

Nonlinear parabolic SPDEs involving Dirichlet operators

by

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Abstract. We study the problem of existence, uniqueness and regularity of probabilistic solutions of the Cauchy problem for nonlinear stochastic partial differential equations involving operators corresponding to regular (nonsymmetric) Dirichlet forms. In the proofs we combine the methods of backward doubly stochastic differential equations with those of probabilistic potential theory and Dirichlet forms.

1. Introduction. In the present paper we are concerned with the problem of existence, uniqueness and regularity of probabilistic solutions of stochastic partial differential equations (SPDEs for short) of the form

$$(1.1) \quad du(t) = (A_t u + f(t, x, u)) dt + \tilde{g}(t, x, u) dB_t, \quad u(0) = \varphi.$$

In (1.1), B is some Q -Wiener process and A_t , $t \in [0, T]$, are operators associated with some family of regular (nonsymmetric) Dirichlet forms satisfying mild regularity assumptions. These assumptions are automatically satisfied if $A_t = A$, $t \in [0, T]$. Therefore our results apply in particular to equations (1.1) with A_t replaced by any operator A corresponding to a regular Dirichlet form. The class of such operators is quite wide. It contains both local operators, whose model example is the Laplacian Δ , and nonlocal operators, whose model example is the fractional Laplacian $\Delta^{\alpha/2}$ with $\alpha \in (0, 2)$. Other interesting examples are to be found for instance in [8, 13, 14, 16]. An important example of the family of operators depending on t and satisfying our regularity assumptions is the family of uniformly elliptic operators of

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the form

$$(1.2) \quad A_t u = \sum_{i,j=1}^d (a_{ij}(t, x) u_{x_i})_{x_j}$$

with ellipticity constant not depending on t . Actually, in case A_t are of the form (1.2), we consider equations more general than (1.1) with coefficients f, \tilde{g} depending on u and its gradient ∇u .

As for φ, f, \tilde{g} , we assume that $\varphi, f(\cdot, \cdot, 0), \tilde{g}(\cdot, \cdot, 0)$ are square-integrable, $f(t, x, \cdot)$ is continuous and monotone (no assumption on the growth of $f(t, x, \cdot)$ is imposed) and $\tilde{g}(t, x, \cdot)$ is Lipschitz continuous. In the case where f, \tilde{g} depend on u and ∇u , we also assume that they are Lipschitz-continuous with respect to ∇u .

To study (1.1) we develop the approach used successfully in [12, 13, 14] to investigate Sobolev space solutions of semilinear PDEs with operators corresponding to Dirichlet forms. In those papers PDEs are studied by the methods of the theory of backward stochastic differential equations (BSDEs for short) combined with those of probabilistic potential theory and Dirichlet forms. In the present paper the strategy for studying SPDEs is similar. The major difference is that now we use backward doubly stochastic differential equations (BDSDEs for short) instead of BSDEs. The idea of studying nonlinear SPDEs via BDSDEs goes back to [20]. In [20] classical solutions of equations involving nondivergence form operators with regular coefficients are considered. Our approach to (1.1) was also motivated by the desire to develop the ideas of [20] to encompass a broader class of operators and to study Sobolev space solutions.

As a matter of fact, we study the following Cauchy problem with terminal condition:

$$(1.3) \quad du(t) = -(A_t u + f(t, x, u)) dt - g(t, x, u) d^\dagger \beta_t, \quad u(T) = \varphi.$$

Here $g = (g_k)$ is a sequence of real functions on $\Omega \times (0, T] \times E \times \mathbb{R}$ determined by \tilde{g} and $Q, \beta = (\beta_k)$ is a sequence of one-dimensional mutually independent standard Wiener processes defined on some probability space (Ω, \mathcal{F}, P) and $g d^\dagger \beta_t = \sum_{k=1}^\infty g_k d^\dagger \beta_t^k$, where $g_k d^\dagger \beta_t^k$ denotes the backward Itô integral. The results for (1.3) can be easily translated into results for (1.1). However, since we heavily rely on the theory of BDSDEs, problem (1.3) is much more convenient to deal with.

Roughly speaking, our strategy for investigating (1.3) consists of two steps. Suppose that the operators A_t are associated with some family $\{B^{(t)}\}$ of Dirichlet forms on $L^2(E; m)$ with common domain V and let \mathcal{E} be the time-dependent Dirichlet form determined by $\{B^{(t)}\}$. Denote by $\mathbf{M} = \{(\mathbf{X}, P_z); z \in \mathbb{R} \times E\}$ a time-space Markov process with life time ζ associated with \mathcal{E} . In the first step we prove that there exists an exceptional set

$N \subset E_{0,T} := (0, T] \times E$ and a pair (Y, M) of processes such that for every $z \in E_{0,T} \setminus N$ the process M is a martingale under $P \otimes P_z$ and (Y, M) is a unique solution of the BDSDE

$$(1.4) \quad Y_t = \varphi(\mathbf{X}_{T_t}) + \int_t^{\zeta \wedge T_t} f(\mathbf{X}_r, Y_r) dr + \int_t^{\zeta \wedge T_t} g(\mathbf{X}_r, Y_r) d^\dagger \beta_r^t - \int_0^{\zeta \wedge T_t} dM_r, \quad t \in [0, \zeta \wedge T_t], \quad P \otimes P_z\text{-a.s.},$$

where $T_t = T - \iota(0)$. Here ι is the uniform motion to the right (in particular, $\iota(0) = s$ under P_z with $z = (s, x)$) and $\beta_t^t = \beta_{t+\iota(0)}$, $t \geq 0$. In fact, we show that

$$Y_t = u(\mathbf{X}_t), \quad t \in [0, \zeta \wedge T_t]$$

for some $u : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ such that $u(\omega; \cdot)$ is quasi-continuous (with respect to the capacity associated with \mathcal{E}). Therefore setting $t = 0$ in (1.4) and taking expectation with respect to P_z we see that u satisfies the nonlinear Feynman–Kac formula

$$(1.5) \quad u(z) = E_z \left(\varphi(\mathbf{X}_{T_t}) + \int_0^{\zeta \wedge T_t} f(\mathbf{X}_t, u(\mathbf{X}_t)) dt + \int_0^{\zeta \wedge T_t} g(\mathbf{X}_t, u(\mathbf{X}_t)) d^\dagger \beta_t^t \right).$$

In fact, a quasi-continuous u satisfying (1.5) is unique and we call it the (probabilistic) solution of (1.1). The second step is to use (1.5) to derive regularity properties of u . Our main result says that there is $c > 0$ such that

$$(1.6) \quad \sup_{0 \leq t \leq T} E \|u(t)\|_{L^2(E;m)}^2 + E \int_0^T \|u(t)\|_V^2 dt \leq cE \left(\|\varphi\|_{L^2(E;m)}^2 \int_0^T \left(\|f(t, \cdot, 0)\|_{L^2(E;m)}^2 + \sum_{k=1}^\infty \|g_k(t, \cdot, 0)\|_{L^2(E;m)}^2 \right) dt \right).$$

The problems of existence and uniqueness of solutions of (1.1) have been the subject of numerous papers. Here let us mention important papers [21, 22] dealing with mild solutions of equations of the form (1.1) with $A_t = \Delta$ and B being a spatially homogeneous Wiener process. We also refer to [21, 22] for many bibliographic comments on (1.1) with $A_t = \Delta$. Note that in [21, 22] the “semigroup approach” to (1.1) is used. For results on (1.1) (with A_t being local operators) which can be obtained by using the “variational approach”, see [15, 25] and the references therein. Solutions in the sense defined by Walsh [29] are considered for instance in [1, 9] (see also the expository paper [5]).

The approach of [20], via BDSDE, was developed in several papers. In [2, 3, 30, 31] it is assumed that the operators A_t are the same as in [20], i.e. second order operators in nondivergence form with coefficients having some regularity properties. In [2, 31] under the assumption that f, \tilde{g} are Lipschitz continuous a stochastic representation of weak solutions of the Cauchy problem for SPDEs in terms of BDSDE is given. In [30] the assumption that f is Lipschitz continuous in u is weakened to monotonicity combined with the linear growth condition on u . In [3] a stochastic representation is given for SPDEs with nonlinear boundary Neumann conditions. Paper [31] also deals with stationary solutions of a SPDEs and related BDSDEs with infinite horizon. In [7] the semigroup method is used to prove that if f, g are Lipschitz continuous then there exists a unique mild solution of a SPDE involving a general nonnegative self-adjoint operator A (not depending on t) and finite-dimensional noise. Then this analytical result is used to get a stochastic representation of the solution in case A is a diffusion operator. It is also worth noting that in [1, 9], in case B is a space-time white noise, $A_t = \partial^2/\partial x^2$ and g is nondegenerate, existence and uniqueness results for (1.1) with irregular f are proved (f is merely measurable and satisfies some integrability condition in case g is constant, or f is measurable and locally bounded and g is nondegenerate and satisfies some regularity conditions).

The novelty of our paper lies in the fact that we prove in a unified way the existence and uniqueness of solutions of (1.1) for much wider classes of operators than those considered in previous papers or under less restrictive assumptions on f (see, however, the one-dimensional results proved in [1, 9]). In particular, in contrast to [7], in our paper the operators A_t may depend on time and we only assume that f is continuous and monotone in u (with no restrictions on the growth of f with respect to u). Secondly, we show in a unified way that the solutions of (1.1) belong to some Sobolev spaces and we prove energy estimates. Moreover, we obtain a stochastic representation of solutions of the form (1.5) for quasi-every (and not just m -a.e.) point in $E_{0,T}$. Let us stress once again, however, that except for some special cases, we have to assume that B is a Q -Wiener process with Q of trace class. Therefore our results do not cover the existence and uniqueness results of [21, 22] obtained for SPDEs with a spatially homogeneous Wiener process.

Our methods of proofs are also new. We think that of particular interest is our method for deriving from (1.5) regularity properties of u . The method is probabilistic in nature. As already mentioned, the general idea comes from our previous papers [12, 13, 14] on PDEs and BSDEs. Here, however, new difficulties and subtleties arise. The idea is as follows. We show that given a solution u of (1.1), i.e. a quasi-continuous function u satisfying (1.5), one can find a process M such that $(Y, M) = (u(\mathbf{X}), M)$ is a solution of BDSDE (1.4). Thus, in fact, (1.4) and (1.5) are equivalent. It is

also worth mentioning here that the regularity of trajectories of the process $u(X)$ (i.e. the fact that $u(X)$ is càdlàg and $[u(X)]_- = u(X_-)$) do not follow directly from the deterministic potential theory, because in general the nest for the quasi-continuous function $u(\omega; \cdot)$ depends on $\omega \in \Omega$. From (1.4) we immediately get

$$(1.7) \quad A_t^{[u]} := u(\mathbf{X}_t) - u(\mathbf{X}_0) = M_t + N_t, \quad t \in [0, T],$$

with

$$(1.8) \quad N_t = - \int_0^t f(\mathbf{X}_r, u(\mathbf{X}_r)) dr - \int_0^t g(\mathbf{X}_r, u(\mathbf{X}_r)) d^\dagger \beta_r^t.$$

The process $A^{[u]}$ in (1.7) is a random additive functional (AF) of the part $\mathbf{M}^{0,T}$ of the process \mathbf{M} on $E_{0,T}$. We show that M is a random martingale AF of $\mathbf{M}^{0,T}$ of finite energy and N is a random continuous AF of $\mathbf{M}^{0,T}$ of finite energy (we introduce these notions in Section 4). However, in most interesting cases N is not of zero energy, because from (1.8) it follows that

$$Ee(N) = \frac{1}{2} E \int_0^T \int_E \|g(t, x, u(t, x))\|^2 dt m(dx).$$

Therefore (1.7) cannot be viewed as Fukushima’s decomposition of $A^{[u]}$. Nevertheless, we are able to prove the following formula for the energy of M :

$$(1.9) \quad Ee(M) = E \left(\|u(0)\|_{L^2(E;m)}^2 + \int_0^T B^{(t)}(u(t), u(t)) dt - \frac{1}{2} \int_{E_{0,T}} |u(z)|^2 k(dz) \right),$$

where k is some killing measure. Roughly speaking, we obtain the energy estimate (1.6) for u by combining a priori estimates for the solution $(u(\mathbf{X}), M)$ of (1.4) with the estimate (1.9). The estimates for $(u(\mathbf{X}), M)$ are proved by using BSDE methods. We also prove that if $Ee(N) > 0$ then

$$(1.10) \quad u \notin \mathbb{W} = \left\{ v \in L^2(\Omega \times (0, T); V) : \frac{\partial v}{\partial t} \in L^2(0, T; V') \text{ } P\text{-a.s.} \right\},$$

which shows the difference between the regularity theory for (1.1) and for usual PDEs.

In the last section of the paper we show a connection between probabilistic and mild solutions of (1.3) in case f is Lipschitz continuous in u and $A_t = A, t \in [0, T]$. Roughly speaking, changing the order of integration in (1.5) and using the fact that

$$E_x f(X_t) = P_t f(x), \quad f \in L^2(E; m), \text{ } m\text{-a.e. } x \in E,$$

where $\{P_t, t \geq 0\}$ is the semigroup on $L^2(E; m)$ generated by A , we find

after some direct calculation that

$$u(s) = P_{T-s}\varphi + \int_s^T P_{t-s}F(t, u(t)) dt + \int_s^T P_{t-s}G(t, u(t)) d^\dagger B_t$$

where F and G are the Nemytskiĭ operators corresponding to f and \tilde{g} .

Finally, note that unlike [12, 13, 14], in the case where A_t are defined by (1.2), we treat regularity of equations with coefficients f, g depending both on the solution and its gradient.

2. General BDSDEs. In this section we consider general (nonMarkovian) BDSDEs with final condition ξ and coefficients f, g (BDSDE(ξ, f, g) for short) of the form

$$(2.1) \quad Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T g(r, Y_r) d^\dagger \beta_r - \int_t^T dM_r, \quad t \in [0, T].$$

To formulate the definition of a solution of (2.1) we need some notation. In what follows, $\beta = (\beta^k)_{k \in \mathbb{N}}$ is a sequence of mutually independent one-dimensional standard Wiener processes defined on some probability space (Ω, \mathcal{F}, P) , and $(\Omega', \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, P')$ is some filtered probability space such that (\mathcal{G}_t) is right continuous and complete. We set

$$\mathcal{F}_{t,T}^\beta = \sigma(\beta_r^k - \beta_t^k, r \in [t, T], k \in \mathbb{N}) \vee \mathcal{N}, \quad \mathcal{F}_t^\beta = \mathcal{F}_{0,t}^\beta, \quad t \in [0, T],$$

where $\mathcal{N} = \{A \subset \Omega : \exists B \in \mathcal{F}_T^\beta \text{ such that } A \subset B, P(B) = 0\}$, and then we set

$$\mathcal{F}_t = \mathcal{F}_{t,T}^\beta \vee \mathcal{G}_t, \quad t \in [0, T].$$

Note that (\mathcal{F}_t) is not increasing, so it is not a filtration. We also set

$$\mathbb{P} = P \otimes P'$$

and we denote by \mathbb{E} (resp. E, E') the expectation with respect to the measure \mathbb{P} (resp. P, P'). Let X be a process defined on Ω , and Y be a process on Ω' . Throughout what follows, without explicit mention we shall freely identify them with processes on $\Omega \times \Omega'$ defined as

$$(2.2) \quad X(\omega, \omega') = X(\omega), \quad Y(\omega, \omega') = Y(\omega').$$

We will need the following spaces.

- M is the space of measurable processes $X = \{(X_t)_{t \in [0, T]}\}$ defined on $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{G}, \mathbb{P})$ such that for a.e. $t \in [0, T]$ the random variable X_t is \mathcal{F}_t -measurable. M^2 is the subspace of M consisting of all processes X such that $(\mathbb{E} \int_0^T |X_t|^2 dt)^{1/2} < \infty$.
- \mathcal{S}^2 is the space of càdlàg processes $Y \in M$ such that $\|Y\|_{\mathcal{S}^2}^2 = \mathbb{E} \sup_{0 \leq t \leq T} |Y_t|^2 < \infty$.

- \mathcal{M}^2 is the space of càdlàg processes $X \in M$ such that X is an $(\mathcal{F}_T^\beta \vee \mathcal{G}_t)$ -martingale with $X_0 = 0$ and $\mathbb{E}[X]_T < \infty$, where $[X]$ denotes the quadratic variation process of X .

We will look for solutions of (2.1) in the space $\mathcal{S}^2 \times \mathcal{M}^2$. Note that \mathcal{S}^2 equipped with the norm $\|\cdot\|_{\mathcal{S}^2}$ is a Banach space. Similarly, \mathcal{M}^2 equipped with the norm $\|M\|_{\mathcal{M}^2} = (\mathbb{E}[M]_T)^{1/2}$ is a Banach space.

REMARK 2.1. By a standard argument, if $M \in \mathcal{M}^2$ then for P -a.e. $\omega \in \Omega$ the process $M(\omega, \cdot)$ is a square integrable (\mathcal{G}_t) -martingale.

Assume that we are given an \mathcal{F}_T -measurable random variable ξ and two families $\{f(t, y), t \geq 0\}_{y \in \mathbb{R}}, \{g_k(t, y), t \geq 0\}_{y \in \mathbb{R}, k \in \mathbb{N}}$ of processes $f(\cdot, y), g_k(\cdot, y) : \Omega \times \Omega' \times [0, T] \rightarrow \mathbb{R}$ of class M (for brevity, in our notation we omit the dependence on $(\omega, \omega') \in \Omega \times \Omega'$). Let $g(\cdot, y) = (g_1(\cdot, y), g_2(\cdot, y), \dots)$ and let $\|x\| = (\sum_{k=1}^\infty |x_k|^2)^{1/2}$ denote the usual norm in l^2 .

DEFINITION. We say that a pair of processes $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}^2$ is a solution of BDSDE(ξ, f, g) if

- (a) $\int_0^T |f(t, Y_t)| dt < \infty, \int_0^T \|g(t, Y_t)\|^2 dt < \infty$ \mathbb{P} -a.s.,
- (b) Equation (2.1) is satisfied \mathbb{P} -a.s. (Note that in (2.1), $\int_t^T g(r, Y_r) d^\dagger \beta_r = \sum_{k=1}^\infty \int_t^T g_k(r, Y_r) d^\dagger \beta_r^k$ and the integrals involving the processes β^k are backward Itô integrals; observe that under (a) the series converges in $L^2(\Omega, \mathcal{F}, P)$ for P' -a.s. $\omega' \in \Omega'$).

REMARK 2.2. Set

$$\hat{\beta} = (\hat{\beta}^k)_{k \in \mathbb{N}}, \quad \hat{\beta}_t^k = \beta_{T-t}^k - \beta_T^k, \quad t \in [0, T],$$

and define $\mathcal{F}_{t,T}^{\hat{\beta}}, \mathcal{F}_t^{\hat{\beta}}$ to be $\mathcal{F}_{t,T}^\beta, \mathcal{F}_t^\beta$ but with β replaced by $\hat{\beta}$. Then $\hat{\beta}$ is a sequence of mutually independent standard Wiener processes. If η_t is $\mathcal{F}_{t,T}^\beta$ -measurable, then η_{T-t} is $\mathcal{F}_t^{\hat{\beta}}$ -measurable and one can check that if $\eta = (\eta^1, \eta^2, \dots)$ is an $(\mathcal{F}_{t,T}^\beta)_{t \in [0, T]}$ -adapted process such that $P(\int_0^T \|\eta_t\|^2 dt < \infty) = 1$ then

$$\int_t^T \eta_s d^\dagger \beta_s = - \int_0^{T-t} \eta_{T-s} d\hat{\beta}_s, \quad t \in [0, T]$$

(see [31, p. 176]).

We are going to show that there exists a unique solution of (2.1) under the following assumptions.

- (A1) $\mathbb{E}|\xi|^2 < \infty, \mathbb{E} \int_0^T |f(t, 0)|^2 dt + \mathbb{E} \int_0^T \|g(t, 0)\|^2 dt < \infty$.
- (A2) For every $y \in \mathbb{R}, \int_0^T |f(t, y)| dt < \infty$ \mathbb{P} -a.s.
- (A3) There exist constants $l, L > 0$ and functions $L_k : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ such that $\sup_{t \leq T} \sum_{k=1}^\infty L_k^2(t) \leq l$ P -a.s. and for a.e. $t \in [0, T]$,

- (a) $(f(t, y) - f(t, y'))(y - y') \leq L|y - y'|^2$ for all $y, y' \in \mathbb{R}$, \mathbb{P} -a.s.,
- (b) $|g_k(t, y) - g_k(t, y')| \leq L_k(t)|y - y'|$ for all $y, y' \in \mathbb{R}$, \mathbb{P} -a.s.

(A4) For a.e. $t \in [0, T]$ the mapping $\mathbb{R} \ni y \mapsto f(t, y)$ is continuous.

The uniqueness for (2.1) follows from the following comparison result.

PROPOSITION 2.3. *Let g satisfy (A3b) and either f or f' satisfy (A3a). Let (Y, M) be a solution of BDSDE(ξ, f, g) and (Y', M') be a solution of BDSDE(ξ', f', g). If $\xi \leq \xi'$ \mathbb{P} -a.s. and $f'(t, y) \leq f(t, y)$ for a.e. $t \in [0, T]$ and every $y \in \mathbb{R}$ then*

$$Y'_t \leq Y_t, \quad t \in [0, T], \mathbb{P}\text{-a.s.}$$

Proof. We assume that f satisfies (A2). In case f' satisfies (A2) the proof is analogous. By the Itô–Meyer formula,

$$\begin{aligned} |(Y'_t - Y_t)^+|^2 &+ \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} d[M' - M]_r \\ &\leq 2 \int_t^T (Y'_r - Y_r)^+ (f'(r, Y'_r) - f(r, Y_r)) dr \\ &\quad + 2 \int_t^T (Y'_{r-} - Y_{r-})^+ (g(r, Y'_r) - g(r, Y_r)) d^\dagger \beta_r \\ &\quad - 2 \int_t^T (Y'_{r-} - Y_{r-})^+ d(M' - M)_r \\ &\quad + \sum_{k=1}^\infty \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |g_k(r, Y'_r) - g_k(r, Y_r)|^2 dr. \end{aligned}$$

By the assumptions of the proposition,

$$\int_t^T (Y'_r - Y_r)^+ (f'(r, Y'_r) - f(r, Y_r)) dr \leq L \int_t^T |(Y'_r - Y_r)^+|^2 dr$$

and

$$\begin{aligned} \sum_{k=1}^\infty \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |g_k(r, Y'_r) - g_k(r, Y_r)|^2 dr \\ \leq \sum_{k=1}^\infty \int_t^T L_k^2(r) \mathbf{1}_{\{Y'_r > Y_r\}} |Y'_r - Y_r|^2 dr \leq l \int_t^T |(Y'_r - Y_r)^+|^2 dr. \end{aligned}$$

Hence

$$\mathbb{E}|(Y'_t - Y_t)^+|^2 \leq 2(L + l) \mathbb{E} \int_t^T |(Y'_r - Y_r)^+|^2 dr, \quad t \in [0, T],$$

so applying Gronwall’s lemma we get the desired result. ■

COROLLARY 2.4. *Let assumption (A3) hold. Then there exists at most one solution of BDSDE(ξ, f, g).*

PROPOSITION 2.5. *Assume that (A1), (A3) are satisfied. Let (Y, M) be a solution of BDSDE(ξ, f, g). Then there exists $c > 0$ depending only on T, l, L such that*

$$\mathbb{E} \left(\sup_{t \leq T} |Y_t|^2 + \int_0^T d[M]_t + \sup_{0 \leq t \leq T} \left| \int_0^t f(r, Y_r) dr \right|^2 \right) \leq c \mathbb{E} \left(|\xi|^2 + \int_0^T (|f(t, 0)|^2 + \|g(t, 0)\|^2) dt \right).$$

Proof. By the Itô–Meyer formula and (A3),

$$\begin{aligned} |Y_t|^2 + \int_t^T d[M]_r &= |\xi|^2 + 2 \int_t^T f(r, Y_r) Y_r dr + \int_t^T \|g(r, Y_r)\|^2 dr \\ &\quad + 2 \int_t^T Y_r g(r, Y_r) d^\dagger \beta_r - 2 \int_t^T Y_{r-} dM_r \\ &\leq |\xi|^2 + (1 + 2L + 2l) \int_t^T |Y_r|^2 dr + \int_t^T (|f(r, 0)|^2 + 2\|g(r, 0)\|^2) dr \\ &\quad + 2 \int_t^T Y_r g(r, Y_r) d^\dagger \beta_r - 2 \int_t^T Y_{r-} dM_r, \quad t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned}$$

From this and Gronwall’s lemma,

$$\mathbb{E}|Y_t|^2 + \mathbb{E} \int_0^T d[M]_t \leq c(L, l, T) \mathbb{E} \left(|\xi|^2 + \int_0^T (|f(t, 0)|^2 + \|g(t, 0)\|^2) dt \right).$$

Therefore applying the Burkholder–Davis–Gundy inequality we get the desired result. ■

THEOREM 2.6. *Let assumptions (A1)–(A4) hold. Then there exists a solution (Y, M) of BDSDE(ξ, f, g).*

Proof. STEP 1. Assume that $g(t, Y_t) = g(t), f(t, Y_t) = f(t), t \in [0, T]$. We first prove that there exists a solution (Y, M) of the linear equation

$$(2.3) \quad Y_t = \xi + \int_t^T f(r) dr + \int_t^T g(r) d^\dagger \beta_r - \int_t^T dM_r, \quad t \in [0, T], \mathbb{P}\text{-a.s.}$$

To this end, set $\mathcal{A}_t = \mathcal{F}_T^\beta \vee \mathcal{G}_t, t \in [0, T]$, and define Y, M by

$$Y_t = \mathbb{E}\left(\xi + \int_t^T f(r) dr + \int_t^T g(r) d^\dagger\beta_r \mid \mathcal{A}_t\right), \quad t \in [0, T],$$

and

$$\begin{aligned} M_t &= \mathbb{E}\left(\xi + \int_0^T f(r) dr + \int_0^T g(r) d\beta_r^\dagger \mid \mathcal{A}_t\right) - Y_0 \\ &= E\left(\xi + \int_0^T f(r) dr + \int_0^T g(r) d\beta_r^\dagger \mid \mathcal{G}_t\right) - Y_0, \quad t \in [0, T]. \end{aligned}$$

One can check that the pair (Y, M) satisfies (2.3). To show that Y, M are (\mathcal{F}_t) -adapted, set

$$(2.4) \quad \Lambda_t = \xi + \int_t^T f(r) dr + \int_t^T g(r) d^\dagger\beta_r, \quad t \in [0, T].$$

With this notation we have

$$(2.5) \quad \begin{aligned} Y_t = \mathbb{E}(\Lambda_t \mid \mathcal{A}_t) &= \mathbb{E}(\Lambda_t \mid \mathcal{G}_t \vee \mathcal{F}_T^\beta) = \mathbb{E}(\Lambda_t \mid \mathcal{G}_t \vee \mathcal{F}_{t,T}^\beta \vee \mathcal{F}_t^\beta) \\ &= \mathbb{E}(\Lambda_t \mid \mathcal{F}_t \vee \mathcal{F}_t^\beta). \end{aligned}$$

Since f, g are measurable processes that are $(\mathcal{G}_T \vee \mathcal{F}_{t,T}^\beta)$ -adapted, they have modifications which are $(\mathcal{G}_T \vee \mathcal{F}_{t,T}^\beta)$ -progressively measurable. Therefore the integrals on the right-hand side of (2.4) are $(\mathcal{G}_T \vee \mathcal{F}_{t,T}^\beta)$ -adapted. Hence

$$\sigma(\Lambda_t) \subset \mathcal{G}_T \vee (\mathcal{G}_T \vee \mathcal{F}_{t,T}^\beta) = \mathcal{G}_T \vee \mathcal{F}_{t,T}^\beta.$$

It follows in particular that $\sigma(\Lambda_t) \vee \mathcal{F}_t$ is independent of \mathcal{F}_t^β . From this, [10, Proposition 5.6] and (2.5),

$$Y_t = \mathbb{E}(\Lambda_t \mid \mathcal{F}_t \vee \mathcal{F}_t^\beta) = \mathbb{E}(\Lambda_t \mid \mathcal{F}_t).$$

Thus Y is (\mathcal{F}_t) -adapted. That M is (\mathcal{F}_t) -adapted now follows from the equality

$$M_t = Y_t - Y_0 + \int_0^t f(r) dr + \int_0^t g(r) d^\dagger\beta_r.$$

STEP 2. Assume that f is Lipschitz continuous in y with Lipschitz constant L . Let $(Y^0, M^0) = (0, 0)$ and let (Y^{n+1}, M^{n+1}) be a solution of the equation

$$(2.6) \quad \begin{aligned} Y_t^{n+1} &= \xi + \int_t^T f(r, Y_r^n) dr + \int_t^T g(r, Y_r^n) d^\dagger\beta_r \\ &\quad + \int_t^T dM_r^{n+1}, \quad t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned}$$

The sequence $\{(Y^n, M^n)\}$ is well defined, i.e. if $(Y^n, M^n) \in \mathcal{S}^2 \otimes \mathcal{M}^2$ then (Y^{n+1}, M^{n+1}) exists and belongs to $\mathcal{S}^2 \otimes \mathcal{M}^2$. To see this, assume that $(Y^n, M^n) \in \mathcal{S}^2 \otimes \mathcal{M}^2$. Then

$$\begin{aligned} \mathbb{E} \int_0^T |f(t, Y_t^n)|^2 dr &\leq 2L^2 \mathbb{E} \int_0^T |Y_t^n|^2 dt + 2\mathbb{E} \int_0^T |f(t, 0)|^2 dt < \infty, \\ \mathbb{E} \int_0^T \|g(t, Y_t^n)\|^2 dt &\leq 2L^2 \mathbb{E} \int_0^T |Y_t^n|^2 dt + 2\mathbb{E} \int_0^T \|g(t, 0)\|^2 dt < \infty. \end{aligned}$$

Therefore by Step 1 there exists a solution $(Y^{n+1}, M^{n+1}) \in \mathcal{S}^2 \otimes \mathcal{M}^2$ of (2.6). By the Itô–Meyer formula and the assumption on f ,

$$\begin{aligned} (2.7) \quad |Y_t^{n+1} - Y_t^n|^2 + \int_t^T d[M^{n+1} - M^n]_r &= 2 \int_t^T (Y_r^{n+1} - Y_r^n)(f(r, Y_r^n) - f(r, Y_r^{n-1})) dr \\ &\quad - 2 \int_t^T (Y_{r-}^{n+1} - Y_{r-}^n) d(M^{n+1} - M^n)_r \\ &\quad + 2 \int_t^T (Y_r^{n+1} - Y_r^n)(g(r, Y_r^n) - g(r, Y_r^{n-1})) d^\dagger \beta_r \\ &\quad + \int_t^T |g(r, Y_r^n) - g(r, Y_r^{n-1})|^2 dr \\ &\leq \int_t^T |Y_r^{n+1} - Y_r^n|^2 dr + L^2 \int_t^T |Y_t^n - Y_r^{n-1}|^2 dr \\ &\quad + l^2 \int_t^T |Y_r^n - Y_r^{n-1}|^2 dr - 2 \int_t^T (Y_{r-}^{n+1} - Y_{r-}^n) d(M^{n+1} - M^n)_r \\ &\quad - 2 \int_t^T (Y_r^{n+1} - Y_r^n)(g(r, Y_r^n) - g(r, Y_r^{n-1})) d^\dagger \beta_r. \end{aligned}$$

Taking the expectation of both sides of the above inequality and using Gronwall’s lemma we get

$$\mathbb{E}|Y_t^{n+1} - Y_t^n|^2 + \mathbb{E} \int_t^T d[M^{n+1} - M^n]_r \leq C \mathbb{E} \int_t^T |Y_r^n - Y_t^{n-1}|^2 dr, \quad t \in [0, T],$$

for some C depending only on T, L, l . Hence

$$\sup_{t \leq r \leq T} \mathbb{E}|Y_r^{n+1} - Y_r^n|^2 + \mathbb{E} \int_t^T d[M^{n+1} - M^n]_r \leq C(T-t) \sup_{t \leq r \leq T} \mathbb{E}|Y_r^n - Y_r^{n-1}|^2.$$

Write

$$\|(Y, M)\|_t^2 = \sup_{t \leq r \leq T} \mathbb{E}|Y_r|^2 + \mathbb{E} \int_t^T d[M]_r.$$

With this notation we have

$$\begin{aligned} \|(Y^{n+1}, M^{n+1}) - (Y^n, M^n)\|_t \\ \leq C(T-t)^{1/2} \|(Y^n, M^n) - (Y^{n-1}, M^{n-1})\|_t, \quad n \geq 1, t \in [0, T]. \end{aligned}$$

From this one can deduce that

$$\|(Y^{n+k}, M^{n+k}) - (Y^n, M^n)\|_{t_1} \leq 2^{-(n+1)} \|(Y^1, M^1)\|_{t_1}$$

for $t_1 \in [0, T]$ such that $C(T-t_1) = 2^{-1}$. Therefore dividing the interval $[0, T]$ into small intervals and using a standard argument one can show that

$$\lim_{n, m \rightarrow \infty} \|(Y^m, M^m) - (Y^n, M^n)\|_0 = 0.$$

Using the Burkholder–Davis–Gundy inequality we conclude from the above convergence and (2.7) with (Y^{n+1}, M^{n+1}) replaced by (Y^m, M^m) that

$$\lim_{n, m \rightarrow \infty} \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 + \int_0^T d[M^n - M^m]_t \right) = 0.$$

Let (Y, M) be the limit of $\{(Y^n, M^n)\}$ in $\mathcal{S}^2 \times \mathcal{M}^2$. Letting $n \rightarrow \infty$ in (2.6) shows that (Y, M) is a solution of $\text{BDSDE}(\xi, f, g)$.

STEP 3. Now we assume that f satisfies the assumptions of the theorem and moreover there exists $\lambda \in \mathbb{R}$ such that $f(t, y) \geq \lambda$ for a.e. $t \in [0, T]$ and every $y \in \mathbb{R}$. Set

$$f_n(t, y) = \inf_{x \in \mathbb{Q}} \{n|y-x| + f(t, x) - Lx\} + Ly, \quad t \in [0, T], y \in \mathbb{R}.$$

It is an elementary check that f_n has the following properties: for a.e. $t \in [0, T]$,

- (a) $|f_n(t, y) - f_n(t, y')| \leq (L+n)|y-y'|$ for all $y, y' \in \mathbb{R}$,
- (b) $\lambda \leq f_n(t, y) \leq f(t, y)$ for every $y \in \mathbb{R}$,
- (c) $f_n(t, \cdot) \nearrow f(t, \cdot)$ uniformly on compact subsets of \mathbb{R} ,
- (d) $(f_n(t, y) - f_n(t, y'))(y - y') \leq L|y - y'|^2$ for all $y, y' \in \mathbb{R}$.

By Step 2, for each $n \geq 1$ there is a solution $(Y^n, M^n) \in \mathcal{S}^2 \otimes \mathcal{M}^2$ of $\text{BDSDE}(\xi, f_n, g)$. By Proposition 2.3,

$$(2.8) \quad Y_t^n \leq Y_t^{n+1}, \quad t \in [0, T], \mathbb{P}\text{-a.s.}, n \geq 1.$$

Set $Y_t = \sup_{n \geq 1} Y_t^n$. By Proposition 2.5 and (b),

$$\begin{aligned}
 (2.9) \quad \mathbb{E} \sup_{t \leq T} |Y_t^n|^2 + \mathbb{E}[M^n]_T &\leq C\mathbb{E} \left(|\xi|^2 + \int_0^T (|f_n(t, 0)|^2 + \|g(t, 0)\|^2) dt \right) \\
 &\leq C\mathbb{E} \left(|\xi|^2 + \lambda^2 + \int_0^T (|f(t, 0)|^2 + \|g(t, 0)\|^2) dt \right).
 \end{aligned}$$

For every $\varepsilon, \eta > 0$ we have

$$\begin{aligned}
 &\mathbb{P} \left(\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt > \varepsilon \right) \\
 &\leq \mathbb{P} \left(\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt > \varepsilon, \sup_{n \geq 1} \sup_{t \leq T} |Y_t^n| \leq \eta \right) \\
 &\quad + \mathbb{P} \left(\sup_{n \geq 1} \sup_{t \leq T} |Y_t^n| \geq \eta \right) \\
 &\leq \mathbb{P} \left(\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt > \varepsilon, \sup_{n \geq 1} \sup_{t \leq T} |Y_t^n| \leq \eta \right) \\
 &\quad + \eta^{-2} \left(\mathbb{E} \sup_{t \leq T} |Y_t^1|^2 + \mathbb{E} \sup_{t \leq T} |Y_t|^2 \right),
 \end{aligned}$$

the last inequality being a consequence of (2.8) and Chebyshev’s inequality. By (b), (c) and (A2) the first term on the right-hand side of the above inequality tends to zero as $n \rightarrow \infty$. The second one tends to zero as $\eta \rightarrow \infty$ thanks to (2.9). Hence

$$(2.10) \quad \sup_{0 \leq t \leq T} \left| \int_0^t (f_n(r, Y_r^n) - f(r, Y_r)) dr \right| \rightarrow 0$$

in probability \mathbb{P} as $n \rightarrow \infty$. By (2.8), (2.9) and (A3) we also have

$$\begin{aligned}
 (2.11) \quad \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t (g(r, Y_r^n) - g(r, Y)) d^\dagger \beta_r \right|^2 \\
 \leq 4\mathbb{E} \sum_{k=1}^\infty \int_0^T |g_k(t, Y_t^n) - g_k(t, Y_t)|^2 dt \leq 4\mathbb{E} \int_0^T |Y_t^n - Y_t|^2 dt,
 \end{aligned}$$

which converges to zero as $n \rightarrow \infty$. By Proposition 2.5,

$$\sup_{n \geq 1} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t f_n(r, Y_r^n) dr \right|^2 < \infty.$$

Therefore letting $n \rightarrow \infty$ in the equation

$$Y_t^n = \mathbb{E} \left(\xi + \int_t^T f_n(r, Y_r^n) dr + \int_t^T g(r, Y_r^n) d^\dagger \beta_r \mid \mathcal{F}_t \right), \quad t \in [0, T],$$

shows that (Y, M) , where

$$M_t = \mathbb{E} \left(\xi + \int_0^T f(r, Y_r) dr + \int_0^T g(r, Y_r) d^\dagger \beta_r \mid \mathcal{F}_t \right) - Y_0,$$

is a solution of $\text{BDSDE}(\xi, f, g)$.

STEP 4. We now show how to dispense with the assumption that f is bounded from below. Let $f_n = f \vee (-n)$. By Step 3, for each $n \geq 1$ there exists a solution (Y^n, M^n) of $\text{BDSDE}(\xi, f_n, g)$. By Proposition 2.3, $Y_t^n \geq Y_t^{n+1}$, $t \in [0, T]$, \mathbb{P} -a.s. for $n \geq 1$, whereas by Proposition 2.5,

$$\mathbb{E} \sup_{t \leq T} |Y_t^n|^2 + \mathbb{E} \int_0^T d[M^n]_t \leq C \mathbb{E} \left(|\xi|^2 + \int_0^T (|f(t, 0)|^2 + \|g(t, 0)\|^2) dt \right).$$

Using the above properties of the processes Y^n one can show in much the same way as in Step 3 that (2.10) and (2.11) hold true and then that there exists a solution (Y, M) of $\text{BDSDE}(\xi, f, g)$. ■

3. SPDEs and Markov-type BDSDEs. In this section we first consider Markov-type BDSDEs. Roughly speaking, these are BDSDEs of the form (2.1) with filtration (\mathcal{G}_t) generated by some Markov process $\mathbf{M} = (\mathbf{X}, P_z)$ and with final condition ξ and coefficients f, g depending on ω' only through $\mathbf{X}(\omega')$. In our paper, \mathbf{M} is a Markov process associated with a time-dependent Dirichlet form. Using results of Section 3 we prove the existence and uniqueness of solutions of BDSDEs associated with \mathbf{M} for ξ, f, g satisfying some “markovian” analogue of conditions (A1)–(A4). Then we use this result to prove the existence and uniqueness of solutions of SPDE of the form (1.1) with operator $\partial/\partial t + A_t$ associated with the underlying Dirichlet form.

3.1. Dirichlet forms and Markov processes. In what follows, E is a locally compact separable metric space and m is an everywhere dense Borel measure on E . We denote by Δ the one-point compactification of E . If E is already compact then we adjoin Δ to E as an isolated point. We set $E^1 = \mathbb{R} \times E$, $E_T = [0, T] \times E$, $E_{0,T} = (0, T] \times E$, $m^1 = l^1 \otimes m$, $m_T = l^1|_{[0,T]} \otimes m$, where l^1 is the one-dimensional Lebesgue measure. We adopt the convention that every function φ on E is extended to E^1 by setting $\varphi(t, x) = \varphi(x)$, $(t, x) \in E^1$, and every function f on E^1 is extended to $E^1 \cup \{\Delta\}$ by setting $f(\Delta) = 0$. Similarly, we extend functions defined on $E_{0,T}$ or on E_T to functions on $E^1 \cup \{\Delta\}$ by setting $f(z) = 0$ outside $E_{0,T}$ or E_T , respectively.

We assume that we are given a family $\{B^{(t)}, t \in \mathbb{R}\}$ of regular Dirichlet forms on $H = L^2(E; m)$ with sector constant independent of t and common

domain $V \subset H$ (see, e.g., [16, 27] for the definition). We also assume that

- (a) $\mathbb{R} \ni t \mapsto B^{(t)}(\varphi, \psi)$ is measurable for every $\varphi, \psi \in V$,
- (b) there are $c_1, c_2 > 0$ such that $c_1 B^{(0)}(\varphi, \varphi) \leq B^{(t)}(\varphi, \varphi) \leq c_2 B^{(0)}(\varphi, \varphi)$ for every $t \in \mathbb{R}$ and $\varphi \in V$.

By assumption, $(B^{(0)}, V)$ is closed, i.e. V is a real Hilbert space with respect to $\tilde{B}_1^{(0)}(\cdot, \cdot)$, where $\tilde{B}^{(0)}(\varphi, \psi) = \frac{1}{2}(B^{(0)}(\varphi, \psi) + B^{(0)}(\psi, \varphi))$ and $\tilde{B}_1^{(0)}(\varphi, \psi) = \tilde{B}^{(0)}(\varphi, \psi) + (\varphi, \psi)_H$ for $\varphi, \psi \in V$. We denote by V' the dual space of V and we set

$$\mathcal{V} = L^2(\mathbb{R}; V), \quad \mathcal{V}' = L^2(\mathbb{R}; V'), \quad \mathcal{W} = \left\{ u \in \mathcal{V} : \frac{\partial u}{\partial t} \in \mathcal{V}' \right\}.$$

We will consider two time dependent Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}^{0,T}, D(\mathcal{E}^{0,T}))$ associated with the families $\{B^{(t)}, t \in \mathbb{R}\}$ and $\{B^{(t)}, t \in [0, T]\}$, respectively. The first one we define by putting $D(\mathcal{E}) = (\mathcal{W} \otimes \mathcal{V}) \cup (\mathcal{V} \otimes \mathcal{W})$ and

$$\mathcal{E}(u, v) = \begin{cases} \left\langle \frac{\partial u}{\partial t}, v \right\rangle + \int_{\mathbb{R}} B^{(t)}(u(t), v(t)) dt, & u \in \mathcal{W}, v \in \mathcal{V}, \\ -\left\langle u, \frac{\partial v}{\partial t} \right\rangle + \int_{\mathbb{R}} B^{(t)}(u(t), v(t)) dt, & u \in \mathcal{V}, v \in \mathcal{W}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between \mathcal{V} and \mathcal{V}' . It is known (see [27, Example I.4.9]) that $(\mathcal{E}, D(\mathcal{E}))$ is a generalized Dirichlet form on $L^2(E^1; m^1)$.

By [17, Theorem 4.2] (see also [19, Theorem 6.3.1]) there exists a Hunt process $\mathbf{M} = (\Omega', (\mathbf{X}_t)_{t \geq 0}, (P_z)_{z \in E^1 \cup \Delta}, (\mathcal{F}_t^{\mathbf{X}})_{t \geq 0})$ with state space E^1 , life time ζ and cemetery state Δ properly associated with $(\mathcal{E}, D(\mathcal{E}))$ in the resolvent sense. Moreover,

$$(3.1) \quad \mathbf{X}_t = (\iota(t), X_{\iota(t)}), \quad t \geq 0,$$

where ι is the uniform motion to the right, i.e. $\iota(t) = \iota(0) + t$, $\iota(0) = s$, P_z -a.s. for $z = (s, x)$. One can also check that $(X, (P_{s,x})_{x \in E})$ is a time inhomogeneous Markov process associated with the family $\{(B^{(t)}, V), t \geq 0\}$, i.e. for every $s \in \mathbb{R}$, $(X_{s+\cdot}, (P_{s,x})_{x \in E})$ is a Hunt process associated with the form $(B^{(s)}, V)$.

Let

$$\mathcal{W}_T = \left\{ u \in L^2(0, T; V) : \frac{\partial u}{\partial t} \in L^2(0, T; V'), u(T) = 0 \right\},$$

$$\mathcal{W}_0 = \left\{ u \in L^2(0, T; V) : \frac{\partial u}{\partial t} \in L^2(0, T; V'), u(0) = 0 \right\}.$$

We define the second form by setting $D(\mathcal{E}^{0,T}) = (\mathcal{W}_T \otimes L^2(0, T; V)) \cup$

$(\mathcal{W}_0 \otimes L^2(0, T; V))$ and

$$\mathcal{E}^{0,T}(u, v) = \begin{cases} \left\langle \frac{\partial u}{\partial t}, v \right\rangle + \mathcal{B}^{0,T}(u, v), & u \in \mathcal{W}_0, v \in L^2(0, T; V), \\ -\left\langle u, \frac{\partial v}{\partial t} \right\rangle + \mathcal{B}^{0,T}(u, v), & v \in \mathcal{W}_T, u \in L^2(0, T; V), \end{cases}$$

where now $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the space $L^2(0, T; V')$ and $L^2(0, T; V)$, and

$$(3.2) \quad \mathcal{B}^{0,T}(u, v) = \int_0^T B^{(t)}(u(t), v(t)) dt, \quad u, v \in L^2(0, T; V).$$

Note that by [27, Example I.4.9], $(\mathcal{E}^{0,T}, D(\mathcal{E}^{0,T}))$ is a generalized Dirichlet form on $L^2(E_{0,T}; m_T)$.

We set

$$R_\alpha f(z) = E_z \int_0^\infty e^{-\alpha t} f(\mathbf{X}_t) dt, \quad z \in E^1, f \in \mathcal{B}(E^1), \alpha \geq 0,$$

$$R_\alpha^{0,T} f(z) = E_z \int_0^{T-\iota(0)} e^{-\alpha t} f(\mathbf{X}_t) dt, \quad z \in E_{0,T}, f \in \mathcal{B}(E_{0,T}), \alpha \geq 0,$$

whenever the integrals exist. Let $\hat{\mathbf{M}} = (\hat{\mathbf{X}}, \hat{P}_z)$ be the dual process of \mathbf{M} (see [19, Section 3.3]). We set (whenever the integrals exist)

$$\hat{R}_\alpha f(z) = \hat{E}_z \int_0^\infty e^{-\alpha t} f(\hat{\mathbf{X}}_t) dt, \quad z \in E^1, f \in \mathcal{B}(E^1), \alpha \geq 0,$$

$$\hat{R}_\alpha^{0,T} f(z) = \hat{E}_z \int_0^{\iota(0)} e^{-\alpha t} f(\hat{\mathbf{X}}_t) dt, \quad z \in E_{0,T}, f \in \mathcal{B}(E_{0,T}), \alpha \geq 0.$$

To shorten notation, we write $R = R_0, \hat{R} = \hat{R}_0, R^{0,T} = R_0^{0,T}, \hat{R}^{0,T} = \hat{R}_0^{0,T}$. It is well known that $G_\alpha = R_\alpha, \hat{G}_\alpha = \hat{R}_\alpha$ on $L^2(E^1; m^1)$ for $\alpha > 0$, where $\{G_\alpha, \alpha > 0\}$ is the resolvent and $\{\hat{G}_\alpha, \alpha > 0\}$ is the dual resolvent associated with the form $(\mathcal{E}, D(\mathcal{E}))$. Similarly, $G_\alpha^{0,T} = R_\alpha^{0,T}, \hat{G}_\alpha^{0,T} = \hat{R}_\alpha^{0,T}$ on $L^2(E_{0,T}; m_T), \alpha \geq 0$, where $\{G_\alpha^{0,T}, \alpha \geq 0\}$ is the resolvent and $\{\hat{G}_\alpha^{0,T}, \alpha \geq 0\}$ is the dual resolvent associated with $(\mathcal{E}^{0,T}, D(\mathcal{E}^{0,T}))$.

We denote by A_t the operator associated with the form $(B^{(t)}, V)$, i.e.

$$(3.3) \quad (-A_t u, v) = B^{(t)}(u, v), \quad u \in D(A_t), v \in V,$$

where $D(A_t) = \{u \in V : v \mapsto B^{(t)}(u, v) \text{ is continuous with respect to } (\cdot, \cdot)^{1/2} \text{ on } V\}$ (see [16, Proposition I.2.16]).

Let cap be the capacity considered in [18] (see also [19, Section 6.2]). We say that a set B is \mathcal{E} -exceptional if $\text{cap}(B) = 0$. We say that some property

is satisfied *quasi-everywhere* (q.e. for short) if the set of those $z \in E^1$ for which it does not hold is \mathcal{E} -exceptional. Note that a nearly Borel set B is \mathcal{E} -exceptional if and only if it is \mathbf{M} -exceptional, i.e. $P_{m_1}(\sigma_B < \infty) = 0$, where $\sigma_B = \inf\{t > 0 : \mathbf{X}_t \in B\}$ (see, e.g., [18, p. 298]).

We say that $f : E_{0,T} \rightarrow \mathbb{R}$ is quasi-integrable ($f \in qL^1$ in notation) if f is Borel measurable and $P_z(\int_0^{T-\iota(0)} |f(\mathbf{X}_t)| dt < \infty) = 1$ for q.e. $z \in E_{0,T}$. We denote by $q\mathbb{L}^1$ the set of measurable functions $f : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ such that $P(\{\omega \in \Omega : f(\omega, \cdot) \in qL^1\}) = 1$.

3.2. Existence and uniqueness of solutions. Let A_t be the operator defined by (3.3). Suppose we are given measurable functions $\varphi : E \rightarrow \mathbb{R}$, $f : \Omega \times E_T \times \mathbb{R} \rightarrow \mathbb{R}$, $g_k : \Omega \times E_T \times \mathbb{R} \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, and let $g = (g_1, g_2, \dots)$. We are going to show that there exists a unique solution of the SPDE

$$(3.4) \quad du(t) = -(A_t u + f(t, x, u)) dt - g(t, x, u) d^\dagger \beta_t, \quad u(T) = \varphi,$$

under hypotheses (H1)–(H5) given below:

(H1) $\|\varphi\|_{L^2(E; m)} + E\|f(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2 + E\sum_{k=1}^\infty \|g_k(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2 < \infty$.

(H2) For every $y \in \mathbb{R}$ the mapping $E_{0,T} \ni z \mapsto f(z, y)$ belongs to $q\mathbb{L}^1$.

(H3) There exist $l, L > 0$ and measurable functions $L_k : E_{0,T} \rightarrow [0, \infty)$ such that $\sup_{z \in E_{0,T}} \sum_{k=1}^\infty L_k^2(z) \leq l$ and for every $z \in E_{0,T}$ and $y, y' \in \mathbb{R}$,

(a) $(f(z, y) - f(z, y'))(y - y') \leq L|y - y'|^2$,

(b) $|g_k(z, y) - g_k(z, y')| \leq L_k(z)|y - y'|$.

(H4) For every $\omega \in \Omega$ and $z \in E_{0,T}$ the mapping $\mathbb{R} \ni y \mapsto f(z, y)$ is continuous.

(H5) For every $y \in \mathbb{R}$ the mappings $f(\cdot, y), g_k(\cdot, y) : \Omega \times E_T \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, are $(\mathcal{F}_{t,T}^\beta)$ -progressively measurable, i.e. for every $t \in [0, T]$ the mappings $\Omega \times [T - t, T] \times E \ni (\omega, s, x) \mapsto f(\omega, s, x, y)$ and $\Omega \times [T - t, T] \times E \ni (\omega, s, x) \mapsto g_k(\omega, s, x, y)$ are $\mathcal{F}_{t,T}^\beta \otimes \mathcal{B}([T - t, T]) \otimes \mathcal{B}(E)$ -measurable.

In what follows

$$(3.5) \quad \beta_t^\iota = \beta_{t+\iota(0)}, \quad T_\iota = T - \iota(0), \quad \mathcal{F}_t = \mathcal{F}_{t, T_\iota}^{\beta^\iota} \vee \mathcal{F}_t^{\mathbf{X}}, \quad t \in [0, T],$$

where $\mathcal{F}_{t, T_\iota}^{\beta^\iota} = \sigma(\beta_{r \wedge T_\iota}^{\beta^\iota} - \beta_t^{\beta^\iota}, r \in [t, T])$ (see our convention (2.2)).

We will say that a random function $u : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ is *adapted* if $u(\mathbf{X}) \in M$, where M is defined as in Section 2 but with respect to (\mathcal{F}_t) defined by (3.5).

REMARK 3.1. (i) Assume (H5). Then for every $y \in \mathbb{R}$ there exists a $P \otimes m_T$ -version $\tilde{f}(\cdot, y)$ of $f(\cdot, y)$ and a $P \otimes m_T$ -version $\tilde{g}_k(\cdot, y)$ of $g_k(\cdot, y)$

such that the processes $\tilde{f}(\mathbf{X}, y), \tilde{g}_k(\mathbf{X}, y)$ are of class M under \mathbb{P}_z for q.e. $z \in E_{0,T}$. To see this, we let h stand for $f(\cdot, y)$ or for $g_k(\cdot, y)$. It is known that any $(\mathcal{F}_{t,T}^\beta)$ -progressively measurable square-integrable $h : \Omega \times E_T \rightarrow \mathbb{R}$ may be approximated in the space $L^2(\Omega \times E_T; P \otimes m_T)$ by linear combinations of processes of the form

$$(3.6) \quad S_j(\omega, t, x) = \mathbf{1}_{A_j}(\omega) \mathbf{1}_{(T-r_{j+1}, T-r_j]}(t) b_j(x),$$

where $0 \leq r_j \leq r_{j+1} \leq T$, $A_j \in \mathcal{F}_{T-r_j, T}^\beta$ and $\{b_j, j \geq 0\}$ is some orthonormal basis of H . It is an elementary check that the process $\{S_j(\mathbf{X}_t), t \geq 0\}$ is $\{\mathcal{F}_{t,T}^{\beta_t}\}$ -adapted. Therefore linear combinations of processes of the form (3.6) are of class M . Hence there is a sequence $\{h_n\}$ such that $h_n \rightarrow h$ in $L^2(\Omega \times E_{0,T}; P \otimes m_T)$ and the processes $h_n(\mathbf{X})$ are of class M . Let $\tilde{h} = \limsup_{n \rightarrow \infty} h_n$. Then $\tilde{h}(\mathbf{X})$ is of class M and \tilde{h} is a $P \otimes m_T$ -version of h . Let \hat{S}_{00} denote the set of all finite zero order integral measures ν on $E_{0,T}$ such that $\nu(E_{0,T}) < \infty$ and $\|\hat{R}^{0,T} \nu\|_\infty < \infty$. For every $\nu \in \hat{S}_{00}$ we have

$$\mathbb{E}_\nu \int_0^{T_i} |(\tilde{h}(\mathbf{X}_t) - h(\mathbf{X}_t))^2| dt \leq E \|\tilde{h} - h\|_{L^2(E_{0,T}; m_T)}^2 \|\hat{R}^{0,T} \nu\|_\infty = 0.$$

From this and [28, Theorem 2.5] (or [18, remark at the end of Section 4]) it follows that $\mathbb{P}_z(\tilde{h}(\mathbf{X}_t) = h(\mathbf{X}_t) \text{ for a.e. } t \in [0, T]) = 1$ for q.e. $z \in E_{0,T}$, which shows the desired result in case h is square-integrable. The general case is handled by approximating h pointwise by square-integrable functions.

(ii) From the fact that Carathéodory functions are jointly measurable it follows that if f, g_k satisfy (H3b), (H4), (H5) and $u : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ is adapted then the processes $t \mapsto \tilde{f}(\mathbf{X}_t, u(\mathbf{X}_t))$ and $t \mapsto \tilde{g}_k(\mathbf{X}_t, u(\mathbf{X}_t))$ are of class M .

We first give the definition of a solution of (3.4) and related Markov-type BDSDE with coefficients f, g . In what follows we always assume that the coefficients $f, g_k, k \geq 1$, satisfy (H5), and we always assume that we are taking their versions having the properties listed in Remark 3.1.

DEFINITION. We say that an adapted function $u : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ is a *solution of SPDE (3.4)* if

- (a) for q.e. $z \in E_{0,T}$ we have $\mathbb{P}_z(\int_0^{T_i} |f(\mathbf{X}_t, u(\mathbf{X}_t))| dt < \infty) = 1$ and $\mathbb{E}_z \sup_{0 \leq t \leq T_i} |\int_0^t f(\mathbf{X}_r, u(\mathbf{X}_r)) dr|^2 < \infty, \mathbb{E}_z \int_0^{T_i} \|g(\mathbf{X}_t, u(\mathbf{X}_t))\|^2 dt < \infty$, where $\mathbb{P}_z = P \otimes P_z$ and \mathbb{E}_z denotes the expectation with respect to \mathbb{P}_z ,

- (b) for P -a.e. $\omega \in \Omega$, $u(\omega; \cdot) : E_{0,T} \rightarrow \mathbb{R}$ is quasi-continuous and for q.e. $z \in E_{0,T}$,

$$u(z) = E_z \left(\varphi(\mathbf{X}_{T_l}) + \int_0^{T_l} f(\mathbf{X}_t, u(\mathbf{X}_t)) dt + \int_0^{T_l} g(\mathbf{X}_t, u(\mathbf{X}_t)) d^\dagger \beta_t^z \right).$$

DEFINITION. A pair $(Y^z, M^z) \in \mathcal{S}^2 \times \mathcal{M}^2$ is a *solution of the BDSDE*

$$(3.7) \quad \begin{aligned} Y_t^z &= \varphi(\mathbf{X}_{T_l}) + \int_t^{T_l} f(\mathbf{X}_r, Y_r^z) dr \\ &\quad + \int_t^{T_l} g(\mathbf{X}_r, Y_r^z) d^\dagger \beta_r^z - \int_0^{T_l} dM_r^z, \quad t \in [0, T_l] \end{aligned}$$

(BDSDE $_z(\varphi, f, g)$ for short) if

- (a) $\int_0^T |f(\mathbf{X}_t, Y_t^z)| dt < \infty$, $\int_0^T \|g(\mathbf{X}_t, Y_t^z)\|^2 dt < \infty$, \mathbb{P}_z -a.s.
- (b) Equation (3.7) is satisfied \mathbb{P}_z -a.s.

We first prove that under (H1)–(H5) for q.e. $z \in E_{0,T}$ there exists a unique solution of (3.7), and moreover one can find a version of the solution which is independent of z . In fact, the solution exists for every $z \in E_{0,T} \setminus N$, where

$$(3.8) \quad N = N_1 \cup N_2$$

with

$$N_1 = \left\{ z \in E_{0,T} : \mathbb{E}_z \left(|\varphi(\mathbf{X}_{T_l})|^2 + \int_0^{T_l} (|f(\mathbf{X}_t, 0)|^2 + \|g(\mathbf{X}_t, 0)\|^2) dt \right) = \infty \right\},$$

$$N_2 = \left\{ z \in E_{0,T} : \exists y \in \mathbb{R} \mathbb{P}_z \left(\int_0^{T_l} |f(\mathbf{X}_r, y)| dr = \infty \right) > 0 \right\}.$$

LEMMA 3.2. *Let (H1)–(H5) hold and let N be defined by (3.8). Then $\text{cap}(N) = 0$ and for every $z \in E_{0,T} \setminus N$ the data*

$$(\xi, \bar{f}(t, y), \bar{g}(t, y)) = (\varphi(\mathbf{X}_{T_l}), f(\mathbf{X}_t, y), g(\mathbf{X}_t, y)), \quad t \in [0, T], y \in \mathbb{R},$$

satisfy assumptions (A1)–(A4) under the measure \mathbb{P}_z .

Proof. Let $N_0 = \{z \in E_{0,T} : \mathbb{E}_z \int_0^{T_l} |f(\mathbf{X}_t, 0)|^2 dt = \infty\}$. We may assume that N_0 is compact. Let σ_{N_0} be the first hitting time of N_0 . Then

$$\mathbb{P}_z(\sigma_{N_0} < \zeta_t) = \mathbb{P}_z(\exists t < \zeta_t \mathbf{X}_t \in N_0) \leq \mathbb{P}_z \left(\mathbb{E}_{\mathbf{X}_{\sigma_{N_0}}} \int_0^{T_l} |f(\mathbf{X}_t, 0)|^2 dt = \infty \right)$$

$$\begin{aligned}
 &= \mathbb{P}_z \left(\mathbb{E}_z \left(\int_{\sigma_{N_0}}^{T_l} |f(\mathbf{X}_t, 0)|^2 dt \mid \mathcal{F}_{\sigma_{N_0}}^{\mathbf{X}} \right) = \infty \right) \\
 &\leq \mathbb{P}_z \left(\mathbb{E}_z \left(\int_0^{T_l} |f(\mathbf{X}_t, 0)|^2 dt \mid \mathcal{F}_{\sigma_{N_0}}^{\mathbf{X}} \right) = \infty \right) = 0
 \end{aligned}$$

for m_T -a.e. $z \in E_{0,T}$, because for any strictly positive Borel function ψ on $E_{0,T}$ such that $\|\hat{R}^{0,T} \psi\|_\infty < \infty$ we have

$$\begin{aligned}
 &\int_{E_{0,T}} \left(\mathbb{E}_z \int_0^{T_l} |f(\mathbf{X}_t, 0)|^2 dt \right) \psi(z) m_T(dz) = E(R^{0,T} f^2(\cdot, 0), \psi)_{L^2(E_{0,T}; m_T)} \\
 &= E(f^2(\cdot, 0), \hat{R}^{0,T} \psi)_{L^2(E_{0,T}; m_T)} \leq E\|f(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2 \cdot \|\hat{R}^{0,T} \psi\|_\infty < \infty.
 \end{aligned}$$

Hence $\text{cap}(N_0) = 0$. In much the same way one can show that $\text{cap}(N_1) = 0$. Moreover, by the definition of the space $q\mathbb{L}^1$, $\text{cap}(N_2) = 0$. Thus $\text{cap}(N) = 0$ and (A1)–(A4) are satisfied under the measure \mathbb{P}_z for z outside the set $N = N_1 \cup N_2$. ■

In what follows, we use the notation

$$\mathcal{A}_t = \mathcal{F}_{T_l}^{\beta^c} \vee \mathcal{F}_t^{\mathbf{X}}.$$

THEOREM 3.3. *Assume that φ, f, g satisfy (H1)–(H5) and define N by (3.8). Then for every $z \in E_{0,T} \setminus N$ there exists a solution (Y^z, M^z) of $\text{BDSDE}_z(\varphi, f, g)$. Moreover, there exists a pair of càdlàg (\mathcal{F}_t) -adapted processes (Y, M) such that for every $z \in E_{0,T} \setminus N$,*

$$(3.9) \quad (Y_t^z, M_t^z) = (Y_t, M_t), \quad t \in [0, T_l], \mathbb{P}_z\text{-a.s.}$$

Proof. The existence of a solution (Y^z, M^z) follows from Theorem 2.6 and Lemma 3.2. The proof of (3.9) will be divided into three steps.

STEP 1. Assume that f, g do not depend on the last variable y . Then

$$(3.10) \quad M_t^z = \mathbb{E}_z \left(\varphi(\mathbf{X}_{T_l}) + \int_0^{T_l} f(\mathbf{X}_r) dr + \int_0^{T_l} g(\mathbf{X}_r) d^\dagger \beta_r^c \mid \mathcal{A}_t \right) - Y_0^z$$

and

$$Y_t^z = \mathbb{E}_z \left(\varphi(\mathbf{X}_{T_l}) + \int_t^{T_l} f(\mathbf{X}_r) dr + \int_t^{T_l} g(\mathbf{X}_r) d^\dagger \beta_r^c \mid \mathcal{A}_t \right).$$

By [8, Lemma A.3.5] there exists a random variable H_0 such that $Y_0^z = H_0$ \mathbb{P}_z -a.s. for every $z \in E_{0,T} \setminus N$, while by [8, Lemma A.3.3, A.3.5] there exists an (\mathcal{A}_t) -adapted càdlàg process M such that $M_t = M_t^z$, $t \in [0, T_l]$ \mathbb{P}_z -a.s.

for $z \in E_{0,T} \setminus N$. Set

$$Y_t = H_0 - \int_0^t f(\mathbf{X}_r) dr - \int_0^t g(\mathbf{X}_r) d^\dagger \beta_r + \int_0^t dM_r, \quad t \in [0, T_l].$$

Then Y is an (\mathcal{F}_t) -adapted càdlàg process such that $Y_t = Y_t^z$, $t \in [0, T_l]$, \mathbb{P}_z -a.s. for every $z \in E_{0,T} \setminus N$.

STEP 2. We now consider general f, g (possibly depending on y) but we assume that f is Lipschitz continuous with respect to y uniformly in t . By Step 2 of the proof of Theorem 2.6,

$$(3.11) \quad (Y^{z,n}, M^{z,n}) \rightarrow (Y^z, M^z) \quad \text{in } S^2 \otimes \mathcal{M}^2,$$

where $(Y^{z,0}, M^{z,0}) = (0, 0)$ and

$$\begin{aligned} Y_t^{z,n+1} &= \varphi(\mathbf{X}_{T_l}) + \int_t^{T_l} f(\mathbf{X}_r, Y_r^{z,n}) dr + \int_t^{T_l} g(\mathbf{X}_r, Y_r^{z,n}) d^\dagger \beta_r \\ &\quad - \int_t^{T_l} dM_r^{z,n+1}, \quad t \in [0, T_l], \mathbb{P}_z\text{-a.s.} \end{aligned}$$

for $z \in E_{0,T} \setminus N$. By Step 1, for every $n \geq 0$ there exists a pair (Y^n, M^n) of (\mathcal{F}_t) -adapted càdlàg processes such that (3.9) holds for (Y^z, M^z) , (Y, M) replaced by $(Y^{z,n}, M^{z,n})$, (Y^n, M^n) . Therefore applying [8, Lemma A.3.3] we show the existence of a pair (Y, M) satisfying (3.9).

STEP 3. *The general case.* From the proof of Theorem 2.6 it follows that (3.11) holds for $(Y^{z,n}, M^{z,n})$ being a solution of $\text{BDSDE}_z(\varphi, f_n, g)$ with some Lipschitz continuous in y function f_n . By Step 2, (3.9) holds for $(Y^{z,n}, M^{z,n})$. Therefore applying [8, Lemma A.3.3] shows that (3.9) is satisfied. ■

Of course the pair (Y, M) of Theorem 3.3 is a solution of $\text{BDSDE}_z(\varphi, f, g)$ for q.e. $z \in E_{0,T}$. Our next goal is to show that the function u defined as $u(z) = E_z Y_0$ for $z \in E_{0,T} \setminus N$ is a solution of SPDE (3.4). We first prove this in Proposition 3.6 for linear equations and then in Theorem 3.7 in the general case. We begin with a simple but useful lemma.

LEMMA 3.4. *Assume that $\Lambda \in \mathcal{A}_{T_l}$ and let N be a properly exceptional subset of $E_{0,T}$. If $\mathbb{P}_z(\Lambda) = 1$ for $z \in E_{0,T} \setminus N$ then for every $z \in E_{0,T} \setminus N$, $\mathbb{P}_z(\theta_\tau^{-1}(\Lambda)) = 1$ for every stopping time τ with respect to (\mathcal{A}_t) such that $0 \leq \tau \leq T_l$.*

Proof. By the strong Markov property and proper exceptionality of N , for every $z \in E_{0,T} \setminus N$ we have

$$\mathbb{P}_z(\theta_\tau^{-1}(\Lambda^c)) = \mathbb{E}_z \mathbb{P}_{\mathbf{X}_\tau}(\Lambda^c) = 0,$$

which proves the lemma. ■

REMARK 3.5. It is clear that if τ is an (\mathcal{A}_t) -stopping time then $\tau(\omega; \cdot)$ is an $(\mathcal{F}_t^{\mathbf{X}})$ -stopping time for every $\omega \in \Omega$.

Now we extend the shift operator θ to $\Omega \times \Omega'$ by setting

$$\theta_t(\omega, \omega') = (\omega, \theta_t(\omega')).$$

It is clear that $\theta_t^{-1}(\mathcal{A}_0) \subset \mathcal{A}_t$.

PROPOSITION 3.6. Assume that φ, f, g satisfy (H1), (H2), (H5) and f, g do not depend on the last variable y . Define N by (3.8) and set

$$(3.12) \quad u(z) = E_z \left(\varphi(\mathbf{X}_{T_i}) + \int_0^{T_i} f(\mathbf{X}_t) dt + \int_0^{T_i} g(\mathbf{X}_t) d^\dagger \beta_t^t \right)$$

for $z \in E_{0,T} \setminus N$ and $u(z) = 0$ otherwise. Then for P -a.s. $\omega \in \Omega$ the function $u(\omega; \cdot)$ is quasi-continuous and for every $z \in E_{0,T} \setminus N$,

$$(3.13) \quad Y_t = u(\mathbf{X}_t), \quad t \in [0, T_i], \mathbb{P}_z\text{-a.s.},$$

where Y is the process of Theorem 3.3.

Proof. By [8, Theorem 4.1.1] we may and will assume that N is properly exceptional. Let (Y, M) be the pair of processes of Theorem 3.3. We shall show that for every $z \in E_{0,T} \setminus N$,

$$(3.14) \quad Y_0 \circ \theta_\tau = Y_\tau \quad \mathbb{P}_z\text{-a.s.}$$

for every (\mathcal{A}_t) -stopping time τ such that $0 \leq \tau \leq T_i$. To this end, first observe that \mathbb{P}_z -a.s. we have

$$(3.15) \quad \left(\int_0^{T_i} f(\mathbf{X}_t) dt \right) \circ \theta_\tau = \int_\tau^{T_i} f(\mathbf{X}_t) dt,$$

$$(3.16) \quad \left(\int_0^{T_i} g(\mathbf{X}_t) d^\dagger \beta_t^t \right) \circ \theta_\tau = \int_\tau^{T_i} g(\mathbf{X}_t) d^\dagger \beta_t^t.$$

Indeed, (3.15) holds since $A_t = \int_0^t f(\mathbf{X}_r) dr$ is a continuous AF of \mathbf{M} for fixed $\omega \in \Omega$. Equality (3.16) can be deduced from the identity

$$\left(\sum_{0 \leq t_i \leq T_i} g(\mathbf{X}_{t_{i+1}})(\beta_{t_{i+1}}^t - \beta_{t_i}^t) \right) \circ \theta_\tau = \sum_{0 \leq t_i \leq T_i - \tau} g(\mathbf{X}_{t_{i+1} + \tau})(\beta_{t_{i+1} + \tau}^t - \beta_{t_i + \tau}^t),$$

which holds \mathbb{P}_z -a.s. Here we have used the fact that $\iota(0) \circ \theta_\tau = \iota(0) + \tau$. It is also clear that

$$(3.17) \quad \varphi(\mathbf{X}_{T_i}) \circ \theta_\tau = \varphi(\mathbf{X}_{T_i}).$$

By [26, Theorem 50.19], $(N_t = (M_{t-\tau} \circ \theta_\tau - M_0 \circ \theta_\tau) \mathbf{1}_{[\tau, \infty)}(t), \mathcal{F}_t^{\mathbf{X}}, t \geq 0)$ is

a martingale for fixed $\omega \in \Omega$. Hence

$$(3.18) \quad \mathbb{E}_z \left(\left(\int_0^{T_l} dM_t \right) \circ \theta_\tau \mid \mathcal{A}_\tau \right) = E_z \left(\left(\int_0^{T_l} dM_t \right) \circ \theta_\tau \mid \mathcal{F}_\tau^{\mathbf{X}} \right) \\ = E_z(N_{T_l+\tau} \mid \mathcal{F}_\tau^{\mathbf{X}}) = 0.$$

We know that for every $z \in E_{0,T} \setminus N$,

$$Y_0 = \varphi(\mathbf{X}_{T_l}) + \int_0^{T_l} f(\mathbf{X}_r) dr - \int_0^{T_l} g(\mathbf{X}_r) d^\dagger \beta_r^t - \int_0^{T_l} dM_r \quad \mathbb{P}_z\text{-a.s.}$$

By Lemma 3.4 and (3.15)–(3.17), for every $z \in E_{0,T} \setminus N$,

$$Y_0 \circ \theta_\tau = \varphi(\mathbf{X}_{T_l}) + \int_\tau^{T_l} f(\mathbf{X}_r) dr - \int_\tau^{T_l} g(\mathbf{X}_r) d^\dagger \beta_r^t - \left(\int_0^{T_l} dM_r \right) \circ \theta_\tau.$$

Since Y_0 is \mathcal{A}_0 -measurable, $Y_0 \circ \theta_\tau$ is \mathcal{A}_τ -measurable. Therefore by (3.18),

$$Y_0 \circ \theta_\tau = \mathbb{E}_z \left(\varphi(\mathbf{X}_{T_l}) + \int_\tau^{T_l} f(\mathbf{X}_r) dr - \int_\tau^{T_l} g(\mathbf{X}_r) d^\dagger \beta_r^t \mid \mathcal{A}_\tau \right) \quad \mathbb{P}_z\text{-a.s.}$$

Thus (3.14) is satisfied for $z \in E_{0,T} \setminus N$. Since $u(z) = E_z Y_0$, using the strong Markov property and (3.14) we see that for every $z \in E_{0,T} \setminus N$,

$$u(\mathbf{X}_\tau) = E_{\mathbf{X}_\tau} Y_0 = E_z(Y_0 \circ \theta_\tau \mid \mathcal{F}_\tau^{\mathbf{X}}) = E_z(Y_\tau \mid \mathcal{F}_\tau^{\mathbf{X}}) = \mathbb{E}_z(Y_\tau \mid \mathcal{A}_\tau) = Y_\tau \quad \mathbb{P}_z\text{-a.s.}$$

A standard approximation argument shows that the process $u(\mathbf{X})$ is optional (with respect to (\mathcal{A}_t)). Therefore the above equality implies (3.13) by the Section Theorem. To prove quasi-continuity of $u(\omega; \cdot)$ for P -a.s. $\omega \in \Omega$, fix $\omega \in \Omega$ such that (3.13) holds P_{m_T} -a.s. Let τ be a predictable $(\mathcal{F}_t^{\mathbf{X}})$ -stopping time such that $0 \leq \tau \leq T_l$. Since \mathbf{X} is a Hunt process, $X_{\tau-} = X_\tau$ P_z -a.s. for every $z \in E_{0,T}$, and moreover the filtration $(\mathcal{F}_t^{\mathbf{X}})$ is quasi-left continuous, so $M_{\tau-} = M_\tau$ P_z -a.s. for every $z \in E_{0,T}$ (see [8, Theorems A.3.2, A.3.6]). By this and (3.13),

$$u(\mathbf{X}_{\tau-}) = u(\mathbf{X}_\tau) = u(\mathbf{X})_{\tau-} \quad P_{m_T}\text{-a.s.}$$

Of course the process $\{u(\mathbf{X})_{t-}, t \geq 0\}$ is predictable. Since the function $u(\omega; \cdot)$ is nearly Borel, the process $\{u(\mathbf{X}_{t-}), t \geq 0\}$ is predictable, too. Therefore applying the Section Theorem yields

$$u(\mathbf{X}_{t-}) = u(\mathbf{X})_{t-}, \quad t \in [0, T_l], \quad P_{m_T}\text{-a.s.},$$

which together with (3.13) and [16, Theorem IV.5.29] shows that $u(\omega; \cdot)$ is quasi-continuous. ■

THEOREM 3.7. *Assume that φ, f, g satisfy (H1)–(H5). Let N be defined by (3.8) and let Y be the process of Theorem 3.3. Then for P -a.s. $\omega \in \Omega$ the function $u(\omega; \cdot) : E_{0,T} \setminus N \rightarrow \mathbb{R}$ defined as*

$$(3.19) \quad u(z) = E_z \left(\varphi(\mathbf{X}_{T_l}) + \int_0^{T_l} f(\mathbf{X}_t, Y_t) dt + \int_0^{T_l} g(\mathbf{X}_t, Y_t) d^\dagger \beta_t^t \right)$$

is quasi-continuous and for every $z \in E_{0,T} \setminus N$,

$$(3.20) \quad Y_t = u(\mathbf{X}_t), \quad t \in [0, T_l], \mathbb{P}_z\text{-a.s.}$$

In particular, u is a solution of (3.4).

Proof. By [8, Theorem 4.1.1] we may and will assume that N is properly exceptional. Let (Y, M) be the pair of processes of Theorem 3.3. By the proof of Theorem 2.6, for every $z \in E_{0,T} \setminus N$ the pair (Y, M) is under \mathbb{P}_z a limit in $S^2 \otimes \mathcal{M}^2$ of solutions (Y^n, M^n) of some linear BSDEs. In particular, $u_n(z) = E_z Y_0^n \rightarrow E_z Y_0 = u(z)$ for every $z \in E_{0,T} \setminus N$, which when combined with the fact that N is properly exceptional implies that $u_n(\mathbf{X}_t) \rightarrow u(\mathbf{X}_t)$, $t \in [0, T_l]$, \mathbb{P}_z -a.s. for every $z \in E_{0,T} \setminus N$. Since we know from Proposition 3.6 that (3.20) holds for solutions of linear equations, we conclude that it holds in the general case. Quasi-continuity of u follows now from Proposition 3.6. Since by (3.19), (3.20) and Theorem 3.3 conditions (a) and (b) of the definition of a solution of SPDE (3.4) are satisfied, u is a solution of (3.4). ■

REMARK 3.8. Assume (H1)–(H5).

(i) From Theorems 3.3 and 3.7 we know that there exists a pair (Y, M) (not depending on z) of (\mathcal{F}_t) -adapted càdlàg processes such that (Y, M) is a solution of $\text{BDSDE}_z(\varphi, f, g)$ for q.e. $z \in E_{0,T}$ and that the solution (Y, M) provides a stochastic representation of the solution u of SPDE (3.4). The representation has the form (3.19), or equivalently,

$$u(z) = Y_0 \quad \mathbb{P}_z\text{-a.s. for q.e. } z \in E_{0,T}.$$

(ii) On the other hand, if u is a solution of (3.4) then there is an (\mathcal{F}_t) -adapted càdlàg process M (not depending on z) such that for q.e. $z \in E_{0,T}$ the pair $(u(\mathbf{X}), M)$ is a solution of $\text{BDSDE}_z(\varphi, f, g)$. Indeed, given a solution u let us define f_u, g_u as $f_u(z, y) = f(z, u(z))$, $g_u(z, y) = g(z, u(z))$, $z \in E_{0,T}$. Let (Y, M) be the pair of processes of Theorem 3.3 such that (Y, M) is a solution of the linear $\text{BDSDE}_z(\varphi, f_u, g_u)$ for q.e. $z \in E_{0,T}$. By Proposition 3.6, $Y = u(\mathbf{X})$, so the pair $(u(\mathbf{X}), M)$ is a solution of $\text{BDSDE}_z(\varphi, f_u, g_u)$, which means that it is a solution of $\text{BDSDE}_z(\varphi, f, g)$. Thus starting from u we can construct a solution $(u(\mathbf{X}), M)$ of the BSDE

$$(3.21) \quad u(\mathbf{X}_t) = \varphi(\mathbf{X}_{T_l}) + \int_t^{T_l} f(\mathbf{X}_r, u(\mathbf{X}_r)) dr + \int_0^{T_l} g(\mathbf{X}_r, u(\mathbf{X}_r)) d^\dagger \beta_r^t - \int_t^{T_l} dM_r, \quad t \in [0, T_l], \mathbb{P}_z\text{-a.s.}$$

PROPOSITION 3.9. Under (H1)–(H5) there exists at most one solution of (3.4).

Proof. Let u_1, u_2 be two solutions of (3.4). By Remark 3.8, for q.e. $z \in E_{0,T}$ the processes $u_1(\mathbf{X}), u_2(\mathbf{X})$ are the first components of solutions of $\text{BDSDE}_z(\varphi, f, g)$. Therefore by Corollary 2.4 and Lemma 3.2, for q.e. $z \in E_{0,T}$ we have $u_1(z) = u_2(z)$ P -a.s. ■

In the next section we prove some results on regularity of the solution u of SPDE (3.4). Here let us only note that our proofs are based on equation (3.21). Clearly (3.21) implies that for q.e. $z \in E_{0,T}$,

$$(3.22) \quad A_t := u(\mathbf{X}_t) - u(\mathbf{X}_0) = M_t + N_t, \quad t \in [0, T], \mathbb{P}_z\text{-a.s.},$$

with N defined by (1.8). We shall see that M is a random martingale AF of finite energy and N is a random continuous AF of finite, but in most cases nonzero energy. This means that (3.22) cannot be viewed as Fukushima’s decomposition of A . Nevertheless, we will prove some estimates for the energy $e(M)$ of M , which when combined with a priori estimates for $(u(\mathbf{X}), M)$ obtained in Proposition 2.5 lead to energy estimates for u .

4. Regularity of solutions. Let \mathcal{H} be some Hilbert space equipped with the norm $|\cdot|_{\mathcal{H}}$, let $\mathcal{B}(\mathcal{H})$ denote the Borel σ -field of subsets of \mathcal{H} and let

- $M^2(0, T; \mathcal{H})$ be the space of all $\mathcal{F} \otimes \mathcal{B}([0, T])/\mathcal{B}(\mathcal{H})$ measurable processes $v : \Omega \times [0, T] \rightarrow \mathcal{H}$ such that for a.e. $t \in [0, T]$ the random variable $v(t)$ is $\mathcal{F}_{t,T}^\beta$ -measurable and

$$E \int_0^T |v(t)|_{\mathcal{H}}^2 dt < \infty,$$

- $S^2(0, T; \mathcal{H})$ be the subspace of $M^2(0, T; \mathcal{H})$ consisting of all processes v such that

$$\sup_{0 \leq t \leq T} E|v(t)|_{\mathcal{H}}^2 < \infty.$$

Let u be a solution of (3.4) of Theorem 3.7 and let $u(t) = u(t, \cdot)$. In this section we show that $t \mapsto u(t)$ belongs to the space $S^2(0, T; H) \cap M^2(0, T; V)$ with H, V defined in Section 3.1, and we prove energy estimates for u , i.e. estimates of u in the norm $\|\cdot\|_{\mathcal{B}^{0,T}}$ defined as $\|u\|_{\mathcal{B}^{0,T}}^2 = \mathcal{B}^{0,T}(u, u)$, where $\mathcal{B}^{0,T}(u, u)$ is defined by (3.2). Note that by assumption (b) in Section 3.1, the norm $\|\cdot\|_{\mathcal{B}^{0,T}}$ is equivalent to the usual norm in $L^2(0, T; V)$. We begin with linear equations.

PROPOSITION 4.1. Assume that f, g do not depend on the last variable y and (H1) is satisfied. Then u defined by (3.12) belongs to $M^2(0, T; V)$ and

there is $c > 0$ such that

$$E\|u\|_{\mathcal{B}^{0,T}}^2 \leq cE\left(\|\varphi\|_{L^2(E;m)}^2 + \|f\|_{L^2(E_{0,T};m_T)}^2 + \sum_{k=1}^{\infty} \|g_k\|_{L^2(E_{0,T};m_T)}^2\right).$$

Proof. Set

$$w(z) = E_z\left(\varphi(\mathbf{X}_{T_l}) + \int_0^{T_l} f(\mathbf{X}_t) dt\right), \quad z \in E_{0,T}.$$

From [12, Theorem 3.7] and standard energy estimates for solutions of PDEs it follows that $w \in M^2(0, T; V)$ and

$$\|w\|_{\mathcal{B}^{0,T}}^2 \leq \|\varphi\|_{L^2(E;m)}^2 + \|f\|_{L^2(E_{0,T};m_T)}^2.$$

Set

$$v(z) = E_z \int_0^{T_l} g(\mathbf{X}_t) d^\dagger \beta_t^t, \quad z \in E_{0,T}.$$

We are going to show that $v \in M^2(0, T; V)$ and

$$(4.1) \quad E\|v\|_{\mathcal{B}^{0,T}}^2 \leq c \sum_{k \geq 1} E\|g_k\|_{L^2(E_{0,T};m_T)}^2.$$

To this end, first observe that by the stochastic Fubini theorem and the Markov property,

$$\begin{aligned} (R_\alpha^{0,T} v)(z) &= E_z \int_0^{T_l} e^{-\alpha r} E_{\mathbf{X}_r} \left(\int_0^{T_l} g(\mathbf{X}_t) d^\dagger \beta_t^t \right) dr \\ &= E_z \int_0^{T_l} e^{-\alpha r} E_z \left(\int_r^{T_l} g(\mathbf{X}_t) d^\dagger \beta_t^t \mid \mathcal{F}_r^{\mathbf{X}} \right) dr \\ &= E_z \int_0^{T_l} e^{-\alpha r} \left(\int_r^{T_l} g(\mathbf{X}_t) d^\dagger \beta_t^t \right) dr \\ &= \frac{1}{\alpha} E_z \int_0^{T_l} (1 - e^{-\alpha t}) g(\mathbf{X}_t) d^\dagger \beta_t^t. \end{aligned}$$

Hence

$$(4.2) \quad v(z) - \alpha(R_\alpha^{0,T} v)(z) = E_z \int_0^{T_l} e^{-\alpha t} g(\mathbf{X}_t) d^\dagger \beta_t^t, \quad z \in E_{0,T}.$$

For given $\alpha > 0$ and $v, u \in M^2(0, T; V)$ write

$$\mathcal{E}^{(\alpha),0,T}(u, v) = \alpha(u - \alpha R_\alpha^{0,T} u, v)_{L^2(E_{0,T};m_T)}.$$

By (4.2),

$$E\mathcal{E}^{(\alpha),0,T}(v, v) = \alpha \int_{E_{0,T}} E \left(E_z \int_0^{T_t} e^{-\alpha t} g(\mathbf{X}_t) d^\dagger \beta_t^t \cdot E_z \int_0^{T_t} g(\mathbf{X}_t) d^\dagger \beta_t^t \right) m_T(dz).$$

Using Itô's isometry and the fact that $\alpha \hat{R}_\alpha^{0,T}$ is a contraction on $L^2(E_{0,T}; m_T)$ we conclude from the above that

$$(4.3) \quad E\mathcal{E}^{(\alpha),0,T}(v, v) \leq \int_{E_{0,T}} E \left(E_z \int_0^{T_t} e^{-\alpha t} |g(\mathbf{X}_t)|^2 dt \right) m_T(dz) = \sum_{k=1}^\infty E \|\alpha R_\alpha^{0,T} g_k^2\|_{L^1(E_{0,T}; m_T)} \leq \sum_{k=1}^\infty E \|g_k\|_{L^2(E_{0,T}; m_T)}^2.$$

By [19, (6.1.28)],

$$E\mathcal{B}^{0,T}(\alpha R_\alpha^{0,T} v, \alpha R_\alpha^{0,T} v) \leq E\mathcal{E}^{(\alpha),0,T}(v, v).$$

When combined with (4.3) and [16, Theorem I.2.13], this shows that $v \in M^2(0, T; V)$. Therefore letting $\alpha \rightarrow \infty$ in (4.3) yields (4.1). ■

LEMMA 4.2. *Let $u : \Omega \times E_{0,T} \rightarrow \mathbb{R}_+$ and let $u_\alpha = \alpha R_\alpha^{0,T} u$, $\alpha > 0$.*

- (i) *If $u \in M^2(0, T; H)$ then $E\|u_\alpha - u\|_{L^2(E_{0,T}; m_T)}^2 \rightarrow 0$ as $\alpha \rightarrow \infty$.*
- (ii) *If $u \in M^2(0, T; V)$ then $E\mathcal{B}^{0,T}(u_\alpha - u, u_\alpha - u) \rightarrow 0$ as $\alpha \rightarrow \infty$.*
- (iii) *If $u(z) = E_z \int_0^{T_t} g(\mathbf{X}_t) d^\dagger \beta_t$ for some g satisfying (H1) and not depending on y , then for every $t \in [0, T)$, $E\|u_\alpha(t) - u(t)\|_{L^2(E; m)}^2 \rightarrow 0$ as $\alpha \rightarrow \infty$.*

Proof. (i) Since $\{R_\alpha^{0,T}\}$ is a strongly continuous contraction resolvent on $L^2(E_{0,T}; m_T)$, $u_\alpha \rightarrow u$ in $L^2(E_{0,T}; m_T)$ and $\|u_\alpha\|_{L^2(E_{0,T}; m_T)} \leq \|u\|_{L^2(E_{0,T}; m_T)}$ for P -a.e. $\omega \in \Omega$. Therefore applying the Lebesgue dominated convergence theorem we get (i).

(ii) By [16, Theorem I.2.13], $\mathcal{B}^{0,T}(u_\alpha - u, u_\alpha - u) \rightarrow 0$ for P -a.e. $\omega \in \Omega$. Moreover, by [16, Lemma I.2.11], there exists $c > 0$ (independent of ω) such that $\mathcal{B}^{0,T}(u_\alpha, u_\alpha) \leq c\mathcal{B}^{0,T}(u, u)$ for P -a.e. $\omega \in \Omega$. Therefore (ii) follows by the Lebesgue dominated theorem.

(iii) By (4.2),

$$E\|u_\alpha(t) - u(t)\|_{L^2(E; m)}^2 \leq \mathbb{E}_{t,m} \int_0^{T_t} e^{-2\alpha r} \|g(\mathbf{X}_r)\|^2 dr,$$

so it suffices to show that the integral on the right-hand side is finite for

every $t \in [0, T)$. But

$$\mathbb{E}_{t,m} \int_0^{T_t} \|g(\mathbf{X}_r)\|^2 dr = E E_{t,m} \int \sum_{k=1}^{\infty} |g_k(r, X_r)|^2 dr \leq E \|g\|_{L^2(E_{0,T};m_T)}^2,$$

which implies the desired conclusion. ■

Let us recall that the energy $e(A)$ of an AF A of the process $\mathbf{M}^{0,T}$ associated with the form $\mathcal{E}^{0,T}$ is defined as

$$e(A) = \frac{1}{2} \lim_{\alpha \rightarrow \infty} \alpha^2 E_{m_T} \int_0^{T_t} e^{-\alpha t} A_t^2 dt,$$

whenever the integral exists (see, e.g., [17, 19]). Also note that if M is a martingale AF of $\mathbf{M}^{0,T}$ then the sharp bracket $\langle M \rangle$ of M is a positive continuous AF of $\mathbf{M}^{0,T}$. Let $\mu_{\langle M \rangle}$ denote the Revuz measure of $\langle M \rangle$. Then

$$(4.4) \quad \mu_{\langle M \rangle}(E_{0,T}) = \lim_{t \rightarrow 0^+} \frac{1}{t} E_{m_T} \langle M \rangle_t = 2e(M)$$

(see [19, Section 5.1.2]).

In what follows we will be interested in AFs of the form

$$(4.5) \quad A_t = \tilde{u}(\mathbf{X}_t) - \tilde{u}(\mathbf{X}_0), \quad t \in [0, T_t],$$

where \tilde{u} is a quasi-continuous m_T -version of $u \in \mathcal{W}_T$. Such AFs admit a unique decomposition (called *Fukushima's decomposition*)

$$A_t = M_t^{[u]} + N_t^{[u]}, \quad t \in [0, T_t],$$

into a martingale AF $M^{[u]}$ of $\mathbf{M}^{0,T}$ of finite energy and a continuous AF $N^{[u]}$ of $\mathbf{M}^{0,T}$ of zero energy (see [17, Theorem 6.4]).

LEMMA 4.3. *Let $k^\beta = \beta(1 - \beta \hat{R}_\beta^{0,T} 1) \cdot m_T$. Then there exists a smooth Radon measure k such that $k^\beta \rightarrow k$ in the vague topology as $\beta \rightarrow \infty$, and for every $u \in \mathcal{W}_T$,*

$$(4.6) \quad e(M^{[u]}) = \|\tilde{u}(0)\|_{L^2(E;m)}^2 + \mathcal{B}^{0,T}(u, u) - \frac{1}{2} \int_{E_{0,T}} |\tilde{u}(z)|^2 k(dz),$$

where \tilde{u} is a quasi-continuous m_T -version of u .

Proof. Let A be defined by (4.5). By an elementary computation we get

$$\beta^2 E_{m_T} \int_0^{T_t} e^{-\beta t} (u(\mathbf{X}_t) - u(\mathbf{X}_0))^2 dt = 2\beta(u, u - \beta R_\beta^{0,T} u) - \beta(u^2, 1 - \beta \hat{R}_\beta^{0,T} 1).$$

Hence

$$(4.7) \quad \beta^2 E_{m_T} \int_0^{T_i} e^{-\beta t} A_t^2 dt = 2\mathcal{E}^{(\beta),0,T}(u, u) - (u^2, k^\beta).$$

By (4.7),

$$(u^2, k^\beta) \leq 2\mathcal{E}^{(\beta),0,T}(u, u).$$

From this we conclude that the sequence $\{k^\beta\}$ is tight in the vague topology. Therefore if $u \in \mathcal{W}_T \cap C_c(E_T)$ then letting $\beta \rightarrow \infty$ in the above inequality we get

$$(4.8) \quad (u^2, k) \leq 2\mathcal{E}^{0,T}(u, u) = \|u(0)\|_{L^2(E;m)}^2 + 2\mathcal{B}^{0,T}(u, u).$$

Since there is a continuous embedding of \mathcal{W}_T into $C([0, T]; H)$, from (4.8) it follows that there is $c > 0$ such that

$$(4.9) \quad (u^2, k) \leq c\|u\|_{\mathcal{W}_T}$$

for all $u \in \mathcal{W}_T \cap C_c(E_T)$. From (4.9) and [23, Theorem 1] it follows that k is a smooth measure. Furthermore, since each quasi-continuous $u \in \mathcal{W}_T$ can be approximated q.e. and in \mathcal{W}_T by functions from the space $\mathcal{W}_T \cap C_c(E_T)$, (4.8) holds true for every quasi-continuous $u \in \mathcal{W}_T$. Let $\{u_n\} \subset \mathcal{W}_T \cap C_c(E_T)$ be such that $u_n \rightarrow u$ in \mathcal{W}_T . Then

$$(4.10) \quad |(u^2, k^\beta)^{1/2} - (u^2, k)^{1/2}| \leq |(u_n^2, k^\beta)^{1/2} - (u_n^2, k)^{1/2}| \\ + \|(u_n - u)(0)\|_{L^2} + (2\mathcal{E}^{(\beta),0,T}(u_n - u, u_n - u))^{1/2} \\ + (2\mathcal{B}^{0,T}(u_n - u, u_n - u))^{1/2}.$$

Since $\mathcal{W}_T \subset C(0, T; L^2(E; m))$ and the embedding is continuous, letting $\beta \rightarrow \infty$ and then $n \rightarrow \infty$ in (4.10) shows that for every $u \in \mathcal{W}_T$,

$$\int_{E_{0,T}} u^2 dk^\beta \rightarrow \int_{E_{0,T}} u^2 dk$$

as $\beta \rightarrow \infty$. By [17, Theorem 6.4], $e(A) = e(M^{[u]})$. Therefore letting $\beta \rightarrow \infty$ in (4.7) yields $2e(M^{[u]}) = 2\mathcal{E}^{0,T}(u, u) - (u^2, k)$, which implies (4.6). ■

DEFINITION. We say that an (\mathcal{F}_t) -adapted process A is a *random additive functional* (random AF for short) of $\mathbf{M}^{0,T}$ if for P -a.e. $\omega \in \Omega$ the process $A(\omega; \cdot)$ is an AF of $\mathbf{M}^{0,T}$. Similarly, we say that a process A is a *random martingale (continuous, positive) AF* of $\mathbf{M}^{0,T}$ if for P -a.e. $\omega \in \Omega$ the process $A(\omega; \cdot)$ is a martingale (continuous, positive) AF of $\mathbf{M}^{0,T}$.

LEMMA 4.4. *Let $\{A^n\}$ be a sequence of random AFs such that*

$$\mathbb{E}_z \sup_{t \leq T_i} |A_t^n - A_t^m| \rightarrow 0 \quad \text{for } m_T\text{-a.e. } z \in E_{0,T} \text{ as } n, m \rightarrow \infty.$$

Then there exists a subsequence $(n_k) \subset (n)$ such that for P -a.s. $\omega \in \Omega$,

$$E_z \sup_{t \leq T_l} |A_t^{n_k} - A_t^{n_l}| \rightarrow 0 \quad \text{for } q.e. \ z \in E_{0,T} \text{ as } k, l \rightarrow \infty.$$

Proof. Let $\rho \in L^1(E_{0,T}; m_T)$ be a strictly positive function chosen so that $\int_{E_{0,T}} \rho \, dm_T = 1$ and

$$\mathbb{E}_\nu \sup_{t \leq T_l} |A_t^n - A_t^m| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

where $\nu = \rho \cdot m_T$. Let $(n_k) \subset (n)$ be a subsequence such that

$$(4.11) \quad \mathbb{E}_\nu \sup_{t \leq T_l} |A_t^{n_{k+1}} - A_t^{n_k}| \leq 2^{-k}, \quad k \geq 1,$$

and let $B = \{z \in E_{0,T} : E_z \sup_{t \leq T_l} |A_t^{n_k} - A_t^{n_l}| \rightarrow 0 \text{ as } k, l \rightarrow \infty\}$. Let us stress that B depends on ω . Write $\tau = \sigma_B$. By the Markov property and additivity of A ,

$$\begin{aligned} P_\nu(\tau < T_l) &\leq \int_{E_{0,T}} P_z \left(\exists s \leq T_l \ E_z \left(\sup_{t \leq T_l \circ \theta_s} |A_t^{n_l} \circ \theta_s - A_t^{n_k} \circ \theta_s| \mid \mathcal{F}_s^{\mathbf{X}} \right) \rightarrow 0 \right) d\nu(z) \\ &= \int_{E_{0,T}} P_z \left(\exists s \leq T_l \ E_z \left(\sup_{t+s \leq T_l} |A_{t+s}^{n_l} - A_s^{n_l} - A_{t+s}^{n_k} + A_s^{n_k}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \rightarrow 0 \right) d\nu(z) \\ &\leq \int_{E_{0,T}} P_z \left(\exists s \leq T_l \ 2E_z \left(\sup_{t \leq T_l} |A_t^{n_l} - A_t^{n_k}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \rightarrow 0 \right) d\nu(z) \\ &\leq \int_{E_{0,T}} P_z \left(\sup_{0 \leq s \leq T_l} \left(E_z \left(\sup_{t \leq T_l} |A_t^{n_l} - A_t^{n_k}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \right)^q \rightarrow 0 \right) d\nu(z) \end{aligned}$$

for every $q \in (0, 1)$. Let $\Pi = P \otimes (\nu \otimes K)$, where $K(z, d\omega') = P_z(d\omega')$. By (4.11) and [4, Lemma 6.1],

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{\Omega \times E_{0,T} \times \Omega'} \sup_{s \leq T_l} \left(E_z \left(\sup_{t \leq T_l} |A_t^{n_{k+1}} - A_t^{n_k}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \right)^q d\Pi(\omega, z, \omega') \\ \leq \sum_{k=1}^{\infty} \frac{1}{1-q} \left(\mathbb{E}_\nu \sup_{t \leq T_l} |A_t^{n_{k+1}} - A_t^{n_k}| \right)^q < \infty. \end{aligned}$$

Since $\Pi(\Omega \times E_{0,T} \times \Omega') = 1$, the Borel–Cantelli lemma shows that

$$\sup_{s \leq T_l} \left(E_z \left(\sup_{t \leq T_l} |A_t^{n_k} - A_t^{n_l}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \right)^q \rightarrow 0 \quad \text{as } k, l \rightarrow \infty, \Pi\text{-a.e.}$$

In particular, for ν -a.e. $z \in E_{0,T}$ and P -a.e. $\omega \in \Omega$,

$$\sup_{s \leq T_l} \left(E_z \left(\sup_{t \leq T_l} |A_t^{n_k} - A_t^{n_l}| \mid \mathcal{F}_s^{\mathbf{X}} \right) \right)^q \rightarrow 0 \quad \text{as } k, l \rightarrow \infty, P_z\text{-a.e.}$$

Hence $\mathbb{P}_\nu(\tau \leq T_l) = 0$, which implies that $\text{cap}(B) = 0$ P -a.e. (see [18, p. 298]). ■

For a given set $A \subset E_{0,T} \times \Omega$ and (z_0, ω_0) we write $A_{z_0} = \{\omega \in \Omega : (z_0, \omega) \in A\}$ and $A_{\omega_0} = \{z \in E_{0,T} : (z, \omega_0) \in A\}$.

LEMMA 4.5. *Let $A \subset E_{0,T} \times \Omega$ be a measurable set. If $\text{cap}(A_\omega) = 0$ for P -a.e. $\omega \in \Omega$, then $P(A_z) = 0$ for cap-q.e. $z \in E_{0,T}$.*

Proof. Let ν be a smooth bounded measure. Then $E \int_{E_{0,T}} \mathbf{1}_A(\omega, z) \nu(dz) = 0$ by the assumption on A . Therefore using Fubini's theorem we obtain

$$0 = \int_{E_{0,T}} E \mathbf{1}_A(\omega, z) \nu(dz) = \int_{E_{0,T}} P(A_z) \nu(dz).$$

Since ν was arbitrary, it follows from [28, Theorem 2.5] (or [18, remark at the end of Section 4]) that $P(A_z) = 0$ for cap-q.e. $z \in E_{0,T}$. ■

PROPOSITION 4.6. *Assume that u is given by (3.19). Then $u(\mathbf{X}_t) = Y_t$, $t \in [0, T]$, for q.e. $z \in E_{0,T}$, where (Y, M) is a solution of $\text{BDSDE}_z(\varphi, f, g)$. Moreover, M is a random martingale AF of $\mathbf{M}^{0,T}$ and*

$$(4.12) \quad Ee(M) = E\left(\|u(0)\|_{L^2(E;m)}^2 + \mathcal{B}^{0,T}(u, u) - \frac{1}{2} \int_{E_{0,T}} |u(z)|^2 k(dz)\right),$$

where k is the killing measure of the form $(\hat{\mathcal{E}}^{0,T}, D(\hat{\mathcal{E}}^{0,T}))$ defined in Lemma 4.3.

Proof. The first assertion follows from Theorem 3.7. If $g \equiv 0$ then (4.12) follows from [17, Lemma 6.1]. Therefore we may and will assume that $\varphi \equiv 0$, $f \equiv 0$. Let $u_\alpha = \alpha R_\alpha^{0,T} u$. Then

$$u_\alpha(\mathbf{X}_t) = \int_0^t \alpha(u - u_\alpha)(\mathbf{X}_r) dr + \int_0^t dM_r^{[u_\alpha]}, \quad t \in [0, T].$$

By Itô's formula,

$$(4.13) \quad \mathbb{E}_z \int_0^{T_t} d[M - M^{[u_\alpha]}]_t \leq \mathbb{E}_z \int_0^{T_t} (-2\alpha|u - u_\alpha|^2(\mathbf{X}_t) + \|g(\mathbf{X}_t)\|^2) dt.$$

By Itô's isometry, for q.e. $z \in E_{0,T}$ we have

$$\begin{aligned} \alpha \mathbb{E}_z \int_0^{T_t} |u - u_\alpha|^2(\mathbf{X}_t) dt &= \alpha \mathbb{E}_z \int_0^{T_t} \left(E_{\mathbf{X}_t} \int_0^{T_t} e^{-\alpha r} g(\mathbf{X}_r) d^\dagger \beta_r \right)^2 dt \\ &= \alpha \mathbb{E}_z \int_0^{T_t} E_{\mathbf{X}_t} \int_0^{T_t} e^{-2\alpha r} \|g(\mathbf{X}_r)\|^2 dr dt \\ &= \frac{1}{2} \mathbb{E}_z \int_0^{T_t} 2\alpha R_{2\alpha}^{0,T} \|g\|^2(\mathbf{X}_t) dt. \end{aligned}$$

This implies that

$$(4.14) \quad 2\alpha \mathbb{E}_z \int_0^{T_i} |u - u_\alpha|^2(\mathbf{X}_t) dt \rightarrow \mathbb{E}_z \int_0^{T_i} \|g(\mathbf{X}_t)\|^2 dt,$$

because

$$\begin{aligned} \mathbb{E}_z \int_0^{T_i} 2\alpha R_{2\alpha}^{0,T} \|g\|^2(\mathbf{X}_t) dt &= ER^{0,T}(2\alpha R_{2\alpha}^{0,T} \|g\|^2)(z) \\ &= ER^{0,T} \|g\|^2(z) - R_{2\alpha}^{0,T} \|g\|^2(z) \\ &= \mathbb{E}_z \int_0^{T_i} \|g(\mathbf{X}_t)\|^2 dt - \mathbb{E}_z \int_0^{T_i} e^{-2\alpha t} \|g(\mathbf{X}_t)\|^2 dt \end{aligned}$$

and by Lemma 3.2, $\mathbb{E}_z \int_0^{T_i} \|g(\mathbf{X}_t)\|^2 dt < \infty$ for q.e. $z \in E_{0,T}$. By (4.13) and (4.14),

$$(4.15) \quad \mathbb{E}_z \int_0^{T_i} d[M - M^{[u_\alpha]}]_t \rightarrow 0$$

for q.e. $z \in E_{0,T}$. From (4.15), the Burkholder–Davis–Gundy inequality and Lemma 4.4 we conclude that there exists a process \tilde{M} such that, up to a subsequence, for P -a.e. $\omega \in \Omega$,

$$(4.16) \quad E_z \sup_{t \leq T_i} |M_t^{[u_\alpha]} - \tilde{M}_t| \rightarrow 0$$

for q.e. $z \in E_{0,T}$. By a standard argument (see the reasoning following (5.2.23) in [8, proof of Theorem 5.2.1]), $\tilde{M}(\omega; \cdot)$ is a martingale AF of $\mathbf{M}^{0,T}$ for P -a.e. $\omega \in \Omega$. By Lemma 4.5, (4.15) and (4.16), \tilde{M} is a \mathbb{P}_z -modification of M for q.e. $z \in E_{0,T}$. By Proposition 4.1 and Lemma 4.2, $u_\alpha \rightarrow u$ in $M^2(0, T; V)$ and $E|u_\alpha(t) - u(t)|_{L^2(E; m)}^2 \rightarrow 0$ for every $t \in [0, T)$ as $\alpha \rightarrow \infty$.

Write $M^\alpha = M^{[u_\alpha]}$. Let $\alpha_n \rightarrow \infty$. By Lemma 4.3,

$$Ee(M^{\alpha_n} - M^{\alpha_m}) \leq E\|(u_{\alpha_n} - u_{\alpha_m})(0)\|_{L^2(E; m)}^2 + E\mathcal{B}^{0,T}(u_{\alpha_n} - u_{\alpha_m}, u_{\alpha_n} - u_{\alpha_m}),$$

which converges to zero as $n, m \rightarrow \infty$. From this and (4.15) it follows that

$$\lim_{n \rightarrow \infty} Ee(M^{\alpha_n}) = Ee(M).$$

By Lemma 4.3,

$$(4.17) \quad \begin{aligned} Ee(M^\alpha) &= E\|u_\alpha(0)\|_{L^2(E; m)}^2 + E\mathcal{B}^{0,T}(u_\alpha, u_\alpha) \\ &\quad - \frac{1}{2} E \int_{E_{0,T}} |u_\alpha(z)|^2 k(dz). \end{aligned}$$

Observe now that

$$(4.18) \quad E \int_{E_{0,T}} |u_\alpha(z)|^2 k(dz) \rightarrow E \int_{E_{0,T}} |u(z)|^2 k(dz).$$

Indeed, since from the proof of Lemma 4.3 we know that (4.8) holds for quasi-continuous elements of \mathcal{W}_T , there exists $v \in L^2(E_{0,T}; k)$ such that $(E|u_\alpha|^2)^{1/2} \rightarrow v$ in $L^2(E_{0,T}; k)$. Therefore the proof of (4.18) is completed by showing that $v^2 = Eu^2$ k -a.e. By (4.2),

$$E|u(z) - u_\alpha(z)|^2 \leq \mathbb{E}_z \int_0^{T_z} e^{-2\alpha t} \|g(\mathbf{X}_t)\|^2 dt, \quad z \in E_{0,T}.$$

By Lemma 3.2, $\mathbb{E}_z \int_0^{T_z} \|g(\mathbf{X}_t)\|^2 dt < \infty$ for q.e. $z \in E_{0,T}$. Hence $Eu_\alpha^2 \rightarrow Eu^2$ q.e. Consequently, $Eu_\alpha^2 \rightarrow Eu^2$ k -a.e. since k is smooth. Thus $v^2 = Eu^2$, which completes the proof of (4.18). Letting $\alpha \rightarrow \infty$ in (4.17) and using (4.18) we get (4.12). ■

REMARK 4.7. Let N be defined by (1.8). A direct calculation shows that

$$(4.19) \quad e(N) = \frac{1}{2} \|gu\|_{L^2(E_{0,T}; m_T)}^2.$$

From this one can conclude that $u \notin \mathbb{W}$, where \mathbb{W} is defined by (1.10). Indeed, suppose that $u \in \mathbb{W}$. We may assume that $\varphi = 0, f = 0$ and g does not depend on y . Let $A^{[u]}$ be defined by (1.7) and u_α be as in the proof of Proposition 4.6. Then by the proof of [8, Theorem 5.2.2] (see also [28, Theorem 4.5]), $N = N^{[u]}$ and $N^{[u_\alpha]}$ converges to N uniformly on compacts in probability \mathbb{P}_{m_T} . Since $N = N^{[u]}$ we have

$$\begin{aligned} Ee(N) &\leq 2Ee(N - N^{[u_\alpha]}) + 2Ee(A^{[u_\alpha]}) = 2Ee(N - N^{[u_\alpha]}) \\ &\leq 4Ee(A^{[u]} - A^{[u_\alpha]}) + 4Ee(M^{[u]} - M^{[u_\alpha]}). \end{aligned}$$

Since $u \in \mathbb{W}$, we have

$$\begin{aligned} Ee(A^{[u]} - A^{[u_\alpha]}) &\leq E\mathcal{E}(u_\alpha - u, u_\alpha - u) \\ &= E\|u_\alpha(0) - u(0)\|_{L^2(E; m)}^2 + EB^{0,T}(u_\alpha - u, u_\alpha - u) \end{aligned}$$

(see, e.g., [28, (13)]). When combined with the previous inequality and Lemma 4.3, this shows that $e(N) = 0$, which contradicts (4.19).

In what follows we denote by \mathcal{M}^+ the set of all positive Borel measures on $E_{0,T}$ and by $\mathcal{M}_{0,b}^+$ the subset of \mathcal{M}^+ consisting of all bounded measures which charge no set of zero capacity associated with the form $(\mathcal{E}^{0,T}, D(\mathcal{E}^{0,T}))$. The total variation norm of $\mu \in \mathcal{M}_{0,b}^+$ will be denoted by $\|\mu\|_{TV}$. To shorten notation, for $\mu \in \mathcal{M}^+$ we write $P_\mu(\cdot) = \int_{E_{0,T}} P_z(\cdot) \mu(dz)$ and denote by E_μ (resp. \mathbb{E}_μ) the expectation with respect to P_μ (resp. $P \otimes P_\mu$).

DEFINITION. We say that $\mu : \Omega \times \mathcal{B}(E_{0,T}) \rightarrow \mathbb{R}$ is a *random measure* if $\mu(\omega; \cdot)$ is a positive Radon measure on $E_{0,T}$ for P -a.s. $\omega \in \Omega$ and $\mu(\omega; B)$ is \mathcal{F}_T^β -measurable for every $B \in \mathcal{B}(E_{0,T})$.

Given a random measure μ such that $\mu(\omega; \cdot) \in \mathcal{M}_{0,b}^+$ for P -a.e. $\omega \in \Omega$ we denote by A^μ the random positive AF of $\mathbf{M}^{0,T}$ such that for P -a.e. $\omega \in \Omega$ the measure $\mu(\omega; \cdot)$ is the Revuz measure of the AF $A(\omega; \cdot)$. We call A^μ the random AF *associated with* μ . Note that if the random AF associated with μ exists, then it is uniquely determined.

In the rest of the paper, writing A^μ for some random measure μ we tacitly assume that this random AF exists.

LEMMA 4.8. *Assume that μ, ν are random measures such that $\mu, \nu \in \mathcal{M}_{0,b}^+(E_{0,T})$ P -a.s. If for some $v \in M^2(0, T; V) \cap C([0, T]; L^2(\Omega \times E; P \otimes m))$,*

$$Ev(z) + \mathbb{E}_z \int_0^{T_t} dA_t^\mu \leq \mathbb{E}_z \int_0^{T_t} dA_t^\nu$$

for m_T -a.e. $z \in E_{0,T}$ then

$$(4.20) \quad E \int_{[0,T] \times E} v(z) k(dz) + E\|\mu\|_{\text{TV}} \leq E\|\nu\|_{\text{TV}}.$$

Proof. Since $Ev(z) + ER^{0,T}\mu(z) \leq ER^{0,T}\nu(z)$ for m_T -a.e. $z \in E_{0,T}$, we have

$$\begin{aligned} E(v, \alpha(1 - \alpha\hat{R}_\alpha^{0,T}1))_{L^2(E_{0,T}; m_T)} + E(R^{0,T}\mu, \alpha(1 - \alpha\hat{R}_\alpha^{0,T}1))_{L^2(E_{0,T}; m_T)} \\ \leq E(R^{0,T}\nu, \alpha(1 - \alpha\hat{R}_\alpha^{0,T}1))_{L^2(E_{0,T}; m_T)}. \end{aligned}$$

Hence

$$\begin{aligned} E(v, \alpha(1 - \alpha\hat{R}_\alpha^{0,T}1))_{L^2(E_{0,T}; m_T)} + E(\alpha(I - \alpha R_\alpha^{0,T})R^{0,T}\mu, 1)_{L^2(E_{0,T}; m_T)} \\ \leq E(\alpha(I - \alpha R_\alpha^{0,T})R^{0,T}\nu, 1)_{L^2(E_{0,T}; m_T)}. \end{aligned}$$

It is an elementary check that

$$\alpha(I - \alpha R_\alpha^{0,T})R^{0,T}\mu(z) = E_z \int_0^{T_t} \alpha e^{-\alpha t} dA_t^\mu, \quad z \in E_{0,T}.$$

Therefore

$$(4.21) \quad \begin{aligned} E(v, \alpha(1 - \alpha\hat{R}_\alpha^{0,T}1))_{L^2(E_{0,T}; m_T)} + \mathbb{E}_{m_T} \int_0^{T_t} \alpha e^{-\alpha t} dA_t^\mu \\ \leq \mathbb{E}_{m_T} \int_0^{T_t} \alpha e^{-\alpha t} dA_t^\nu. \end{aligned}$$

Letting $\alpha \rightarrow \infty$ in (4.21) we get (4.20). Indeed, directly from the definition of the Revuz duality it follows that the integrals involving A^μ and A^ν converge to $E\|\mu\|_{\text{TV}}$ and $E\|\nu\|_{\text{TV}}$, respectively. To show the convergence of the first term on the left-hand side of (4.21), assume first that $v(T) = 0$. Set $v_m = \beta_m R_{\beta_m}^{0,T} v$. Then by (4.10),

$$\begin{aligned} \limsup_{n \rightarrow \infty} |(v^2, k^{\alpha_n})^{1/2} - (v^2, k)^{1/2}| &\leq 4\|(v_m - v, v_m - v)(0)\|_{L^2(E; m)} \\ &\quad + 4(\mathcal{B}^{0,T}(v_m - v, v_m - v))^{1/2}. \end{aligned}$$

Since $v_m(t) \rightarrow v(t)$ in $L^2(E; m)$ for every $t \in [0, T]$ and $v_m \rightarrow v$ in $M^2(0, T; V)$, this shows the desired convergence of the first term. In general, if $v(T) \neq 0$, we consider a sequence $\{v_n\} \subset M^2(0, T; V) \cap C([0, T]; L^2(\Omega \times E; P \otimes m))$ such that $v_n \leq v$, $v_n(T) = 0$ and $v_n \nearrow v$ on $[0, T] \times E$. Of course, v_n satisfies the assumptions of the lemma, so by what has already been proved, (4.20) is satisfied with v replaced by v_n . Letting $n \rightarrow \infty$ we get (4.20) for v . ■

THEOREM 4.9. *Assume (H1)–(H3). Let u be a solution of SPDE (3.4). Then $u \in S^2(0, T; L^2(E; m)) \cap M^2(0, T; V)$ and there is $c > 0$ depending only on T, L, l such that*

$$\begin{aligned} (4.22) \quad &\sup_{0 \leq t \leq T} E\|u(t)\|_{L^2(E; m)}^2 + E\mathcal{B}^{0,T}(u, u) \\ &\leq cE\left(\|\varphi\|_{L^2(E; m)}^2 + \|f(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2 + \sum_{k=1}^\infty \|g_k(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2\right). \end{aligned}$$

Proof. By Proposition 2.5,

$$\begin{aligned} \mathbb{E}_z \sup_{0 \leq t \leq T_i} |u(\mathbf{X}_t)|^2 + \mathbb{E}_z \int_0^{T_i} d[M]_t \\ \leq c\mathbb{E}_z \left(|\varphi(\mathbf{X}_{T_i})|^2 + \int_0^{T_i} (|f(\mathbf{X}_t, 0)|^2 + \|g(\mathbf{X}_t, 0)\|^2) dt \right). \end{aligned}$$

Hence

$$E|u(z)|^2 + \mathbb{E}_z \int_0^{T_i} d\langle M \rangle_t \leq c\mathbb{E}_z \int_0^{T_i} dA'_t,$$

where

$$\nu(dz) = (\delta_{\{T\}} \otimes |\varphi|^2 \cdot m)(dz) + (|f(z, 0)|^2 + \|g(z, 0)\|^2) m_T(dz).$$

Let $\mu_{\langle M \rangle}$ denote the random smooth measure associated with the random

continuous AF $\langle M \rangle$ of \mathbf{M} . By Lemma 4.8,

$$(4.23) \quad E \int_{E_0, T} |u(z)|^2 k(dz) + E \|\mu_{\langle M \rangle}\|_{\text{TV}} \leq c \|\nu\|_{\text{TV}}.$$

Since by (4.4), $\|\mu_{\langle M \rangle}\|_{\text{TV}} = 2e(M)$ P -a.s., it follows from Proposition 4.6 and (4.23) that

$$E \|u(0)\|_{L^2(E; m)}^2 + E \mathcal{B}^{0, T}(u, u) \leq c \|\nu\|_{\text{TV}}.$$

Since the same estimate can be obtained on any interval $[t, T]$ with $t \in (0, T)$, and $c \|\nu\|_{\text{TV}}$ is equal to the right-hand side of (4.22), the theorem is proved. ■

5. BDSDEs with Brownian filtration. In the present section and in Section 6 we assume that the filtration (\mathcal{G}_t) of Section 2 is generated by a d -dimensional Wiener process W on Ω' . This will allow us to treat in Section 6 equations dependent on the gradient of a solution.

We also assume that we are given an \mathcal{F}_T -measurable random variable ξ and two families $\{f(t, y, z), t \geq 0\}_{y \in \mathbb{R}, z \in \mathbb{R}^d}$, $\{g_k(t, y, z), t \geq 0\}_{y \in \mathbb{R}, z \in \mathbb{R}^d, k \in \mathbb{N}}$ of processes of class M (as in Section 2, in our notation we omit the dependence on $(\omega, \omega') \in \Omega \times \Omega'$). We set $g(\cdot, y, z) = (g_1(\cdot, y, z), g_2(\cdot, y, z), \dots)$.

Let us consider the following hypotheses:

- (B1) $\mathbb{E}|\xi|^2 + \mathbb{E} \int_0^T |f(t, 0, 0)|^2 dt + \mathbb{E} \int_0^T \|g(t, 0, 0)\|^2 dt < \infty$.
- (B2) $\int_0^T |f(t, y, 0)| dt < \infty$ \mathbb{P} -a.s. for every $y \in \mathbb{R}$.
- (B3) There exist $l, L > 0, m \in (0, 1)$ and $M_k, L_k : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ such that $\sup_{0 \leq t \leq T} \sum_{k=1}^\infty L_k^2(t) \leq l, \sup_{0 \leq t \leq T} \sum_{k=1}^\infty M_k^2(t) \leq m$ P -a.s. and for a.e. $t \in [0, T]$,
 - (a) $(f(t, y, z) - f(t, y', z))(y - y') \leq L|y - y'|^2$ for all $y, y' \in \mathbb{R}$ and $z \in \mathbb{R}^d$,
 - (b) $|f(t, y, z) - f(t, y, z')| \leq L|z - z'|$ for all $y \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$,
 - (c) $|g_k(t, y, z) - g_k(t, y', z')| \leq L_k(t)|y - y'| + M_k(t)|z - z'|$ for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$.
- (B4) For a.e. $t \in [0, T]$ and every $z \in \mathbb{R}^d$ the mapping $\mathbb{R} \ni y \mapsto f(t, y, z)$ is continuous.

DEFINITION. We say that a pair of processes $(Y, Z) \in \mathcal{S}^2 \times M^2$ is a *solution of BDSDE* (ξ, f, g) if

- (a) $\mathbb{P}(\int_0^T (|f(t, Y_t, 0)| + \|g(t, Y_t, 0)\|^2) dt < \infty) = 1$,
- (b) $Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T g(r, Y_r, Z_r) d^\dagger \beta_r - \int_t^T Z_r dW_r, t \in [0, T]$, \mathbb{P} -a.s.

REMARK 5.1. If the coefficients f, g do not depend on z and (Y, Z) is a solution of $\text{BDSDE}(\xi, f, g)$ then the pair

$$(Y_t, M_t) := \left(Y_t, \int_0^t Z_r dW_r \right), \quad t \in [0, T],$$

is a solution of $\text{BDSDE}(\xi, f, g)$ in the sense of Section 3. The difference between Section 3 and Section 5 is that in the present section we have additional information on the filtration (\mathcal{G}_t) , which gives us additional information on the component M of the solution.

PROPOSITION 5.2. *Let g satisfy condition (B3c) and either f or f' satisfy (B3a), (B3b). If $\xi \leq \xi'$ \mathbb{P} -a.s. and $f'(t, y, z) \leq f(t, y, z)$ for a.e. $t \in [0, T]$ and every $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$ then*

$$Y'_t \leq Y_t, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.},$$

where $(Y, M), (Y', M')$ are solutions of $\text{BDSDE}(\xi, f, g)$ and $\text{BSDE}(\xi', f', g)$, respectively.

Proof. Assume that f satisfies (B3a), (B3b). By Itô's formula,

$$\begin{aligned} |(Y'_t - Y_t)^+|^2 &+ \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |Z'_r - Z_r|^2 dr \\ &\leq 2 \int_t^T (Y'_r - Y_r)^+ (f'(r, Y'_r, Z'_r) - f(r, Y_r, Z_r)) dr \\ &\quad - 2 \int_t^T (Y'_r - Y_r)^+ (Z'_r - Z_r) dW_r \\ &\quad + \sum_{k=1}^{\infty} \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |g_k(r, Y'_r, Z'_r) - g_k(r, Y_r, Z_r)|^2 dr. \end{aligned}$$

By the assumptions of the proposition,

$$\int_t^T (Y'_r - Y_r)^+ (f'(r, Y'_r, Z'_r) - f(r, Y_r, Z_r)) dr \leq 2L \int_t^T |(Y'_r - Y_r)^+|^2 dr$$

and

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |g_k(r, Y'_r, Z'_r) - g_k(r, Y_r, Z_r)|^2 dr \\ &\leq \sum_{k=1}^{\infty} \int_t^T (L_k^2(r) \mathbf{1}_{\{Y'_r > Y_r\}} |Y'_r - Y_r|^2 + M_k^2(r) \mathbf{1}_{\{Y'_r > Y_r\}} |Z'_r - Z_r|^2) dr \\ &\leq l \int_t^T |(Y'_r - Y_r)^+|^2 dr + m \int_t^T \mathbf{1}_{\{Y'_r > Y_r\}} |Z'_r - Z_r|^2 dr. \end{aligned}$$

By the above estimates,

$$\mathbb{E}|(Y'_t - Y_t)^+|^2 \leq 2(L + l)\mathbb{E} \int_t^T |(Y'_r - Y_r)^+|^2 dr, \quad t \in [0, T],$$

so Gronwall’s lemma yields the desired result. ■

COROLLARY 5.3. *Let assumption (B3) hold. Then there exists at most one solution of BDSDE(ξ, f, g).*

THEOREM 5.4. *Let assumptions (B1)–(B4) hold. Then there exists a solution (Y, Z) of BDSDE(ξ, f, g).*

Proof. Let M_k^2 denote the set of k -dimensional processes $X = (X^1, \dots, X^k)$ such that $X^i \in M^2, i = 1, \dots, k$. Define the mapping $\Phi : M_1^2 \otimes M_d^2 \rightarrow M_1^2 \otimes M_d^2$ by letting $\Phi(U, V)$ be the solution (Y, Z) of the BSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, V_r) dr + \int_t^T g(r, Y_r, V_r) d^\dagger B_r - \int_t^T Z_r dW_r, \quad t \in [0, T].$$

From Corollary 5.3, Theorem 2.6 and the representation theorem for Brownian filtration it follows that Φ is well defined. Let $(U^1, V^1), (U^2, V^2) \in M_1^2 \otimes M_d^2$ and $(Y^i, Z^i) = \Phi(U^i, V^i), i = 1, 2$. By Itô’s formula,

$$\begin{aligned} \mathbb{E}e^{\beta t}|Y_t^1 - Y_t^2|^2 + \beta \mathbb{E} \int_t^T e^{\beta r}|Y_r^1 - Y_r^2|^2 dr + \mathbb{E} \int_t^T e^{\beta r}|Z_r^1 - Z_r^2|^2 dr \\ = 2\mathbb{E} \int_t^T e^{\beta r}(f(r, Y_r^1, V_r^1) - f(r, Y_r^2, V_r^2))(Y_r^1 - Y_r^2) dr \\ + \mathbb{E} \int_t^T e^{\beta r}\|g(r, Y_r^1, V_r^1) - g(r, Y_r^2, V_r^2)\|^2 dr \\ \leq 2L\mathbb{E} \int_t^T e^{\beta r}(|Y_r^1 - Y_r^2|^2 + |V_r^1 - V_r^2| \cdot |Y_r^1 - Y_r^2|) dr \\ + l\mathbb{E} \int_t^T e^{\beta r}|Y_r^1 - Y_r^2|^2 dr + m\mathbb{E} \int_t^T |V_r^1 - V_r^2|^2 dr. \end{aligned}$$

Hence, for every $\alpha > 0$,

$$c\mathbb{E} \int_t^T e^{\beta r}|Y_r^1 - Y_r^2|^2 dr + \mathbb{E} \int_t^T e^{\beta r}|Z_r^1 - Z_r^2|^2 dr \leq (\alpha + m)\mathbb{E} \int_t^T |V_r^1 - V_r^2|^2 dr$$

with $c = \beta - 2L - (2L)^2\alpha^{-1} - l > 0$. Let α, β be chosen so that $c > 0$ and $\alpha + m < 1$. Then Φ is a contraction if we equip $M_1^2 \times M_d^2$ with the norm

$$(5.1) \quad \|(Y, Z)\|_\beta^2 = \mathbb{E} \int_0^T e^{\beta r}(c|Y_r|^2 + |Z_r|^2) dr.$$

By Banach’s contraction principle, Φ has a fixed point. Of course, it solves BDSDE(ξ, f, g). ■

6. SPDEs with a divergence form operator. In this section we consider equations of the form (1.1) with A being a uniformly elliptic divergence form operator. We allow, however, the coefficients f, g to depend on the gradient of a solution. More precisely, we assume that $E = D$ is a nonempty bounded open subset of \mathbb{R}^d and

$$B^{(t)}(\varphi, \psi) = \sum_{i,j=1}^d \int_D a_{ij}(t, x) \varphi_{x_i}(x) \psi_{x_j}(x) dx, \quad \varphi, \psi \in V = H_0^1(D),$$

where $a_{ij} : [0, T] \times D \rightarrow \mathbb{R}$ are measurable functions such that for every $(t, x) \in [0, T] \times D$,

$$a_{ij} = a_{ji}, \quad \lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(t, x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $0 < \lambda \leq \Lambda$. In this case the operator A_t associated with $(B^{(t)}, V)$ is given by (1.2). Suppose we are given measurable functions $\varphi : D \rightarrow \mathbb{R}$ and $f, g_k : \Omega \times [0, T] \times D \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$. We consider equations of the form

(6.1) $du(t) = -(A_t u + f(t, x, u, \sigma \nabla u)) dt - g(t, x, u, \sigma \nabla u) d^\dagger \beta_t, \quad u(T) = \varphi,$
 where σ is such that $\sigma \cdot \sigma^T = a$. We are going to show that (6.1) has a unique solution under the following assumptions:

- (D1) $\| \varphi \|_{L^2(E; m)}^2 + E \| f(\cdot, 0, 0) \|_{L^2(E_{0,T}; m_T)}^2 + E \sum_{k=1}^\infty \| g_k(\cdot, 0, 0) \|_{L^2(E_{0,T}; m_T)}^2 < \infty.$
- (D2) For all $y \in \mathbb{R}$ and $e \in \mathbb{R}^d$ the mapping $E_{0,T} \ni z \mapsto f(z, y, e)$ belongs to $q\mathbb{L}^1$.
- (D3) There exist $l, L > 0, m \in (0, 1)$ and functions $M_k, L_k : E_{0,T} \rightarrow \mathbb{R}_+$ such that $\sup_{z \in E_{0,T}} \sum_k L_k(r) \leq l, \sup_{z \in E_{0,T}} \sum_k M_k(r) \leq m$ and for every $z \in E_{0,T}$ we have
 - (a) $(f(z, y, e) - f(z, y', e))(y - y') \leq L|y - y'|^2$ for all $y, y' \in \mathbb{R}$ and $e \in \mathbb{R}^d,$
 - (b) $|f(z, y, e) - f(z, y, e')| \leq L|e - e'|$ for all $y \in \mathbb{R}$ and $e, e' \in \mathbb{R}^d,$
 - (c) $|g_k(z, y, e) - g_k(z, y', e')| \leq L_k(z)|y - y'| + M_k(z)|e - e'|$ for all $y, y' \in \mathbb{R}$ and $e, e' \in \mathbb{R}^d.$
- (D4) For every $z \in E_{0,T}$ and $e \in \mathbb{R}^d$ the mapping $\mathbb{R} \ni y \mapsto f(z, y, e)$ is continuous.
- (D5) For every $y \in \mathbb{R}$ and $e \in \mathbb{R}^d$ the mappings $f(\cdot, y, e), g_k(\cdot, y, e) : \Omega \times E_T \rightarrow \mathbb{R}, k \in \mathbb{N},$ are $(\mathcal{F}_{t,T}^\beta)$ -progressively measurable.

The process $\mathbf{M}^{0,T}$ associated with the operator $\partial/\partial t - A_t$ has the following unique Fukushima decomposition:

$$\mathbf{X}_t = \mathbf{X}_0 + \mathbf{M}_t + \mathbf{A}_t, \quad t \in [0, T], \quad P_z\text{-a.s.}, \quad z \in E_{0,T},$$

where \mathbf{M} is a martingale AF of $\mathbf{M}^{0,T}$ of finite energy and \mathbf{A} is a continuous AF of $\mathbf{M}^{0,T}$ of zero energy. It is well known that W defined as

$$W_t = \int_0^t \sigma^{-1}(\mathbf{X}_r) d\mathbf{M}_r^\pi, \quad t \geq 0,$$

where $\mathbf{M}^\pi = \pi(\mathbf{M})$ and $\pi(x_1, \dots, x_{d+1}) = (x_2, \dots, x_{d+1})$ for $x_i \in \mathbb{R}$, $i = 1, \dots, d+1$, is a standard (\mathcal{G}_t) -Brownian motion under P_z .

PROPOSITION 6.1. *Assume that φ, f, g satisfy (D1)–(D5) and f, g do not depend on the last variable e . Let (Y, M) be the pair of processes of Theorem 3.3 and let u be the function defined by (3.19). Then $u \in M^2(0, T; H_0^1(D))$ and for q.e. $z \in E_{0,T}$,*

$$\left(u(\mathbf{X}_t), \int_0^t \sigma \nabla u(\mathbf{X}_r) dW_r \right) = (Y_t, M_t), \quad t \in [0, T], \quad \mathbb{P}_z\text{-a.s.}$$

Proof. That $Y_t = u(\mathbf{X}_t), t \in [0, T]$, follows from Theorem 3.7. Observe that $M = M^1 + M^2$, where M^1, M^2 are the processes of Theorem 3.3 associated with the data $(\varphi, f(\cdot, u), 0)$ and $(0, 0, g(\cdot, u))$, respectively. The desired representation for M^1 follows from [11, Proposition 3.6]. Therefore we may and will assume that $\varphi = 0, f = 0$ and g does not depend on y . By Theorem 4.9, $u \in M^2(0, T; H_0^1(D))$. Let u_α be defined as in the proof of Proposition 4.6. Then by (4.15) and the Burkholder–Davis–Gundy inequality,

$$(6.2) \quad \lim_{\alpha \rightarrow \infty} \mathbb{E}_z \sup_{0 \leq t \leq T} |M_t - M_t^{[u_\alpha]}|^2 = 0.$$

On the other hand, it is well known (see [28, Theorem 5.6]) that

$$M_t^{[u_\alpha]} = \int_0^t \sigma \nabla u_\alpha(\mathbf{X}_r) dW_r, \quad t \in [0, T].$$

Hence

$$\begin{aligned} & \mathbb{E}_\nu \sup_{0 \leq t \leq T} \left| M_t^{[u_\alpha]} - \int_0^t \sigma \nabla u(\mathbf{X}_r) dW_r \right|^2 \\ & \leq 4\Lambda \mathbb{E}_\nu \int_0^T |\nabla u - \nabla u_\alpha|^2(\mathbf{X}_t) dt \leq 4\Lambda E \|\nabla u - \nabla u_\alpha\|_{L^2(E_{0,T}; m_T)}^2 \|\hat{R}^{0,T} \nu\|_\infty \end{aligned}$$

for every $\nu \in \hat{S}_{00}$ (for the definition of \hat{S}_{00} see Remark 3.1). Since $u_\alpha \rightarrow u$ in $M^2(0, T; H_0^1(D))$ (see Lemma 4.2), using standard arguments (see the reasoning in [8, proof of Theorem 5.2.1]) we conclude from the above inequality and [28, Theorem 2.5] (see also [18, remark at the end of Section 4]) that,

up to a subsequence,

$$\lim_{\alpha \rightarrow \infty} \mathbb{E}_z \sup_{0 \leq t \leq T_\alpha} \left| M_t^{[u_\alpha]} - \int_0^t \sigma \nabla u(\mathbf{X}_r) dW_r \right|^2 = 0$$

for q.e. $z \in E_{0,T}$. When combined with (6.2), this completes the proof. ■

DEFINITION. We say that a measurable function $u : E_{0,T} \rightarrow \mathbb{R}$ is a *solution of (6.1)* if

- (a) for q.e. $z \in E_{0,T}$ we have $\mathbb{P}_z(\int_0^{T_i} |f(\mathbf{X}_t, u(\mathbf{X}_t), 0)| dt < \infty) = 1$ and $\mathbb{E}_z(\sup_{0 \leq t \leq T_i} |\int_0^t f(\mathbf{X}_r, u(\mathbf{X}_r), 0) dr|^2 + \int_0^{T_i} \|g(\mathbf{X}_t, u(\mathbf{X}_t), 0)\|^2 dt) < \infty$,
- (b) u is quasi-continuous, $u \in M^2(0, T; H_0^1(D))$ and for q.e. $z \in E_{0,T}$,

$$(6.3) \quad u(z) = E_z \left(\varphi(\mathbf{X}_{T_i}) + \int_0^{T_i} f(\mathbf{X}_t, u(\mathbf{X}_t), \sigma \nabla u(\mathbf{X}_t)) dt + \int_0^{T_i} g(\mathbf{X}_t, u(\mathbf{X}_t), \sigma \nabla u(\mathbf{X}_t)) d^\dagger \beta_t^\dagger \right).$$

DEFINITION. A pair $(Y^z, Z^z) \in \mathcal{S}^2 \times M^2$ is a *solution of the BDSDE*

$$(6.4) \quad Y_t^z = \varphi(\mathbf{X}_{T_i}) + \int_t^{T_i} f(\mathbf{X}_r, Y_r^z, Z_r^z) dr + \int_t^{T_i} g(\mathbf{X}_r, Y_r^z, Z_r^z) d^\dagger \beta_r^\dagger - \int_0^{T_i} Z_r^z dW_r, \quad t \in [0, T_i],$$

if $\mathbb{P}_z(\int_0^T (|f(\mathbf{X}_t, Y_t^z, 0)| + \|g(\mathbf{X}_t, Y_t^z, 0)\|^2) dt < \infty) = 1$ and (6.4) is satisfied \mathbb{P}_z -a.s.

In much the same way as in the proof of Lemma 3.2 one can show that if (D1)–(D5) are satisfied then for q.e. $z \in E_{0,T}$ the data

$$(\xi, \bar{f}(t, y, e), \bar{g}(t, y, e)) = (\varphi(\mathbf{X}_{T_i}), f(\mathbf{X}_t, y, e), g(\mathbf{X}_t, y, e)),$$

where $t \in [0, T]$, $y \in \mathbb{R}$, $e \in \mathbb{R}^d$, satisfy assumptions (B1)–(B4) under the measure \mathbb{P}_z . Therefore by Corollary 5.3 and Theorem 5.4, for q.e. $z \in E_{0,T}$ there exists a unique solution (Y^z, Z^z) of (6.4).

THEOREM 6.2. *Assume that (D1)–(D5) are satisfied. Then there exists a unique solution u of SPDE (6.1). Moreover, for q.e. $z \in E_{0,T}$,*

$$Y_t^z = u(\mathbf{X}_t), \quad t \in [0, T_i], \mathbb{P}_z\text{-a.s.},$$

$$\sigma \nabla u(\mathbf{X}) = Z^z, \quad l^1 \otimes \mathbb{P}_z\text{-a.e. on } [0, T_i] \times \Omega \times \Omega',$$

where (Y^z, Z^z) is a solution of (6.4).

Proof. The proof of uniqueness is similar to the proof of Proposition 3.9, with obvious modifications in Remark 3.8. Therefore we only show the

existence of a solution. Let $(Y^0, Z^0) = (0, 0)$ and for $n \geq 0$ let (Y^{n+1}, Z^{n+1}) be a solution of the BDSDE

$$Y_t^{n+1} = \varphi(\mathbf{X}_{T_t}) + \int_t^{T_t} f(\mathbf{X}_r, Y_r^{n+1}, Z_r^n) dr + \int_t^{T_t} g(\mathbf{X}_r, Y_r^{n+1}, Z_r^n) d^\dagger \beta_r - \int_t^{T_t} Z_r^{n+1} dW_r, \quad t \in [0, T_t], \mathbb{P}_z\text{-a.s.}$$

By Proposition 6.1,

$$(6.5) \quad Y_t^n = u^n(\mathbf{X}_t), \quad t \in [0, T_t], \mathbb{P}_z\text{-a.s.},$$

and

$$(6.6) \quad \sigma \nabla u^n(\mathbf{X}) = Z^n \quad l^1 \otimes \mathbb{P}_z\text{-a.e. on } [0, T_t] \times \Omega \times \Omega'$$

for q.e. $z \in E_{0,T}$, where $u^0 = 0$ and $u^n, n \geq 1$, is a solution of the SPDE

$$\frac{\partial u^n}{\partial t} + A_t u^n = -f(t, x, u^n, \sigma \nabla u^{n-1}) - g(t, x, u^n, \sigma \nabla u^{n-1}) d^\dagger \beta_t, \quad u^n(T) = \varphi.$$

We know that for q.e. $z \in E_{0,T}$ there exists a unique solution (Y^z, Z^z) of (6.4) and from the proof of Theorem 5.4 it follows that

$$(6.7) \quad \lim_{n \rightarrow \infty} \|(Y^n, Z^n) - (Y^z, Z^z)\|_{\beta, z} = 0,$$

where the norm $\|\cdot\|_{\beta, z}$ is defined by (5.1) with \mathbb{E} replaced by \mathbb{E}_z . Applying the Burkholder–Davis–Gundy inequality and standard arguments we get

$$(6.8) \quad \lim_{n \rightarrow \infty} \mathbb{E}_z \left(\sup_{0 \leq t \leq T_t} |Y_t^n - Y_t^z|^2 + \int_0^{T_t} |Z_t^n - Z_t^z|^2 dt \right) = 0$$

for q.e. $z \in E_{0,T}$. Set $u(z) = \lim_{n \rightarrow \infty} u_n(z)$ for those $z \in E_{0,T}$ for which the limit exists in probability \mathbb{P}_z , and $u(z) = 0$ otherwise. Then from (6.5) and (6.8) we conclude that for q.e. $z \in E_{0,T}$,

$$Y_t^z = u(\mathbf{X}_t), \quad t \in [0, T_t], \mathbb{P}_z\text{-a.s.}$$

By (6.6), for any $\nu \in \hat{S}_{00}$ (for the definition of \hat{S}_{00} see Remark 3.1) we have

$$E \int_{E_{0,T}} (|\sigma \nabla(u_n - u_m)(z)|^2 \wedge 1) \hat{R}^{0,T} \nu(z) dz = \mathbb{E}_\nu \int_0^{T_t} (|Z_t^n - Z_t^m|^2 \wedge 1) dt,$$

which by (6.8) converges to zero as $n, m \rightarrow \infty$. Therefore using [28, Theorem 2.5] (see also [18, remark at the end of Section 4]) and applying a standard argument (see [8, proof of Theorem 5.2.1]) shows that there is a measurable $v : \Omega \times E_{0,T} \rightarrow \mathbb{R}$ and a subsequence, still denoted by n , such that for q.e. $z \in E_{0,T}$, $\sigma \nabla u_n(z) \rightarrow v(z)$ in probability P . From this and (6.8) it follows that

$$v(\mathbf{X}) = Z^z \quad l^1 \otimes \mathbb{P}_z\text{-a.e. on } [0, T_t] \times \Omega \times \Omega'$$

for q.e. $z \in E_{0,T}$. Hence $(u(\mathbf{X}), v(\mathbf{X}))$ is a solution of (6.4) for q.e. $z \in E_{0,T}$. Consequently, (6.3) with $\sigma \nabla u$ replaced by v is satisfied for q.e. $z \in E_{0,T}$. Proposition 6.1 now shows that $u \in M^2(0, T; H_0^1(D))$ and for q.e. $z \in E_{0,T}$,

$$0 = \mathbb{E}_z \left\langle \int_0^{\cdot} (\sigma \nabla u - v)(\mathbf{X}_t) dW_t \right\rangle_{T_t} = \mathbb{E}_z \int_0^{T_t} |(\sigma \nabla u - v)(\mathbf{X}_t)|^2 dt,$$

which completes the proof of the theorem. ■

7. Probabilistic and mild solutions of SPDEs. In this section we adopt the following notation.

- U is a separable real Hilbert space, $\{e_k\}_{k \geq 1}$ is an orthonormal basis of U , and Q is a symmetric nonnegative trace class operator on U such that $Qe_k = \lambda_k e_k$, $k \geq 1$. We assume that $U \subset H = L^2(E; m)$. Then $U_0 = Q^{1/2}(U) \subset U$ is a separable Hilbert space with inner product $\langle u, v \rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U$ (note that $\{f_k = \sqrt{\lambda_k} e_k\}$ is an orthonormal basis of U_0).
- $L(U_0, \mathbb{R})$ is the Banach space of all bounded operators from U_0 into \mathbb{R} endowed with the supremum norm and $L_2(U_0, \mathbb{R})$, $L_2(U_0, H)$ are the spaces of Hilbert–Schmidt operators from U_0 into \mathbb{R} and H , respectively.
- B is a Q -Wiener process defined on some complete probability space (Ω, \mathcal{F}, P) with values in U .

It is known (see, e.g., [6, Chapter 4]) that B has the expansion

$$(7.1) \quad B_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_t^k e_k = \sum_{k=1}^{\infty} \beta_t^k f_k, \quad t \geq 0,$$

where $\beta_t^k = \lambda_k^{-1/2} \langle B_t, e_k \rangle_U$ are real valued mutually independent standard Wiener processes on (Ω, \mathcal{F}, P) (the series above converges in $L^2(\Omega, \mathcal{F}, P; U)$ and P -a.s. in $C([0, T]; U)$).

Suppose we are given $\tilde{g} : \Omega \times E_T \times \mathbb{R} \rightarrow \mathbb{R}$ and A_t, φ, f as in Section 3. In this section we consider SPDE of the form

$$(7.2) \quad du(t) = -(A_t u + f(t, x, u)) dt - \tilde{g}(t, x, u) d^\dagger B_t, \quad u(T) = \varphi.$$

In what follows,

$$(7.3) \quad g = (g_1, g_2, \dots), \quad g_k(t, x, y) = \tilde{g}(t, x, y) \cdot f_k(x), \quad (t, x) \in E_T, y \in \mathbb{R}.$$

7.1. Probabilistic solutions of (7.2). We assume that φ, f, g satisfy (H1)–(H5). We denote by u the unique solution of SPDE (3.4) of Theorem 3.7.

Let $t \mapsto G(\mathbf{X}_t, u(\mathbf{X}_t))$ be the process with values in $L(U_0, \mathbb{R})$ defined as

$$G(\mathbf{X}_t, u(\mathbf{X}_t))\psi = \tilde{g}(\mathbf{X}_t, u(\mathbf{X}_t)) \cdot \psi(\mathbf{X}_t), \quad t \geq 0,$$

for $\psi \in U_0$. Since g satisfies (H3b), (H4), (H5), and $u(\mathbf{X})$ is of class M , it follows from Remark 3.1 that the process $t \mapsto g_k(\mathbf{X}_t, u(\mathbf{X}_t))$ is of class M . Since the family $\{f_k \otimes 1, k \geq 1\}$, where $(f_k \otimes 1)(\psi) = \langle \psi, f_k \rangle_{U_0}$, $\psi \in U_0$, is linearly dense in $L_2(U_0, \mathbb{R})$ and

$$\begin{aligned} \langle G(t, \mathbf{X}_t, u(\mathbf{X}_t)), f_k \otimes 1 \rangle_{L_2(U_0, \mathbb{R})} &= \sum_{i=1}^{\infty} G(t, \mathbf{X}_t, u(\mathbf{X}_t)) f_i \cdot f_k \otimes 1(f_i) \\ &= \tilde{g}(t, \mathbf{X}_t, u(\mathbf{X}_t)) \cdot f_k(\mathbf{X}_t) = g_k(t, \mathbf{X}_t, u(\mathbf{X}_t)), \end{aligned}$$

it follows from [6, Lemma 4.8] that $t \mapsto G(\mathbf{X}_t, u(\mathbf{X}_t))$ is of class M . In particular, for every ω' the process $t \mapsto G(\mathbf{X}_t, u(\mathbf{X}_t))$ is $(\mathcal{F}_{t, T_t}^{\beta_t})$ -adapted. Moreover,

$$(7.4) \quad \mathbb{E}_z \int_0^{T_t} |G(\mathbf{X}_t, u(\mathbf{X}_t))|_{L_2(U_0, \mathbb{R})}^2 dt = \mathbb{E}_z \int_0^{T_t} \|g(\mathbf{X}_t, u(\mathbf{X}_t))\|^2 dt < \infty,$$

so for every ω' , $t \mapsto G(\mathbf{X}_t, u(\mathbf{X}_t))$ is an $(\mathcal{F}_{t, T_t}^{\beta_t})$ -adapted $L_2(U_0; \mathbb{R})$ -valued process. One can also check (see [5, Proposition 3.4]) that if we set

$$B_t^\iota = \sum_{k=1}^{\infty} \beta_{t+\iota(0)}^k f_k = B_{t+\iota(0)}, \quad t \in [0, T - \iota(0)],$$

then

$$(7.5) \quad \int_0^t G(\mathbf{X}_r, u(\mathbf{X}_r)) d^\dagger B_r^\iota = \int_0^t g(\mathbf{X}_r, u(\mathbf{X}_r)) d^\dagger \beta_r^\iota$$

for every $t \in [0, T]$. Therefore u of Theorem 3.7 is a probabilistic solution of (7.2) in the sense that for q.e. $z \in E_{0, T}$,

$$(7.6) \quad u(z) = E_z \left(\varphi(\mathbf{X}_{T_t}) + \int_0^{T_t} f(\mathbf{X}_t, u(\mathbf{X}_t)) dt + \int_0^{T_t} G(\mathbf{X}_t, u(\mathbf{X}_t)) d^\dagger B_t^\iota \right) \quad P\text{-a.s.}$$

By (3.1), this can be written in the form

$$(7.7) \quad u(s, x) = E_{s, x} \left(\varphi(X_T) + \int_0^{T-s} f(s+t, X_{s+t}, u(s+t, X_{s+t})) dt + \int_0^{T-s} G(s+t, X_{s+t}, u(s+t, X_{s+t})) d^\dagger B_t^s \right) \quad P\text{-a.s.}$$

REMARK 7.1. By [19, (6.2.24)] and [18, (4.4)], if $m(B) > 0$ for some Borel set $B \subset E$ then $\text{cap}(\{s\} \times B) > 0$ for every $s \in \mathbb{R}$. From this and the fact that (7.7) holds for q.e. $(s, x) \in E_{0, T}$ it follows that for every $s \in (0, T]$ equality (7.7) holds true for m -a.e. $x \in E$.

7.2. Mild solutions of (7.2). In this subsection we assume additionally that $B^{(t)} = B^{(0)}$ for all $t \in [0, T]$, and that f is Lipschitz continuous with respect to y , i.e. we assume (H1), (H2), (H3)(a), (H4), (H5) and the following condition: there exists $L > 0$ such that for every $z \in E_{0,T}$,

$$(7.8) \quad |f(z, y) - f(z, y')| \leq L|y - y'| \quad \text{for } y, y' \in \mathbb{R}.$$

Let A denote the operator corresponding to the form $B^{(0)}$, $\{P_t, t > 0\}$ denote the semigroup of linear operators on H associated with $B^{(0)}$, and let

$$F : \Omega \times [0, T] \times H \rightarrow H, \quad G : \Omega \times [0, T] \times H \rightarrow L_2(U_0, H)$$

be operators defined by

$$F(\omega, t, v)(x) = f(\omega, t, x, v(x)), \quad (G(\omega, t, v)\psi)(x) = \tilde{g}(\omega, t, x, v(x)) \cdot \psi(x)$$

for $t \in [0, T]$, $v \in H$, $\psi \in U_0$, $x \in E$. We shall show that u is a mild solution of equation (7.2) interpreted as an abstract evolution equation of the form

$$(7.9) \quad u(s) = \varphi + \int_s^T (Au(t) + F(t, u(t))) dt + \int_s^T G(t, u(t)) d^\dagger B_t, \quad u(T) = \varphi,$$

on the Hilbert space H , i.e. if we set $u(s)(x) = u(s, x)$ for $(s, x) \in E_{0,T}$ then

$$(7.10) \quad u(s) = P_{T-s}\varphi + \int_s^T P_{t-s}F(t, u(t)) dt + \int_s^T P_{t-s}G(t, u(t)) d^\dagger B_t.$$

Here the integral involving F is a Bochner integral. To see this, first note that from (7.8) and (H3b) it follows that $F(t, \cdot)$ and $G(t, \cdot)$ are Lipschitz continuous. Using this and [6, Lemma 4.8] one can show that F (resp. G) is an $(\mathcal{F}_{t,T}^\beta)$ -progressively measurable mapping from $\Omega \times [0, T] \times H$ into $(H, \mathcal{B}(H))$ (resp. into $(L^2(U_0, H), \mathcal{B}(L^2(U_0, H)))$). Since $\{P_t\}$ is a contraction on H , it follows from (H1), (H3)(b) and (7.8) that there is $c > 0$ such that

$$|P_t F(t, v)|_H \leq c(1 + |v|_H), \quad |P_t G(t, v)|_{L_2(U_0, H)} \leq c(1 + |v|_H), \quad t \in [0, T].$$

As $u \in M^2(0, T; H)$ we have $E \int_s^T (|P_t F(t, u(t))|_H + |P_t G(t, u(t))|_{L_2(U_0, H)}^2) dt < \infty$, so the integrals in (7.10) involving F and G are well defined. By [24, Proposition A.2.2], for $h \in H$ we have

$$\begin{aligned} \left\langle \int_s^T P_{t-s}F(t, u(t)) dt, h \right\rangle_H &= \int_s^T \langle P_{t-s}F(t, u(t)), h \rangle_H dt \\ &= \int_0^{T-s} \langle E_{s, \cdot} f(s+t, X_{s+t}, u(s+t, X_{s+t})), h \rangle_H dt \\ &= \left\langle \int_0^{T-s} E_{s, \cdot} f(s+t, X_{s+t}, u(s+t, X_{s+t})) dt, h \right\rangle_H. \end{aligned}$$

By the above and Fubini’s theorem,

$$(7.11) \quad \left\langle \int_s^T P_{t-s} F(t, u(t)) dt, h \right\rangle_H = \left\langle E_{s,\cdot} \int_0^{T-s} f(s+t, X_{s+t}, u(s+t, X_{s+t})) dt, h \right\rangle_H.$$

Similarly, by [24, Lemma 2.4.1] and the fact that the process X is time-homogeneous,

$$\begin{aligned} \left\langle \int_s^T P_{t-s} G(t, u(t)) d^\dagger B_t, h \right\rangle_H &= \sum_{k=1}^\infty \int_s^T \langle P_{t-s} G(t, u(t))(f_k), h \rangle_H d^\dagger \beta_t^k \\ &= \sum_{k=1}^\infty \int_0^{T-s} \langle E_{s,\cdot} g_k(s+t, X_{s+t}, u(s+t, X_{s+t})), h \rangle_H d^\dagger \beta_{t+s}^k \\ &= \sum_{k=1}^\infty \left\langle \int_0^{T-s} E_{s,\cdot} g_k(s+t, X_{s+t}, u(s+t, X_{s+t})) d^\dagger \beta_{t+s}^k, h \right\rangle_H. \end{aligned}$$

By the above and the stochastic Fubini theorem (see [6, Theorem 4.18]),

$$(7.12) \quad \left\langle \int_s^T P_{t-s} G(t, u(t)) d^\dagger B_t, h \right\rangle_H = \sum_{k=1}^\infty \left\langle E_{s,\cdot} \int_0^{T-s} g_k(s+t, X_{s+t}, u(s+t, X_{s+t})) d^\dagger \beta_{t+s}^k, h \right\rangle_H = \left\langle E_{s,\cdot} \sum_{k=1}^\infty \int_0^{T-s} g_k(s+t, X_{s+t}, u(s+t, X_{s+t})) d^\dagger \beta_{t+s}^k, h \right\rangle_H = \left\langle E_{s,\cdot} \int_0^{T-s} G(s+t, X_{s+t}, u(s+t, X_{s+t})) d^\dagger B_t^s, h \right\rangle_H.$$

From Remark 7.1 and (7.11), (7.12) it follows that $\langle u(s), h \rangle_H = \langle v(s), h \rangle_H$ for $h \in H$, where $v(s)$ is defined by the right-hand side of (7.10). This shows that u satisfies (7.10). Thus we have proved the following proposition.

PROPOSITION 7.2. *Assume that $B^{(t)} = B^{(0)}$ for all $t \in [0, T]$, and (H1)–(H5), (7.8) are satisfied. Let u be the unique solution of SPDE (3.4). Then (7.10) holds true for every $s \in (0, T]$.*

REMARK 7.3. Let

$$\hat{B}_t = B_{T-t} - B_T, \quad t \in [0, T].$$

One can check that

$$\int_{T-s}^T P_{t-(T-s)}G(t, u(t)) d^\dagger B_t = - \int_0^s P_{s-t}G(T-t, u(T-t)) d\hat{B}_t, \quad s \in [0, T].$$

Therefore from (7.10) it follows that \bar{u} defined by $\bar{u}(t) = u(T-t)$, $t \in [0, T]$, is a mild solution of the problem

$$d\bar{u}(t) = (A\bar{u} + f(T-t, x, \bar{u})) dt - \tilde{g}(T-t, x, \bar{u}) d\hat{B}_t, \quad \bar{u}(0) = \varphi.$$

REMARK 7.4. Note that in case A_t is defined by (1.2) one can generalize (7.6) to equations of the form (7.2) with f, \tilde{g} also depending on the gradient of a solution (see (6.1)) and satisfying (D1)–(D5). If, in addition, $A_t = A_0$ for $t \in [0, T]$, and we replace (D3)(a) by Lipschitz continuity in y , then in much the same way as in the proof of (7.10) one can prove that u of Theorem 6.2 is a mild solution of (7.9) with the mappings $F : \Omega \times [0, T] \times H_0^1(D) \rightarrow H$ and $G : \Omega \times [0, T] \times H_0^1(D) \rightarrow L_2(U_0, H)$ defined by

$$\begin{aligned} F(\omega, t, v)(x) &= f(\omega, t, x, v(x), \sigma \nabla v(x)), \\ (G(\omega, t, v)\psi)(x) &= \tilde{g}(\omega, t, x, v(x), \sigma \nabla v(x)) \cdot \psi(x) \end{aligned}$$

for $t \in [0, T]$, $v \in H_0^1(D)$, $\psi \in U_0$, $x \in D$.

7.3. Examples. Assume that $e_k \in L^\infty(E; m)$ and e_k are bounded uniformly in k , or, more generally,

$$(7.13) \quad \sup_{x \in E} \sum_{k=1}^\infty \lambda_k |e_k(x)|^2 < \infty.$$

Below we show that under (7.13) the results of Sections 7.1 and 7.2 apply to equation (7.2) with \tilde{g} Lipschitz continuous such that $\tilde{g}(\cdot, 0)$ is bounded or square integrable. For simplicity we assume that \tilde{g} does not depend on ω .

EXAMPLE 7.5. Assume that

$$\tilde{g}(\cdot, 0) \in L^2(E_{0,T}; m_T) \cup L^\infty(E_{0,T}; m_T)$$

and \tilde{g} is Lipschitz continuous in y with some constant L' , i.e.

$$|\tilde{g}(t, x, y_1) - \tilde{g}(t, x, y_2)| \leq L'|y_1 - y_2|$$

for all $(t, x) \in E_{0,T}$ and $y_1, y_2 \in \mathbb{R}$. Then g defined by (7.3) satisfies (H1) and (H3). Indeed, we have

$$\|g_k(\cdot, 0)\|_{L^2(E_{0,T}; m_T)}^2 = \lambda_k \|\tilde{g}(\cdot, 0)e_k(\cdot)\|_{L^2(E_{0,T}; m_T)}^2$$

and

$$|g_k(t, x, y) - g_k(t, x, y')| \leq L' \sqrt{\lambda_k} |e_k(x)|.$$

By the last inequality, (H3) is satisfied with $L_k(x) = L' \sqrt{\lambda_k} |e_k(x)|$. In case $\tilde{g}(\cdot, 0) \in L^2(E_{0,T}; m_T)$ assumption (7.13) immediately forces g to satisfy hypothesis (H1). If $\tilde{g}(\cdot, 0) \in L^\infty(E_{0,T}; m_T)$ then $\|\tilde{g}(\cdot, 0)e_k(\cdot)\|_{L^2(E_{0,T}; m_T)}^2 \leq T^2 \|\tilde{g}(\cdot, 0)\|_\infty \|e_k\|_H$, so g satisfies (H3), because $\text{Tr } Q < \infty$ and we assume that $U \subset H$.

EXAMPLE 7.6. Let D be a nonempty bounded open subset of \mathbb{R}^d and let $\tilde{g} : D_{0,T} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. If $\tilde{g}(\cdot, 0, 0) \in L^2(D_{0,T}; m_T) \cup L^\infty(D_{0,T}; m_T)$ and

$$|\tilde{g}(t, x, y_1, e_1) - \tilde{g}(t, x, y_2, e_2)| \leq L'(|y_1 - y_2| + |e_1 - e_2|)$$

for every $(t, x) \in E_{0,T}$ and $y_1, y_2 \in \mathbb{R}$, $e_1, e_2 \in \mathbb{R}^d$ then $g = (g_1, g_2, \dots)$ defined by $g_k(t, x, y, e) = \tilde{g}(t, x, y, e) \cdot f_k(x)$ satisfies (D1) and (D3) with $L_k(x) = M_k(x) = L' \sqrt{\lambda_k} |e_k(x)|$, $x \in D$.

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