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Unitary closure and Fourier algebra of a topological group

by

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Abstract. This is a sequel to our recent work (2012) on the Fourier–Stieltjes algebra $B(G)$ of a topological group G. We introduce the unitary closure $\overline{G}$ of G and use it to study the Fourier algebra $A(G)$ of G. We also study operator amenability and fixed point property as well as other related geometric properties for $A(G)$.

1. Introduction. Let G be a topological group, i.e. a group with a Hausdorff topology such that the mappings $x \mapsto x^{-1}$ from G to G and $(x, y) \mapsto xy$ from $G \times G$ to G are continuous. Let $P(G)$ denote the collection of all continuous *positive definite* functions on G, i.e. continuous complex-valued functions φ on G such that for any complex numbers $\lambda_1, \ldots, \lambda_n$ and any $a_1, \ldots, a_n$ in G, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\lambda_i} \lambda_j \varphi(a_i^{-1} a_j) \geq 0.$$

Let $B(G)$ denote the linear span of $P(G)$. As shown in [La1], $B(G)$ can be identified with the predual of a von Neumann algebra $W^*(G) \subset \mathcal{B}(\mathcal{H}_\omega)$, where ω is a $*$-homomorphism of G into the group of unitary operators in $\mathcal{B}(\mathcal{H}_\omega)$, the space of bounded linear operators from a Hilbert space $\mathcal{H}_\omega$ into $\mathcal{H}_\omega$. Furthermore, $B(G)$, with the predual norm of $W^*(G)$, is a commutative Banach algebra called the *Fourier–Stieltjes* algebra of G.

In a recent paper [La-Lu], we study $B(G)$ when G has a host algebra or a group C^*-algebra, the analogue of the group C^*-algebra of a locally compact group G. Our main challenge is that a topological group cannot have a positive regular Borel measure which is left translation invariant unless G is locally compact.

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In Section 3, we introduce the unitary cover and the unitary closure of a topological group and establish their universal properties (Theorems 3.3, 3.6 and 3.7). The unitary closure is then used in Section 4 to define the Fourier algebra $A(G)$. When G is locally compact, $A(G)$ was introduced by P. Eymard in his classical paper [Ey]. We study operator amenability of $B(G)$ and $A(G)$ and the weak fixed point property of $A(G)$ for non-expansive mappings.

In Section 5, we study the universal C^*-algebra $C^*_\Omega(G)$ generated by the continuous representations of G and we derive some of its properties.

In Section 6, we study functions in $B(G)$ arising from the set of left invariant means on an amenable topological group and its extreme points. Some open problems are posed in Section 7.

2. Preliminaries and notation. Let A be a subset of a linear space E. Then $\langle A\rangle$ will denote the linear span of A. If E is also normed, then the closure of A and the closed linear span of A will be denoted by $\overline{A}$ and $\overline{\langle A\rangle}$ respectively if the closure is taken with respect to the norm topology, or by $\overline{A}^\tau$ and $\overline{\langle A\rangle}^\tau$ if the closure is taken with respect to some other topology τ on E.

The continuous dual of a normed space E will be denoted by E^*. If $x\in E$ and $\varphi\in E^*$, then the value of φ on x will be denoted by $\varphi(x)$ or $\langle\varphi,x\rangle$. Also if $F\subset E^*$, then $\sigma(E,F)$ will denote the locally convex topology on E determined by the seminorms $\{p_\varphi;\ \varphi\in F\}$, where $p_\varphi(x) := |\varphi(x)|$ for all $x\in E$. If $F=E^*$, then $\sigma(E,E^*)$ is called the *weak topology* of E. The *weak* topology* on E^* is the locally convex topology determined by the seminorms $p_x(\varphi)=|\langle\varphi,x\rangle|$ for $\varphi\in E^*$ and $x\in E$.

If M is a *W^*-algebra* (i.e. a C^*-algebra with a predual), then M_* will denote the (unique) predual of M. For each $\varphi\in M_*$, write φ^* for the functional in M_* defined by $\varphi^*(y) := \overline{\varphi(y^*)}$ for $y\in M$. Also the *ultra-weak topology* on M (i.e. the $\sigma(M,M_*)$ topology) will often be referred to as the *σ-topology.*

Let G be a topological group and let $CB(G)$ be the Banach algebra of bounded continuous complex-valued functions on G. For each $a\in G$, define the left and right translation operators l_a, r_a on $CB(G)$ by

$$l_af(g) := f(ag),\qquad r_af(g) := f(ga),$$

for $g\in G$, and for f in $CB(G)$ define the supremum norm by

$$\|f\|_\infty := \sup\{|f(x)|;\ x\in G\}.$$

DEFINITION 2.1. Let G be a topological group. Define the space $WAP(G)$ of *weakly almost periodic functions* to be the set of all $f\in CB(G)$ such that $LO(f) := \{l_af;\ a\in G\}$ is relatively compact in the weak topology of $CB(G)$.

It is well known that $WAP(G)$ is a closed and translation invariant $*$-subalgebra of $CB(G)$. Furthermore, $f \in WAP(G)$ if and only if $RO(f) := \{r_a f;\ a \in G\}$ is relatively compact in the weak topology of $CB(G)$ (see [B-J-M]).

DEFINITION 2.2. Let $LUC(G)$ be the space of bounded left uniformly continuous functions on G, i.e. all $f \in CB(G)$ such that the map $a \mapsto l_a f$ from G to $(CB(G), \|\ \|_\infty)$ is continuous.

REMARK 2.3. Let G be a topological group. Then $WAP(G) \subset LUC(G)$ (see [M-P-U] for a proof).

The collection $P(G)$ of continuous positive definite functions is a cone in $CB(G)$, closed under conjugation, involution and product.

We denote

$$P(G)_1 := \{\varphi \in P(G);\ \varphi(e) = 1\}.$$

It is clear that $P(G)_1$ is a convex subset and a subsemigroup of $P(G)$ with pointwise multiplication. When G is a locally compact group, $P(G)$ corresponds to the set of positive linear functionals on $C^*(G)$, the group C^*-algebra of G.

By a *representation* or *unitary representation* $(\pi, \mathcal{H}_\pi)$ of a topological group we shall mean a continuous homomorphism of G into the group of unitary operators in $\mathcal{B}(\mathcal{H}_\pi)$, when $\mathcal{B}(\mathcal{H}_\pi)$ has the weak operator topology. If $(\pi, \mathcal{H}_\pi)$ is a continuous unitary representation of G and $M_\pi := \overline{\langle \pi(a) :\ a \in G\rangle}^\sigma$ is the W^*-algebra determined by π, then $\pi : G \to M_\pi$ is a σ-continuous homomorphism of G into the group of unitary elements of M_π, where σ denotes the ultra-weak topology on $\mathcal{B}(\mathcal{H}_\pi)$.

A subspace $\mathcal{F}$ of the representation space $\mathcal{H}_\pi$ is called *G-invariant* if $\pi(g)\xi \in \mathcal{F}$ for all $g \in G$ and $\xi \in \mathcal{F}$.

The representation $(\pi, \mathcal{H}_\pi)$ is called *irreducible* if the only closed G-invariant subspaces of $\mathcal{H}_\pi$ are the two trivial ones.

A unitary representation $(\pi, \mathcal{H}_\pi)$ is called *cyclic* with cyclic vector $\xi \in \mathcal{H}_\pi$ if the subspace spanned by $\{\pi(a)\xi;\ a \in G\}$ is dense in $\mathcal{H}_\pi$. If π is irreducible, then every non-zero vector in $\mathcal{H}_\pi$ is cyclic.

A *coefficient* of the representation $(\pi, \mathcal{H}_\pi)$ is by definition the continuous function

$$c^\pi_{\xi,\eta}(g) := \langle \pi(g)\xi, \eta\rangle, \qquad g \in G,$$

where ξ, η are two elements in $\mathcal{H}_\pi$. If $\xi = \eta$, we also write c^π_ξ instead of $c^\pi_{\xi,\xi}$.

We say that a continuous positive definite function φ is *pure* if the corresponding representation $(\pi_\varphi, \mathcal{H}_\varphi)$ is irreducible.

The following proposition follows easily from [Li-Ma, Theorem 3.2]:

Proposition 2.4. *Let G be a topological group. Then $\varphi \in P(G)_1$ if and only if there exists a cyclic unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ with cyclic vector $\xi \in \mathcal{H}_\varphi$ of length 1 such that $\varphi = c_\xi^{\pi_\varphi}$.*

A function $\varphi \in P(G)_1$ is pure if and only φ is an extremal point in $P(G)_1$. If G is abelian, then every irreducible unitary representation π of G is one-dimensional, i.e. $\pi : G \to \mathbb{T}$ is a character of G.

Let G be a topological group. Let

$$B(G) := \langle P(G) \rangle.$$

Then it follows readily from Proposition 2.4 that $\varphi \in B(G)$ if and only if there exists a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G and vectors $\xi, \eta \in \mathcal{H}_\pi$ such that $\varphi(a) = \langle \pi(a)\xi, \eta \rangle$ for all $a \in G$. Furthermore, $B(G) \subset WAP(G) \subset LUC(G)$ by Remark 2.3.

Definition 2.5. Let G be a topological group. The group G acts on the Banach space $A := LUC(G)$ of left uniformly continuous functions on G by left translation. This action is strongly continuous. Hence G also acts on the dual space $LUC(G)^*$ and this action is jointly continuous for the weak* topology on bounded subsets of $LUC(G)^*$. We say that a bounded linear functional n of $LUC(G)$ is a *mean* if n is positive and $\langle n, 1 \rangle = 1$. The mean n is called *left invariant* if $\langle n, l_g f \rangle = \langle n, f \rangle$ for all $g \in G$ and $f \in LUC(G)$. We say that G is *amenable* if there exists a left invariant mean on $LUC(G)$. This is equivalent to the existence, for each $f \in LUC(G)$), of a mean n such that $\langle n, l_g f \rangle = \langle n, f \rangle$ for all $g \in G$ (see [Mi2], [Gre] and [La4]).

We denote by $LIM(G)$ the set of left invariant means on $LUC(G)$ (which is convex and weak*-closed).

A topological group G is called *extremely amenable* if for every jointly continuous action of G on a compact Hausdorff set X, there exists a fixed point in X for this action. If G is extremely amenable, then there exists an element in $LIM(G)$ which is multiplicative on $LUC(G)$, since the set of characters $X = \sigma(LUC(G))$ is compact in the weak* topology and G acts jointly continuously on this compact Hausdorff space. Hence it has a fixed point δ. This character δ is then a left invariant mean on $LUC(G)$. The converse is also true (see [Mi3] and [La3]).

Of course extremely amenable groups are amenable. It has been shown in [Gra-La] that the only extremely amenable locally compact group is the trivial one.

By a *σ-continuous representation* of G in a W^*-algebra M we shall mean a pair (ω, M) such that ω is a homomorphism of G into $M_u := \{x \in M;\ x^*x = xx^* = 1\}$, the group of unitaries in M, where 1 is the identity of M, and ω is continuous when M has the σ-topology.

Following [La1], we define $\Omega(G)$ to be the collection of all σ-continuous representations $\alpha = (\omega, M)$ of G such that $\overline{\langle\omega(G)\rangle}^{\sigma} = M$. Then $B(G)$ is precisely the collection of all complex-valued functions φ on G such that $\varphi = \hat{f}(\alpha)$ for some $f \in M_*$ and some $\alpha = (\omega, M) = (\omega_\alpha, M_\alpha)$ in $\Omega(G)$, where $\hat{f}(\alpha)(a) = \langle\omega(a), f\rangle$ for all $a \in G$. For each φ in $B(G)$, define

$$\begin{aligned}\|\varphi\| &:= \|\varphi\|_{B(G)} \\ &:= \inf\{\|\hat{f}(\alpha)\|;\ f \in M_*,\ \varphi = \hat{f}(\alpha) \text{ and } \alpha = (\omega, M) \in \Omega(G)\}.\end{aligned}$$

Also let $M_\Omega := \sum \oplus M_{\omega_\alpha}$, the direct sum of the W^*-algebras $M_\alpha := M_{\omega_\alpha}$ for $\alpha \in \Omega(G)$ (see [Sa, p. 2]). Define a σ-continuous homomorphism of G into M_Ω by $\omega_G(a)(\alpha) := \omega_\alpha(a)$ for each $\alpha = (\omega_\alpha, M_\alpha)$ in $\Omega(G)$. Write

$$W^*(G) := \overline{\langle\omega_G(G)\rangle}^{\sigma}.$$

Then

$$\|x\| = \sup\{\|x_\alpha\|;\ \alpha \in \Omega(G)\}$$

for each

$$x = \sum_i \lambda_i \omega_G(a_i) = \Big(x_\alpha = \sum_i \lambda_i \omega_\alpha(a_i)\Big)_{\alpha \in \Omega(G)} \in \langle\omega_G(G)\rangle.$$

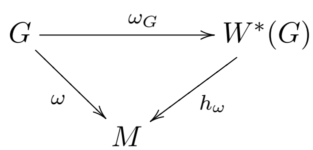
We call ω_G the *universal representation* of G.

The following theorem follows from [La1, Theorems 3.2 and 4.1].

THEOREM 2.6.

(a) *$B(G)$ is a subalgebra of $WAP(G)$ containing the constant functions. Furthermore $\|\ \|$ is a norm on $B(G)$ and $(B(G), \|\ \|)$ is a commutative Banach algebra isomorphic to the predual $W^*(G)_*$ of $W^*(G)$. More specifically, the map $\rho : W^*(G)_* \to B(G)$ defined by $\rho(f) := \hat{f}$, $f \in W^*(G)_*$, is a linear isometry from $W^*(G)_*$ onto $B(G)$. Furthermore, $\rho(f)$ is positive definite if and only if f is positive.*

(b) *If $\alpha = (\omega, M)$ is any σ-continuous representation of G, then there is a w^*-homomorphism h_ω from $W^*(G)$ into M such that the diagram*



is commutative. Also if $f \in M_*$, then $\hat{f}(\alpha)(a) = \langle h_\omega(\omega_G(a)), f\rangle$ for all $a \in G$.

(c) *If $\varphi \in B(G)$ and $a \in G$, then the functions $l_a\varphi$, $r_a\varphi$, φ^*, $\overline{\varphi}$ are all in $B(G)$ and $\|l_a\varphi\| = \|\varphi\|$, $\|r_a\varphi\| = \|\varphi\|$, $\|\varphi^*\| = \|\varphi\|$, $\|\overline{\varphi}\| = \|\varphi\|$.*

The following theorem has been proved in [La-Lu, Theorem 4.2].

THEOREM 2.7. *Let G be a topological group.*

(i) *The spectrum $\sigma(B(G))$ of the algebra $B(G)$ consists of all the non-zero elements $T \in W^*(G)$ such that*

$$\pi \otimes \rho(T) = \pi(T) \otimes \rho(T) \text{ for all unitary representations } \pi, \rho \text{ of } G.$$

(ii) *The spectrum $\sigma(B(G))$ is a compact semitopological semigroup contained in $W^*(G)$ with the weak* topology. Moreover if $T \in \sigma(B(G))$, then $T^* \in \sigma(B(G))$.*

EXAMPLE 2.8. Examples of amenable topological groups which are not locally compact include:

(1) The group $\mathrm{Aut}(\mathbb{Q}, \leq)$ of all order preserving self-bijections of the set $\mathbb{Q}$ of rational numbers with the usual order, equipped with the Polish topology of simple convergence on the set $\mathbb{Q}$ viewed as discrete. This group is extremely amenable (see [Pe]).
(2) Let A be a C^*-algebra with unit and $U(A)$ be its unitary group with the relative weak topology. Then $U(A)$ is amenable if and only if A is nuclear. This is equivalent to the existence of a left invariant mean on the space of right uniformly continuous bounded complex-valued functions on $U(A)$ (see [Pe] and [Pa]).
(3) In [He-Ch] a topological group which is abelian, metrizable and admits no non-trivial strongly continuous representations is constructed. This group is also extremely amenable.
(4) It is known that if a von Neumann algebra $\mathcal{M}$ has property (P) (for example $VN(G)$ of an amenable $[IN]$ group G, see [La-Pa]), then the topological group $(\mathcal{M}_u, \mathrm{sot})$ is the direct product of a compact group and an extremely amenable topological group (see [Gi-Pe]), where sot denotes the strong operator topology on $\mathcal{B}(L^2(G))$ and $VN(G)$ is the von Neumann algebra in $\mathcal{B}(L^2(G))$ generated by left translations (see [Ey]).
(5) $(\mathcal{B}(\mathcal{H})_u, \mathrm{sot})$ ($\mathcal{H}$ separable) is extremely amenable (see [Gro-Mi]) .

REMARK 2.9. Let G be a topological group. Let $p \in P(G)_1$ and let $N_p := \{g \in G; p(g) = 1\}$. Choose a unitary representation of G with a cyclic vector ξ of length 1 such that $p = c^\pi_\xi$. Then

$$G_p = \{g \in G; \pi(g)\xi = \xi\}$$

is a closed subgroup of G. Let now

$$\begin{aligned} N_G &= \bigcap_{p \in P(G)_1} G_p \\ &= \{g \in G; \pi(g) = \mathbb{I}_{\mathcal{H}} \text{ for any unitary representation } (\pi, \mathcal{H}) \text{ of } G\}. \end{aligned}$$

Hence

$$B(G) \equiv B(G/N_G)$$

and $B(G/N_G)$ separates the points of G/N_G.

3. The unitary closure group

DEFINITION 3.1. Let G be a topological group. Let

$$\tilde{G} := \sigma(B(G)) \cap W^*(G)_u, \qquad \overline{G} := \overline{\omega_G(G)}^{w^*} \cap W^*(G)_u.$$

Here $W^*(G)_u$ denotes the unitary group of the von Neumann algebra $W^*(G)$ of G. We call $\tilde{G}$ the *unitary cover* of G, and $\overline{G}$ the *unitary closure* of G.

PROPOSITION 3.2. *The subset $\tilde{G}$ equipped with the weak* topology is a topological group contained in a compact semitopological semigroup. The canonical mapping ω_G of G into $\tilde{G}$ is continuous. Furthermore $\overline{G}$ is a closed subgroup of $\tilde{G}$.*

Proof. It is clear that $\tilde{G}$ is a subgroup in the algebra $W^*(G)$ since $\sigma(B(G))$ is a semigroup and the inverse of every unitary element in $\sigma(B(G))$ is also contained in $\sigma(B(G))$ by [La1, Proposition 5.4]. Since the weak* topology and the strong operator topology coincide on the unitary elements of $W^*(G)$, multiplication in $\tilde{G}$ is weak*-continuous and the same is true for taking the inverse. The representations of G being strongly continuous, it follows that the mapping ω_G of G into $\tilde{G}$ is continuous. Since $\overline{\omega_G(G)}^{w^*}$ is also contained in $\sigma(B(G))$, it follows that $\overline{G} = \overline{\omega_G(G)}^{w^*} \cap W^*(G)_u \subset \sigma(B(G)) \cap W^*(G)_u = \tilde{G}$. ∎

THEOREM 3.3. *The Banach algebras $B(G)$ and $B(\overline{G})$ are isomorphic. In particular $\overline{\overline{G}}$ is isomorphic to $\overline{G}$.*

Proof. Every unitary representation $(\pi, \mathcal{H}_\pi)$ of G extends to a weak*-continuous representation of the von Neumann algebra $W^*(G)$ and hence to a representation $\overline{\pi}$ of the group $\overline{G} \subset W^*(G)_u$. If on the other hand $(\overline{\pi}, \mathcal{H}_{\overline{\pi}})$ is a continuous representation of $\overline{G}$, then we obtain a unitary representation $\pi := \overline{\pi} \circ \omega_G$ of G which is continuous and thus π defines a weak*-continuous representation $\tilde{\pi}'$ of $W^*(G)$. The representation $\tilde{\pi}'$ coincides with $\overline{\pi}$ since $\omega(G)$ is weak*-dense in $\overline{G}$. This shows that $\mathrm{Rep}(G)$ and $\mathrm{Rep}(\overline{G})$ are in bijection and so the mapping $\theta : B(\overline{G}) \to B(G)$, $\varphi \mapsto \varphi \circ \omega_G$, is an isometric Banach algebra isomorphism.

To see that $\overline{\overline{G}} \simeq \overline{G}$, consider the image $G^\omega := \omega_G(G)$ of G in $W^*(G)$ under the universal representation ω_G. Every unitary representation π of G^ω extends to a unitary representation $\overline{\pi}$ of $\overline{G}$, and every unitary representation $\overline{\pi}$ of $\overline{G}$ is the extension of the restriction $\overline{\pi}_{|G^\omega}$. In particular the universal representation $\omega_{\overline{G}}$ is unitarily equivalent to the extension $\overline{\omega_{G^\omega}}$ of the universal representation ω_{G^ω}, and this equivalence identifies $W^*(G^\omega)$ with $W^*(\overline{G})$. In this way the group $\overline{\overline{G}}$ is the intersection of the weak*-closure of $\overline{G}$, i.e. of $\omega_G(G)$, with $W^*(G)_u$, which means that $\overline{\overline{G}} = \overline{G}$. ∎

Definition 3.4. Let $B_c(\tilde{G})$ be the space of the coefficients of unitary representations of $\tilde{G}$ which are restrictions of weak*-continuous representations of $W^*(G)$.

Proposition 3.5. *$B_c(\tilde{G})$ is isometrically isomorphic to $B(G)$.*

Since $B_c(\tilde{G})$ is left and right $\tilde{G}$-invariant, we have a central projection $z_{B(G)} \in W^*(\tilde{G})$ such that

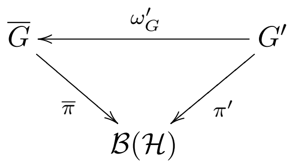
$$B_c(\tilde{G}) = z_{B(G)} B(\tilde{G}).$$

Theorem 3.6. *Let G_1, G_2 be topological groups such that the Banach algebras $B(G_1)$ and $B(G_2)$ are isometrically isomorphic. Then $\tilde{G}_1$ is isomorphic or anti-isomorphic to $\tilde{G}_2$.*

Proof. It follows from the proof of [La-Lu, Lemma 5.3] that if G_1 and G_2 are topological groups such that $B(G_1)$ and $B(G_2)$ are isometrically isomorphic, then the von Neumann algebras $W^*(G_1)$ and $W^*(G_2)$ are either isomorphic or anti-isomorphic. Furthermore the groups $\tilde{G}_1$ and $\tilde{G}_2$ are then isomorphic or anti-isomorphic, since the elements in $\tilde{G}_i$, $i = 1, 2$, are the characters of the Fourier–Stieltjes algebras contained in the unitary groups of the corresponding W^*-algebras. ∎

Theorem 3.7. *Let G be a topological group.*

(1) *The group $\overline{G}$ has the following property $(*)$: For any continuous unitary representation $(\pi, \mathcal{H})$ of G there is a unique continuous unitary representation $(\overline{\pi}, \mathcal{H})$ of $\overline{G}$ such that $\pi = \overline{\pi} \circ \omega_G$ and $\overline{\pi}(\overline{G}) \subset \overline{\pi(G)}^{w^*}$.*
(2) *If G' is another topological group and $\psi : G \to G'$ is a continuous homomorphism satisfying condition $(*)$ (for ψ), then there is a continuous homomorphism $\omega'_G : G' \to \overline{G}$ such that for every unitary representation $(\pi, \mathcal{H}_\pi)$ of G there exists a unique unitary representation $(\pi', \mathcal{H}_\pi)$ of G' such that the following diagram commutes:*



Proof. (1) is clear by Theorem 3.3. To prove (2), by condition $(*)$, we know that there exists a unique unitary representation $(\omega'_G, \mathcal{H}_{\omega_G})$ of G' such that $\omega_G = \omega'_G \circ \psi$ and $\omega'_G(G') \subset W^*(G)_u \cap \overline{\omega_G(G)}^{w^*} = \overline{G}$. For every continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, we then have by $(*)$ a unique representation $(\pi', \mathcal{H}_\pi)$ of G' such that $\pi = \pi' \circ \psi$ and therefore

$$\pi' \circ \psi = \pi = \overline{\pi} \circ \omega_G = (\overline{\pi} \circ \omega'_G) \circ \psi.$$

The fact that $\overline{\pi}(\overline{G}) \subset \overline{\pi(G)}^{w^*}$ and the uniqueness of π' tell us then that $\pi' = \overline{\pi} \circ \omega'_G$. ∎

REMARK 3.8. If G is locally compact, then $\tilde{G} = \overline{G} = G$ (see [Wa1] and [Wa2]).

4. The Fourier algebra $A(G)$ and its basic properties. In this section, we shall define the Fourier algebra of a topological group.

4.1. The Fourier algebra of a topological group. Let G be a topological group.

DEFINITION 4.1. We define the ideal $A(G)$ inside $B(G)$ as

$$A(G) := \bigcap_{\delta \in \sigma(B(G)),\, \delta \notin \tilde{G}} \ker \delta.$$

An *F-algebra* is a Banach algebra A such that A^* is a W^*-algebra and the identity in A^* is multiplicative on A. In this case the set of positive elements in A^* with norm 1 is a semigroup (see [La2]).

THEOREM 4.2. *Let G be a topological group. Then $A(G)$ is a closed translation invariant ideal in $B(G)$. Furthermore, $A(G)^* = z_A W^*(G)$ for a central projection z_A in $W^*(G)$. In particular $A(G)$ is an F-algebra. If $A(G) \neq \{0\}$, then there is a net (φ_α) in $P_1(G) \cap A(G)$ with $\|\varphi\varphi_\alpha - \varphi\| \to 0$ for all $\varphi \in P_1(G)$.*

Proof. It is easy to see that $A(G)$ is translation invariant. Indeed, for $s, t \in G$, $\varphi \in B(G)$, and $\delta \in \sigma(B(G))$, we have

$$\langle \delta, l_s r_t \varphi \rangle = \langle \delta_s \delta \delta_t, \varphi \rangle.$$

Since $\tilde{G}$ is a group inside $\sigma(B(G))$, it follows that if $\delta \notin \tilde{G}$, then $\delta_s \delta \delta_t$ is not in $\tilde{G}$ either. Hence $\langle \delta, l_s r_t \varphi \rangle = 0$ for all $\delta \notin \tilde{G}$ and $s, t \in G$, $\varphi \in A(G)$. This means that $l_s r_t \varphi \in A(G)$ whenever $\varphi \in A(G)$. By [Ta1, p. 123, Theorem 2.7], we now have a central projection $z_A \in W^*(G)$ such that $A(G) = z_A B(G)$. Consequently, $A(G)^* = z_A W^*(G)$, which is a W^*-algebra with identity z_A, which is multiplicative on $A(G)$. In particular $A(G)$ is an F-algebra. The last statement follows from [La2, Theorem 4.6]. ∎

We are now ready to prove one of our main results:

THEOREM 4.3. *The ideal $A(G)$ of $B(G)$ is different from $\{0\}$ if and only if there exists a continuous homomorphism $i : G \to H$ into a locally compact group H such that the canonical homomorphism $i_* : B(H) \to B(G)$,*

$i_*(\psi) =: \psi \circ i$, is an isometric isomorphism. In this case, $\overline{G} = \tilde{G}$, $\overline{G}$ is a locally compact group and $A(G)$ is isometrically isomorphic to $A(\overline{G})$.

Proof. Suppose that $A(G) \neq \{0\}$. Take $\varphi \in A(G)$ such that $\varphi(e) = 1$. Let $1/4 > \varepsilon > 0$ and consider the closed neighborhood

$$\begin{aligned} U = U_\varepsilon &:= \{\delta \in \sigma(B(G));\ |\langle \delta, \varphi\rangle - \langle \delta_e, \varphi\rangle| \le \varepsilon\} \\ &= \{\delta \in \sigma(B(G));\ |\langle \delta, \varphi\rangle - 1| \le \varepsilon\} \end{aligned}$$

of δ_e in $\sigma(B(G))$. Since every $\delta \in \sigma(B(G)) \setminus \tilde{G}$ vanishes on $A(G)$, we see that $U \subset \tilde{G}$. This shows that $\tilde{G}$ contains the compact neighborhood U of $\delta_e \in \sigma(B(G))$ and therefore $\tilde{G}$ is open in $\sigma(B(G))$ and is a locally compact group. Hence also the closure $\overline{G}$ of $G^\omega = \omega_G(G)$ in $\tilde{G}$ is locally compact and the canonical mapping $i := \omega_G$ of G into $\overline{G}$ is continuous. Every $\varphi \in B(G)$ has a continuous extension $\overline{\varphi}$ to $\overline{G}$ defined by $\overline{\varphi}(\delta) := \langle \delta, \varphi\rangle$ for $\delta \in \overline{G}$, i.e. if $\varphi = c^\pi_{\xi,\eta}$, then for $\delta = \lim_i \delta_{g_i} \in \overline{G}$,

$$\overline{\varphi}(\delta) = \langle \delta, \varphi\rangle = \langle \pi(\delta)\xi, \eta\rangle = \lim_i \langle \pi(g_i)\xi, \eta\rangle.$$

Every element $\overline{\varphi} \in B(\overline{G})$ restricts to a continuous function φ on G:

$$\varphi(g) = \overline{\varphi}(\delta_g) = \overline{\varphi} \circ i(g), \qquad g \in G.$$

Hence the mapping $\Theta : B(\overline{G}) \to B(G)$, $\tilde{\varphi} \mapsto i_*(\tilde{\varphi}) = \tilde{\varphi} \circ i$, is an isometric isomorphism of Banach algebras.

Furthermore, every continuous representation $(\tilde{\pi}, \mathcal{H})$ of $\overline{G}$ is determined by its restriction to G. The representation $\pi = \tilde{\pi} \circ i$ of G is continuous, since the mapping $i : G \to \overline{G}$, $g \mapsto \delta_g$, is continuous, and for $\overline{G} \ni \delta = \lim_j \delta_{g_j}$,

$$\tilde{\pi}(\delta) = \lim_j \tilde{\pi}(\delta_{g_j}) = \lim_j \pi(g_j) \qquad \text{weakly.}$$

Conversely, if π is a continuous representation of G, then for $\delta = \lim_j \delta_{g_j} \in \overline{G}$,

$$\langle \tilde{\pi}(\delta)\xi, \eta\rangle = \lim_j \langle \pi(g_j)\xi, \eta\rangle, \qquad \xi, \eta \in \mathcal{H}_\pi,$$

defines a continuous unitary representation of $\overline{G}$ such that $\pi = \tilde{\pi} \circ \omega_G$. Therefore the W^*-algebras $W^*(G)$ and $W^*(\overline{G})$ also coincide and Θ is isometric. Hence

$$\overline{G} = \sigma(B(\overline{G})) \cap W^*(\overline{G})_u = \sigma(B(G)) \cap W^*(G)_u = \tilde{G}$$

using [Wa1, Theorem 1]. Let $H = \tilde{G}$.

Conversely, suppose that there exists a continuous homomorphism $i : G \to H$ of G into a locally compact group H such that the canonical mapping $i_* : B(H) \to B(G)$, $i_*(\tilde{\varphi}) := \tilde{\varphi} \circ i$, is an isometric isomorphism. Let K be the closure of $i(G)$ in H. Let $R : B(H) \to B(K)$ be the restriction map and denote by j the mapping $j : G \to K$, $j(g) := i(g)$, $g \in G$. Then $i_* = j_* \circ R$. Hence $j_* : B(K) \to B(G)$ is bijective and is thus an isometric algebra isomorphism. Since H is locally compact, we have $B(K) = C^*(K)^*$ and therefore by [La-Lu, Lemma 5.5], there exists an isometric linear mapping $\Psi : W^*(G) \to W^*(K)$, which is either an isomorphism or an anti-isomorphism, such that $\Psi_* : B(K) \to B(G)$ is an isometric isomorphism. Hence $\Psi(W^*(G)_u) = W^*(K)_u$ and also

$$\begin{aligned}\Psi(\tilde{G}) &= \Psi(\sigma(B(G)) \cap W^*(G)_u) = \Psi(\sigma(B(G))) \cap \Psi(W^*(G)_u)\\ &= \sigma(B(K)) \cap W^*(K)_u = K\end{aligned}$$

(by [Wa1, Theorem 1]) since K is locally compact. This shows that $\tilde{G}$ and K are isomorphic or anti-isomorphic as topological groups. In particular $\omega_G(G)$ is dense in $\tilde{G}$ and therefore $\tilde{G} = \overline{G}$. Furthermore

$$\begin{aligned}\Psi_*^{-1}(A(G)) &= \Psi_*^{-1}\Big(B(G) \cap \bigcap_{\delta \in \sigma(B(G))\setminus \tilde{G}} \ker \delta\Big)\\ &= B(K) \cap \bigcap_{\delta \in \sigma(B(K))\setminus K} \ker \delta = A(K).\end{aligned}$$

This shows that $A(G) \neq \{0\}$. ∎

Corollary 4.4. *For every dense subgroup H of a locally compact group equipped with the relative topology, the algebra $A(H)$ is different from $\{0\}$.*

Proof. This follows from Theorem 4.3 and [La-Lu, Proposition 3.5]). ∎

Theorem 4.5. *Let G be a topological group.*

(1) *For every $x \in \tilde{G} \setminus \overline{G}$, the annihilator of the subset*

$$\delta_G x = \{\delta_s x;\, s \in G\}$$

in $B(G)$ is reduced to $\{0\}$.

(2) *The central element z_A is contained in $\sigma(A(G))$.*

(3) *The algebra $A(G)$ has bounded approximate units if and only if its ideal $I_0(G) = \{\varphi \in A(G);\, \varphi(e) = 0\}$ has bounded approximate units.*

Proof. (1) Let $x \in \tilde{G} \setminus \overline{G}$ and take $\varphi \in B(G)$ such that $\langle \delta_s \delta_x, \varphi\rangle = 0$ for every $s \in G$. We can describe φ as a coefficient of the cyclic representation $(\pi, \mathcal{H})$ with cyclic vector η, i.e. $\varphi(g) = \langle \pi(g)\xi, \eta\rangle$, $g \in G$, for some $\xi \in \mathcal{H}$. Then

$$0 = \langle \delta_s x, \varphi\rangle = \langle \pi(s)\pi(x)\xi, \eta\rangle, \quad s \in G.$$

This implies that $\pi(x)\xi = 0$ and so $\xi = 0$, since $\pi(x)$ is invertible. Hence $\varphi = 0$.

(2) For $\varphi, \psi \in A(G)$, we have $\varphi = z_A\varphi$, $\psi = z_A\psi$, $\varphi\psi = z_A(\varphi\psi)$, and so

$$\langle z_A, \varphi\rangle\langle z_A, \psi\rangle = \varphi(e)\psi(e) = \varphi\psi(e) = \langle z_A, \varphi\psi\rangle.$$

(3) Since $A(G) = A(\overline{G})$, where $\overline{G}$ is a locally compact group, if $A(G) \neq \{0\}$ we can assume that G is locally compact. But then assertion (3) is well known (see [La2, Theorem 4.10]. ∎

We now proceed to give an example of a topological group such that $A(G) = \{0\}$.

Lemma 4.6. *Let G be a topological group containing a closed normal subgroup N such that in $\sigma(B(N))$ there exists an element w which is not invertible in $W^*(N)$ and such that $B(N) = B(G)_{|N}$. Then $A(G) \cap B(G/N) = \{0\}$.*

Proof. We can consider w to be an element $\tilde{w} \in \sigma(B(G))$, since the restriction mapping $B(G) \to B(N)$, $\psi \mapsto \psi_{|N}$, is a surjective continuous homomorphism and so

$$\langle \tilde{w}, \psi \rangle := \langle w, \psi_{|N} \rangle, \qquad \psi \in B(G).$$

Since w is not invertible in $W^*(N)$, its counterpart $\tilde{w}$ is not invertible in $W^*(G)$ either and so $\langle \tilde{w}, A(G) \rangle = \{0\}$, by the definition of $A(G)$. Let now $\varphi \in P(G/N) \cap A(G)$ with $\varphi(e) = 1$. Then $\varphi_{|N} = 1_N$ and so

$$1 = \langle w, 1_N \rangle = \langle \tilde{w}, \varphi \rangle = 0.$$

This contradiction tells us that $P(G/N) \cap A(G) = \{0\}$. Since every element of $B(G/N) \cap A(G)$ is a finite linear combination of elements of $P(G/N) \cap A(G)$, it follows that $B(G/N) \cap A(G) = \{0\}$. ∎

Lemma 4.7. *Let G be a locally compact group which is not compact. Then $\sigma(B(G))$ contains elements which are not invertible in $W^*(G)$.*

Proof. The algebra $B(G)$ contains a unit element, hence its spectrum $\sigma(B(G))$ is a compact space. We know from [Wa1] that $G \simeq \sigma(B(G)) \cap W^*(G)_r$, where $W^*(G)_r$ denotes the invertible elements in $W^*(G)$. Hence $\sigma(B(G)) \cap W^*(G)_r$ is not compact and so different from $\sigma(B(G))$. ∎

Theorem 4.8. *Let $(G_\alpha)_{\alpha \in \mathcal{A}}$ be an infinite family of locally compact, non-compact groups. Then the direct product $G = \prod_{\alpha \in \mathcal{A}} G_\alpha$ is a topological group such that $A(G) = \{0\}$.*

Proof. It is clear that G is a topological group containing the locally compact groups $G_F = \prod_{\alpha \in F} G_\alpha$ (F a finite non-empty subset of $\mathcal{A}$) as closed subgroups such that $B(G_F) = B(G)_{|G_F}$. Let also $G^F := \prod_{\alpha \notin F} G_\alpha$. The groups G_F and G/G^F are isomorphic and the G_F's are locally compact but not compact. Hence we know from Lemmas 4.6 and 4.7 that $A(G) \cap B(G/G_F) = \{0\}$. Let $\varphi = c^\pi_\xi \in A(G)$. Since φ is continuous, the subset $U_\varepsilon := \{g \in G;\ \|\pi(g)\xi - \xi\| < \varepsilon\}$, $\varepsilon > 0$, is an open neighborhood of e. Hence there exists a finite subset $F' \subset \mathcal{A}$ such that $G^{F'} \subset U_\varepsilon$. Then by [La-Lu, Lemma 6.3], there exists a $\delta \in \mathcal{H}_\pi$ which is $G^{F'}$-invariant and $\|\delta - \xi\| < \varepsilon$. Since $\varphi \in A(G)$, we have $z_A\varphi = \varphi$, and since z_A is a central projection, we can assume that also $\pi(z_A)\delta = \delta$. Hence $\psi \in A(G)$. Furthermore, for any finite non-empty subset $F \subset \mathcal{A}$ with $F \cap F' = \emptyset$ we have $G_F \subset G^{F'}$ and so

ψ is G_F-invariant. Hence $\psi \in B(G/G_F) \cap A(G)=\{0\}$. Therefore $\delta = 0$ and finally we see that $\xi = 0$ and so $\varphi = 0$. This shows that $A(G) = \{0\}$. ■

4.2. The enveloping von Neumann algebra $W^*(G)$ and operator amenability

Definition 4.9. A Banach algebra which is also an operator space is *completely contractive* if the multiplication map $A \times A \to A$, $(a, b) \mapsto ab$, is completely contractive.

Given two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we have the von Neumann algebra tensor product $\mathcal{M} \overline{\otimes} \mathcal{N}$ generated by the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$. The von Neumann algebras $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \overline{\otimes} \mathcal{N}$ have unique predual spaces $\mathcal{M}_*$, $\mathcal{N}_*$ and $(\mathcal{M} \overline{\otimes} \mathcal{N})_*$ respectively. Since von Neumann algebras have a (concrete) operator space structure, so do their duals and their preduals. Hence we may form the operator space tensor product $\mathcal{M}_* \overline{\otimes} \mathcal{N}_*$. Furthermore $\mathcal{M}_* \overline{\otimes} \mathcal{N}_*$ and $(\mathcal{M} \overline{\otimes} \mathcal{N})_*$ are completely isometric (see [Ru2]).

A *Hopf–von Neumann algebra* is a pair $(\mathcal{M}, \Gamma^*)$, where $\mathcal{M}$ is a von Neumann algebra and Γ^* is a co-multiplication, i.e. a unital, injective weak*-weak*-continuous $*$-homomorphism $\mathcal{M} \to \mathcal{M} \overline{\otimes} \mathcal{M}$ which is co-associative, i.e. the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \Gamma^*\ } & \mathcal{M} \overline{\otimes} \mathcal{M} \\ {\scriptstyle \Gamma^*}\downarrow & & \downarrow{\scriptstyle \Gamma \otimes \mathbb{I}_{\mathcal{M}}} \\ \mathcal{M} \overline{\otimes} \mathcal{M} & \xrightarrow{\ \mathbb{I}_{\mathcal{M}} \otimes \Gamma^*\ } & \mathcal{M} \overline{\otimes} \mathcal{M} \overline{\otimes} \mathcal{M} \end{array}$$

is commutative (see [Ta2]). Let $(\mathcal{M}, \Gamma^*)$ be a Hopf–von Neumann algebra. Since $\Gamma^* : \mathcal{M} \to \mathcal{M} \overline{\otimes} \mathcal{M}$ is weak*-continuous, it must be the adjoint of an operator $\Gamma : \mathcal{M}_* \overline{\otimes} \mathcal{M}_* \to \mathcal{M}_*$. Since Γ^* as $*$-homomorphism is a complete contraction, so is Γ (see [Ru2]).

Consequently, $\Gamma : \mathcal{M}_* \overline{\otimes} \mathcal{M}_* \to \mathcal{M}_*$ induces a completely contractive, bilinear map. The commutativity of the diagram above ensures that this bilinear map is an associative multiplication on $\mathcal{M}_*$. In particular, $\mathcal{M}_*$ equipped with this product is a completely contractive Banach algebra.

For a topological group G, let $W^*(G) = \overline{\langle \omega_G(G) \rangle}^{\sigma} = B(G)^*$ as in Theorem 2.6, where ω_G denotes the universal representation of G.

Consider the σ-continuous representation of $G \to W^*(G) \overline{\otimes} W^*(G)$ defined by $x \mapsto \omega_G(x) \otimes \omega_G(x)$ for $x \in G$. By Theorem 2.6, there exists a w^*-homomorphism

$$\Gamma^* : W^*(G) \to W^*(G) \overline{\otimes} W^*(G)$$

such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\omega_G\quad} & W^*(G) \\ & {\scriptstyle\omega_G\otimes\omega_G}\searrow \qquad \swarrow{\scriptstyle\Gamma^*} & \\ & W^*(G)\,\overline{\otimes}\,W^*(G) & \end{array}$$

commutes. Clearly, if $\varphi, \psi \in B(G)$, then for $x \in G$,

$$\begin{aligned}\langle \Gamma(\varphi \otimes \psi), \omega_G(x)\rangle &= \langle \varphi \otimes \psi, \Gamma^*(\omega_G(x))\rangle = \langle \varphi \otimes \psi, \omega_G(x) \otimes \omega_G(x)\rangle \\ &= \varphi(x)\psi(x) = (\varphi\psi)(x).\end{aligned}$$

Consequently, the bilinear $\Gamma : B(G) \times B(G) \to B(G)$ agrees with $(\varphi, \psi) \mapsto \varphi \cdot \psi$. Hence $B(G)$ is a completely contractive Banach algebra.

THEOREM 4.10. *For any topological group G, $B(G)$ with the operator space structure as the* (*unique*) *predual of $W^*(G)$ is a completely contractive Banach algebra.*

REMARK 4.11. Let G be a topological group such that $A(G) \neq \{0\}$. Then by Theorem 4.3 and its proof, there is a locally compact group $\overline{G}$ such that there is an isometric isomorphism from $B(\overline{G})$ onto $B(G)$ and the von Neumann algebras of G and of $\overline{G}$ coincide. Consequently, the operator structures of $B(G)$ and of $B(\overline{G})$ coincide too.

Let $A(G)$ be equipped with the operator space structure from $B(G)$. Then $A(G)$ is also a completely contractive Banach algebra completely isometric to $A(\overline{G})$.

DEFINITION 4.12. A bimodule X over a completely contractive Banach algebra A is called an *operator* Banach A-module if X is also an operator space and the module action $A \times X \to X$, $(a, x) \mapsto a \cdot x$, is completely bounded.

A completely contractive Banach algebra is called *operator amenable* if for each operator Banach A-module X, each completely bounded derivation $D : A \to X^*$ is inner; and A is called *operator weakly amenable* if every completely bounded derivation $D : A \to A^*$ is inner when A is regarded as an operator Banach A-module by left and right multiplication.

REMARK 4.13. Let G be a topological group and let H be a dense subgroup of G. The algebras $LUC(G)$ and $LUC(H)$ are isometrically isomorphic as Banach algebras. Indeed, the restriction map $R : LUC(G) \to LUC(H)$, $R(\varphi) := \varphi_{|H}$, $\varphi \in LUC(G)$, is an isometric homomorphism. Since every $\psi \in LUC(H)$ extends to a unique element $\varphi \in LUC(G)$, we see that the mapping R is also surjective.

DEFINITION 4.14. Let G be a topological group and let τ be the weakest topology on G such that all the functions in $B(G)$ are continuous for this topology. Then τ turns the group G into a topological group G_τ and $B(G) = B(G_\tau)$.

LEMMA 4.15. *Let H be a dense subgroup of the topological group G. Then G is amenable if and only if H is so.*

Proof. If H is amenable, take an H-left invariant mean m on $LUC(H)$. Then the mean $\overline{m}$ defined on $LUC(G)$ by $\overline{m}(\varphi) := m(R(\varphi))$ for $\varphi \in LUC(G)$, where R denotes the restriction map $LUC(G) \to LUC(H)$, is H-left invariant and so by left uniform continuity also G-left invariant. Hence G is amenable.

Conversely, if G is amenable, then every G-left invariant mean on $LUC(G)$ defines an H-left invariant mean on $LUC(H)$, since every $\psi \in LUC(H)$ extends in a unique way to an element in $LUC(G)$. Hence H is amenable too. ∎

THEOREM 4.16. *Let G be a topological group such that $B(G)$ separates the elements of G and $A(G) \neq \{0\}$. Then G_τ is amenable if and only if $A(G)$ is operator amenable.*

Proof. If $A(G)$ is operator amenable, then $A(\overline{G})$ is operator amenable by Theorem 4.3 and Remark 4.11. Hence the locally compact group $\overline{G}$ is amenable by a result of Ruan [Ru1]. Therefore G_τ, which is homeomorphic to $\omega_G(G) \subset \overline{G}$, is also amenable by Lemma 4.15.

Conversely, if G_τ is amenable, then $\overline{G}$ is amenable too by Lemma 4.15, since G_τ and $\omega_G(G)$ are homeomorphic. Again by the result of Ruan, $A(\overline{G})$ is operator amenable. Therefore, since by Theorem 4.3, $A(G) \simeq A(\overline{G})$, we see that $A(G)$ is operator amenable. ∎

THEOREM 4.17. *For every topological group G, the algebra $A(G)$ is weakly operator amenable.*

Proof. If $A(G) = \{0\}$, then $A(G)$ is trivially weakly operator amenable. Now if $A(G) \neq \{0\}$, then $A(G) \simeq A(\overline{G})$, where $\overline{G}$ is a locally compact group. Hence $A(\overline{G})$ is weakly operator amenable by [Sp]. So $A(G)$ is weakly operator amenable. ∎

An F algebra A is called *left* (resp. *right*) *amenable* if for each two-sided Banach A-module X such that $\varphi \cdot x = \varphi(1)x$ (resp. $x \cdot \varphi = \varphi(1)x$) for all $\varphi \in A$ and $x \in X$, every bounded derivation from A into X^* is inner (see [La2, p. 167]). It was shown in [La2, Theorem 4.1 and Corollary 4.3] that a locally compact group is amenable if and only if the measure algebra $M(G)$ (or the group algebra $L^1(G)$) is left amenable. The following is a consequence of [La2, Example (i), p. 168]:

THEOREM 4.18. *For any topological group G, both F-algebras $B(G)$ and $A(G)$ are left (and right) amenable.*

4.3. Fixed point property for the Fourier algebra. Let E be a Banach space and K a non-empty bounded closed convex subset of E. We say that K has the *fixed point property* (or simply fpp) if every non-expansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for all x and y in E) has a fixed point. We say that E has the (resp. weak) fixed point property if every bounded closed (resp. weakly compact) convex subset $K \subset E$ has the fixed point property.

It is well known that l^1 has the weak fixed point property (see for instance [Go-Ki] and [Li]), but not the fixed point property. A well known result of Browder (see [Go-Ki]) asserts that if E is uniformly convex, then E has the weak fpp. As shown by Alspach [Al], the Banach space $L^1([0,1])$ does not have the weak fpp (hence not the fpp). In fact, he exhibited a weakly compact convex subset K of $L^1([0,1])$ and an isometry $T : K \to K$ (i.e. $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in K$) without a fixed point. In particular, the Fourier algebra $A(\mathbb{Z}) \simeq L^1(\varPi)$ does not have the weak fpp. Here $\varPi = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$ is the circle group with multiplication and $\mathbb{Z}$ is the additive group of the integers. On the other hand, $A(\varPi) \simeq l^1(\mathbb{Z})$ has the weak fpp but not the fpp.

We say that a topological group G is a $[SIN]$-*group* if the left and right uniformities agree. In the case of a locally compact group this is equivalent to the existence of a basis of the identity e consisting of compact sets V such that $xVx^{-1} = V$ for all $x \in G$ (see [Mil]).

THEOREM 4.19. *Let G be a $[SIN]$-group. Then $A(G)$ has the weak fixed point property if and only if either $A(G) = \{0\}$, or there exists a continuous embedding $i : G/N_G \to H$ into a compact group such that $i_* : A(H) \to A(G/N_G)$, $i_*(\psi) := \psi \circ i$, is an isometric isomorphism.*

Proof. We can assume for the proof that $N_G = \{e\}$. Suppose that $A(G)$ has the weak fixed point property and $A(G) \neq \{e\}$. Then there exists a continuous embedding $i : G \to H$ into a locally compact group such that $i(G)$ is dense in H and $i_* : C^*(H)^* = B(H) \to B(G)$ is an isometric isomorphism. Hence by [La-Lu, Lemma 5.3], there exists a linear isometric mapping $\psi : W^*(G) \to W^*(H)$ which is either an isomorphism or an anti-isomorphism such that $\psi_*(A(G)) = A(H)$. In particular $A(H)$ has the weak fixed point property. Now by [La-Lu, Remark 3.4], the restriction map $R : LUC(H) \to LUC(G), R(\varphi) = \varphi_{|G}$, is a surjective isometric isomorphism. Since G is a $[SIN]$-group, we have $LUC(G) = RUC(G)$ and therefore $LUC(H) = RUC(H)$. Hence by [Mil], H is also a $[SIN]$-group. But then by [La-Le, Corollary 4.2], H must be a compact group.

Conversely, if $A(G) = \{0\}$, then clearly $A(G)$ has the weak fpp. Otherwise there is a continuous embedding $i : G \to H$ into a compact group such that $i_* : A(H) \to A(G)$, $i_*(\varphi) = \varphi \circ i$, is an isometric isomorphism. By [La-Le], $A(H)$ has the weak fpp. Hence $A(G)$ has the weak fpp as well. ∎

Remark 4.20. If there exists a continuous embedding $i : G/N_G \to H$ into a compact group such that $i_* : A(H) \to A(G)$ is an isometric isomorphism, then $A(H) = C^*(H)^*$ and $A(G)$ have the weak* fpp also (i.e. every weak*-compact convex subset of $A(H)$ has a fixed point for non-expansive self-maps, by using again [La-Ma, Theorem 5]). In particular $A(H)$ regarded as the dual of $C^*(H)$ also has the weak* fpp.

Using [La-Le, Theorem 5.7], we can also prove by an argument similar to that for Theorem 4.19:

Theorem 4.21. *Let G be a topological group. Then $A(G)$ has the fixed point property if and only if G/N_G is finite.*

Definition 4.22. A Banach space E is said to have *UKK* (*uniform Kadec–Klee property*) if for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $(x_n)_n$ is a sequence in the unit ball of E converging weakly to x and satisfying $\inf \|x_n - x_m\| > \varepsilon$ then $\|x\| \leq \delta$ (see [Hu]). As is known [Du-Si], if E has *UKK*, then E has weak fpp.

Definition 4.23. A Banach space E is said to have the *Radon–Nikodym property* (or *RNP*) if each closed convex subset D of E is dentable, i.e. for any $\varepsilon > 0$, there exists an $x \in D$ such that $x \notin \overline{\mathrm{co}}(D \setminus B_\varepsilon(x))$, where $B_\varepsilon(x) = \{y \in E;\ \|x - y\| < \varepsilon\}$ and $\overline{\mathrm{co}}(K)$ is the closed convex hull of a set K in E. It was shown in [L-M-U] that for a von Neumann algebra $\mathcal{M}$, if its predual $\mathcal{M}_*$ has the *RNP* then $\mathcal{M}_*$ has the fpp.

Definition 4.24. A Banach space E is said to have the *Krein–Milman property* (or *KMP*) if every closed bounded convex subset of E is the closed convex hull of its set of extreme points.

Using [La-Le, Corollary 4.2] (see also [La-Ma] and [L-M-U]) and the arguments of Theorem 4.19, we have

Theorem 4.25. *Let G be a [SIN]-group. Then $A(G)$ has RNP (resp. KMP, UKK) if and only if either $A(G) = \{0\}$, or there exists a continuous embedding $i : G/N_G \to H$ into a compact group such that $i_* : A(H) \to A(G)$ is an isometric isomorphism.*

5. The universal C^*-algebra generated by the continuous representations

Definition 5.1. Let G be a topological group.

(1) Let $(\omega, \mathcal{H}_\omega)$ be a unitary representation of G. Define $C^*_{\delta,\omega}(G) := \overline{\langle \omega(G) \rangle}$, i.e. the C^*-algebra generated by the group $\omega(G)$ in $\mathcal{B}(\mathcal{H}_\omega)$.

(2) Denote by $(\omega_d, \mathcal{H}_d)$ the universal representation of the group G_d.
(3) Let $C^*_\Omega(G)$ be the completion of the algebra $l^1(G_d)$ with respect to the norm

$$\|a\|_{*,c} := \sup_{\pi\in\mathrm{Rep}(G)} \|\pi(a)\|_{\mathrm{op}}, \qquad a \in l^1(G_d),$$

where $\|\pi(a)\|_{\mathrm{op}}$ denotes the operator norm of $\pi(a)$. Then of course $C^*_\Omega(G)$ is isomorphic to $C^*_{\delta,\omega_G}(G)$, an isomorphism being given by the universal representation ω_G.

Definition 5.2. For every continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, we obtain a unitary representation $(\pi', \mathcal{H}_\pi)$ of $C^*_\Omega(G)$ defined by

$$\pi'\Big(\sum_g c_g\delta_g\Big) := \sum_g c_g\pi(g), \qquad f = \sum_g c_g\delta_g \in l^1(G_d).$$

On the other hand, every unitary representation $(\pi', \mathcal{H}_{\pi'})$ of $C^*_\Omega(G)$ restricts to a unitary representation $(\overline{\pi'}, \mathcal{H}_{\pi'})$ of the group G_d, i.e. the group G equipped with the discrete topology, since G can be considered as being a subgroup of the unitary group of the unital C^*-algebra $C^*_\Omega(G)$. This gives us two injective mappings:

$$\iota_\delta : \mathrm{Rep}(G) \to \mathrm{Rep}(C^*_\Omega(G)),\ (\pi, \mathcal{H}_\pi) \mapsto (\pi', \mathcal{H}_\pi),$$
$$\overline{\iota'_\delta} : \mathrm{Rep}(C^*_\Omega(G)) \to \mathrm{Rep}(G_d),\ (\pi', \mathcal{H}_{\pi'}) \mapsto (\overline{\pi'}, \mathcal{H}'_\pi).$$

Denote, for a unitary representation $(\pi, \mathcal{H}_\pi)$ of G_d, its canonical extension to $W^*(G_d)$ by $(\overline{\pi}, \mathcal{H}_\pi)$, i.e. $\pi(g) = \overline{\pi}(\omega_d(g))$ for $g \in G$.

Let $P_\Omega(G) \subset B(G_d)$ be the set of positive linear functionals defined on $C^*_\Omega(G)$ and let $P_{\Omega,1}(G)$ be the elements in $P_\Omega(G)$ of length 1.

Proposition 5.3. *The subset $P_{\Omega,1}(G)$ of $B(G_d)$ is the weak*-closure of the convex subset $P_1(G)$ in $B(G_d)$.*

Proof. By definition of $C^*_\Omega(G)$, the ideal $I_{\mathrm{Rep}(G)} := \bigcap_{\pi\in\mathrm{Rep}(G)} \ker\pi'$ in $C^*_\Omega(G)$ is $\{0\}$. But then every representation ρ' of $C^*_\Omega(G)$ is weakly contained in the set $\{\pi';\ \pi \in \mathrm{Rep}(G)\}$, and therefore by [Di, Theorem 3.4.4], every $p' \in P_{\delta,1}(G)$ is a weak*-limit of a net contained in $P_1(G)$. ∎

Proposition 5.4. *The C^*-algebra $C^*_\Omega(G)$ is a $B(G)$-module.*

Proof. Let $a = \sum_g c_g\delta_g$ be a finite sum in $C^*_\Omega(G)$, let $u = c^\pi_{\xi,\eta} \in B(G)$ and let

$$u\cdot a := \sum_g u(g)c_g\delta_g,$$

which is also in $C^*_\Omega(G)$. For any continuous unitary representation $(\omega, \mathcal{H}_\omega)$

of G and $x, y \in \mathcal{H}_\omega$ it follows that

$$\begin{aligned}\langle\omega(u\cdot a)x,y\rangle &= \sum_g u(g)c_g\langle\omega(g)x,y\rangle = \sum_g c_g\langle\pi(g)\xi,\eta\rangle\langle\omega(g)x,y\rangle\\ &= \Big\langle\sum_g c_g(\omega\otimes\pi)(g)(x\otimes\xi), y\otimes\eta\Big\rangle\\ &= \langle(\omega\otimes\pi)'(a)(x\otimes\xi), y\otimes\eta\rangle.\end{aligned}$$

This shows that $\|\omega(u\cdot a)\|_{\mathrm{op}} \le \|u\|_{B(G)}\|a\|_{*,c}$. Consequently, the multiplication $B(G)\times C^*_\Omega(G)\to C^*_\Omega(G)$, $(u,a)\mapsto u\cdot a$, is well defined and the multiplication in $B(G)$ induces a $B(G)$-module structure on $C^*_\Omega(G)$. ∎

Definition 5.5. Let G be a topological group.

(1) For a unitary representation $(\pi, \mathcal{H}_\pi)$ of G_d let

$$\mathcal{H}^c_\pi := \{\xi\in\mathcal{H}_\pi;\ \text{the map } G\to\mathcal{H}_\pi,\ g\mapsto\pi(g)\xi,\ \text{is continuous}\}.$$

Then the subspace $\mathcal{H}^c_\pi$ of $\mathcal{H}_\pi$ is closed and G_d-invariant. We call an element of $\mathcal{H}^c$ a *continuous vector*.

(2) In particular, the restriction π^c of π to the invariant subspace $\mathcal{H}^c_\pi$ is continuous and all the associated coefficients $u = c^\pi_{\xi,\eta}$, $\xi,\eta\in\mathcal{H}^c_\pi$, are continuous functions on G, i.e. $u\in B(G)$.
The orthogonal complement $\mathcal{H}^{c,\perp}_\pi$ contains only elements $\xi\ne 0$ for which the mapping $g\mapsto\pi(g)\xi$ is not continuous. We say that $\mathcal{H}^{c,\perp}_\pi$ is the *totally discontinuous part* of π.

Remark 5.6. (1) Let G be a topological group and let $(\pi,\mathcal{H}_\pi)$ be an irreducible unitary representation of the group G_d. Then π is either continuous or totally discontinuous.

(2) For the universal representation $(\omega_d,\mathcal{H}_d) := (\omega_{G_d},\mathcal{H}_{\omega_{G_d}})$ of G_d, we obtain the orthogonal decomposition

$$\mathcal{H}_d = \mathcal{H}^c_d \oplus \mathcal{H}^{c,\perp}_d.$$

Then $\mathcal{H}^c_d$ is an orthogonal sum $\mathcal{H}^c_d = \sum^\oplus_{p\in C}\mathcal{H}_p$ for a certain subset C of $P(G)_1$. On the other hand, the subspace $\sum^\oplus_{p\in P(G)_1}\mathcal{H}_p$ is contained in $\mathcal{H}^c_d$. This means that

$$\mathcal{H}_{\omega_G} = \sum^\oplus_{p\in P(G)_1}\mathcal{H}_p = \mathcal{H}^c_d,$$

i.e. the restriction of ω_d to $\mathcal{H}^c_d$ is our universal representation ω_G. Let z^c be the orthogonal projection of $\mathcal{H}_d$ onto $\mathcal{H}^c_d$. Then z^c is central in $W^*(G_d)$.

(3) Note that $B(G)$ is a closed translation invariant subspace of $B(G_d)$. By [Ta1, Theorem 2.7(i), p. 127], $B(G)$ is invariant in $B(G_d)$ as a predual of $W^*(G_d)$, i.e. for all $a\in W^*(G_d)$ and $\varphi\in B(G)$, we have $a\cdot\varphi$, $\varphi\cdot a\in B(G)$, where $\langle a\cdot\varphi, b\rangle = \langle\varphi, ba\rangle$ and $\langle\varphi\cdot a, b\rangle = \langle\varphi, ab\rangle$ for all $b\in W^*(G)$. So, by

[Ta1, Theorem 2.7(iii), p. 127], there is a central projection $z \in W^*(G_d)$ such that

$$zB(G_d) = B(G).$$

Hence the polar of $zB(G_d)$,

$$(zB(G_d))^0 = \{a \in W^*(G_d);\ \langle a, \varphi\rangle = 0 \text{ for all } \varphi \in B(G)\}$$

is the ideal $(1 - z)W^*(G)$ of $W^*(G_d)$ and $zW^*(G_d) \simeq W^*(G)$. We write $z = z_{B(G)}$.

PROPOSITION 5.7. *Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of the group G_d. Then $\pi(z_{B(G)})$ is the orthogonal projection onto the subspace $\mathcal{H}^c_\pi$. In particular*

$$z^c = \omega_d(z_{B(G)}).$$

Proof. Take $\xi, \eta \in \mathcal{H}^c_\pi$. Then the function $u = c^\pi_{\xi,\eta}$ of G_d is in $B(G)$. Hence $z_{B(G)}u = u$ and so for every $g \in G$,

$$\langle \pi(g)\xi, \eta\rangle = \langle \pi(g)\pi(z_{B(G)})\xi, \eta\rangle.$$

Thus $\pi(z_{B(G)})\xi = \xi$.

If now $\xi \in \mathcal{H}^{c,\perp}_\pi$, then for any $\eta \in \mathcal{H}_\pi$ and $u = c^\pi_{\xi,\eta}$ we have $z_{B(G)}u \in B(G)$, hence $z_{B(G)}u$ is a continuous function on G. Therefore the function $G_d \to \mathbb{C}$, $g \mapsto \langle \pi(g)\pi(z_{B(G)})(\xi), \eta\rangle$, is continuous for every $\eta \in \mathcal{H}_\pi$. Hence the vector $\pi(z_{B(G)})(\xi)$ is weakly continuous and so also continuous. This shows that $\pi(z_{B(G)})(\xi) \in \mathcal{H}^c_\pi \cap \mathcal{H}^{c,\perp}_\pi = \{0\}$. Hence $\pi(z_{B(G)})(\mathcal{H}^{c,\perp}_\pi) = \{0\}$ and $\pi(z_{B(G)})$ is the orthogonal projection onto $\mathcal{H}^c_\pi$.

In particular, for the universal representation ω_{G_d}, we get

$$\omega_{G_d}(z_{B(G)}) = \text{orthogonal projection onto } \mathcal{H}^c_d = z^c. \ ■$$

DEFINITION 5.8. We denote

$$\mathrm{Rep}(C^*_\Omega(G))^c := \{(\pi', \mathcal{H}_{\pi'}) \in \mathrm{Rep}(C^*_\Omega(G));\ \overline{\pi'}(z_{B(G)}) = \mathbb{I}_{\mathcal{H}_{\pi'}}\}.$$

THEOREM 5.9. *The mapping*

$$\mathrm{Rep}(G) \to \mathrm{Rep}(C^*_\Omega(G))^c, \qquad (\pi, \mathcal{H}_\pi) \mapsto (\pi', \mathcal{H}_\pi),$$

is a bijection between the space $\mathrm{Rep}(G)$ *of continuous unitary representations of the topological group G and the subspace* $\mathrm{Rep}(C^*_\Omega(G))^c$ *of the space* $\mathrm{Rep}(C^*_\Omega(G))$ *of unitary representations of $C^*_\Omega(G)$.*

Proof. Let $(\pi, \mathcal{H}_\pi) \in \mathrm{Rep}(G)$. Then π defines a representation π' of $C^*_\Omega(G)$, and also a representation $(\overline{\pi}, \mathcal{H}_\pi)$ of the von Neumann algebra $W^*(G_d)$. But then $\overline{\pi}(z_{B(G)}) = \mathbb{I}_{\mathcal{H}_{\pi'}}$, since $\mathcal{H}_\pi = \mathcal{H}^c_\pi$ by Proposition 5.7.

If on the other hand $(\pi', \mathcal{H}_{\pi'})$ is a cyclic unitary representation of $C^*_\Omega(G)$ such that $\overline{\pi}(z_{B(G)}) = \mathbb{I}_{\mathcal{H}_{\pi'}}$, then $\mathcal{H}_{\pi'} = \mathcal{H}^c_{\pi'}$ by Proposition 5.7 and $\pi := \overline{\pi}_{|G}$ is a continuous representation of G. Therefore π' is the extension of π to $C^*_\Omega(G)$. ■

Definition 5.10 (see [La-Lu]). Let G be a topological group. We say that G *has a group* C^**-algebra* if there exists a C^*-algebra A and a linear isometric isomorphism $\Phi : A^* \to B(G)$ such that multiplication in $B(G)$ induces a $B(G)$-module structure on A. This means the following: for every $a \in A$ and $\varphi \in B(G)$, we have an element $\varphi \cdot a \in A^{**}$ defined by

$$\langle \varphi \cdot a, \delta \rangle := \langle \Phi^{-1}(\varphi \Phi(\delta)), a \rangle, \qquad \delta \in A^*.$$

We demand now that $\varphi \cdot a \in A$ for every $a \in A$ and $\varphi \in B(G)$. In particular then

$$\|\varphi \cdot a\| \le \|a\| \, \|\varphi\|.$$

Proposition 5.11. *Let G be a topological group. Then $C^*_\Omega(G)$ is a group C^*-algebra of G if and only if $B_\Omega(G) = B(G)$.*

Proof. We already know that $C^*_\Omega(G)$ is a $B(G)$-module by Proposition 5.4 and that $B(G) \subset B_\delta(G)$. Hence $C^*_\Omega(G)$ is a group C^*-algebra of G if and only if the canonical injection $B(G) \to B_\Omega(G)$ is a bijection. ∎

Remark 5.12. Let G be a topological group. Suppose G admits a group C^*-algebra A. Let $\Phi : A^* \to B(G)$ be a linear isometric isomorphism. Then, according to [La-Lu, Lemma 5.2], the adjoint map Φ^t is an isomorphism or an anti-isomorphism of $W^*(G)$ onto the von Neumann algebra $\tilde{A}$ of A and Φ maps positive linear functionals on A to continuous positive definite functions on G. In particular the irreducible representations of A correspond to irreducible continuous representations of G. This shows that the topological group mentioned in [Gl], i.e. the group $L^0(X, \mu; \Pi)$ of all measurable maps from a standard Lebesgue measure space X into the circle rotation group with pointwise multiplication and the $L^1(\mu)$-norm, which admits a faithful unitary representation by multiplication operators, yet no non-trivial unitary irreducible representations, cannot have a group C^*-algebra.

6. Functions in $B(G)$ arising from invariant means. In this section we study functions in $B(G)$ arising from the set of left invariant means on an amenable topological group and its extreme points.

The following lemma has been proved in [Hu]. We present another proof for completeness.

Lemma 6.1. *Let G be a topological group. Suppose that G contains an increasing net $(G_i)_{i \in I}$ of closed amenable locally compact subgroups such that $G_0 = \bigcup_{i \in I} G_i$ is dense in G. Then G is amenable.*

Proof. Choose for every $i \in I$ a left invariant mean m_i on the space $CB(G_i)$. It defines a mean n_i on $LUC(G)$ by

$$n_i(\varphi) := m_i(\varphi_{|G_i}), \qquad \varphi \in LUC(G).$$

The mean n_i is left G_i-invariant. Let n be a cluster point in $(LUC(G))^*$ of the net $(n_i)_i$. Then n is a *LIM* on $LUC(G)$. Indeed, there exists a subnet $(n'_j := n_{i_j})_j$ of (n_i) such that $n = \lim_j n'_j$. Let $x_i \in G_i$, $i \in I$. There exists an index j such that $i_j \geq i$ and therefore $x_i \in G_{i_k}$ for every $k \geq j$. Hence the means n'_k, $k > j$, are x_i-invariant. Consequently, n is G_0-invariant. Since G acts strongly on $LUC(G)$, n must be in $LIM(G)$, since G_0 is dense in G. ∎

DEFINITION 6.2. Let G be an amenable topological group and let $n \in LIM(G)$. We define an inner product $\langle\,,\rangle_n$ on $LUC(G)$ by letting

$$\langle \varphi, \psi\rangle_n := n(\varphi\overline{\psi}), \qquad \varphi, \psi \in LUC(G).$$

Let $I_n := \{\varphi \in LUC(G);\ n(|\varphi|^2) = 0\}$. Then the quotient space $LUC(G)/I_n$ is a pre-Hilbert space with $\langle[\varphi],[\psi]\rangle_n := n(\varphi\overline{\psi})$, where $[\varphi]$ denotes the equivalence class of φ in $LUC(G)/I_n \subset \mathcal{H}_n$, with $\mathcal{H}_n$ denoting the completion of $LUC(G)/I_n$.

For $\varphi \in LUC(G)$, we can define a bounded operator $\pi_n(\varphi)$ on $\mathcal{H}_n$ by

$$\pi_n(\varphi)[\psi] := [\varphi\psi], \qquad \psi \in LUC(G).$$

Then of course $\|\pi_n(\varphi)[\psi]\|^2 = n(|\varphi\psi|^2) \leq \|\varphi\|_\infty^2 n(|\psi|^2) = \|\varphi\|_\infty^2\|[\psi]\|^2$. Hence $\|\pi_n(\varphi)\|_{\mathrm{op}} \leq \|\varphi\|_\infty$.

We can extend the mean n to a bounded linear functional $\tilde{n}$ on $\mathcal{H}_n$. It suffices to remark that $n(\varphi) = \langle[\varphi],[1]\rangle$. Hence if we take $\tilde{n}(\xi) := \langle \xi, [1]\rangle_n$, we have such an extension, which has the property that $|\tilde{n}(\xi)| \leq \|\xi\|$, by Cauchy's inequality.

We can also define a unitary representation π_n of the group G on the Hilbert space $\mathcal{H}_n$ by setting

$$\pi_n(g)[\psi] := [l_{g^{-1}}\psi], \qquad g \in G,\ \psi \in LUC(G).$$

Since n is a left invariant mean, the action of $g \in G$ on $\mathcal{H}_n$ is isometric, and because G acts strongly continuously on $LUC(G)$, the representation π_n of G is strongly continuous.

Since $B(G)$ is contained in $LUC(G)$, we have:

LEMMA 6.3. *The function $G \ni g \mapsto n(l_{g^{-1}}\varphi\psi) =: h(g)$ $(\varphi, \psi \in LUC(G))$ is in $LUC(G)$.*

THEOREM 6.4. *Let G be an amenable topological group and let $n \in LIM(G)$. Let $\pi := \pi_n$. The following conditions are equivalent:*

(1) *The mean n is an extreme point in $LIM(G)$.*
(2) *For each $m \in LIM(G)$ and all $\xi, \eta \in \mathcal{H}_\pi$, we have $m(c^\pi_{\xi,\eta}) = \tilde{n}(\xi)\overline{\tilde{n}(\eta)}$.*
(3) *For each $\xi \in \mathcal{H}_\pi$, $\tilde{n}(\xi)[1] \in \overline{\mathrm{co}}^{\|\,\|}\{\pi(x)\xi;\ x \in G\}$.*

(4) *For each* $\varphi \in LUC(G)$*, there exists a sequence*

$$\Big(\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})\Big)_k$$

of convex combinations of operators $\pi(x)$, $x \in G$, *such that*

$$\Big\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})\varphi\Big\|_n^2 \to |n(\varphi)|^2.$$

Proof. (1)⇒(2). Let $0 \le \psi \le 1$, $\psi \in LUC(G)$ and $m \in LIM(G)$. For $\varphi \in LUC(G)$, define $\varphi^n \in LUC(G)$ by

$$\varphi^n(x) := n((l_x\varphi)\psi) = \langle \pi(x^{-1})[\varphi], [\overline{\psi}]\rangle = \overline{\langle \pi(x)[\overline{\psi}]), [\varphi]\rangle}, \qquad x \in G.$$

Then for $x, y \in G$ we have

$$\begin{aligned}(l_y\varphi)^n(x) &= n((l_x(l_y\varphi))\psi)\\ &= n((l_{yx}\varphi)\psi) = \varphi^n(yx) = l_y(\varphi^n)(x),\end{aligned}$$

i.e. $(l_y\varphi)^n = l_y(\varphi^n)$. Let

$$\theta(\varphi) := m(\varphi^n) - n(\varphi)n(\psi).$$

Since m and n are left invariant, $\theta(l_y\varphi) = \theta(\varphi)$ for every $y \in G$. Also $n + \theta$ and $n - \theta$ are in $LIM(G)$, since both are G-invariant and for $\varphi \ge 0$ we have

$$\begin{aligned}(n+\theta)(\varphi) &= n(\varphi) + m(\varphi^n) - n(\varphi)n(\psi)\\ &= n(\varphi)(1 - n(\psi)) + m(\varphi^n) \ge 0,\\ (n-\theta)(\varphi) &= n(\varphi) - m(\varphi^n) + n(\varphi)n(\psi)\\ &\ge n(\varphi)n(\psi) \ge 0.\end{aligned}$$

Furthermore

$$\begin{aligned}(n+\theta)(1) &= n(1) + m(n(1\psi)1) - n(1)n(\psi) = n(1) = 1,\\ (n-\theta)(1) &= n(1) - m(n(1\psi)1) + n(1)n(\psi) = n(1) = 1.\end{aligned}$$

But n is extreme. It follows that $\theta = 0$, i.e. (2) holds for all $0 \le \psi \le 1$. Consequently, (2) must hold for all $\xi, \eta \in \mathcal{H}_\pi$.

(2)⇒(3). Let $(n_\alpha = \sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\delta_{x_{i,\alpha}})_\alpha$ be a net of convex combinations of point evaluations such that $\langle n_\alpha, \varphi\rangle \to \langle n, \varphi\rangle$ for all $\varphi \in LUC(G)$. So for $\xi, \eta \in \mathcal{H}_\pi$, the function $G \ni y \mapsto \langle \pi(y)\xi, \eta\rangle_n =: c_{\xi,\eta}(y)$ being a coefficient of a unitary representation of G, we have

$$\Big\langle \sum_{i=1}^{k_\alpha} \lambda_{i,\alpha}\pi(x_{i,\alpha})\xi, \eta\Big\rangle_n = \langle n_\alpha, c_{\xi,\eta}\rangle \to n(c_{\xi,\eta}) = \tilde{n}(\xi)\tilde{n}(\eta) = \langle \tilde{n}(\xi)[1], \eta\rangle$$

by (2). Hence the vector $\tilde{n}(\xi)[1]$ is in the weak closed convex hull of $\{\pi(x)\xi;\ x \in G\}$, hence also in the norm closed convex hull of this set.

(3)⇒(4). If (3) holds, then for $\varphi \in LUC(G)$, we can find a sequence $(\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k}))_k$ of convex linear combinations such that

$$\Big\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi] - \tilde{n}([\varphi])[1]\Big\|_n \to 0.$$

Since

$$\begin{aligned}\Big\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi] - \tilde{n}([\varphi])[1]\Big\|_n^2 \\ &= \Big\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi]\Big\|_n^2 - 2\operatorname{Re}\Big(\overline{n}(\varphi)\Big\langle\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi],[1]\Big\rangle_n\Big) + |n(\varphi)|^2 \\ &= \Big\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi]\Big\|_n^2 - 2\operatorname{Re}(\overline{n}(\varphi)n(\varphi)) + |n(\varphi)|^2,\end{aligned}$$

we see that $\|\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})[\varphi]\|_n^2 - |n(\varphi)|^2 \to 0$.

(4)⇒(1). Suppose that n is not extreme. Then there exist two means n_1, n_2 in $LIM(G)$ such that $n \neq n_1$ and $n = \frac{1}{2}(n_1 + n_2)$. Let $\varphi \in LUC(G)$. There exists a sequence of convex linear combinations $\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})$ such that for $\varphi_k := \sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})\varphi$, we have $\|\varphi_k\| \to |n(\varphi)|$. But then

$$\begin{aligned}|n_1(\varphi)|^2 &= \Big|n_1\Big(\sum_{i=1}^{n_k} \lambda_{i,k}\pi(x_{i,k})\varphi\Big)\Big|^2 = |n_1(\varphi_k)|^2 \\ &\le n_1(1)n_1(|\varphi_k|^2) \le 2n(|\varphi_k|^2) \to 2|n(\varphi)|^2.\end{aligned}$$

Thus $n(\varphi) = 0$ implies $n_1(\varphi) = 0$. Therefore $n_1 = cn$ for some complex number c. But since $n_1(1) = 1 = n(1)$, necessarily $n = n_1$, a contradiction. ∎

Corollary 6.5. *Suppose that the topological group G is extremely amenable. Then there exists a net $(x_\alpha)_\alpha$ in G such that for every extreme point $n \in LIM(G)$,*

$$n((l_{x_\alpha}\varphi)\psi) \to n(\varphi)n(\psi), \quad \varphi, \psi \in LUC(G).$$

Proof. Since G is extremely amenable, there exists a net $(x_\alpha)_\alpha$ in G such that the point evaluations $\delta_{x_\alpha^{-1}}$ converge pointwise to a multiplicative LIM m on $LUC(G)$.

Hence for every $\varphi, \psi \in LUC(G)$, we have

$$\begin{aligned}n((l_{x_\alpha}\varphi)\psi) &= c^{\pi_n}_{[\varphi],[\overline{\psi}]}(x_\alpha^{-1}) = \langle \delta_{x_\alpha^{-1}}, c^{\pi_n}_{[\varphi],[\overline{\psi}]}\rangle \\ &\to \langle m, c^{\pi_n}_{[\varphi],[\overline{\psi}]}\rangle = n(\varphi)n(\psi)\end{aligned} \tag{6.1}$$

by condition (2) in Theorem 6.4. ∎

Remark 6.6. Condition (3) of the preceding theorem is an analogue of the following result (due to [C-N-P] for discrete semigroups). Let S be a semitopological semigroup. Then $LUC(S)$ has a *LIM* if and only if for any $f \in LUC(S)$, the pointwise closure $\overline{\mathrm{co}}^{\mathrm{p}}\{r_s f;\ s \in S\}$ of the convex hull of the right orbit of f contains a constant function $\lambda 1$. Furthermore, if $\lambda 1 \in \overline{\mathrm{co}}^{\mathrm{p}}\{r_s f;\ s \in S\}$, then there exists a *LIM* on $LUC(G)$ such that $m(f) = \lambda$ (see [Mi1] and [La4]).

In order to combine these two actions, we form the cross-product

$$C_n(G) := G \ltimes LUC(G) \subset \mathcal{B}(\mathcal{H}_n),$$

which is the uniform closure of the set of operators of the form $\sum_j \pi_n(g_j) \circ \pi_n(\varphi_j)$, and we obtain a bounded representation σ_n of the algebra $C_n(G)$ on $\mathcal{H}_n$.

Proposition 6.7. *Let G be an amenable topological group. Let n be an extremal point in LIM(G). Then the representation π_n of the algebra $C_n(G)$ is irreducible.*

Proof. Let $\xi \in \mathcal{H}_n \setminus \{0\}$. We must show that ξ is cyclic. Let V be the $C_n(G)$-invariant subspace generated by ξ and suppose that $V \neq \mathcal{H}_n$. Choose a vector η in $\mathcal{H}_n$ orthogonal to V. Then

$$\langle \pi_n(a)\eta, \pi_n(g)\pi_n(b)\xi\rangle_n = 0, \qquad a, b \in LUC(G),\ g \in G.$$

Let $c^{\pi_n}_{\pi_n(b)\xi,\pi_n(a)\eta}$ be the coefficient of π_n associated to the vectors $\pi_n(b)\xi$, $\pi_n(a)\eta$. By Theorem 6.4,

$$\tilde{n}(\pi_n(b)\xi)\overline{\tilde{n}(\pi_n(a)\eta)} = n(c^{\pi_n}_{\pi_n(b)\xi,\pi_n(a)\eta}) = n(0) = 0.$$

Hence $\tilde{n}(\pi_n(b)\xi)\tilde{n}(\pi_n(a)\eta) = 0$ for all $a, b \in LUC(G)$. Since $\xi \neq 0$, we have $\tilde{n}(\pi_n(a)\eta) = 0$ for every a and so $\eta = 0$. Hence π_n is an irreducible representation of the algebra $C_n(G)$. ∎

7. Remarks and open problems. (1) For G locally compact, let ρ be the left regular representation of G and let $C^*_{\delta,\rho}(G) = \overline{\langle \rho(G)\rangle} \subset W^*_\rho(G)$. Then we have a canonical surjection

$$C^*_{\delta,\rho}(G) \to C^*_\rho(G_d),$$

and by [B-K-L-S] the following relations hold:

(a) $C^*_{\delta,\omega_G}(G) \simeq C^*_\rho(G_d) \Leftrightarrow G$ contains an open subgroup which is amenable as discrete ([B-K-L-S, Theorem 1]).

(b) $C^*_d(G) \simeq C^*_{\delta,\omega_G}(G) \Leftrightarrow G$ is amenable.

From the above, for G amenable as discrete, we have

$$B(G_d) = C^*_{\delta,\rho}(G)^* = C^*_{\delta,\omega_G}(G)^*.$$

So

$$B(G) = C^*_{\delta,\omega_G}(G)^* \;\Leftrightarrow\; G \text{ is discrete}$$

in this case.

(2) If G is a compact group such that $C^*_\rho(G) \subset C^*_\delta(G)$, where $C^*_\delta(G)$ denotes the C^*-algebra generated by $\{\rho(g);\ g \in G\}$ in $\mathcal{B}(L^2(G))$ (see [C-L-R] for examples), then

$$C^*_\delta(G) \simeq C^*_{\delta,\omega_G}(G)$$

since G is amenable (see [B-K-L-S]). But in general $C^*_\delta(G)$ is only a homomorphic image of $C^*_{\delta,\omega}(G)$.

(3) Let $B_{\delta,\omega_G}(G) := C_{\delta,\omega_G}(G)^*$. Then $B_{\delta,\omega_G}(G)$ is always a commutative Banach algebra, since $\omega \otimes \omega \simeq \omega$ [La-Lu, Theorem 3.3].

(4) When is $B(G) = C^*_{\delta,\omega_G}(G)^*$, i.e., when is $C^*_\Omega(G)$ a group C^*-algebra of G?

(5) Let $\Theta : C^*(G_d) \to C^*_{\delta,\omega_G}(G)$ be the canonical projection (G locally compact). How big is $\ker \Theta$? For which groups is Θ injective?

(6) Can we characterize extremely amenable groups in terms of $B(G)$? Note that for G locally compact: G extremely amenable $\Leftrightarrow B(G) \equiv \mathbb{C}$. Also: G extremely amenable $\Rightarrow G$ has no non-trivial finite-dimensional representations (see [Gra-La]).

(7) (see [C-L-R]) If $C^*_\rho(G) \cap C^*_{\delta,\rho}(G) \neq \{0\}$ (G locally compact) then $C^*_\rho(G) \subset C^*_{\delta,\rho}(G)$. What can be said in the topological group case?

(8) (see [C-L-R]) If a nondiscrete locally compact group G contains a dense subgroup with property (T) (as a discrete group), does it follow that $C^*_\rho(G) \cap C^*_{\delta,\rho}(G) \neq \{0\}$?

(9) Let G be a topological group. We know that $\tilde{G} \subset \mathcal{M}_u, \mathcal{M} = W^*(G)$. If G is locally compact, we have $\tilde{G} = G$. How big is $\tilde{G}$ in the topological group case?

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References

[Al] D. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. 82 (1981), 423–424.

[B-K-L-S] M. B. Bekka, E. Kaniuth, A. T. M. Lau and G. Schlichting, *On C^*-algebras associated with locally compact groups*, Proc. Amer. Math. Soc. 124 (1996), 3151–3158.

[B-J-M] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups*, Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York, 1989.

[C-L-R] C. Chou, A. T. M. Lau and J. Rosenblatt, *Approximation of compact operators by sums of translations*, Illinois J. Math. 29 (1985), 340–350.

[C-N-P] G. Converse, I. Namioka and R. R. Phelps, *Extreme invariant positive operators*, Trans. Amer. Math. Soc. 137 (1969), 375–385.

[Di] J. Dixmier, *C^*-algebras*, North-Holland Math. Library 15, North-Holland, Amsterdam, 1977.

[Du-Si] D. van Dulst and B. Sims, *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type* (*KK*), in: Banach Space Theory and its Applications (Bucharest, 1981), Lecture Notes in Math. 991, Springer, Berlin, 1983, 35–43.

[Ey] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.

[Gi-Pe] T. Giordano and V. Pestov, *Some extremely amenable groups related to operator algebras and ergodic theory*, J. Inst. Math. Jussieu 6 (2007), 279–315.

[Gl] E. Glasner, *On minimal actions of Polish groups*, in: 8th Prague Topological Symposium on General Topology and Its Relations to Modern Analysis and Algebra, Topology Appl. 85 (1998), 119–125.

[Go-Ki] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Stud. Adv. Math. 28, Cambridge Univ. Press, Cambridge, 1990.

[Gra-La] E. Granirer and A. T. M. Lau, *Invariant means on locally compact groups*, Illinois J. Math. 19 (1971), 249–257.

[Gre] F. P. Greanleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand, Princeton, NJ, 1964.

[Gro-Mi] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. 105 (1983), 843–854.

[He-Ch] W. Herer, J. Christensen and R. Peter, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. 213 (1975), 203–210.

[Hu] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. 10 (1980), 743–749.

[La1] A. T. M. Lau, *The Fourier–Stieltjes algebra of a topological semigroup with involution*, Pacific J. Math. 77 (1978), 165–181.

[La2] A. T. M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math. 118 (1983), 161–175.

[La3] A. T. M. Lau, *Topological semi groups with invariant means in the convex hull of multiplicative means*, Trans. Amer. Math. Soc. 148 (1970), 69–84.

[La4] A. T. M. Lau, *Semigroup of operators on dual Banach spaces*, Proc. Amer. Math. Soc. 54 (1976), 393–396.

[La-Le] A. T. M. Lau and M. Leinert, *Fixed point property and the Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. 360 (2008), 6389–6402.

[La-Lu] A. T. M. Lau and J. Ludwig, *Fourier–Stieltjes algebra of a topological group*, Adv. Math. 229 (2012), 2000–2023.

[La-Ma] A. T. M. Lau and P. Mah, *Normal structure in dual Banach spaces associated with a locally compact group*, Trans. Amer. Math. Soc. 310 (1988), 341–353.

[L-M-U] A. T. M. Lau, P. Mah and A. Ülger, *Fixed point property and normal structure for Banach spaces associated to locally compact groups*, Proc. Amer. Math. Soc. 125 (1997), 2021–2027.

[La-Pa] A. T. M. Lau and A. L. T. Paterson, *Inner amenable locally compact groups*, Trans. Amer. Math. Soc. 325 (1991), 155–169.

[Li] T. C. Lim, *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. 90 (1980), 135–143; *On asymptotic centers and fixed points of nonexpansive mappings*, Canad. J. Math. 32 (1980), 421–430.

[Li-Ma] R. J. Lindahl and P. H. Maserick, *Positive-definite functions on involution semigroups*, Duke Math. J. 38 (1971), 771–782.

[M-P-U] M. G. Megrelishvili, V. G. Pestov and V. V. Uspenskij, *A note on the precompactness of weakly almost periodic groups*, in: Nuclear Groups and Lie Groups (Madrid, 1999), Res. Exp. Math. 24, Heldermann, Lemgo, 2001, 209–216.

[Mil] P. Milnes, *Uniformity and uniformly continuous functions for locally compact groups*, Proc. Amer. Math. Soc. 109 (1990), 567–570.

[Mi1] T. Mitchell, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc. 119 (1965), 244–261.

[Mi2] T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math. 14 (1970), 630–641.

[Mi3] T. Mitchell, *Fixed points and multiplicative left invariant means*, Trans. Amer. Math. Soc. 122 (1966), 195–202.

[Pa] A. L. T. Paterson, *Nuclear C^*-algebras have amenable unitary groups*, Proc. Amer. Math. Soc. 114 (1992), 719–721.

[Pe] V. Pestov, *Dynamics of Infinite-Dimensional Groups. The Ramsey–Dvoretzky–Milman Phenomena*, Univ. Lecture Ser. 40, Amer. Math. Soc., 2005.

[Ru1] Z.-J. Ruan, *The operator amenability of $A(G)$*, Amer. J. Math. 117 (1995), 1449–1474.

[Ru2] Z.-J. Ruan, *Amenability of Hopf von Neumann algebras and Kac algebras*, J. Funct. Anal. 139 (1996), 466–499.

[Sa] S. Sakai, *C^*-algebras and W^*-algebras*, Ergeb. Math. Grenzgeb. 60, Springer, New York, 1971.

[Sp] N. Spronk, *Operator weak amenability of the Fourier algebra*, Proc. Amer. Math. Soc. 130 (2002), 3609–3617.

[Ta1] M. Takesaki, *Theory of Operator Algebras. I*, Springer, New York, 1979.

[Ta2] M. Takesaki, *Duality and von Neumann algebras*, in: Lectures on Operator Algebras; Tulane Univ. Ring and Operator Theory Year, 1970–1971, Vol. II; Lecture Notes in Math. 247, Springer, Berlin, 1972, 665–786.

[Wa1] M. E. Walter, *W^*-algebras and nonabelian harmonic analysis*, J. Funct. Anal. 11 (1972), 17–38.

[Wa2] M. E. Walter, *On the structure of the Fourier–Stieltjes algebra*, Pacific J. Math. 58 (1975), 267–281.

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