s_m,n)\sup_{j\in\mathbb{Z}}\|\sigma_j\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} < \infty.$$

The structure of this paper is as follows: Section 2 contains preliminaries and known results. In Section 3, we regularize the multiplier to be able to work with a nicer operator and thus facilitate the passage of infinite sums in and out of the operator in the proof of the main result given in Section 4. In Section 5, we construct examples to justify the minimality of conditions (1.5) claimed in the main theorem. Section 6 presents some results about the boundedness of our operator in the endpoint cases where we need the coordinate-type Hörmander integral conditions. The last section contains the detailed proofs of some technical lemmas used through the paper.

We use the notation $A \lesssim B$ to indicate that $A \le CB$, where the constant C is independent of any essential parameters, and $A \approx B$ if simultaneously $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries and known results. Now fix $0 < p < \infty$ and a Schwartz function Φ with $\widehat{\Phi}(0) \neq 0$. Then the Hardy space H^p contains all tempered distributions f on $\mathbb{R}^n$ such that

$$\|f\|_{H^p} := \Big\|\sup_{0<t<\infty}|\Phi_t * f|\Big\|_{L^p} < \infty.$$

It is well known that the definition of the Hardy space does not depend on the choice of Φ. Note that $H^p = L^p$ for all $p > 1$. When $0 < p \le 1$, one of nice features of Hardy spaces is the *atomic decomposition*. More precisely, any function $f \in H^p$ $(0 < p \le 1)$ can be decomposed as $f = \sum_j \lambda_j a_j$, where a_j's are L^∞-atoms for H^p supported in cubes Q_j such that $\|a_j\|_{L^\infty} \le |Q_j|^{-1/p}$

and $\int x^\gamma a_j(x)\,dx = 0$ for all $|\gamma| \le \lfloor n(1/p-1)\rfloor + 1$, and the coefficients λ_j satisfy $\sum_j |\lambda_j|^p \le 2^p \|f\|_{H^p}^p$.

The following two lemmas are essentially contained in [15] modulo some minor modifications.

LEMMA 2.1 ([15]). *Let k, l be positive integers, $0 < s_1, \dots, s_{k+l} < \infty$, and let $1 < \rho < \infty$. Assume that σ is a bounded function defined on $\mathbb{R}^{kn} \times \mathbb{R}^{ln}$, supported in $\{(x,y) \in \mathbb{R}^{kn} \times \mathbb{R}^{ln} : |x|^2 + |y|^2 \le 4\}$, where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_l)$ with $x_1, \dots, x_k, y_1, \dots, y_l \in \mathbb{R}^n$, and set $K = \sigma^\vee$, the inverse Fourier transform of σ. Then there exists a constant $C > 0$ such that*

$$\|\langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} K(x,y)\|_{L^\infty(\mathbb{R}^{ln},dy)} \le C \|\langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} K(x,y)\|_{L^\rho(\mathbb{R}^{ln},dy)}$$

for all $x \in \mathbb{R}^{kn}$.

Proof. Take φ a Schwartz function on $\mathbb{R}^{ln}$ such that $\widehat{\varphi}(y) = 1$ for all $y \in \mathbb{R}^{ln}$, $|y| \le 2$. Then $\sigma(x,y) = \sigma(x,y)\widehat{\varphi}(y)$. Using the inverse Fourier transform we have

$$\begin{aligned} K(x,y) = (K * (\delta_0 \otimes \varphi))(x,y) &= \int_{\mathbb{R}^{kn}\times\mathbb{R}^{ln}} K(x-u, y-v)\delta_0(u)\varphi(v)\,du\,dv \\ &= \int_{\mathbb{R}^{ln}} K(x, y-v)\varphi(v)\,dv, \end{aligned}$$

where δ_0 is the Dirac distribution. Therefore,

$$\begin{aligned} &\langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} |K(x,y)| \\ &\quad = \langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} \Big| \int_{\mathbb{R}^{ln}} K(x_1, \dots, x_k, y_1 - v_1, \dots, y_l - v_l)\varphi(v)\,dv \Big| \\ &\quad \lesssim \int_{\mathbb{R}^{ln}} \Big(\prod_{j=1}^{l} \langle y_j - v_j\rangle^{s_j} \Big) \\ &\qquad \times |K(x_1, \dots, x_k, y_1 - v_1, \dots, y_l - v_l)| \langle v_1\rangle^{s_1} \cdots \langle v_l\rangle^{s_l} |\varphi(v)|\,dv \\ &\quad \le C_1 \|\langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} K(x,y)\|_{L^\rho(\mathbb{R}^{ln},dy)} \|\langle v_1\rangle^{s_1} \cdots \langle v_l\rangle^{s_l} |\varphi(v)|\|_{L^{\rho'}(\mathbb{R}^{ln},dv)} \\ &\quad \le C_2 \|\langle y_1\rangle^{s_1} \cdots \langle y_l\rangle^{s_l} K(x,y)\|_{L^\rho(\mathbb{R}^{ln},dy)}, \end{aligned}$$

where we used Hölder's inequality in the second to last line. ∎

LEMMA 2.2 ([15]). *Let $s_k > n/2$ for $1 \le k \le m$, and let $\widehat{\zeta}$ be a smooth function which is supported in an annulus centered at zero. Suppose that Φ is a smooth function away from zero that satisfies the estimates*

$$|\partial_\xi^\alpha \Phi(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

for all $\xi \in \mathbb{R}^{mn}$, $x \ne 0$ and all multi-indices α. Then there exists a constant C such that

$$\sup_{j\in\mathbb{Z}} \|\sigma(2^j(\cdot))\Phi(2^j(\cdot))\widehat{\zeta}\,\|_{W^{(s_1,\dots,s_m)}} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j(\cdot))\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}}.$$

Adapting the Calderón and Torchinsky interpolation techniques in the multilinear setting (for details on this we refer to [8, p. 318]) allows us to interpolate between two endpoint estimates for multilinear multiplier operators from a product of some Hardy spaces to Lebesgue spaces.

THEOREM 2.3 ([8]). *Let $0 < p_1, p_2, p_{1k}, p_{2k} \le \infty$ and $n/2 < s_{1k}, s_{2k} < \infty$ and $1 \le k \le m$. For $0 < \theta < 1$, set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/p_k = (1-\theta)/p_{1k} + \theta/p_{2k}$, and $s_k = (1-\theta)s_{1k} + \theta s_{2k}$. Assume that the multilinear operator T_σ defined in* (1.4) *satisfies the estimates*

$$\|T_\sigma\|_{H^{p_{l1}}\times\cdots\times H^{p_{lm}}\to L^{p_l}} \le C_l \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_{l1},\dots,s_{lm})}} \qquad (l=1,2).$$

Then

$$\|T_\sigma\|_{H^{p_1}\times\cdots\times H^{p_m}\to L^{p}} \le C \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}.$$

The following result is due to Fujita and Tomita [4] for $2 < p < \infty$, while the extension to $p > 1$ and the endpoint case where all but one indices are equal to infinity is due to Grafakos, Miyachi and Tomita [8].

THEOREM 2.4 ([4], [8]). *Let $1 < p_1, \dots, p_m \le \infty, 1 < p < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. If σ satisfies* (1.6)*, then the multilinear multiplier operator T_σ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p with constant at most a multiple of A.*

Finally, we will need the following lemma from [7].

LEMMA 2.5 ([7, Lemma 2.1]). *Let $0 < p \le 1$ and let $(f_Q)_{Q\in\mathcal{J}}$ be a family of nonnegative integrable functions with $\operatorname{supp}(f_Q) \subset Q$ for all $Q \in \mathcal{J}$, where $\mathcal{J}$ is a family of finite or countable cubes in $\mathbb{R}^n$. Then*

$$\Big\|\sum_{Q\in\mathcal{J}} f_Q\Big\|_{L^p} \lesssim \Big\|\sum_{Q\in\mathcal{J}} \Big(\frac{1}{|Q|}\int_Q f_Q(x)\,dx\Big)\chi_{Q^*}\Big\|_{L^p},$$

with the implicit constant depending only on p. Here Q^* is a dimensional dilate of the cube Q.

3. Regularizing the multiplier. In this section, we show that the operator defined in (1.1) with enough smoothness of the multiplier can be approximated by a family of very nice operators.

THEOREM 3.1. *Let σ be a function on $\mathbb{R}^{mn}$ satisfying* (1.6) *and let $s_k > n/2$ for $1 \le k \le m$. Then there exists a family $(\sigma^\epsilon)_{0<\epsilon<1/2}$ of functions such that $K^\epsilon := (\sigma^\epsilon)^\vee$ is smooth and compactly supported for every $0 < \epsilon < 1/2$; also*

$$\sup_{0<\epsilon<1/2}\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \tag{3.1}$$

and

$$\lim_{\epsilon\to 0}\|T_\epsilon(f_1,\dots,f_m)-T_\sigma(f_1,\dots,f_m)\|_{L^2}=0 \tag{3.2}$$

for all functions $f_k\in L^{2m}$, $1\le k\le m$, *where* T_ϵ *are multilinear singular integral operators of convolution type associated to* K^ϵ.

The following lemma, whose proof will be given in the last section, is the first step in constructing a family of functions σ^ϵ as stated in Theorem 3.1.

LEMMA 3.2. *Let* φ *be a Schwartz function. Suppose* σ *is a function on* $\mathbb{R}^{mn}$ *satisfying* (1.6) *for* $s_k>n/2$. *Then*

$$\sup_{\epsilon>0}\sup_{j\in\mathbb{Z}}\big\|[(\varphi_\epsilon*\sigma)(2^j\cdot)]\widehat{\psi}\big\|_{W^{(s_1,\dots,s_m)}}\lesssim\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}},$$

with $\varphi_\epsilon(x_1,\dots,x_m)=\epsilon^{-mn}\varphi(\epsilon^{-1}x_1,\dots,\epsilon^{-1}x_m)$ *for all* $x_k\in\mathbb{R}^n$, $1\le k\le m$.

Proof of Theorem 3.1. Fix $0<\epsilon<1/2$. Choose a smooth function φ such that $\widehat{\varphi}$ is supported in the unit ball and $\widehat{\varphi}(0)=1$. Set $\sigma^\epsilon=\varphi_\epsilon*(\sigma\phi^\epsilon)$, where $\phi^\epsilon=\theta(\epsilon^{-1}\cdot)-\theta(\epsilon\cdot)$, and θ is a smooth function satisfying $\theta(x)=0$ for all $|x|\le 1$ and $\theta(x)=1$ for all $|x|\ge 2$. We note that these functions are suitable regularized versions of the multiplier in Theorem 3.1. Indeed, let $K^\epsilon=(\sigma^\epsilon)^\vee=(\sigma\phi^\epsilon)^\vee\widehat{\varphi}(\epsilon(\cdot))$; then K^ϵ are smooth functions with compact support for all $0<\epsilon<1/2$.

Using the fact that

$$|\partial^\alpha\phi^\epsilon(\xi)|\le C_{\alpha,\theta}|\xi|^{-\alpha},\qquad \xi\neq 0,\ 0<\epsilon<1/2,$$

Lemma 3.2 applied to the function $\sigma\phi^\epsilon$ combined with Lemma 2.2 gives

$$\begin{aligned}\sup_{0<\epsilon<1/2}\sup_{j\in\mathbb{Z}}\|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}&\lesssim\sup_{0<\epsilon<1/2}\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\phi^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}\\&\lesssim\sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}},\end{aligned}$$

which yields (3.1). Thus, we are left with establishing (3.2). For $\epsilon>0$, now recall that

$$\begin{aligned}T_\epsilon(f_1,\dots,f_m)(x)&=\int K^\epsilon(x-y_1,\dots,x-y_m)f_1(y_1)\cdots f_m(y_m)\,dy\\&=\int\sigma^\epsilon(\xi_1,\dots,\xi_m)\widehat{f}_1(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi ix(\xi_1+\cdots+\xi_m)}\,d\xi.\end{aligned}$$

Invoking estimate (3.1) with Theorem 2.4, we can see that T_σ and T_ϵ are uniformly bounded from $L^{2m}\times\cdots\times L^{2m}$ to L^2 for all $0<\epsilon<1/2$. By density, it suffices to verify (3.2) for all functions in the Schwartz class.

Now fix Schwartz functions f_k for $1 \le k \le m$. The Fourier transform of $T_\sigma(f_1, \dots, f_m)$ can be written as

$$\int\limits_{\mathbb{R}^{n(m-1)}} \sigma\Big(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l\Big) \times \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}\Big(\xi - \sum_{l=1}^{m-1} \xi_l\Big)\, d\xi_1 \cdots d\xi_{m-1}.$$

Similarly, the Fourier transform of $T_\epsilon(f_1, \dots, f_m)$ is

$$\int\limits_{\mathbb{R}^{n(m-1)}} \sigma^\epsilon\Big(\xi_1, \dots, \xi_{m-1}, \xi - \sum_{l=1}^{m-1} \xi_l\Big) \times \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{f_m}\Big(\xi - \sum_{l=1}^{m-1} \xi_l\Big)\, d\xi_1 \cdots d\xi_{m-1}.$$

We now claim that σ^ϵ converges pointwise to σ. Taking this claim for granted, we have

$$(T_\epsilon(f_1, \dots, f_m))\widehat{\ }(\xi) \to (T_\sigma(f_1, \dots, f_m))\widehat{\ }(\xi) \quad \text{as } \epsilon \to 0$$

for a.e. $\xi \in \mathbb{R}^n$. Notice that

$$\|T_\epsilon(f_1, \dots, f_m) - T_\sigma(f_1, \dots, f_m)\|_{L^2} = \|(T_\epsilon(f_1, \dots, f_m))\widehat{\ } - (T_\sigma(f_1, \dots, f_m))\widehat{\ }\|_{L^2}.$$

Since $\|\sigma^\epsilon\|_{L^\infty} \lesssim \|\sigma\|_{L^\infty} < \infty$ for all $\epsilon > 0$, Lebesgue's dominated convergence theorem implies that

$$(T_\epsilon(f_1, \dots, f_m))\widehat{\ } \to (T_\sigma(f_1, \dots, f_m))\widehat{\ } \quad \text{as } \epsilon \to 0$$

in L^2, and this establishes (3.2).

It remains to prove the above claim about pointwise convergence of σ^ϵ as $\epsilon \to 0$. So fix $j_0 \in \mathbb{Z}$; we want to show that $\sigma^\epsilon(x) \to \sigma(x)$ for a.e. $2^{j_0} \le |x| \le 2^{j_0+1}$. Indeed, let $0 < \epsilon < \min\{2^{2j_0-2}, 2^{-|j_0|-2}\}$ be a small positive number. Then

$$\begin{aligned} |\sigma^\epsilon(x) - \sigma(x)| \le & \int\limits_{|y| \le \sqrt{\epsilon}} |\varphi_\epsilon(y)|\, |\sigma(x-y)| \sup_{2^{j_0} \le |x| \le 2^{j_0+1}} |\phi^\epsilon(x-y) - 1|\, dy \\ & + \int\limits_{|y| \le \sqrt{\epsilon}} |\varphi_\epsilon(y)|\, |\sigma(x-y) - \sigma(x)|\, dy \\ & + \int\limits_{|y| > \sqrt{\epsilon}} |\varphi_\epsilon(y)|\, |\sigma(x-y)\phi^\epsilon(x-y) - \sigma(x)|\, dy. \end{aligned}$$

The first integral vanishes since $\phi^\epsilon(x) = 1$ for all $2\epsilon \le |x| \le 1/\epsilon$. To estimate the second integral, we denote

$$\widehat{\Psi}(x) = \sum_{|l|\le 2} \widehat{\psi}(2^{-l}x).$$

Then $\widehat{\Psi}(x) = 1$ for all $1/4 \le |x| \le 4$. Therefore

$$\widehat{\Psi}(2^{-j_0}(x-y)) = \widehat{\Psi}(2^{-j_0}x) = 1$$

for all $2^{j_0} \le |x| \le 2^{j_0+1}$ and $|y| \le 2^{j_0-1}$. Now recall $\sigma_j(x) = \sigma(2^j x)$; we estimate

$$\begin{aligned}
&\int_{|y|\le\sqrt{\epsilon}} |\varphi_\epsilon(y)|\,|\sigma(x-y)-\sigma(x)|\,dy \\
&\quad= \int_{|y|\le\sqrt{\epsilon}} |\varphi_\epsilon(y)|\,|\sigma(x-y)\widehat{\Psi}(2^{-j_0}(x-y)) - \sigma(x)\widehat{\Psi}(2^{-j_0}x)|\,dy \\
&\quad\le \|\varphi\|_{L^1} \sup_{|y|\le\sqrt{\epsilon}} \|\sigma(\cdot-y)\widehat{\Psi}(2^{-j_0}(\cdot-y)) - \sigma\widehat{\Psi}(2^{-j_0}\cdot)\|_{L^\infty} \\
&\quad\le \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y|\le\sqrt{\epsilon}} \|(\sigma_j\widehat{\psi})(\cdot-2^{-j}y) - (\sigma_j\widehat{\psi})\|_{L^\infty} \\
&\quad= \|\varphi\|_{L^1} \sum_{j=j_0-2}^{j_0+2} \sup_{|y|\le 2^{-j}\sqrt{\epsilon}} \|(\sigma_j\widehat{\psi})(\cdot-y) - (\sigma_j\widehat{\psi})\|_{L^\infty}.
\end{aligned}$$

We would like to show

$$\lim_{\epsilon\to 0} \sup_{|y|\le 2^{-j}\sqrt{\epsilon}} \|(\sigma_j\widehat{\psi})(\cdot-y) - (\sigma_j\widehat{\psi})\|_{L^\infty} = 0.$$

The preceding limit is equal to 0 because $\sigma_j\widehat{\psi} \in W^{(s_1,\dots,s_m)}$ for $s_k > n/2$, $1 \le k \le m$. The last term of the sum at the bottom of page 8 is majorized by

$$C\|\sigma\|_{L^\infty} \int_{|y|\ge 1/\sqrt{\epsilon}} |\varphi(y)|\,dy,$$

which tends to 0 as $\epsilon \to 0$.

Thus $\sigma^\epsilon(x) \to \sigma(x)$ as $\epsilon \to 0$ for a.e. $2^{j_0} \le |x| \le 2^{j_0+1}$. Hence, σ^ϵ converges to σ pointwise on $\mathbb{R}^{mn}$. Also $\|\sigma^\epsilon\|_{L^\infty(\mathbb{R}^{mn})} \lesssim \|\sigma\|_{L^\infty(\mathbb{R}^{mn})}$ uniformly for all $\epsilon > 0$. The proof of Theorem 3.1 is complete. ∎

For a Schwartz function K, denote the multilinear singular integral operator of convolution type associated with the kernel K by

$$T^K(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} K(x-y_1,\dots,x-y_m) f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m.$$

PROPOSITION 3.3. *Let K be a smooth function on $\mathbb{R}^{mn}$ with compact support. Then*

$$\|T^K\|_{H^{p_1}\times\cdots\times H^{p_m}\to L^p} \le C_K < \infty$$

for all $0 < p_1,\dots,p_m,p < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$.

Proof. The boundedness of T^K can be deduced from [6, Lemma 4.2], which provides the estimate (for some sufficiently large integer N)

$$|T^K(f_1,\dots,f_m)(x)| \lesssim \prod_{k=1}^m \mathcal{M}_N(f_k)(x) \tag{3.3}$$

for all $f_k \in L^2 \cap H^{p_k}$, where

$$\mathcal{M}_N(f)(x) = \sup_{\varphi\in\mathfrak{F}_N}\sup_{t>0}\sup_{y\in B(x,t)} |(\varphi_t * f)(y)|$$

is the grand maximal function with respect to N, and

$$\mathfrak{F}_N := \Big\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|\,dx \le 1\Big\}.$$

Taking the L^p quasinorm, applying Holder's inequality to (3.3), and using the quasinorm equivalence of some maximal functions [5, Theorem 6.4.4] yields

$$\|T^K(f_1,\dots,f_m)\|_{L^p} \lesssim \prod_{k=1}^m \|\mathcal{M}_N(f_k)\|_{L^{p_k}} \le C_K \prod_{k=1}^m \|f_k\|_{H^{p_k}}. \ \blacksquare$$

Working with smooth kernels K with compact support comes handy when dealing with infinite sums of atoms, since we are able to freely interchange the action of T^K with infinite sums of atoms. Precisely, a consequence of the boundedness of T^K, given in Proposition 3.3, is the following result.

PROPOSITION 3.4. *Let $0 < p_1,\dots,p_m \le 1$ and let K be a smooth function with compact support. Then for every $f_k \in H^{p_k}$ with atomic representation $f_k = \sum_{j_k} \lambda_{k,j_k} a_{k,j_k}$, where a_{k,j_k} are L^∞-atoms for H^{p_k} and $\sum_{j_k} |\lambda_{k,j_k}|^{p_k} \le 2^{p_k}\|f_k\|_{H^{p_k}}^{p_k}$ for $1 \le k \le m$, we have*

$$T^K(f_1,\dots,f_m)(x) = \sum_{j_1=1}^\infty \cdots \sum_{j_m=1}^\infty \lambda_{1,j_1}\cdots\lambda_{m,j_m} T^K(a_{1,j_1},\dots,a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$.

Proof. Let $0 < p < \infty$ be such that $1/p = 1/p_1 + \cdots + 1/p_m$. For any positive integers $N_1,\dots,N_m$, Proposition 3.3 gives the estimate

$$\Big\|T^K(f_1,\dots,f_m) - \sum_{j_1=1}^{N_1}\cdots\sum_{j_m=1}^{N_m} \lambda_{1,j_1}\cdots\lambda_{m,j_m} T^K(a_{1,j_1},\dots,a_{m,j_m})\Big\|_{L^p}$$
$$\le C_K \sum_{k=1}^m \Big\|f_k - \sum_{j_k=1}^{N_k} \lambda_{k,j_k} a_{k,j_k}\Big\|_{H^{p_k}} \prod_{l\ne k} \|f_l\|_{H^{p_l}}.$$

Now passing to the limit, we obtain the asserted equality. ∎

4. The proof of the main result. We first consider the case where σ is smooth and its Fourier transform is compactly supported; then, by regularization, we prove the result for any σ.

Proof of Theorem 1.1. By regularization, we may assume that the inverse Fourier transform of σ is smooth and compactly supported. If this case is established, then Theorem 3.1 yields the existence of a family $(T_\epsilon)_{0<\epsilon<1/2}$ of multilinear multiplier operators associated with a family of multipliers $(\sigma^\epsilon)_{0<\epsilon<1/2}$ such that $K^\epsilon = (\sigma^\epsilon)^\vee$ are smooth functions with compact supports for all $0 < \epsilon < 1/2$, and (3.1), (3.2) hold. Fix $f_k \in H^{p_k} \cap L^{2m}$ $(1 \le k \le m)$. The L^2 convergence in (3.2) implies that we can find a sequence $(\epsilon_j)_j$ of positive numbers convergent to 0 such that

$$\lim_{j\to\infty} T_{\epsilon_j}(f_1,\dots,f_m)(x) = T_\sigma(f_1,\dots,f_m)(x)$$

for a.e. $x \in \mathbb{R}^n$. Fatou's lemma together with (3.1) gives

$$\begin{aligned}\|T_\sigma(f_1,\dots,f_m)\|_{L^p} &\le \liminf_{j\to\infty} \|T_{\epsilon_j}(f_1,\dots,f_m)\|_{L^p} \lesssim \sup_{0<\epsilon<1/2} \|T_\epsilon(f_1,\dots,f_m)\|_{L^p}\\ &\lesssim \sup_{0<\epsilon<1/2}\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \|f_1\|_{H^{p_1}}\cdots\|f_m\|_{H^{p_m}}\\ &\lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}} \|f_1\|_{H^{p_1}}\cdots\|f_m\|_{H^{p_m}},\end{aligned}$$

thus establishing the claimed estimate for a general multiplier σ.

In view of this deduction, we suppose $\sigma^\vee$ is smooth and compactly supported. The aim is to show that

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^p} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}} \|f_1\|_{H^{p_1}}\cdots\|f_m\|_{H^{p_m}}. \tag{4.1}$$

Fix $f_k \in H^{p_k}$. Using atomic representations for H^{p_k} functions, write

$$f_k = \sum_{j_k\in\mathbb{Z}} \lambda_{k,j_k} a_{k,j_k} \qquad (1 \le k \le m),$$

where a_{k,j_k} are L^∞ atoms for H^{p_k} satisfying

$$\operatorname{supp}(a_{k,j_k}) \subset Q_{k,j_k}, \qquad \|a_{k,j_k}\|_{L^\infty} \le |Q_{k,j_k}|^{-1/p_k}, \qquad \int_{Q_{k,j_k}} x^\alpha a_{k,j_k}(x)\,dx = 0$$

for all $|\alpha|$ large enough, and $\sum_{j_k} |\lambda_{k,j_k}|^{p_k} \le 2^{p_k}\|f_k\|_{H^{p_k}}^{p_k}$.

For a cube Q, denote by Q^* the dilation with factor $2\sqrt{n}$. Since $K = \sigma^\vee$ is smooth and compactly supported, Proposition 3.4 yields

$$T_\sigma(f_1,\dots,f_m)(x) = \sum_{j_1}\dots\sum_{j_m} \lambda_{1,j_1}\cdots\lambda_{m,j_m} T_\sigma(a_{1,j_1},\dots,a_{m,j_m})(x)$$

for a.e. $x \in \mathbb{R}^n$. Now we can split $T_\sigma(f_1,\dots,f_m)$ into two parts and estimate

$$|T_\sigma(f_1,\dots,f_m)(x)| \le G_1(x) + G_2(x),$$

where

$$G_1(x)=\sum_{j_1}\cdots\sum_{j_m}|\lambda_{1,j_1}|\cdots|\lambda_{m,j_m}|\,|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})|\chi_{Q^*_{1,j_1}\cap\cdots\cap Q^*_{m,j_m}}(x),$$

$$G_2(x)=\sum_{j_1}\cdots\sum_{j_m}|\lambda_{1,j_1}|\cdots|\lambda_{m,j_m}|\,|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})|\chi_{(Q^*_{1,j_1}\cap\cdots\cap Q^*_{m,j_m})^c}(x).$$

First we estimate the L^p norm of G_1, in which we repeat the arguments in [7] for the sake of completeness. Without loss of generality, suppose that $Q^*_{1,j_1}\cap\cdots\cap Q^*_{m,j_m}\neq\emptyset$ and let Q_{1,j_1} have the smallest length among $Q_{1,j_1},\dots,Q_{m,j_m}$. Since $Q^*_{k,j_k}, 1\le k\le m$, have nonempty intersection, we can pick a cube $R_{j_1,\dots,j_m}$ such that

$$Q^*_{1,j_1}\cap\cdots\cap Q^*_{m,j_m}\subset R_{j_1,\dots,j_m}\subset R^*_{j_1,\dots,j_m}\subset Q^\sharp_{1,j_1}\cap\cdots\cap Q^\sharp_{m,j_m}$$

and $|Q_{1,j_1}|\lesssim|R_{j_1,\dots,j_m}|$, where the implicit constant depends only on n, and $Q^\sharp_{k,j_k}$ denotes for a suitable dilation of Q_{k,j_k}. For $s_k>n/2$, it was showed in [8] that

$$\|T_\sigma\|_{L^2\times L^\infty\times\cdots\times L^\infty\to L^2}\lesssim A.$$

Therefore, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\int\limits_{R_{j_1,\dots,j_m}}|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})(x)|\,dx\\
&\quad\le\|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})\|_{L^2}|R_{j_1,\dots,j_m}|^{1/2}\\
&\quad\lesssim A|R_{j_1,\dots,j_m}|^{1/2}\|a_{1,j_1}\|_{L^2}\prod_{k=2}^m\|a_{k,j_k}\|_{L^\infty}\\
&\quad\lesssim A|R_{j_1,\dots,j_m}|^{1/2}|Q_{1,j_1}|^{1/2}\prod_{k=1}^m|Q_{k,j_k}|^{-1/p_k}\lesssim A|R_{j_1,\dots,j_m}|\prod_{k=1}^m|Q_{k,j_k}|^{-1/p_k}.
\end{aligned}$$

The last inequality implies that

$$\frac{1}{|R_{j_1,\dots,j_m}|}\int\limits_{R_{j_1,\dots,j_m}}|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})(x)|\,dx\lesssim A\prod_{k=1}^m|Q_{k,j_k}|^{-1/p_k}.$$

Now the trivial estimate

$$G_1(x)\le\sum_{j_1}\cdots\sum_{j_m}|\lambda_{1,j_1}|\cdots|\lambda_{m,j_m}|\,|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})|\chi_{R_{j_1,\dots,j_m}}(x)$$

combined with Lemma 2.5 yields

$$\begin{aligned}
\|G_1\|_{L^p} &\le \Big\| \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}|\, |T_\sigma(a_{1,j_1},\dots,a_{m,j_m})| \chi_{R_{j_1,\dots,j_m}} \Big\|_{L^p} \\
&\lesssim A \Big\| \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| \Big(\prod_{k=1}^m |Q_{k,j_k}|^{-1/p_k} \Big) \chi_{R^*_{j_1,\dots,j_m}} \Big\|_{L^p} \\
&\le A \Big\| \sum_{j_1} \cdots \sum_{j_m} |\lambda_{1,j_1}| \cdots |\lambda_{m,j_m}| \prod_{k=1}^m (|Q_{k,j_k}|^{-1/p_k} \chi_{Q^\sharp_{k,j_k}}) \Big\|_{L^p} \\
&= A \Big\| \prod_{k=1}^m \Big(\sum_{j_k} |\lambda_{k,j_k}|\, |Q_{k,j_k}|^{-1/p_k} \chi_{Q^\sharp_{k,j_k}} \Big) \Big\|_{L^p} \\
&\le A \prod_{k=1}^m \Big\| \sum_{j_k} |\lambda_{k,j_k}|\, |Q_{k,j_k}|^{-1/p_k} \chi_{Q^\sharp_{k,j_k}} \Big\|_{L^{p_k}} \lesssim A \prod_{k=1}^m \|f_k\|_{H^{p_k}}.
\end{aligned}$$

Thus

$$\|G_1\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \tag{4.2}$$

For the harder part, $G_2(x)$, we first restrict $x \in (\bigcap_{k\notin J} Q^*_{k,j_k}) \setminus \bigcup_{k\in J} Q^*_{k,j_k}$ for some nonempty subset $J \subset \{1,\dots,m\}$. To continue, we need the following lemma whose proof will be given in the last section.

LEMMA 4.1 (The key lemma). *Let $n/2 < s_1,\dots,s_m < \infty$, $0 < p_1,\dots,p_m$, $p \le 1$ and let σ be a function satisfying* (1.5) *and* (1.6). *Suppose a_k are atoms supported in a cube Q_k $(k = 1,\dots,m)$ such that*

$$\|a_k\|_{L^\infty} \le |Q_k|^{-1/p_k}, \qquad \int_{Q_k} x^\alpha a_k(x)\,dx = 0$$

for all $|\alpha| \le N_k$ with $N_k = [n(1/p_k - 1)] + 1$. Fix a nonempty subset $J_0 \subset \{1,\dots,m\}$. Then there exist positive functions $b_1,\dots,b_m$ such that

$$|T_\sigma(a_1,\dots,a_m)(x)| \lesssim A b_1(x) \cdots b_m(x) \tag{4.3}$$

for all $x \in (\bigcap_{k\notin J_0} Q^*_k) \setminus \bigcup_{k\in J_0} Q^*_k$, and $\|b_k\|_{L^{p_k}} \lesssim 1$ for all $1 \le k \le m$.

Lemma 4.1 guarantees the existence of positive functions $b^J_{1,j_1},\dots,b^J_{m,j_m}$ depending on $Q_{1,j_1},\dots,Q_{m,j_m}$ respectively, such that

$$|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})| \lesssim A b^J_{1,j_1} \cdots b^J_{m,j_m} \tag{4.4}$$

for all $x \in (\bigcap_{k\notin J} Q^*_{k,j_k}) \setminus \bigcup_{k\in J} Q^*_{k,j_k}$ and $\|b^J_{k,j_k}\|_{L^{p_k}} \lesssim 1$. Now set

$$b_{k,j_k} = \sum_{\emptyset \ne J \subset \{1,\dots,m\}} b^J_{k,j_k}.$$

Then

$$|T_\sigma(a_{1,j_1},\dots,a_{m,j_m})| \chi_{(Q^*_{1,j_1} \cap \cdots \cap Q^*_{m,j_m})^c} \lesssim A b_{1,j_1} \cdots b_{m,j_m} \tag{4.5}$$

and $\|b_{k,j_k}\|_{L^{p_k}} \lesssim 1$. Estimate (4.5) yields

$$G_2(x) \lesssim A \prod_{k=1}^{m} \Big(\sum_{j_k} |\lambda_{k,j_k}| b_{k,j_k}(x) \Big).$$

Then we apply Hölder's inequality to deduce that

$$\|G_2\|_{L^p} \lesssim A \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}. \tag{4.6}$$

Combining (4.2) and (4.6) yields (4.1). The proof of Theorem 1.1 is now complete. ∎

5. Minimality of conditions. In this section we will show that conditions (1.5) and $s_k > n/2$ are minimal in general that guarantee boundedness for multilinear multiplier operators. We fix a smooth function ψ whose Fourier transform is supported in $\{2^{-3/4} \le |\xi| \le 2^{3/4}\}$, satisfies $\widehat{\psi}(\xi) = 1$ for all $2^{-1/4} \le |\xi| \le 2^{1/4}$, and for some nonzero constant c,

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = c, \qquad \xi \neq 0.$$

THEOREM 5.1. *Let $0 < p_k \le \infty$, $0 < p < \infty$, and $s_k > 0$ for $1 \le k \le m$. Suppose that*

$$\|T_\sigma(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \prod_{k=1}^{m} \|f_k\|_{H^{p_k}}$$

for all $f_k \in H^{p_k}$ and $\sigma \in L^\infty$ such that $\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j)\widehat{\psi}\big\|_{W^{(s_1,\dots,s_m)}} < \infty$. Then

$$s_k \ge \frac{n}{2}, \qquad \forall 1 \le k \le m, \tag{5.1}$$

and

$$\sum_{k\in J} \left(\frac{s_k}{n} - \frac{1}{p_k} \right) \ge -\frac{1}{2} \tag{5.2}$$

for every nonempty $J \subset \{1, \dots, m\}$.

The following lemma is obvious by changing variables, so its proof is omitted.

LEMMA 5.2. *Let φ be a nontrivial Schwartz function and $s > 0$. Then*

$$\Big(\int |\varphi(\epsilon y)|^2 (1 + |y|^2)^s \, dy \Big)^{1/2} \approx \epsilon^{-n/2-s}$$

for all $0 < \epsilon \le 1$.

Proof of Theorem 5.1. We first show (5.1) for $1 \le k \le m$. Without loss of generality, we will show $s_1 \ge n/2$. To do this, we need to construct functions

σ^ϵ $(0 < \epsilon \ll 1)$ and $f_k \in H^{p_k}$ such that $\|f_k\|_{H^{p_k}} = 1$ for all $1 \le k \le m$, and $\|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \approx 1$, and further that

$$\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \lesssim \epsilon^{n/2-s_1}.$$

Once these functions are constructed, one observes that

$$1 \approx \|T_{\sigma^\epsilon}(f_1, \dots, f_m)\|_{L^p} \lesssim \sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \prod_{k=1}^m \|f_k\|_{H^{p_k}} \lesssim \epsilon^{n/2-s_1}$$

for all $0 < \epsilon \ll 1$. Therefore we must have $s_1 \ge n/2$.

Let φ be a nontrivial Schwartz function such that $\widehat{\varphi}$ is supported in the unit ball, and let $\phi_2 = \cdots = \phi_{m-1}$ be a Schwartz function whose Fourier transform, $\widehat{\phi_2}$, is supported in an annulus $\frac{1}{17m} \le |\xi| \le \frac{1}{13m}$, and identical to 1 on $\frac{1}{16m} \le |\xi| \le \frac{1}{14m}$. Similarly, fix a Schwartz function ϕ_m with $\widehat{\phi_m} \subset \left\{\xi \in \mathbb{R}^n : \frac{12}{13} \le |\xi| \le \frac{14}{13}\right\}$ and $\widehat{\phi_m} \equiv 1$ on the annulus $\frac{25}{26} \le |\xi| \le \frac{27}{26}$. Take $a, b \in \mathbb{R}^n$ with $|a| = \frac{1}{15m}$ and $|b| = 1$.

For $0 < \epsilon < \frac{1}{240m}$, set

$$\sigma^\epsilon(\xi_1, \dots, \xi_m) = \widehat{\varphi}\left(\frac{\xi_1 - a}{\epsilon}\right)\widehat{\phi_2}(\xi_2)\cdots\widehat{\phi_m}(\xi_m).$$

It is easy to check that $\operatorname{supp}\sigma^\epsilon \subset \{2^{-1/4} \le |\xi| \le 2^{1/4}\}$; hence, $\sigma^\epsilon(2^j\cdot)\widehat{\psi} = \sigma^\epsilon$ for $j = 0$ and $\sigma^\epsilon(2^j\cdot)\widehat{\psi} = 0$ for $j \ne 0$. This directly implies that

$$\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1,\dots,s_m)}}.$$

Taking the inverse Fourier transform of σ^ϵ gives

$$(\sigma^\epsilon)^\vee(x_1, \dots, x_m) = \epsilon^n e^{2\pi i a\cdot x_1}\varphi(\epsilon x_1)\phi_2(x_2)\cdots\phi_m(x_m).$$

Now apply Lemma 5.2 to obtain

$$\|\sigma^\epsilon\|_{W^{(s_1,\dots,s_m)}} \lesssim \epsilon^{n/2-s_1}.$$

Thus

$$\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \lesssim \epsilon^{n/2-s_1}.$$

Now choose

$$\widehat{f_k}(\xi) = \epsilon^{n/p_k-n}\widehat{\varphi}\left(\frac{\xi - a}{\epsilon}\right) \quad \text{for } 1 \le k \le m-1, \quad \widehat{f_m}(\xi) = \epsilon^{n/p_m-n}\widehat{\varphi}\left(\frac{\xi - b}{\epsilon}\right).$$

We will show that these functions are as desired.

In the following estimates, we will use the fact, whose proof can be done by using the Littlewood–Paley characterization for Hardy spaces, that if f is a function whose Fourier transform is supported in a fixed annulus centered at the origin, then $\|f\|_{H^p} \approx \|f\|_{L^p}$ for $0 < p < \infty$ (cf. [4, Remark 7.1]).

Indeed, using the above fact and checking that each $\widehat{f_k}$ is supported in an annulus centered at zero and not depending on ϵ allow us to estimate H^p-norms via L^p-norms, namely

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} = 1 \quad (1 \le k \le m).$$

Thus, we are left with showing that $\|T_\sigma(f_1,\dots,f_m)\|_{L^p} \approx 1$. Notice that $\widehat{\phi_k}(\xi) = 1$ on the support of $\widehat{f_k}$ for $2 \le k \le m$. Therefore,

$$\begin{aligned}
&T_{\sigma^\epsilon}(f_1,\dots,f_m)(x)\\
&\quad= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right)\epsilon^{n/p_1-n}\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right)\right)^\vee(x)(\widehat{\phi_2}\widehat{f_2})^\vee(x)\cdots(\widehat{\phi_m}\widehat{f_m})^\vee(x)\\
&\quad= \left(\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right)\epsilon^{n/p_1-n}\widehat{\varphi}\left(\frac{\cdot - a}{\epsilon}\right)\right)^\vee(x)(\widehat{f_2})^\vee(x)\cdots(\widehat{f_m})^\vee(x)\\
&\quad= \epsilon^{n/p_1+\cdots+n/p_m}e^{2\pi i[(m-1)a+b]\cdot x}(\varphi*\varphi)(\epsilon x)[\varphi(\epsilon x)]^{m-1}\\
&\quad= \epsilon^{n/p}e^{2\pi i[(m-1)a+b]\cdot x}(\varphi*\varphi)(\epsilon x)[\varphi(\epsilon x)]^{m-1},
\end{aligned}$$

which obviously gives $\|T_{\sigma^\epsilon}(f_1,\dots,f_m)\|_{L^p} \approx 1$. So far, we have proved that $s_1 \ge n/2$; hence, by symmetry, we have $s_k \ge n/2$ for all $1 \le k \le m$.

It now remains to show (5.2). By symmetry, we only need to prove

$$\sum_{k=1}^{r}\left(\frac{s_k}{n} - \frac{1}{p_k}\right) \ge -\frac{1}{2} \tag{5.3}$$

for some fixed $1 \le r \le m$. To achieve our goal, we construct a multiplier σ^ϵ such that

$$\sup_{j\in\mathbb{Z}}\|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \lesssim \epsilon^{n/2-s_1-\cdots-s_r}$$

for $0 < \epsilon \ll 1$, and functions f_k satisfying $\|f_k\|_{H^{p_k}} \approx 1$ for $1 \le k \le m$ and

$$\|T_{\sigma^\epsilon}(f_1,\dots,f_m)\|_{L^p} \approx \epsilon^{n-n/p_1-\cdots-n/p_r}.$$

Then the inequalities

$$\begin{aligned}
\epsilon^{n-n/p_1-\cdots-n/p_r} &\approx \|T_{\sigma^\epsilon}(f_1,\dots,f_m)\|_{L^p}\\
&\le \sup_{j\in\mathbb{Z}}\|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}\prod_{k=1}^{m}\|f_k\|_{H^{p_k}} \lesssim \epsilon^{n/2-s_1-\cdots-s_r},
\end{aligned}$$

for all small positive numbers ϵ, yield (5.3).

We construct functions that give us enough ingredients to establish the multiplier σ^ϵ and functions f_k $(1 \le k \le m)$ as above. Take smooth functions φ, ϕ such that $\varphi(0) \ne 0$, $\widehat{\varphi}$ is supported in $\left\{\xi \in \mathbb{R}^n : |\xi| \le \frac{1}{19mr}\right\}$ and $\widehat{\varphi}(\xi) = 1$ for all $|\xi| \le \frac{1}{30mr}$, and $\widehat{\phi}$ is supported in the annulus $\frac{1}{23m} \le |\xi| \le \frac{1}{19m}$ and $\widehat{\phi}(\xi) = 1$ for all $\frac{1}{22m} \le |\xi| \le \frac{1}{20m}$. Fix $a, b \in \mathbb{R}^n$ such that $|a| = r^{-1/2}$, $|b| = \frac{1}{21m}$.

For $0 < \epsilon < \frac{1}{462m}$, define

$$\sigma^\epsilon(\xi_1,\dots,\xi_m) = \widehat{\varphi}\Big(\frac{1}{r\epsilon}\sum_{l=1}^{r}(\xi_l - a)\Big)\widehat{\varphi}\Big(\frac{1}{r}\sum_{l=1}^{r}(\xi_l-\xi_2)\Big)\cdots\widehat{\varphi}\Big(\frac{1}{r}\sum_{l=1}^{r}(\xi_l-\xi_r)\Big) \times \widehat{\phi}(\xi_{r+1})\cdots\widehat{\phi}(\xi_m).$$

Once again, $\operatorname{supp}\sigma^\epsilon \subset \{\xi\in\mathbb{R}^n : 2^{-1/4}\le|\xi|\le 2^{1/4}\}$, which, as in the previous case, implies that

$$\sup_{j\in\mathbb{Z}}\|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1,\dots,s_m)}}.$$

By changing variables, the inverse Fourier transform of σ^ϵ is

$$(\sigma^\epsilon)^\vee(x_1,\dots,x_m) = re^{2\pi i a\cdot\sum_{l=1}^r x_l}\epsilon^n\varphi\Big(\epsilon\sum_{l=1}^{r}x_l\Big)\varphi(x_1-x_2)\cdots\varphi(x_1-x_r)\phi(x_{r+1})\cdots\phi(x_m).$$

Taking Sobolev norm leads to

$$\|\sigma^\epsilon\|_{W^{(s_1,\dots,s_m)}} = C\epsilon^n\Big(\int_{\mathbb{R}^{nr}}\Big|\varphi\Big(\epsilon\sum_{l=1}^{r}x_l\Big)\prod_{l=2}^{r}\varphi(x_1-x_l)\Big|^2\prod_{l=1}^{r}(1+|x_l|^2)^{s_l}\,dx_1\cdots dx_r\Big)^{1/2},$$

where $C = r\|\phi\|_{W^{s_{r+1}}}\cdots\|\phi\|_{W^{s_m}}$.

Next, we show that

$$\int_{\mathbb{R}^{nr}}\Big|\varphi\Big(\epsilon\sum_{l=1}^{r}x_l\Big)\varphi(x_1-x_2)\cdots\varphi(x_1-x_r)\Big|^2\prod_{l=1}^{r}(1+|x_l|^2)^{s_l}\,dx_1\cdots dx_r \lesssim \epsilon^{-n-2(s_1+\cdots+s_r)}. \tag{5.4}$$

In fact, changing variables in the above integral and taking into account Lemma 5.2 we have

$$\int_{\mathbb{R}^{nr}}\Big|\varphi\Big(\epsilon\sum_{l=1}^{r}x_l\Big)\varphi(x_1-x_2)\cdots\varphi(x_1-x_r)\Big|^2\prod_{l=1}^{r}(1+|x_l|^2)^{s_l}\,dx_1\cdots dx_r = \frac{1}{r}\int_{\mathbb{R}^{nr}}|\varphi(\epsilon y_1)\varphi(y_2)\cdots\varphi(y_r)|^2\Big(1+\frac{1}{r^2}\Big|\sum_{l=1}^{r}y_l\Big|^2\Big)^{s_1} \times\prod_{l=2}^{r}\Big(1+\Big|-y_l+\frac{1}{r}\sum_{l=1}^{r}y_l\Big|^2\Big)^{s_l}dy_1\cdots dy_r$$

$$\lesssim \int_{\mathbb{R}^{nr}} |\varphi(\epsilon y_1)\varphi(y_2)\cdots\varphi(y_r)|^2 \prod_{l=1}^{r}(1+|y_l|^2)^{s_1+\cdots+s_r}\,dy_1\cdots dy_r$$
$$\lesssim \int_{\mathbb{R}^{nr}} |\varphi(\epsilon y_1)|^2(1+|y_1|^2)^{s_1+\cdots+s_r}\,dy_1 \lesssim \epsilon^{-n-2s_1-\cdots-2s_r},$$

where the implicit constants do not depend on ϵ. Inequality (5.4) gives

$$\sup_{j\in\mathbb{Z}} \|\sigma^\epsilon(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} = \|\sigma^\epsilon\|_{W^{(s_1,\dots,s_m)}} \lesssim \epsilon^{n/2-s_1-\cdots-s_r}.$$

To construct functions f_k, we fix a smooth function ζ such that $\widehat{\zeta}$ is supported in $\{\xi\in\mathbb{R}^n : |\xi-a|\le \frac{1}{3m}\}$ and is identical to 1 on $\{\xi\in\mathbb{R}^n : |\xi-a|\le \frac{3}{19m}\}$. Now set

$$f_1=\cdots=f_r=\zeta, \qquad \widehat{f_k}(\xi)=\epsilon^{n/p_k-n}\widehat{\varphi}\left(\frac{\xi-b}{\epsilon}\right) \quad \text{for } r+1\le k\le m.$$

It is clear that

$$\|f_k\|_{H^{p_k}} \approx \|f_k\|_{L^{p_k}} \approx 1, \quad 1\le k\le m.$$

Moreover, $\widehat{f_1}(\xi_1)\cdots\widehat{f_r}(\xi_r)=1$ on the support of the function

$$\widehat{\varphi}\left(\frac{1}{r\epsilon}\sum_{l=1}^{r}(\xi_l-a)\right)\widehat{\varphi}\left(\frac{1}{r}\sum_{l=1}^{r}(\xi_l-\xi_2)\right)\cdots\widehat{\varphi}\left(\frac{1}{r}\sum_{l=1}^{r}(\xi_l-\xi_r)\right),$$

and also $\widehat{\phi}(\xi)=1$ on the support of $\widehat{f_k}$ for all $r+1\le k\le m$. Therefore

$$T_{\sigma^\epsilon}(f_1,\dots,f_m)(x)$$
$$= re^{2\pi i(ra+(m-r)b)\cdot x}\epsilon^n\varphi(\epsilon rx)[\varphi(0)]^{r-1}\epsilon^{n/p_{r+1}+\cdots+n/p_m}[\varphi(\epsilon x)]^{m-r}.$$

Take L^p-norm to get

$$\|T_{\sigma^\epsilon}(f_1,\dots,f_m)\|_{L^p} \approx \epsilon^{n-n/p_1-\cdots-n/p_r},$$

which is the last thing we want to achieve. Notice that the above argument also works for $p_k=\infty$. ∎

6. Endpoint estimates. In this section we consider two endpoint estimates for multilinear singular integral operators. In the first case all indices are equal to infinity, and in the second case one index is 1 and the others are infinity.

For $x\in\mathbb{R}^n$ and $1\le k\le m$, define

$$\Gamma_x^k=\{(y_1,\dots,y_m)\in\mathbb{R}^{mn} : |y_k|>2|x|\}.$$

We say that a locally integrable function $K(y_1,\dots,y_m)$ on $\mathbb{R}^{mn}\setminus\{0\}$ *satisfies a coordinate-type Hörmander condition* if for some finite constant A we have

$$\sum_{k=1}^{m}\int_{\Gamma_x^k} |K(y_1,\dots,y_{k-1},x-y_k,y_{k+1},\dots,y_m)-K(y_1,\dots,y_m)|\,d\vec{y}\le A \tag{6.1}$$

for all $0 \neq x \in \mathbb{R}^n$. Another type of (bi-)linear Hörmander condition of geometric nature appeared in Pérez and Torres [16].

Denote by $\Lambda_p = \{(p, \infty, \dots, \infty), (\infty, p, \infty, \dots, \infty), \dots, (\infty, \dots, \infty, p)\}$ the set of all m-tuples with $m-1$ entries infinity and one entry p. The following result provides a version of the classical multilinear Calderón–Zygmund theorem in which the kernel satisfies a coordinate-type Hörmander condition under the initial assumption that the operator is bounded on Lebesgue spaces with indices in Λ_2. We denote by L^∞_c the space of all compactly supported bounded functions.

THEOREM 6.1. *Suppose that an m-linear singular integral operator of convolution type T with kernel K is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^2 with norm at most B for all $(q_1, \dots, q_m) \in \Lambda_2$. If K satisfies the coordinate-type Hörmander condition* (6.1)*, then*

$$\|T(f_1, \dots, f_m)\|_{\mathrm{BMO}} \lesssim (A+B)\|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \tag{6.2}$$

for all f_j in L^∞_c. Moreover, T has a bounded extension which satisfies

$$\|T(f_1, \dots, f_m)\|_{L^{1,\infty}} \lesssim (A+B)\|f_l\|_{L^1} \prod_{\substack{l=1 \\ k \neq l}}^{m} \|f_k\|_{L^\infty} \tag{6.3}$$

for all $1 \leq k \leq m$, $f_l \in L^1$, and $f_k \in L^\infty_c$ for $k \neq l$.

Proof. Fix a cube Q. To prove (6.2) we show that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T(f_1, \dots, f_m)(x) - C_Q|\, dx \lesssim (A+B)\|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \tag{6.4}$$

We decompose $f_l = f_l^0 + f_l^1$, where $f_l^0 = f_l \chi_{Q^*}$ and $f_l^1 = f_l \chi_{(Q^*)^c}$. Let F be the set of the 2^m sequences of length m consisting of zeros and ones. We claim that for each sequence $\vec{k} = (k_1, \dots, k_m)$ in F there is a constant $C_{\vec{k}}$ such that

$$\frac{1}{|Q|} \int_Q |T(f_1^{k_1}, \dots, f_m^{k_m})(x) - C_{\vec{k}}|\, dx \lesssim (A+B)\|f_1\|_{L^\infty} \cdots \|f_m\|_{L^\infty}. \tag{6.5}$$

Assuming the validity of this claim we obtain (6.4) with $C_Q = \sum_{\vec{k} \in F} C_{\vec{k}}$.

Next, we want to establish (6.5) for each $\vec{k} \in F$. If $\vec{k} = (k_1, \dots, k_m)$ has at least one zero entry we pick $C_{\vec{k}} = 0$. Without loss of generality, we may

assume that $k_1 = 0$. Since T maps $L^2 \times L^\infty \times \cdots \times L^\infty$ to L^2, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |T(f_1^{k_1},\dots,f_m^{k_m})(x)|\,dx &\le \left(\frac{1}{|Q|}\int_Q |T(f_1^{k_1},\dots,f_m^{k_m})(x)|^2\,dx\right)^{1/2}\\ &\le \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} |T(f_1^{k_1},\dots,f_m^{k_m})(x)|^2\,dx\right)^{1/2}\\ &\le B|Q|^{-1/2}\|f_1^0\|_{L^2}\|f_2^{k_2}\|_{L^\infty}\cdots\|f_m^{j_m}\|_{L^\infty}\\ &\le B|Q|^{-1/2}|Q^*|^{1/2}\|f_1\|_{L^\infty}\cdots\|f_m\|_{L^\infty}\\ &\lesssim B\|f_1\|_{L^\infty}\cdots\|f_m\|_{L^\infty}.\end{aligned}$$

Now suppose that $\vec{k} = (1,\dots,1)$. Set $C_{\vec{k}} = T(f_1^{k_1},\dots,f_m^{k_m})(x_Q)$, where x_Q is the center of the cube Q. Then, by the coordinate-type Hörmander condition (6.1), we have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T(f_1^1,\dots,f_m^1)(x) - C_{\vec{k}}|\,dx\\ &\quad\le \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^{nm}} |K(x-y_1,\dots,x-y_m) - K(x_Q-y_1,\dots,x_Q-y_m)|\\ &\qquad\qquad\times\prod_{k=1}^m |f_k^1(y_k)|\,d\vec{y}\,dx\\ &\quad\le \frac{\prod_{k=1}^m\|f_k\|_{L^\infty}}{|Q|}\\ &\qquad\times\int_Q\sum_{k=1}^m\int_{\Gamma^k_{x-x_Q}} |K(y_1,\dots,(x-x_Q)-y_k,\dots,y_m) - K(y_1,\dots,y_m)|\,d\vec{y}\,dx\\ &\quad\lesssim A\|f_1\|_{L^\infty}\cdots\|f_m\|_{L^\infty}.\end{aligned}$$

This completes the proof of (6.2) and we are left with establishing (6.3). Fix $\lambda > 0$. It is enough to show that

$$\begin{aligned}&|\{x\in\mathbb{R}^n : |T(f_1,\dots,f_m)(x)| > 2\lambda\}|\\ &\qquad\lesssim (A+B)\frac{1}{\lambda}\|f_1\|_{L^1}\|f_2\|_{L^\infty}\cdots\|f_m\|_{L^\infty}.\end{aligned}$$

By scaling, we may assume that $\|f_1\|_{L^1} = \|f_2\|_{L^\infty} = \cdots = \|f_m\|_{L^\infty} = 1$. Let δ be a positive number to be chosen later and let $f_1 = g_1 + b_1$ be the Calderón–Zygmund decomposition at height $\delta\lambda$, and $b_1 = \sum_j b_{1,j}$, where

$b_{1,j}$ are functions supported in the (pairwise disjoint) cubes Q_j such that

$$\operatorname{supp}(b_{1,j}) \subset Q_j, \quad \int b_{1,j}(x)\,dx = 0, \qquad \|b_{1,j}\|_{L^1} \le 2^{n+1}\delta\lambda|Q_j|,$$
$$\sum_j |Q_j| \le \frac{1}{\delta\lambda}, \qquad \|g_1\|_{L^\infty} \le 2^n\delta\lambda, \quad \|g_1\|_{L^1} \le 1.$$

Now we can estimate

$$\begin{aligned}|\{x\in\mathbb{R}^n : |T(f_1,\dots,f_m)(x)| > 2\lambda\}| &\le |\{x\in\mathbb{R}^n : |T(g_1,f_2,\dots,f_m)(x)| > \lambda\}| \\ &\quad + |\{x\in\mathbb{R}^n : |T(b_1,f_2,\dots,f_m)(x)| > \lambda\}|.\end{aligned}$$

Since T maps $L^2\times L^\infty\times\cdots\times L^\infty$ to L^2, the first part can be controlled by

$$\begin{aligned}|\{x\in\mathbb{R}^n : |T(g_1,f_2,\dots,f_m)(x)| > \lambda\}| &\le \frac{1}{\lambda^2}\int_{\mathbb{R}^n} |T(g_1,f_2,\dots,f_m)(x)|^2\,dx \\ &\le \frac{B^2}{\lambda^2}\|g_1\|_{L^2}^2\|f_2\|_{L^\infty}^2\cdots\|f_m\|_{L^\infty}^2 \le \frac{2^nB^2\delta}{\lambda}.\end{aligned}$$

To estimate the second part, we set $G = \bigcup_j Q_j^*$. Then

$$\begin{aligned}&|\{x\in\mathbb{R}^n : |T(b_1,f_2,\dots,f_m)(x)| > \lambda\}| \\ &\qquad \le |G| + |\{x\in G^c : |T(b_1,f_2,\dots,f_m)(x)| > \lambda\}| \\ &\qquad \le |G| + \frac{1}{\lambda}\sum_j \int_{(Q_j^*)^c} |T(b_{1,j},f_2,\dots,f_m)(x)|\,dx.\end{aligned}$$

Notice that

$$|G| \le \sum_j |Q_j^*| \lesssim \sum_j |Q_j| \le \frac{1}{\delta\lambda}.$$

Denote by c_j the center of the cube Q_j. Invoking condition (6.1) yields

$$\begin{aligned}&\int_{(Q_j^*)^c} |T(b_{1,j},f_2,\dots,f_m)(x)|\,dx \\ &\qquad \le \int_{(Q_j^*)^c} \Big|\int K(x-y_1,\dots,x-y_m)b_{1,j}(y_1)f_2(y_2)\cdots f_m(y_m)\,d\vec{y}\,dx\Big| \\ &\qquad \le \int_{(Q_j^*)^c} \Big|\int [K(x-y_1,y_2,\dots,y_m) - K(x-c_j,y_2,\dots,y_m)] \\ &\qquad\qquad \times b_{1,j}(y_1)\prod_{k=2}^m f_k(x-y_k)\,d\vec{y}\,dx\Big|\end{aligned}$$

$$\le \prod_{k=2}^{m} \|f_k\|_{L^\infty}$$
$$\times \int_{(Q_j^*)^c} \int |K(x-y_1,y_2,\dots,y_m)-K(x-c_j,y_2,\dots,y_m)|\,|b_{1,j}(y_1)|\,d\vec{y}\,dx$$
$$\le \int_{Q_j} \Big\{ \int_{\Gamma^1_{y_1-c_j}} |K(y_1-z_1,z_2\dots,z_m)-K(z_1,z_2,\dots,z_m)|\,d\vec{z}\Big\}|b_{1,j}(y_1)|\,dy_1$$
$$\le A\|b_{1,j}\|_{L^1}.$$

Therefore

$$\frac{1}{\lambda}\sum_j \int_{(Q_j^*)^c} |T(b_{1,j},f_2,\dots,f_m)(x)|\,dx \le \frac{A}{\lambda}\sum_j \|b_{1,j}\|_{L^1} \le \frac{2^{n+1}A}{\lambda}.$$

Choosing $\delta = B^{-1}$ and combining the preceding inequalities we obtain

$$|\{x\in\mathbb{R}^n : |T(f_1,\dots,f_m)(x)|>2\lambda\}| \le \frac{1}{\lambda}(2^nB+B+2^{n+1}A) \le 2^{n+1}(A+B)\frac{1}{\lambda},$$

which yields (6.3). ∎

This result allows us to obtain intermediate estimates between the results in [4] (in which $2<p_k,p<\infty$) and the results in [8] (in which $1<p_k\le\infty$ and $1<p\le 2$).

COROLLARY 6.2. *Let $1<p_k\le\infty$ and $1<p<\infty$ satisfy $1/p_1+\cdots+1/p_m=1/p$. Assume that* (1.6) *holds for a function σ on $\mathbb{R}^{mn}$ where $s_k>n/2$ for all k. Then the multilinear Fourier multiplier operator T_σ maps $L^{p_1}\times\cdots\times L^{p_m}$ to L^p.*

Proof. Note that the Sobolev condition (1.6) for σ implies the Hörmander condition (6.1) for $K=\sigma^\vee$. The proof is standard in the linear case, and in the m-linear case, it follows by freezing all but one variable (in the bilinear case it is contained in [15]). We are now able to apply Theorem 6.1 to T_σ, and hence Corollary 6.2 follows: Interpolating between (6.2) and (6.3) implies that T_σ maps $L^p\times L^\infty\times\cdots\times L^\infty$ to L^p for all $1<p<\infty$. By symmetry, we deduce that T_σ is bounded from $L^{q_1}\times\cdots\times L^{q_m}$ to L^p for all $(q_1,\dots,q_m)\in\Lambda_p$ and $1<p<\infty$. Once again, by interpolation, we find that T_σ maps from $L^{p_1}\times\cdots\times L^{p_m}$ to L^p for all $1<p_1,\dots,p_m\le\infty$ and $1<p<\infty$ such that $1/p_1+\cdots+1/p_m=1/p$ with norm at most a multiple of A. ∎

COROLLARY 6.3. *Let σ be a bounded function on $\mathbb{R}^{mn}\setminus\{0\}$ which satisfies* (1.6) *with $s_k>n/2$ for all $k=1,\dots,m$. Then*

$$\|T_\sigma(f_1,\dots,f_m)\|_{\mathrm{BMO}} \lesssim A\|f_1\|_{L^\infty}\cdots\|f_m\|_{L^\infty} \tag{6.6}$$

for all $f_k\in L^\infty_c$.

Proof. As before, condition (1.6) for σ implies (6.1) for $K = \sigma^\vee$. Applying Theorem 6.1 to T_σ yields the assertion. ∎

7. Proofs of some technical lemmas. In this section, we will give the detailed proofs of some lemmas that were used in previous sections.

7.1. The proof of Lemma 3.2. For $\rho \in \mathbb{Z}$, $\rho \geq 2$, denote

$$F_\rho = \{y \in \mathbb{R}^{mn} : 2^{\rho-1} - 2 \leq |y| \leq 2^{\rho+1} + 2\}.$$

Fix $x = (x_1, \dots, x_m) \in \mathbb{R}^{mn}$. Then

$$\begin{aligned}
(\varphi_\epsilon * \sigma)(2^j x)\widehat{\psi}(x) &= \Big\{ \int \epsilon^{-mn} \varphi(\epsilon^{-1} y) \sigma(2^j x - y)\, dy \Big\} \widehat{\psi}(x) \\
&= \Big\{ \int \epsilon^{-mn} 2^{jmn} \varphi(\epsilon^{-1} 2^j y) \sigma(2^j (x - y))\, dy \Big\} \widehat{\psi}(x) \\
&= \sum_{\rho \in \mathbb{Z}} \Big\{ \int \varphi_{\epsilon 2^{-j}}(y) \sigma(2^j (x - y)) \widehat{\psi}(2^{-\rho}(x - y))\, dy \Big\} \widehat{\psi}(x) \\
&= \sum_{\rho \leq -3} \Big\{ \int \varphi_{\epsilon 2^{-j}}(x - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y)\, dy \Big\} \widehat{\psi}(x) \qquad (7.1) \\
&\quad + \sum_{|\rho| \leq 2} \int \varphi_{\epsilon 2^{-j}}(y) \sigma(2^j (x - y)) \widehat{\psi}(2^{-\rho}(x - y)) \widehat{\psi}(x)\, dy \qquad (7.2) \\
&\quad + \sum_{\rho \geq 3} \Big\{ \int \varphi_{\epsilon 2^{-j}}(x - y) \sigma(2^j y) \widehat{\psi}(2^{-\rho} y)\, dy \Big\} \widehat{\psi}(x). \qquad (7.3)
\end{aligned}$$

The $W^{(s_1,\dots,s_m)}$ norm of the term (7.2) can easily be estimated by

$$\begin{aligned}
&\sum_{|\rho| \leq 2} \int |\varphi_{\epsilon 2^{-j}}(y)|\, \|\sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y)) \widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}\, dy \\
&\quad \leq \sum_{|\rho| \leq 2} \int |\varphi_{\epsilon 2^{-j}}(y)|\, \|\sigma(2^j(\cdot - y)) \widehat{\psi}(2^{-\rho}(\cdot - y))\|_{W^{(s_1,\dots,s_m)}} \|\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}\, dy \\
&\quad \lesssim \sum_{|\rho| \leq 2} \|\sigma(2^{j+\rho}\cdot) \widehat{\psi}\|_{W^{(s_1,\dots,s_m)}} \int |\varphi_{\epsilon 2^{-j}}(y)|\, dy \\
&\quad \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \widehat{\psi}\|_{W^{(s_1,\dots,s_m)}},
\end{aligned}$$

in which the second last inequality follows from the fact [4, Proposition A.2], [9] that

$$\|fg\|_{W^{(s_1,\dots,s_m)}} \lesssim \|f\|_{W^{(s_1,\dots,s_m)}} \|g\|_{W^{(s_1,\dots,s_m)}}$$

when $f, g \in W^{(s_1,\dots,s_m)}$ for $s_1, \dots, s_m > n/2$.

Now fix integers $N_k \geq s_k$ $(1 \leq k \leq m)$ and set $N = N_1 + \cdots + N_m$. Since $\|f\|_{W^{(s_1,\dots,s_m)}} \leq \|f\|_{W^N}$, the Sobolev norm of the term (7.1) is bounded by

$$
\begin{aligned}
&\sum_{\rho\leq -3} \Big\| \Big\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y)\sigma(2^j y)\widehat{\psi}(2^{-\rho}y)\,dy \Big\}\widehat{\psi}\Big\|_{W^N} \\
&\quad \lesssim \sum_{\rho\leq -3}\sum_{|\alpha|+|\beta|\leq N} \Big\| \Big\{ \int (\epsilon 2^{-j})^{-|\alpha|}(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(\cdot - y)\sigma(2^j y)\widehat{\psi}(2^{-\rho}y)\,dy \Big\}\partial^\beta\widehat{\psi}\Big\|_{L^2} \\
&\quad = \sum_{\rho\leq -3}\sum_{|\alpha|+|\beta|\leq N} \Big\| \Big\{ \int\limits_{1/4\leq|y|\leq 9/4} \frac{(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)}{(\epsilon 2^{-j})^{|\alpha|}} \\
&\qquad\qquad \times \sigma(2^j(\cdot - y))\widehat{\psi}(2^{-\rho}(\cdot - y))\,dy \Big\}\partial^\beta\widehat{\psi}\Big\|_{L^2} \\
&\quad \lesssim \sum_{\rho\leq -3}\sum_{|\alpha|\leq N} \int \left(\frac{|y|}{\epsilon 2^{-j}}\right)^{|\alpha|} |(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,\|\sigma(2^j(\cdot - y))\widehat{\psi}(2^{-\rho}(\cdot - y))\|_{L^2}\,dy \\
&\quad \lesssim \sum_{\rho\leq -3} 2^{\rho n/2}\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\|_{L^2} \sum_{|\alpha|\leq N}\int \left(\frac{|y|}{\epsilon 2^{-j}}\right)^{|\alpha|} |(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,dy \\
&\quad \lesssim \sum_{\rho\leq -3} 2^{\rho n/2}\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\|_{W^s} \sum_{|\alpha|\leq N}\int |y|^{|\alpha|}|(\partial^\alpha\varphi)(y)|\,dy \\
&\quad \lesssim \sup_{j\in\mathbb{Z}} \|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}.
\end{aligned}
$$

Finally, we deal with the term (7.3). We have

$$
\begin{aligned}
&\Big\|\sum_{\rho\geq 3} \Big\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y)\sigma(2^j y)\widehat{\psi}(2^{-\rho}y)\,dy \Big\}\widehat{\psi}\,\Big\|_{W^{(s_1,\dots,s_m)}} \\
&\quad \leq \sum_{\rho\geq 3} \Big\| \Big\{ \int \varphi_{\epsilon 2^{-j}}(\cdot - y)\sigma(2^j y)\widehat{\psi}(2^{-\rho}y)\,dy \Big\}\widehat{\psi}\,\Big\|_{W^N} \\
&\quad \lesssim \sum_{\rho\geq 3}\sum_{|\alpha|+|\beta|\leq N} \Big\| \Big\{ \int (\epsilon 2^{-j})^{-|\alpha|}(\partial^\alpha \varphi)_{\epsilon 2^{-j}}(\cdot - y)\sigma(2^j y)\widehat{\psi}(2^{-\rho}y)\,dy \Big\}\partial^\beta\widehat{\psi}\Big\|_{L^2} \\
&\quad = \sum_{\rho\geq 3}\sum_{|\alpha|+|\beta|\leq N} \Big\| \Big\{ \int\limits_{F_\rho} (\epsilon 2^{-j})^{-|\alpha|}(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y) \\
&\qquad\qquad \times \sigma(2^j(\cdot - y))\widehat{\psi}(2^{-\rho}(\cdot - y))\,dy \Big\}\partial^\beta\widehat{\psi}\,\Big\|_{L^2} \\
&\quad \lesssim \sum_{\rho\geq 3}\sum_{|\alpha|+|\beta|\leq N} \int\limits_{F_\rho} \left(\frac{|y|}{2^\rho\epsilon 2^{-j}}\right)^{|\alpha|} |(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)| \\
&\qquad\qquad \times \|\sigma(2^j(\cdot - y))\widehat{\psi}(2^{-\rho}(\cdot - y))\partial^\beta\widehat{\psi}\,\|_{L^2}\,dy
\end{aligned}
$$

$$= \sum_{\rho\ge 3}\sum_{|\alpha|+|\beta|\le N}\int_{F_\rho}\Big(\frac{|y|}{2^\rho\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|$$
$$\times\,\|\sigma(2^j\cdot)\widehat{\psi}(2^{-\rho}\cdot)(\partial^\beta\widehat{\psi})(\cdot+y)\|_{L^2}\,dy$$
$$\le \sum_{\rho\ge 3}\sum_{|\alpha|+|\beta|\le N}\int_{F_\rho}2^{\rho n/2}\Big(\frac{|y|}{\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|$$
$$\times\,\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}(\partial^\beta\widehat{\psi})(2^\rho\cdot+y)\|_{L^2}\,dy$$
$$\le \sum_{\rho\ge 3}\sum_{|\alpha|\le N}\int_{F_\rho}2^{\rho n/2}\Big(\frac{|y|}{\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\|_{L^2(B(-2^{-\rho}y,2^{1-\rho}))}\,dy$$
$$\lesssim \sum_{|\alpha|\le N}\sum_{\rho\ge 3}\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\,\|_{L^\infty}\int_{F_\rho}\Big(\frac{|y|}{\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,dy$$
$$\le \sum_{|\alpha|\le N}\sum_{\rho\ge 3}\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}}\int_{F_\rho}\Big(\frac{|y|}{\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,dy$$
$$\lesssim \sup_{\rho\in\mathbb{Z}}\|\sigma(2^{j+\rho}\cdot)\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}}\sum_{|\alpha|\le N}\sum_{\rho\ge 3}\int_{F_\rho}\Big(\frac{|y|}{\epsilon 2^{-j}}\Big)^{|\alpha|}|(\partial^\alpha\varphi)_{\epsilon 2^{-j}}(y)|\,dy$$
$$\lesssim \sup_{j\in\mathbb{Z}}\|\sigma(2^j\cdot)\widehat{\psi}\,\|_{W^{(s_1,\dots,s_m)}}.$$

The proof of Lemma 3.2 is now complete.

7.2. The proof of Lemma 4.1. Before verifying the lemma, we mention approaches that were used by other authors. First, with the assumption on the kernel

$$|\partial^\alpha_{y_0,\dots,y_m}K(y_0,\dots,y_m)|\le A\Big(\sum_{k,l=0}^m|y_k-y_l|\Big)^{-mn-|\alpha|}$$

for all $|\alpha|\le N$, Grafakos and Kalton [7] showed that estimate (4.3) holds for the corresponding multilinear singular integral operator with

$$b_k(x)=\frac{|Q_k|^{1-1/p_k+(N+1)/(mn)}}{(|x-c_k|+\ell(Q_k))^{n+(N+1)/m}}.$$

Miyachi and Tomita [15] constructed functions b_k satisfying Lemma 4.1 in the bilinear case. We adapt these techniques to prove the key lemma in the multilinear setting.

Now we start the proof of Lemma 4.1. We may assume that $J_0 = \{1,\dots,r\}$ for some $1 \le r \le m$. Fix

$$x \in \Big(\bigcap_{k=r+1}^{m} Q_k^*\Big) \setminus \bigcup_{k=1}^{r} Q_k^*$$

(when $r = m$, just fix $x \in \mathbb{R}^n \setminus \bigcup_{k=1}^m Q_k^*$). Now we rewrite $T_\sigma(a_1,\dots,a_m)(x)$ as

$$T_\sigma(a_1,\dots,a_m)(x) = \sum_{j\in\mathbb{Z}} g_j(x),$$

where

$$g_j(x) = \int_{\mathbb{R}^{mn}} 2^{jmn} K_j\big(2^j(x-y_1),\dots,2^j(x-y_m)\big) a_1(y_1)\cdots a_m(y_m)\,dy_1\cdots dy_m$$

with $K_j = \big(\sigma(2^j\cdot)\widehat{\psi}\big)^\vee$. Let c_k be the center of the cube Q_k ($1 \le k \le m$). For $1 \le k \le r$, since $x \notin Q_k^*$ and $y_k \in Q_k$, we have $|x-c_k| \approx |x-y_k|$. Fix $1 \le l \le r$. Lemma 2.1 with $s_k > n/2$ and the Cauchy–Schwarz inequality yield

$$\begin{aligned}
&\prod_{k=1}^{r} \langle 2^j(x-c_k)\rangle^{s_k} |g_j(x)| \qquad (7.4)\\
&\lesssim 2^{jmn} \int_{Q_1\times\cdots\times Q_m} \Big(\prod_{k=1}^{r}\langle 2^j(x-y_k)\rangle^{s_k}\Big)\\
&\qquad\qquad \times |K_j(2^j(x-y_1),\dots,2^j(x-y_m))| \prod_{k=1}^{m} \|a_k\|_{L^\infty}\,d\vec{y}\\
&\le 2^{jmn} \int_{Q_1\times\cdots\times Q_m} \Big(\prod_{k=1}^{r}\langle 2^j(x-y_k)\rangle^{s_k}\Big)\\
&\qquad\qquad \times |K_j(2^j(x-y_1),\dots,2^j(x-y_m))| \prod_{k=1}^{m} |Q_k|^{-1/p_k}\,d\vec{y}\\
&\le 2^{jrn} \prod_{k=1}^{m} |Q_k|^{-1/p_k} \int_{Q_1\times\cdots\times Q_r\times\mathbb{R}^{(m-r)n}} \Big(\prod_{k=1}^{r}\langle 2^j(x-y_k)\rangle^{s_k}\Big)\\
&\quad \times |K_j(2^j(x-y_1),\dots,2^j(x-y_r),y_{r+1},\dots,y_m)|\,dy_1\cdots dy_r\,dy_{r+1}\cdots dy_m\\
&\le 2^{jrn} \prod_{k=1}^{r} |Q_k|^{1-1/p_k} \prod_{k=r+1}^{m} |Q_k|^{-1/p_k} \int_{\mathbb{R}^{(m-r)n}} \int_{Q_l} |Q_l|^{-1}\langle 2^j(x-y_l)\rangle^{s_l}\\
&\quad \times \Big\| \Big(\prod_{\substack{k=1\\k\neq l}}^{r}\langle y_k\rangle^{s_k}\Big) K_j(y_1,\dots,y_{l-1},2^j(x-y_l),y_{l+1},\dots,y_m)\Big\|_{L^\infty(dy_1\cdots\widehat{dy_l}\cdots dy_r)}\\
&\qquad\qquad \times dy_l\,dy_{r+1}\cdots dy_m
\end{aligned}$$

$$\lesssim 2^{jrn}\prod_{k=1}^{r}|Q_k|^{1-1/p_k}\prod_{k=r+1}^{m}|Q_k|^{-1/p_k}\int\limits_{\mathbb{R}^{(m-r)n}}\int\limits_{Q_l}|Q_l|^{-1}\langle 2^j(x-y_l)\rangle^{s_l}$$
$$\times\Big\|\Big(\prod_{\substack{k=1\\k\neq l}}^{r}\langle y_k\rangle^{s_k}\Big)K_j(y_1,\dots,y_{l-1},2^j(x-y_l),y_{l+1},\dots,y_m)\Big\|_{L^2(dy_1\cdots\widehat{dy_l}\cdots dy_r)}$$
$$\times\, dy_l\, dy_{r+1}\cdots dy_m$$
$$\lesssim 2^{jrn}\prod_{k=1}^{r}|Q_k|^{1-1/p_k}\prod_{k=r+1}^{m}|Q_k|^{-1/p_k}\int\limits_{Q_l}|Q_l|^{-1}\langle 2^j(x-y_l)\rangle^{s_l}$$
$$\times\Big\|\Big(\prod_{\substack{k=1\\i\neq k}}^{m}\langle y_k\rangle^{s_l}\Big)K_j(y_1,\dots,y_{l-1},2^j(x-y_l),y_{l+1},\dots,y_m)\Big\|_{L^2(dy_1\cdots\widehat{dy_l}\cdots dy_m)}\,dy_l$$
$$=2^{jrn}\Big(\prod_{k=1}^{r}|Q_k|^{1-1/p_k}\Big)h_j^{(l,0)}(x)\prod_{k=r+1}^{m}b_k(x)$$

for all $x\in(\bigcap_{k=r+1}^{m}Q_k^*)\setminus\bigcup_{k=1}^{r}Q_k^*$, where

$$h_j^{(l,0)}(x)=\frac{1}{|Q_l|}\int\limits_{Q_l}\langle 2^j(x-y_l)\rangle^{s_l}$$
$$\times\Big\|\Big(\prod_{\substack{k=1\\k\neq l}}^{m}\langle y_k\rangle^{s_k}\Big)K_j(y_1,\dots,y_{l-1},2^j(x-y_l),y_{l+1},\dots,y_m)\Big\|_{L^2(dy_1\cdots\widehat{dy_l}\cdots dy_m)}\,dy_l$$

and $b_k(x)=|Q_k|^{-1/p_k}\chi_{Q_k^*}(x)$ for $r+1\le k\le m$. A direct computation gives

$$\|h_j^{(l,0)}\|_{L^2}\le 2^{-jn/2}\|\sigma(2^j\cdot)\widehat{\psi}\|_{W^{(s_1,\dots,s_m)}}=A2^{-jn/2}.$$

Using the vanishing moment condition of a_k and Taylor's formula, we write

$$g_j(x)=2^{jmn}\sum_{|\alpha|=N_l}C_\alpha\int\limits_{\mathbb{R}^{mn}}\Big\{\int\limits_0^1(1-t)^{N_l-1}$$
$$\times\partial_{y_l}^{\alpha}K_j\big(2^j(x-y_1),\dots,2^j(x-c_l-t(y_l-c_l)),\dots,2^j(x-y_m)\big)$$
$$\times(2^j(y_l-c_l))^{\alpha}a_1(y_1)\cdots a_m(y_m)\,dt\Big\}\,dy_1\cdots dy_m.$$

Repeat the preceding argument to obtain

$$\prod_{k=1}^{r}\langle 2^j(x-c_k)\rangle^{s_k}|g_j(x)|\lesssim 2^{jrn}\Big(\prod_{k=1}^{r}|Q_k|^{1-1/p_k}\Big)h_j^{(l,1)}(x)\prod_{k=r+1}^{m}b_k(x)\tag{7.5}$$

for all $x\in(\bigcap_{k=r+1}^{m}Q_k^*)\setminus\bigcup_{k=1}^{r}Q_k^*$, where $b_k(x)=|Q_k|^{-1/p_k}\chi_{Q_k^*}(x)$ for $r+1\le$

$k \le m$ and

$$h_j^{(l,1)}(x) = \sum_{|\alpha|=N_l} \int_{Q_l} \Big\{ \int_0^1 \langle 2^j x^t_{c_l,y^l} \rangle^{s_l} \times \Big\| \Big(\prod_{\substack{k=1 \\ k\neq l}}^m \langle y_k \rangle^{s_k} \Big) \partial_{y_l}^\alpha K_j(y_1, \dots, y_{l-1}, 2^j x^t_{c_l,y_l}, y_{l+1}, \dots, y_m) \Big\|_{L^2(dy_1 \cdots \widehat{dy_l} \cdots dy_m)} \times (2^j \ell(Q_l))^{N_l} |Q_l|^{-1} \, dt \Big\} \, dy_l,$$

with $x^t_{c_l,y_l} = x - c_l - t(y_l - c_l)$. Applying Minkowski's inequality together with Lemma 2.2 implies that

$$\|h_j^{(l,1)}\|_{L^2} \le A 2^{-jn/2} (2^j \ell(Q_l))^{N_l}.$$

Combine inequalities (7.4) and (7.5) to get

$$\prod_{k=1}^r \langle 2^j (x - c_k) \rangle^{s_k} |g_j(x)| \le 2^{jrn} \Big(\prod_{k=1}^r |Q_k|^{1-1/p_k} \Big) \min\{h_j^{(l,0)}(x), h_j^{(l,1)}(x)\} \prod_{k=r+1}^m b_k(x) \tag{7.6}$$

for all $1 \le l \le r$. The inequalities in (7.6) imply that

$$|g_j(x)| \le 2^{jrn} \prod_{k=1}^r |Q_k|^{1-1/p_k} \prod_{k=1}^r \langle 2^j (x - c_k) \rangle^{-s_k} \min_{1\le l\le r} \{h_j^{(l,0)}(x), h_j^{(l,1)}(x)\} \prod_{k=r+1}^m b_k(x) \tag{7.7}$$

for all $x \in (\bigcap_{k=r+1}^m Q_k^*) \setminus \bigcup_{k=1}^r Q_k^*$.

Now we need to construct functions u_j^k $(1 \le k \le r)$ such that

$$g_j(x) \lesssim A \prod_{k=1}^r u_j^k(x) \prod_{k=r+1}^m b_k(x)$$

for all $x \in (\bigcap_{k=r+1}^m Q_k^*) \setminus \bigcup_{k=1}^r Q_k^*$ and $\|\sum_j u_j^k\|_{L^{p_k}} \lesssim 1$ for all $1 \le k \le r$. Then the lemma follows by taking $b_k = \sum_j u_j^k$ $(1 \le k \le r)$ and $b_k = |Q_k|^{-1/p_k} \chi_{Q_k^*}$ $(r+1 \le k \le m)$.

Indeed, we can choose $0 < \lambda_k < \min\{1/2, s_k/n - 1/p_k + 1/2\}$ such that

$$\sum_{k=1}^r \lambda_k = \frac{r-1}{2}.$$

This is suitable since conditions (1.5) imply that

$$\sum_{k=1}^{r} \min\left\{\frac{1}{2}, \frac{s_k}{n} - \frac{1}{p_k} + \frac{1}{2}\right\} > \frac{r-1}{2}.$$

Set $\alpha_k = 1/p_k - 1/2 + \lambda_k$ and $\beta_k = 2(1/p_k - \alpha_k)$. Then

$$\sum_{k=1}^{r} \alpha_k = \sum_{k=1}^{r} \frac{1}{p_k} - \frac{1}{2},$$

$\beta_k > 0$ and $\beta_1 + \cdots + \beta_r = 1$. Now define

$$u_j^k = A^{-\beta_k} 2^{jn} |Q_k|^{1-1/p_k} \langle 2^j(\cdot - c_k)\rangle^{-s_k} \chi_{(Q_k^*)^c} \min\{h_j^{(k,0)}, h_j^{(k,1)}\}^{\beta_k}$$

for $1 \le k \le r$. Then, from (7.7), it is easy to see that

$$g_j(x) \lesssim A \prod_{k=1}^{r} u_j^k(x) \prod_{k=r+1}^{m} b_k(x)$$

for all $x \in (\bigcap_{k=r+1}^{m} Q_k^*) \setminus \bigcup_{k=1}^{r} Q_k^*$. It remains to check that

$$\sum_j \int_{\mathbb{R}^n} |u_j^k(x)|^{p_k}\, dx \lesssim 1. \tag{7.8}$$

Since $1/p_k = \alpha_k + \beta_k/2$, setting $1/p_k' = 1 - 1/p_k$, from Hölder's inequality we get

$$\begin{aligned}\|u_j^k\|_{L^{p_k}} &\le A^{-\beta_k} 2^{jn} |Q_k|^{1/p_k'} \\ &\quad \times \|\langle 2^j(\cdot - c_k)\rangle^{-s_k} \chi_{(Q_k^*)^c}\|_{L^{1/\alpha_k}} \|\min\{h_j^{(k,0)}, h_j^{(k,1)}\}^{\beta_k}\|_{L^{2/\beta_k}}\end{aligned}$$

for all $1 \le k \le r$. Notice that $n < s_k/\alpha_k$, so we have

$$\begin{aligned}\|\langle 2^j(\cdot - c_k)\rangle^{-s_k} \chi_{(Q_k^*)^c}\|_{L^{1/\alpha_k}} &\lesssim 2^{-jn\alpha_k} \min\{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\}, \\ \|(\min\{h_j^{(k,0)}, h_j^{(k,1)}\})^{\beta_k}\|_{L^{2/\beta_k}} &\le \min\{\|h_j^{(k,0)}\|_{L^2}^{\beta_k}, \|h_j^{(k,1)}\|_{L^2}^{\beta_k}\} \\ &\lesssim \big(A 2^{-jn/2} \min\{1, (2^j \ell(Q_k))^{N_k}\}\big)^{\beta_k}.\end{aligned}$$

Therefore

$$\begin{aligned}\|u_j^k\|_{L^{p_k}} &\lesssim 2^{jn} |Q_k|^{1-1/p_k} 2^{-jn/p_k} \min\{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min\{1, (2^j \ell(Q_k))^{N_k \beta_k}\} \\ &\lesssim (2^j \ell(Q_k))^{n - n/p_k} \min\{1, (2^j \ell(Q_k))^{\alpha_k n - s_k}\} \min\{1, (2^j \ell(Q_k))^{N_k \beta_k}\}.\end{aligned}$$

This inequality is enough to establish (7.8). The proof of Lemma 4.1 is complete.

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REFERENCES

[1] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, Adv. Math. 24 (1977), 101–171.

[2] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), 315–331.

[3] R. R. Coifman et Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) 28 (1978), no. 3, 177–202.

[4] M. Fujita and N. Tomita, *Weighted norm inequalities for multilinear Fourier multipliers*, Trans. Amer. Math. Soc. 364 (2012), 6335–6353.

[5] L. Grafakos, *Modern Fourier Analysis*, 3rd ed., Springer, New York, 2014.

[6] L. Grafakos and D. He, *Multilinear Calderón–Zygmund operators on Hardy spaces, II*, J. Math. Anal. Appl. 416 (2014), 511–521.

[7] L. Grafakos and N. Kalton, *Multilinear Calderón–Zygmund operators on Hardy spaces*, Collect. Math. 52 (2001), 169–179.

[8] L. Grafakos, A. Miyachi and N. Tomita, *On multilinear Fourier multipliers of limited smoothness*, Canad. J. Math. 65 (2013), 299–330.

[9] L. Grafakos and S. Oh, *The Kato–Ponce inequality*, Comm. Partial Differential Equations 39 (2014), 1128–1157.

[10] L. Grafakos and Z. Si, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angew. Math. 668 (2012), 133–147.

[11] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. 165 (2002), 124–164.

[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.

[13] Y. Meyer and R. R. Coifman, *Wavelets: Calderón–Zygmund and Multilinear Operators*, Cambridge Univ. Press, Cambridge, 1997.

[14] S. G. Mikhlin, *On the multipliers of Fourier integrals*, Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 701–703 (in Russian).

[15] A. Miyachi and N. Tomita, *Minimal smoothness conditions for bilinear Fourier multipliers*, Rev. Mat. Iberoamer. 29 (2013), 495–530.

[16] C. Pérez and R. H. Torres, *Minimal regularity conditions for the end-point estimate of bilinear Calderón–Zygmund operators*, Proc. Amer. Math. Soc. Ser. B 1 (2014), 1–13.

[17] N. Tomita, *A Hörmander type multiplier theorem for multilinear operators*, J. Funct. Anal. 259 (2010), 2028–2044.

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