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The Bohr–Pál theorem and the Sobolev space $W_2^{1/2}$

by

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Abstract. The well-known Bohr–Pál theorem asserts that for every continuous real-valued function f on the circle $\mathbb{T}$ there exists a change of variable, i.e., a homeomorphism h of $\mathbb{T}$ onto itself, such that the Fourier series of the superposition $f \circ h$ converges uniformly. Subsequent improvements of this result imply that actually there exists a homeomorphism that brings f into the Sobolev space $W_2^{1/2}(\mathbb{T})$. This refined version of the Bohr–Pál theorem does not extend to complex-valued functions. We show that if $\alpha < 1/2$, then there exists a complex-valued f that satisfies the Lipschitz condition of order α and at the same time has the property that $f \circ h \notin W_2^{1/2}(\mathbb{T})$ for every homeomorphism h of $\mathbb{T}$.

For every integrable function f on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (where $\mathbb{R}$ is the real line and $\mathbb{Z}$ is the group of integers) consider its Fourier series

$$f(t) \sim \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikt}, \qquad t \in \mathbb{T}.$$

Recall that the Sobolev space $W_2^{1/2}(\mathbb{T})$ is the space of all (integrable) functions f with

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2|k| < \infty.$$

Let $C(\mathbb{T})$ be the space of all continuous functions on $\mathbb{T}$.

It is well-known that certain properties of continuous functions related to Fourier series can be considerably improved by a change of variable, i.e., by a homeomorphism of the circle onto itself. The first significant result in this area is the Bohr–Pál theorem that states that for every real-valued $f \in C(\mathbb{T})$ there exists a homeomorphism h of $\mathbb{T}$ onto itself such that the superposition $f \circ h$ belongs to the space $U(\mathbb{T})$ of functions with uniformly convergent

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Fourier series. (The theorem was obtained in a somewhat weaker form by J. Pál [11], and in the final form by H. Bohr [2].) The original method of proof of this result uses conformal mappings and in fact allows us (see [9, Sec. 3]) to obtain the following representation:

$$f \circ h = g + \psi, \qquad g \in W_2^{1/2} \cap C(\mathbb{T}),\ \psi \in V \cap C(\mathbb{T}), \tag{1}$$

where $V(\mathbb{T})$ is the space of functions of bounded variation on $\mathbb{T}$. It is well-known that both $W_2^{1/2} \cap C(\mathbb{T})$ and $V \cap C(\mathbb{T})$ are subsets of $U(\mathbb{T})$, thus (1) implies $f \circ h \in U(\mathbb{T})$(1).

A substantial improvement of the Bohr–Pál theorem was obtained by A. A. Sahakian [12, Corollary 1], who showed that if $a(n)$, $n = 0, 1, 2, \dots$, is a positive sequence satisfying $\sum_n a(n) = \infty$ and a certain condition of regularity, then for every real-valued $f \in C(\mathbb{T})$ there is a homeomorphism h such that $\widehat{f \circ h}(k) = O(a(|k|))$. An immediate consequence of Sahakian's result is that the term ψ in (1) can be omitted, i.e., the following refined version of the Bohr–Pál theorem holds: for every real-valued $f \in C(\mathbb{T})$ there exists a homeomorphism h of $\mathbb{T}$ onto itself such that $f \circ h \in W_2^{1/2}(\mathbb{T})$. This refined version also follows from a result on conjugate functions, obtained by W. Jurkat and D. Waterman [4] (see also [3, Theorem 9.5]). We note that Sahakian's result is obtained by purely real analysis techniques, whereas Jurkat and Waterman use an approach similar to the one of Bohr and Pál. A very short proof of the refined version of the Bohr–Pál theorem was communicated to the author by A. Olevskiĭ (see [7, Sec. 3]).

Another improvement of the Bohr–Pál theorem was obtained by J.-P. Kahane and Y. Katznelson [6] (see also [9], [5]). These authors showed that if K is a compact family of functions in $C(\mathbb{T})$, then there exists a homeomorphism h of $\mathbb{T}$ such that $f \circ h \in U(\mathbb{T})$ for all $f \in K$. This result naturally leads to the question whether it is possible to attain the condition $f \circ h \in W_2^{1/2}(\mathbb{T})$ for all $f \in K$. This question was posed by A. Olevskiĭ [10]. A negative answer was obtained by the present author [7, Theorem 4]: it turns out that, given a real-valued $u \in C(\mathbb{T})$, the property that for every real-valued $v \in C(\mathbb{T})$ there is a homeomorphism h such that both $u \circ h$ and $v \circ h$ are in $W_2^{1/2}(\mathbb{T})$ is equivalent to the boundedness of the variation of u. Thus, in general, there is no single change of variable which will bring two real-valued functions in $C(\mathbb{T})$ into $W_2^{1/2}(\mathbb{T})$. Certainly this amounts to the existence of a complex-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin W_2^{1/2}(\mathbb{T})$ for every homeomorphism h of $\mathbb{T}$.

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(1) Assuming $g \in W_2^{1/2}(\mathbb{T})$, it follows that $\sum_{|k| \le N} |k\widehat{g}(k)| = o(N)$, which for a function $g \in C(\mathbb{T})$ implies $g \in U(\mathbb{T})$ (see, e.g., [1, Ch. I, Sec. 64]). The inclusion $V \cap C(\mathbb{T}) \subseteq U(\mathbb{T})$ is due to Jordan (see [1, Ch. I, Sec. 39]).

The purpose of this work is to show that there exists a complex-valued function f that is *very smooth* but at the same time has the property that $f \circ h \notin W_2^{1/2}(\mathbb{T})$ for every homeomorphism h of $\mathbb{T}$.

Note that, as one can easily verify (see, e.g., [7, Sec. 3]), the two seminorms

$$\text{(2)}\qquad \begin{aligned} \|f\|_{W_2^{1/2}(\mathbb{T})} &= \Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 |k|\Big)^{1/2}, \\ |||f|||_{W_2^{1/2}(\mathbb{T})} &= \Big(\int_0^{2\pi} \frac{1}{\theta^2} \int_0^{2\pi} |f(t+\theta) - f(t)|^2 \, dt \, d\theta\Big)^{1/2} \end{aligned}$$

are equivalent on $W_2^{1/2}(\mathbb{T})$, i.e., f is in $W_2^{1/2}(\mathbb{T})$ if and only if $|||f|||_{W_2^{1/2}(\mathbb{T})} < \infty$, and

$$c_1 \|f\|_{W_2^{1/2}(\mathbb{T})} \le |||f|||_{W_2^{1/2}(\mathbb{T})} \le c_2 \|f\|_{W_2^{1/2}(\mathbb{T})} \quad \text{for all } f \in W_2^{1/2}(\mathbb{T}),$$

where $c_1, c_2 > 0$ do not depend on f. Thus, every function that satisfies the Lipschitz condition of order greater than $1/2$ belongs to $W_2^{1/2}(\mathbb{T})$. We shall show that, in general, there is no change of variable which will bring a complex-valued function that satisfies the Lipschitz condition of order less than $1/2$ into $W_2^{1/2}(\mathbb{T})$. The author does not know if the same holds for functions satisfying the Lipschitz condition of order $1/2$ (see remarks at the end of the paper).

Let ω be a *modulus of continuity*, i.e., a nondecreasing continuous function on $[0, \infty)$ such that $\omega(0) = 0$ and $\omega(x + y) \le \omega(x) + \omega(y)$. We denote by $\mathrm{Lip}_\omega(\mathbb{T})$ the class of all complex-valued functions f on $\mathbb{T}$ with $\omega(f, \delta) = O(\omega(\delta))$, $\delta \to +0$, where

$$\omega(f, \delta) = \sup_{|t_1 - t_2| \le \delta} |f(t_1) - f(t_2)|, \qquad \delta \ge 0,$$

is the modulus of continuity of f. For $0 < \alpha \le 1$ we just write Lip_α instead of $\mathrm{Lip}_{\delta^\alpha}$.

THEOREM. *Suppose that* $\limsup_{\delta \to +0} \omega(\delta)/\sqrt{\delta} = \infty$. *Then there exists a complex-valued function* $f \in \mathrm{Lip}_\omega(\mathbb{T})$ *such that* $f \circ h \notin W_2^{1/2}(\mathbb{T})$ *for every homeomorphism* h *of the circle* $\mathbb{T}$ *onto itself. In particular, if* $\alpha < 1/2$, *then there exists a function of class* $\mathrm{Lip}_\alpha(\mathbb{T})$ *with this property.*

Ideologically the method of the proof of this theorem is close to the one used by the author to prove Theorem 4 in [7].

We shall need certain preliminary constructions and lemmas. The simple lemma below is purely technical.

Lemma 1. *Under the assumption of the Theorem on ω, there exists a sequence $\delta_k > 0$, $k = 1, 2, \ldots$, such that*

$$\sum_{k=1}^{\infty} \delta_k < 2\pi/6, \tag{3}$$

$$\sum_{k=1}^{\infty} (\omega(\delta_k))^2 = \infty. \tag{4}$$

Proof. For each $j = 1, 2, \ldots$ we can find ε_j such that $0 < \varepsilon_j < 2^{-(j+1)}$ and

$$\frac{(\omega(\varepsilon_j))^2}{\varepsilon_j} \geq 2^j.$$

Choose positive integers n_j satisfying

$$\frac{1}{2^{j+1}\varepsilon_j} \leq n_j < \frac{1}{2^j \varepsilon_j}, \qquad j = 1, 2, \ldots.$$

Let $N_0 = 1$ and let $N_j = N_{j-1} + n_j$ for $j = 1, 2, \ldots$. We define the sequence δ_k, $k = 1, 2, \ldots$, by setting $\delta_k = \varepsilon_j$ if $N_{j-1} \leq k < N_j$, $j = 1, 2, \ldots$. This yields

$$\sum_{k=1}^{\infty} \delta_k = \sum_{j=1}^{\infty} \sum_{N_{j-1} \leq k < N_j} \delta_k = \sum_{j=1}^{\infty} n_j \varepsilon_j \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,$$

and at the same time

$$\sum_{N_{j-1} \leq k < N_j} (\omega(\delta_k))^2 = n_j (\omega(\varepsilon_j))^2 \geq n_j \varepsilon_j 2^j \geq \frac{1}{2},$$

which proves the lemma. ∎

For a closed interval $I = [a, b] \subseteq (0, 2\pi)$ let $\varDelta_I$ denote the "triangle" function supported on I, i.e., a continuous function on $[0, 2\pi]$ such that $\varDelta_I(t) = 0$ for all $t \in [0, a] \cup [b, 2\pi]$, $\varDelta_I(c) = 1$, where $c = (a + b)/2$ is the center of I, and $\varDelta_I$ is linear on $[a, c]$ and on $[c, b]$.

Let δ_k, $k = 1, 2, \ldots$, be the sequence from Lemma 1. Consider intervals $I_k = [a_k, b_k] \subseteq (0, 2\pi)$ of length $b_k - a_k = 6\delta_k$, where $a_k < b_k < a_{k+1}$, $k = 1, 2, \ldots$ (see (3)). For each k let J_k denote the left half of I_k, i.e., $J_k = [a_k, (a_k + b_k)/2]$, $k = 1, 2, \ldots$.

Everywhere below we use u and v to denote the real-valued functions on $\mathbb{T}$ defined by

$$u(t) = \sum_{k=1}^{\infty} \omega(\delta_k) \varDelta_{I_k}(t), \qquad v(t) = \sum_{k=1}^{\infty} \omega(\delta_k) \varDelta_{J_k}(t), \qquad t \in [0, 2\pi].$$

We shall show that the function $f = u + iv$ satisfies the assertion of the theorem.

Lemma 2. *The functions u and v are of class $\mathrm{Lip}_\omega(\mathbb{T})$.*

Proof. It is clear that for every (closed) interval $I \subseteq (0, 2\pi)$, the function Δ_I satisfies

$$(5)\qquad |\Delta_I(t_1) - \Delta_I(t_2)| \le \frac{2}{|I|}|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, 2\pi],$$

where $|I|$ is the length of I.

Note also that if $0 < x \le y$, then $\omega(y)/y \le 2\omega(x)/x$. Indeed, let $n = [y/x] + 1$, where $[\alpha]$ denotes the integer part of α; then $y \le nx \le 2y$, so

$$\frac{\omega(y)}{y} \le \frac{\omega(nx)}{y} \le \frac{n\omega(x)}{y} \le 2\frac{\omega(x)}{x}.$$

Let us show that $u \in \mathrm{Lip}_\omega(\mathbb{T})$; for v the proof is similar. It is easy to see that it suffices to verify that for all $t_1, t_2 \in \bigcup_k I_k$ we have

$$|u(t_1) - u(t_2)| \le c\omega(|t_1 - t_2|),$$

where $c > 0$ does not depend on t_1 or t_2.

First we consider the case when t_1 and t_2 belong to the same interval I_k. Then, since $|t_1 - t_2| \le |I_k| = 6\delta_k$, we have

$$\frac{\omega(6\delta_k)}{6\delta_k} \le 2\frac{\omega(|t_1 - t_2|)}{|t_1 - t_2|},$$

so (see (5))

$$\begin{aligned} |u(t_1) - u(t_2)| &= \omega(\delta_k)|\Delta_{I_k}(t_1) - \Delta_{I_k}(t_2)| \\ &\le \omega(\delta_k)\frac{2}{6\delta_k}|t_1 - t_2| \le 2\frac{\omega(6\delta_k)}{6\delta_k}|t_1 - t_2| \le 4\omega(|t_1 - t_2|). \end{aligned}$$

Consider now the case when $t_1 \in I_{k_1}$, $t_2 \in I_{k_2}$, $k_1 \ne k_2$. We can assume that $t_1 < t_2$, and hence $0 < t_1 < b_{k_1} < a_{k_2} < t_2 < 2\pi$. Using the previous estimate, we obtain

$$\begin{aligned} |u(t_1) - u(t_2)| &\le |u(t_1)| + |u(t_2)| = |u(t_1) - u(b_{k_1})| + |u(t_2) - u(a_{k_2})| \\ &\le 8\omega(|t_1 - t_2|), \end{aligned}$$

proving the lemma. ∎

For $n = 1, 2, \ldots$ we define

$$u_n(t) = \max\{u(t), 1/n\}, \quad t \in \mathbb{T}.$$

As above, $V(\mathbb{T})$ stands for the class of functions of bounded variation on $\mathbb{T}$.

Lemma 3. *The functions u_n, $n=1,2,\dots$, have the following properties:*

(6) $$|u_n(t_1)-u_n(t_2)| \le |u(t_1)-u(t_2)| \quad \text{for all } t_1,t_2\in\mathbb{T} \text{ and all } n;$$

(7) $$u_n\in V(\mathbb{T}) \quad \text{for all } n;$$

(8) $$\sup_n \Big|\int_{\mathbb{T}} v(t)\,du_n(t)\Big| = \infty.$$

Proof. Properties (6) and (7) are obvious. Let us verify (8). To this end consider the middle thirds of the intervals J_k, namely, the intervals $J_k^*=[a_k+\delta_k, a_k+2\delta_k]$, $k=1,2,\dots$. Note that if

(9) $$\frac{\omega(\delta_k)}{3}\ge\frac{1}{n},$$

then u_n coincides with u on J_k^*. So, if (9) holds, then u_n is increasing on J_k^*, and at the endpoints of J_k^* we have

$$u_n(a_k+\delta_k)=\omega(\delta_k)/3, \quad u_n(a_k+2\delta_k)=2\omega(\delta_k)/3.$$

It is easily seen that for each k,

$$\min_{J_k^*} v = 2\omega(\delta_k)/3.$$

Taking into account that u, and hence u_n, is nondecreasing on each J_k, we see that for all n and k satisfying (9),

$$\int_{J_k} v\,du_n \ge \int_{a_k+\delta_k}^{a_k+2\delta_k} v\,du_n \ge \frac{2}{3}\omega(\delta_k)\int_{a_k+\delta_k}^{a_k+2\delta_k} du_n$$
$$=\frac{2}{3}\omega(\delta_k)\frac{1}{3}\omega(\delta_k)=\frac{2}{9}(\omega(\delta_k))^2.$$

In addition (since u_n is nondecreasing on each J_k) we have

$$\int_{J_k} v\,du_n\ge 0$$

for all n and k. Thus, taking into account that v vanishes outside $\bigcup_{k=1}^\infty J_k$, we obtain

$$\int_{\mathbb{T}} v\,du_n=\sum_{k=1}^\infty\int_{J_k} v\,du_n \ge \sum_{k\,:\,\omega(\delta_k)\ge 3/n}\int_{J_k} v\,du_n \ge \sum_{k\,:\,\omega(\delta_k)\ge 3/n}\frac{2}{9}(\omega(\delta_k))^2.$$

Applying (4) we see that (8) holds. ∎

We shall also need the following auxiliary lemma.

Lemma 4. *If $x,y\in W_2^{1/2}\cap C(\mathbb{T})$ and $y\in V(\mathbb{T})$, then*

$$\Big|\frac{1}{2\pi}\int_{\mathbb{T}} x(t)\,dy(t)\Big|\le \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.$$

Proof. Integration by parts yields

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\,dy(t) = -\frac{1}{2\pi}\int_0^{2\pi} y(t)\,de^{ikt} = -ik\widehat{y}(-k).$$

So, if x is a trigonometric polynomial, using the Cauchy inequality we obtain

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{\mathbb{T}} x(t)\,dy(t)\right| &= \left|\sum_k \widehat{x}(k)\frac{1}{2\pi}\int_{\mathbb{T}} e^{ikt}\,dy(t)\right| \\ &= \left|\sum_k \widehat{x}(k)(-ik)\widehat{y}(-k)\right| \le \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.\end{aligned}$$

To see that the assertion holds in the general case, consider the Fejér sums

$$\sigma_N(x)(t) = \sum_{|k|\le N}\left(1-\frac{|k|}{N}\right)\widehat{x}(k)e^{ikt}.$$

Since $|\widehat{\sigma_N(x)}(k)| \le |\widehat{x}(k)|$ for all $k\in\mathbb{Z}$, we have $\|\sigma_N(x)\|_{W_2^{1/2}(\mathbb{T})} \le \|x\|_{W_2^{1/2}(\mathbb{T})}$. Hence,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{\mathbb{T}} \sigma_N(x)(t)\,dy(t)\right| &\le \|\sigma_N(x)\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})} \\ &\le \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.\end{aligned}$$

At the same time, since y is of bounded variation and $\sigma_N(x)$ converges uniformly to x, it is clear that

$$\frac{1}{2\pi}\int_{\mathbb{T}} \sigma_N(x)(t)\,dy(t) \to \frac{1}{2\pi}\int_{\mathbb{T}} x(t)\,dy(t)$$

as $N\to\infty$, which yields the assertion. ■

Proof of the Theorem. Let $f = u+iv$. Lemma 2 yields $f\in \mathrm{Lip}_\omega(\mathbb{T})$, so it remains to show that $f\circ h \notin W_2^{1/2}(\mathbb{T})$ for every homeomorphism h of $\mathbb{T}$. It is obvious that if a function is in $W_2^{1/2}(\mathbb{T})$, then so are its real and imaginary parts. Assume, contrary to the assertion, that $f\circ h\in W_2^{1/2}(\mathbb{T})$ for a certain homeomorphism h. Then $u\circ h\in W_2^{1/2}(\mathbb{T})$ and $v\circ h \in W_2^{1/2}(\mathbb{T})$.

Note that (6) implies $|u_n\circ h(t_1) - u_n\circ h(t_2)| \le |u\circ h(t_1) - u\circ h(t_2)|$ for all $t_1,t_2\in\mathbb{T}$. Using the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}(\mathbb{T})}$ and $|||\cdot|||_{W_2^{1/2}(\mathbb{T})}$ (see (2)), we infer that $u_n\circ h\in W_2^{1/2}(\mathbb{T})$ for all $n=1,2,\ldots,$ and

$$\|u_n\circ h\|_{W_2^{1/2}(\mathbb{T})} \le c\|u\circ h\|_{W_2^{1/2}(\mathbb{T})}, \qquad n=1,2,\ldots, \tag{10}$$

where $c>0$ does not depend on n.

The property of a function to be of bounded variation is invariant under homeomorphic changes of variable, hence (7) implies that $u_n \circ h \in V(\mathbb{T})$ for all n. Certainly we also have $u_n \circ h \in C(\mathbb{T})$. Applying Lemma 4, and taking (10) into account, we obtain

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{\mathbb{T}} v(t)\,du_n(t)\right| &= \left|\frac{1}{2\pi}\int_{\mathbb{T}} v\circ h(t)\,du_n\circ h(t)\right| \\ &\le \|v\circ h\|_{W_2^{1/2}(\mathbb{T})}\|u_n\circ h\|_{W_2^{1/2}(\mathbb{T})} \\ &\le c\|v\circ h\|_{W_2^{1/2}(\mathbb{T})}\|u\circ h\|_{W_2^{1/2}(\mathbb{T})},\end{aligned}$$

which contradicts (8). ∎

Remarks. 1. For $s > 0$ consider the Sobolev space $W_2^s(\mathbb{T})$ of all (integrable) functions f with

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 |k|^{2s} < \infty.$$

As shown in [7, Corollary 3], for each compact family K in $C(\mathbb{T})$ (or equivalently for each class $\mathrm{Lip}_\omega(\mathbb{T})$) there exists a homeomorphism h of $\mathbb{T}$ such that $f\circ h \in \bigcap_{s<1/2} W_2^s(\mathbb{T})$ for all $f \in K$ (resp. for all $f \in \mathrm{Lip}_\omega(\mathbb{T})$).

2. There exists a real-valued $f \in C(\mathbb{T})$ such that $f\circ h \notin \bigcup_{s>1/2} W_2^s(\mathbb{T})$ for every homeomorphism h of $\mathbb{T}$. This is a simple consequence of the inclusion $\bigcup_{s>1/2} W_2^s \cap C(\mathbb{T}) \subseteq A(\mathbb{T})$, where $A(\mathbb{T})$ is the Wiener algebra of absolutely convergent Fourier series, and of a well-known result of Olevskiĭ that provides a negative answer to Lusin's rearrangement problem: there exists a real-valued $f \in C(\mathbb{T})$ such that $f\circ h \notin A(\mathbb{T})$ for every homeomorphism h ([8], see also [9]).

3. The function $f(t) = \sum_{k\ge 0} 2^{-k/2} e^{i2^k t}$ is in $\mathrm{Lip}_{1/2}(\mathbb{T})$ (see, e.g., [1, Ch. XI, Sec. 6]), but it is obvious that $f \notin W_2^{1/2}(\mathbb{T})$; thus $\mathrm{Lip}_{1/2}(\mathbb{T}) \nsubseteq W_2^{1/2}(\mathbb{T})$. The author does not know if the assertion of the Theorem holds for $\omega(\delta) = \delta^{1/2}$. At the same time there is no change of variable which will bring the whole class $\mathrm{Lip}_{1/2}(\mathbb{T})$ into $W_2^{1/2}(\mathbb{T})$: a proof will be presented in another paper.

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