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L^p boundedness of Riesz transforms for orthogonal polynomials in a general context

by

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Abstract. Nowak and Stempak (2006) proposed a unified approach to the theory of Riesz transforms and conjugacy in the setting of multi-dimensional orthogonal expansions, and proved their boundedness on L^2. Following them, we give easy to check sufficient conditions for their boundedness on L^p, $1 < p < \infty$. We also discuss the symmetrized version of these transforms.

1. Introduction. The investigation of the boundedness on L^p, $1 < p < \infty$, of the Riesz transforms associated with the ultra-spherical polynomials, a particular case of a nontrigonometric orthogonal expansion, was initiated and extensively studied in the seminal article of Muckenhoupt and Stein [15]. It was followed by the no less fundamental articles of Muckenhoupt [14, 13], where the boundedness of the Riesz transform was treated in a one-dimensional setting for the case of Hermite and Laguerre polynomials.

Later on, for $1 < p < \infty$, the L^p-boundedness of these transforms for orthogonal polynomial-like expansions was studied in a high-dimensional setting. Starting in a series of articles concerning Hermite polynomial expansions by P. A. Meyer [12], Gundy [6], Pisier [21], Urbina [25], Gutiérrez [7] and Gutiérrez, Segovia and Torrea [9] (see also the survey article by Sjögren [22]), the investigation was extended to other orthogonal expansions such as Laguerre [8], [16], ultraspherical [2], [3] and Jacobi [18] polynomials. Recently, Mauceri and Spinelli [11] investigated the L^p-boundedness of the Riesz transforms and spectral multipliers for the Hodge–Laguerre operator, a generalization to differential forms of the Laguerre operator on functions.

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Associated with each system of orthogonal polynomials there are a measure and an elliptic differential equation whose solution is a semigroup satisfying certain Cauchy–Riemann equations. These equations define two derivatives, a conjugate semigroup and a conjugate elliptic differential equation. Together, all these define the Riesz transforms in each particular system. To prove the L^p-boundedness, the papers listed in the previous paragraph introduce appropriate Littlewood–Paley–Stein square functions [23], [4] that relate a function to its Riesz transform, and prove two-sided L^p-inequalities for these square functions.

These two-sided inequalities, established for each particular system, involve two key steps. The first step relates the elliptic operator applied to a positive power of the solution to the square of its *gradient*.

The second step consists in proving a pointwise estimate between both semigroups. This is done in the papers mentioned above by computing the kernels of the corresponding semigroups and then comparing them. These computations involve finding the sum of a certain series that has to be done ad-hoc for each particular case, and their comparison requires a highly analytical technique.

In 2006 Nowak and Stempak [20] presented a fairly general and unified approach to the theory of Riesz transforms and conjugacy in the setting of high-dimensional orthogonal expansions. The resulting scheme, emerging from observations furnished in numerous articles, allowed them to show the L^2-boundedness of the Riesz transforms in a rather effortless and general way. This scheme, however, was not used for the boundedness of these transforms on L^p for $p \neq 2$.

The goal of this paper is to use this unified presentation in order to clearly specify the conditions required of an orthogonal system to yield the L^p-boundedness of its Riesz transforms. In particular, we will see that the first step mentioned above is true for any elliptic differential operator. As for the second step, we give clear and easy to check hypotheses which involve only one-dimensional boundedness of functions. More importantly, we do not need the existence and computation of the associated kernel, but use the L^2-expansion given by Nowak and Stempak [20] and results on the maximum principle for differential equations that will hopefully be applied in the future to prove boundedness in more general settings.

This paper is organized as follows: Section 2 describes the general setting, terminology and the main results. Section 3 introduces the assumptions for a system to get the boundedness of its Riesz transforms. Sections 4 and 5 contain the proofs of the main results. Sections 6 and 7 present all the auxiliary proofs. Finally, in Section 8, some orthogonal systems are presented so as to display how one can easily check the hypotheses in the most common semigroups.

2. Main results. For $X = (b,c)$ with $-\infty \le b < c \le \infty$, and $\mathcal{X} = \prod_{i=1}^d X$, we define the elliptic differential operator

$$\begin{aligned} Lu &= -\sum_{i=1}^d \left(p_i^2(x_i)u_{x_ix_i} + \left[(p_i^2(x_i))' + p_i^2(x_i)\frac{w_i'(x_i)}{w_i(x_i)} \right] u_{x_i} \right) \\ &= \sum_{i=1}^d \delta_i^*\delta_i u =: \sum_{i=1}^d L_i u, \end{aligned}$$

with

$$\delta_i = p_i(x_i)\partial_{x_i} \quad \text{and} \quad \delta_i^* = -\left(p_i(x_i)\partial_{x_i} + p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} + p_i'(x_i) \right),$$

where $p_i \in C^2(X)$ does not vanish on X, and $w_i \in C^1(X)$, $w_i > 0$ on X.

We also define the elliptic differential operator

$$M_i = L + [\delta_i, \delta_i^*],$$

where $[\delta_i, \delta_i^*] := \delta_i\delta_i^* - \delta_i^*\delta_i$. Notice that L and M_i are self-adjoint with respect to

$$d\mu(x) = \prod_{i=1}^d d\mu_i(x_i) = \prod_{i=1}^d w_i(x_i)dx_i =: w(x)dx.$$

Formally, we define the ith first-order *Riesz transform* associated to L as

$$R_i = \delta_i L^{-1/2},$$

and the ith *conjugate Riesz transform* as in [20] by

$$R_i^* = \delta_i^* M_i^{-1/2},$$

for $i = 1, \dots, d$. The first definition is correct if $\ker L$ is trivial, otherwise we must compose it with the orthogonal projection of L^2 onto $(\ker L)^\perp$. A similar modification has to be made for the second definition. Then we have

THEOREM 2.1. *Under the assumptions given in Section* 3*, the Riesz transforms R_j for $j = 1, \dots, d$ and the conjugate Riesz transforms R_j^* are bounded on $L^p(\mathcal{X}, d\mu)$, $1 < p < \infty$, with constants independent of dimension.*

For $b = 0$, Nowak and Stempak [19] created the following symmetrization process. Given $X = (0,c)$ define $\mathbb{X} := -X \cup X$, and extend p_i and w_i to $\mathbb{X}$ as even functions.

With those new definitions L, M_i, δ_i and δ_i^* are extended to $\mathcal{X} = \mathbb{X}^d$. On the other hand, the eigenfunctions $\{\varphi_n\}$ defined in **H2** of Section 3 are extended to $\mathcal{X}$ as even functions, i.e., $\varphi_n(\sigma_j x) = \varphi_n(x)$ for $j = 1, \dots, d$ and

$n \in \mathbb{N}_0^d$, where σ_j denotes the reflection on $\mathcal{X}$ with respect to the hyperplane orthogonal to the jth coordinate axis.

Following [19] the *symmetrization* L_S of the operator L is defined as

$$L_S = -\sum_{i=1}^{d} D_i^2$$

with

$$D_i f(x) = p_i(x_i)\frac{\partial f}{\partial x_i}(x) + \left[p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} + p_i'(x_i)\right]\frac{f(x) - f(\sigma_i x)}{2}.$$

In [19] it is proved that the eigenfunctions of L_S are Φ_n as given in **H2** of Section 3.

For $1 < p < \infty$, let $\mathcal{E}_p$ be the L^p-closure of $\operatorname{span}\{\Phi_n : n_j \text{ even for all } j\}$ and for each $i = 1, \ldots, d$, let $\mathcal{O}_{i,p}$ be the L^p-closure of $\operatorname{span}\{\Phi_n : n_i \text{ odd and } n_j \text{ even for } j \neq i\}$. Then $\mathcal{E}_p \cap \mathcal{O}_{i,p} = \langle 0 \rangle$.

Formally we define the ith *symmetric Riesz transform* to be

$$R_{S,i} = D_i L_S^{-1/2},$$

and then we have

THEOREM 2.2. *Under the same assumptions given in Theorem* 2.1, *the operator $R_{S,i}$ is bounded on $\mathcal{E}_p \oplus \mathcal{O}_{i,p}$ with constant independent of dimension.*

Let us remark that we do not get boundedness on the whole space $L^p(\mathcal{X})$ since the operator we are dealing with is not differential but differential-difference, and therefore we cannot apply the results of this paper—we are considering differential operators. But we can restrict this operator to the regions just defined where it turns out to be differential and apply all what is known about boundedness of Riesz transforms to these two regions.

REMARK 2.3. Observe that when $d = 1$, every function can be written as the sum of an even one plus an odd one; then $L^p(\mathcal{X}) = \mathcal{E}_p \oplus \mathcal{O}_{i,p}$ and therefore R_S is bounded on L^p for every $1 < p < \infty$.

Nowak and Stempak [19] proved that the derivatives D_i commute with L_S. As a consequence, for $d = 1$, the higher order Riesz transforms defined by them are also bounded on L^p for $1 < p < \infty$.

3. Definitions and assumptions for Theorems 2.1 and 2.2. The assumptions will be in italics. Any direct consequence of the assumptions that we need later will be given in roman type.

H1 (On the coefficients of differential operators). *For $t > 0$ sufficiently large, let $\mathcal{X}_t = \prod_{i=1}^d X_t$ with $X_t = [(b + e^{-t}) \vee (-t), (c - e^{-t}) \wedge t] =: [b_t, c_t]$.*

(a) *For every* $i = 1, \dots, d$, *either* $p_i^2(b_t)w_i(b_t)$ *and* $p_i^2(c_t)w_i(c_t) \to 0$ *as* $t \to \infty$; *or* $\lim_{t\to\infty} p_i^2(b_t)w_i(b_t) = \lim_{t\to\infty} p_i^2(c_t)w_i(c_t) \neq 0$, $\varphi_{n_i}^i(b) = \varphi_{n_i}^i(c)$ *and* $(\varphi_{n_i}^i)'(b) = (\varphi_{n_i}^i)'(c)$ *for all* $n_i \in \mathbb{N}_0$ *whenever* b *and* c *are finite; or* $|\varphi_{n_i}^i(x_i)| \le C_{n_i}\gamma_i^1(x_i)$ *and* $|(\varphi_{n_i}^i)'(x_i)| \le C_{n_i}\gamma_i^2(x_i)$ *with* $\gamma_i^j \in L^1(X, d\mu_i(x_i)) \cap C(X)$, $j = 1, 2$, *and* $\gamma_i^j(b_t)$ *or* $\gamma_i^j(c_t) \to 0$ *as* $t \to \infty$, *whenever* $b = -\infty$ *or* $c = \infty$.
For each $i = 1, \dots, d$ *the functions* $\varphi_{n_i}^i(x_i)$ *are defined in* **H2**.

(b) *There exists a* C^1 *function* $\phi : [1, \infty) \to (0, \infty)$ *such that an antiderivative* Θ *of* $1/\phi$ *satisfies* $\Theta \ge 1$ *on* $[1, \infty)$, *and for* $\tau \ge 1$,

$$\tau\Theta''(\tau) \le m_1\Theta(\tau)\Theta'(\tau), \qquad \tau\Theta'(\tau) \le m_2[\Theta(\tau)]^{2-\mu}$$

for some positive constants μ, m_1 *and* m_2 *and for every* $i = 1, \dots, d$ *there are positive constants* C_1 *and* C_2 *such that*

$$p_i^2(x_i) \le C_1\phi(1 + x_i^2),$$

$$\left|\left[(p_i^2(x_i))' + p_i^2(x_i)\frac{w_i'(x_i)}{w_i(x_i)}\right]\right| \le C_2\phi(1 + x_i^2)\frac{\Theta(1 + x_i^2)}{\sqrt{1 + x_i^2}}, \qquad \forall x_i \in X.$$

(c) *For every* $i = 1, \dots, d$,

$$[\delta_i, \delta_i^*] = -p_i(x_i)\left[p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} + p_i'(x_i)\right]' \ge 0$$

on X.

H2 (On the eigenvalues and eigenfunctions of the differential operators)

(a) *For each* i *there is an orthonormal basis* $\{\varphi_{n_i}^i\}_{n_i \ge 0}$ *of* $L^2(X, d\mu_i)$, *consisting of eigenfunctions of* L_i *corresponding to a discrete ordered set* $\{\lambda_{n_i}^i\}_{n_i \ge 0} \subset \mathbb{R}$ *of nonnegative eigenvalues going to infinity, i.e.*

$$L_i\varphi_{n_i}^i = \lambda_{n_i}^i\varphi_{n_i}^i.$$

The eigenfunctions and their first derivatives can be continuously extended to ∂X *whenever* b *or* c *is finite. For* $n \in \mathbb{N}_0^d$, *let* $\varphi_n(x) = \prod_{i=1}^d \varphi_{n_i}^i(x_i)$ *and* $\lambda_n = \sum_{i=1}^d \lambda_{n_i}^i$. *Then*

$$L\varphi_n = \lambda_n\varphi_n.$$

Let us remark that the eigenfunctions of L belong to $C^\infty(\mathcal{X})$ (see [20]).

From the definition of L, the constant functions are eigenfunctions associated to the eigenvalue 0, and the assumption on the eigenvalues implies $\lambda_0 = 0$. Also, since these eigenfunctions have to be in the Lebesgue spaces, the measure μ has to be finite; without loss of generality, we may assume it is a probability measure.

From [20] we know that $\{\delta_i\varphi_n\}$ are eigenfunctions of M_i with eigenvalues λ_n, i.e.,

$$M_i(\delta_i\varphi_n) = \lambda_n\delta_i\varphi_n,$$

and they can be extended to (or already are) an orthogonal basis on L^2 (see [20, Lemma 6]). On the other hand, it is not hard to prove that for each $n \in \mathbb{N}_0^d$ with $n_i > 0$, $\partial\varphi_n/\partial x_i$ is an eigenfunction of the elliptic differential operator

$$N_i = L - (p_i^2(x_i))'\partial_{x_i} - \left[(p_i^2(x_i))^2\frac{w_i'(x_i)}{w_i(x_i)}\right]'$$

associated to the eigenvalue λ_n, i.e.,

$$N_i\left(\frac{\partial\varphi_n}{\partial x_i}\right) = \lambda_n\frac{\partial\varphi_n}{\partial x_i}.$$

As was done in [20] with $\{\delta_i\varphi_n\}$, the sequence $\{\partial\varphi_n/\partial x_i\}$ can be extended to (or already is) an orthogonal basis of $L^2(\mathcal{X}, d\nu_i)$ with $d\nu_i = p_i^2(x_i)d\mu$. Notice that $E = \operatorname{span}\{\varphi_n\}$ and F_i, the completion of $\operatorname{span}\{\delta_i\varphi_n\}$, $1 \le i \le d$, are dense in $L^2(\mathcal{X}, d\mu)$, while G_i, the completion of $\operatorname{span}\{\partial\varphi_n/\partial x_i\}$, is dense in $L^2(\mathcal{X}, d\nu_i)$.

Furthermore, we will assume that for $1 < p < \infty$, E *is also dense in* $L^p(\mathcal{X}, d\mu)$. *And for every* $1 \le i \le d$, $F_i \subset L^p(\mathcal{X}, d\mu)$ *and* $G_i \subset L^p(\mathcal{X}, d\mu)$.

For the symmetrized process the eigenfunctions of L_S are defined as $\Phi_n(x) = \prod_{i=1}^d \Phi_{n_i}^i(x_i)$ with

$$\Phi_{n_i}^i(x_i) = \begin{cases} 2^{-1/2}\varphi_{n_i/2}^i(x_i), & n_i \text{ even},\\ -2^{-1/2}(\lambda_{(n_i+1)/2})^{-1/2}\delta_i\varphi_{(n_i+1)/2}^i(x_i), & n_i \text{ odd},\end{cases}$$

where for $i = 1, \dots, d$ and $m \in \mathbb{N}_0$, φ_m^i is extended to $\mathbb{X}$ as an even function. The eigenvalues of L_S are $\lambda_{\langle n\rangle}$ where $\langle n\rangle = (\lfloor (n_j + 1)/2\rfloor)_{j=1}^d$ with $\lfloor x\rfloor$ the integer part of x.

(b) *There exists* $N \in \mathbb{N}$ *such that* $\sum_{k\in\mathbb{N}_0^d\setminus\{0\}} \lambda_k^{-N} < \infty$ *and for every compact set* $K \subset \mathcal{X}$ *there exists* $C = C_K$ *such that*

$$\max_{1\le i\le d}\sup_{x\in K}\left|\frac{\partial^n\varphi_k}{\partial x_i^n}(x)\right| \le C\lambda_k^N$$

for every $k \in \mathbb{N}_0^d \setminus \{0\}$ *and* $n = 0, 1, 2, 3$.

H3 (On the semigroups). We will consider some semigroups such as $T^t = e^{-Lt}$,

$$P^t = e^{-L^{1/2}t} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}T^{t^2/4u}\,du = \int_0^\infty \frac{1}{t^2}\phi\left(\frac{s}{t^2}\right)T^s\,ds \tag{3.1}$$

where $\phi(s) = \frac{e^{-1/4s}}{2\sqrt{\pi}\, s^{3/2}}$ (note that $\phi(s)$ and $s\phi'(s)$ are in $L^1((0,\infty), ds)$),

$$\widetilde{T}_i^t = e^{-M_i t}$$

and

$$\widetilde{P}_i^t = e^{-M_i^{1/2} t}. \tag{3.2}$$

The spectral decomposition of T^t on $L^2(\mathcal{X}, d\mu)$ is

$$T^t f(x) = \sum_{n=0}^{\infty} \sum_{|k|=n} e^{-\lambda_k t} \langle f, \varphi_k \rangle_\mu \varphi_k(x),$$

for every $f \in L^2(\mathcal{X}, d\mu)$, and

$$P^t f(x) = \sum_{n=0}^{\infty} \sum_{|k|=n} e^{-\sqrt{\lambda_k} t} \langle f, \varphi_k \rangle_\mu \varphi_k(x)$$

with $\langle f, h \rangle_\mu = \int_{\mathcal{X}} f(x) h(x)\, d\mu(x)$. On the other hand, the spectral decomposition of $\widetilde{T}_i^t$ is similar to that of T^t. In this case, φ_n is replaced by the normalization of $\delta_i \varphi_n$. Note that $P^t f$ satisfies the equation $\mathcal{L} P^t f = 0$ with $\mathcal{L} = \partial_t^2 - L$ and as a consequence of the spectral decomposition we find that P^t and $\tilde{P}^t$ are symmetric on $L^2(\mathcal{X}, d\mu)$.

(a) *For $f \in C_c^2(\mathcal{X})$ with $f \geq 0$, we have $T^t f, \tilde{T}_i^t f \geq 0$ on $\mathcal{X}$. Moreover, for every $1 \leq p \leq \infty$ and $f \in L^p(\mathcal{X}, d\mu)$, we have $\|P^t f\|_p \leq \|f\|_p$ and $\|\tilde{P}_i^t f\|_p \leq \|f\|_p$.*

(b) *For every $f \in E$, we have $(P^t f)^2 \leq C P^t f^2$ and $(\widetilde{P}_i^t f)^2 \leq C \widetilde{P}_i^t f^2$.*

(c) *For $f \in L^2(X, w_i(x_i) dx_i)$ with $f \geq 0$ and $x_i \in X$, and for*

$$u_1(t, x_i) = \sum_{n=0}^{\infty} e^{-\lambda_n^i t} \langle f, \varphi_n^i \rangle \varphi_n^i(x_i),$$
$$u_2(t, x_i) = \sum_{n=1}^{\infty} e^{-\lambda_n^i t} \frac{\langle f, \delta_i \varphi_n^i \rangle}{\|\delta_i \varphi_n^i\|_2^2} \delta_i \varphi_n^i(x_i),$$

the one-parameter semigroups T^t and $\widetilde{T}_i^t$ respectively have the property that u_1, u_2 are bounded and

$$u_2(t, b) \leq u_1(t, b) \quad \text{and} \quad u_2(t, c) \leq u_1(t, c) \tag{3.3}$$

for all $t > 0$ whenever b or c is finite.

(d) *For every $i = 1, \ldots, d$, let $T_i^t := e^{-N_i t}$. This semigroup is bounded on L^∞.*

4. Proof of Theorem 2.1. According to the definition of the ith Riesz transform associated to L we have

$$R_i\varphi_n = \frac{1}{\lambda_n^{1/2}}\delta_i\varphi_n$$

if $\lambda_n > 0$, and 0 otherwise. By **H2**(a), $\{\varphi_n\}$ is an orthonormal basis in $L^2(\mathcal{X}, d\mu)$ and therefore these operators naturally extend to L^2 and turn out to be bounded there (see [20, p. 681, Proposition 1]).

The L^p-boundedness of R_i will be obtained from the following proposition which is interesting per se; its proof is in Section 6.

Proposition 4.1. *For every $1 < p < \infty$, there exist positive constants c_p and C_p, depending only on p (not on dimension), such that for every $f \in E$,*

$$\|g(f)\|_{L^p(d\mu)} \le C_p \|f\|_{L^p(d\mu)}, \tag{4.1}$$

where

$$g(f)(x) = \Big(\int_0^\infty t|\nabla P^t f(x)|^2\,dt\Big)^{1/2},$$

with $\nabla = (\partial_t, \delta_1, \ldots, \delta_d)$, and for every $i = 1, \ldots, d$,

$$c_p\|f\|_{L^p(d\mu)} \le \|\tilde{g}_i(f)\|_{L^p(d\mu)} + \|E_i(f)\|_{L^p(d\mu)}, \tag{4.2}$$

where $E_i(f) = \lim_{t\to\infty} \widetilde{P}_i^t(f)$ and

$$\tilde{g}_i(f)(x) = \Big(\int_0^\infty t|\partial_t \tilde{P}_i^t f(x)|^2\,dt\Big)^{1/2},$$

and the semigroups P^t and $\tilde{P}_i^t$ are defined in **H3**.

Proof of Theorem 2.1. For every $f \in E$, according to the spectral decompositions of P^t and $\widetilde{P}_i^t$ and taking into account that

$$\|\delta_i\varphi_n\|^2_{L^2(\mathcal{X},d\mu)} = \lambda^i_{n_i}, \quad \forall n_i \ge 1$$

(see [20, pp. 678–679]) we have the identity

$$\partial_t\tilde{P}_i^t(R_i f) = -\delta_i P^t(f). \tag{4.3}$$

Thus for $f \in E$, taking into account (4.3), we obtain

$$|\tilde{g}_i(R_i f)(x)| \le g(f)(x).$$

On the other hand, $E_i(R_i f) = 0$.

Thus, by Proposition 4.1 the first part of the theorem follows for the operator R_i restricted to E. Since E is dense in L^p, R_i extends boundedly to the whole L^p with constant independent of dimension.

Now, from [20, p. 687] we have

$$\langle R_i^* f, h\rangle_\mu = \langle f, R_i h\rangle_\mu$$

for every $f, h \in E$ and $i = 1, \ldots, d$. For every $f \in E$, taking into account that for every $1 < p < \infty$, E is dense in L^q and R_i is bounded on $L^q(\mathcal{X}, d\mu)$ with $q = p/(p-1)$, we have

$$\begin{aligned}\|R_i^* f\|_{L^p(d\mu)} &= \sup_{h\in E,\, \|h\|_q\le 1} \langle R_i^* f, h\rangle_\mu \\ &\le \sup_{h\in E,\, \|h\|_q\le 1} \|R_i h\|_q \|f\|_{L^p(d\mu)} \\ &\le C_q \|f\|_{L^p(d\mu)}.\end{aligned}$$

Hence R_i^* extends to the whole $L^p(\mathcal{X}, d\mu)$ by density. This ends the proof of Theorem 2.1. ∎

5. Proof of Theorem 2.2. First, notice that for every $i = 1, \ldots, d$, $L_i \Phi_{n_i}^i(x_i) = \lambda_{n_i/2}^i \Phi_{n_i}^i(x_i)$ if n_i is even and $\delta_i \delta_i^* \Phi_{n_i}^i(x_i) = \lambda_{(n_i+1)/2} \Phi_{n_i}^i(x_i)$ if n_i is odd.

In order to prove the theorem we need a couple of remarks.

Remark 5.1. The extensions $R_i = \delta_i L^{-1/2}$ and $R_i^* = \delta_i^* M_i^{-1/2}$ are bounded on $L^p(\mathcal{X}, d\mu)$ for $1 < p < \infty$, with $\mathcal{X} = \mathbb{X}^d$ and $\mathbb{X} = (-c, 0) \cup (0, c)$.

Indeed, observe that

$$R_i \varphi_n(x) = \begin{cases} \dfrac{1}{\lambda_n^{1/2}} p_i(x_i) \dfrac{\partial \varphi_n}{\partial x_i}(x) & \text{for } x_i \in (0, c), \\ -\dfrac{1}{\lambda_n^{1/2}} p_i(-x_i)(\varphi_{n_i}^i)'(-x_i) \prod_{j\ne i} \varphi_{n_j}^j(x_j) & \text{for } x_i \in (-c, 0). \end{cases}$$

Now for $f = \sum_{|n|\le N} c_n \varphi_n$, we have

$$\begin{aligned}\|R_i f\|_{L^p(\mathcal{X}, d\mu)}^p &= \int_{\mathcal{X}} |R_i f(x)|^p \, d\mu(x) \\ &= \int_{\mathbb{X}^{d-1}} d\mu^i(x^i) \bigg[\int_0^c \bigg| \sum_{0<|n|\le N} \frac{c_n}{\lambda_n^{1/2}} p_i(x_i)(\varphi_{n_i}^i)'(x_i) \prod_{j\ne i} \varphi_{n_j}^j(x_j) \bigg|^p \\ &\quad + \int_{-c}^0 \bigg| \sum_{0<|n|\le N} \frac{c_n}{\lambda_n^{1/2}} p_i(-x_i)(\varphi_{n_i}^i)'(-x_i) \prod_{j\ne i} \varphi_{n_j}^j(x_j) \bigg|^p d\mu_i(x_i) \bigg] \\ &= 2^d \int_{X^d} |R_i(f\chi_{X^d})(x)|^p \, d\mu(x) \\ &\le C_p^p 2^d \|f\chi_{X^d}\|_{L^p(X^d, d\mu)}^p = C_p^p \|f\|_{L^p(\mathcal{X}, d\mu)}^p. \ ∎\end{aligned}$$

Remark 5.2. We have

$$R_{S,i}\Phi_n = \begin{cases} R_i\Phi_n, & n_j \text{ even for all } j = 1,\dots,d,\\ -R_i^*\Phi_n, & n_i \text{ odd and } n_j \text{ even for all } j \neq i. \end{cases} \tag{5.1}$$

Indeed, observe that

$$L_S f = \sum_{j=1}^{d} D_j^2 f = Lf + \sum_{j=1}^{d}[\delta_j,\delta_j^*]\frac{f(x)-f(\sigma_j x)}{2}$$

where

$$D_j f(x) = p_j(x_j)\frac{\partial f}{\partial x_j}(x) + \left[p_j(x_j)\frac{w_j'(x_j)}{w_j(x_j)} + p_j'(x_j)\right]\frac{f(x)-f(\sigma_j x)}{2}$$

(see [19]).

So if n_j is even for every j, then $\Phi_n(\sigma_j x) = \Phi_n(x)$ for all x, $L_S\Phi_n = L\Phi_n = \lambda_{\langle n\rangle}\Phi_n$, and $D_i\Phi_n = \delta_i\Phi_n$. On the other hand, if n_i is odd and for $j \neq i$, n_j is even, then $\Phi_n(\sigma_i x) = -\Phi_n(x)$, $\Phi_n(\sigma_j x) = \Phi_n(x)$ for all x, $L_S\Phi_n = M_i\Phi_n = \lambda_{\langle n\rangle}\Phi_n$, and $D_i\Phi_n = -\delta_i^*\Phi_n$.

Thus if the components of n are all even then

$$\begin{aligned} R_{S,i}\Phi_n &= D_i L_S^{-1/2}\Phi_n = \frac{1}{\sqrt{\lambda_{\langle n\rangle}}}D_i\Phi_n \\ &= \frac{1}{\sqrt{\lambda_{\langle n\rangle}}}\delta_i\Phi_n = \delta_i L^{-1/2}\Phi_n = R_i\Phi_n. \end{aligned}$$

On the other hand, if n_i is odd and the other entries of n are even we have

$$\begin{aligned} R_{S,i}\Phi_n &= D_i L_S^{-1/2}\Phi_n = \frac{1}{\sqrt{\lambda_{\langle n\rangle}}}D_i\Phi_n \\ &= -\frac{1}{\sqrt{\lambda_{\langle n\rangle}}}\delta_i^*\Phi_n = -\delta_i^* M_i^{-1/2}\Phi_n = -R_i^*\Phi_n. \end{aligned}$$

Thus, for $f \in L^2(\mathcal{X}) \cap (\mathcal{E}_p \oplus \mathcal{O}_{i,p})$, $f = f_e + f_{o,i}$ with $f_e \in \mathcal{E}_p$ and $f_{o,i} \in \mathcal{O}_{i,p}$. Then, by setting $I_1 := \{n \in \mathbb{N}_0 : n_j \text{ is even for all } j = 1,\dots,d\}$, $I_2 := \{n \in \mathbb{N}_0 : n_i \text{ is odd and } n_j \text{ is even for all } j \neq i\}$ and taking into account Remark 5.2, we have

$$\begin{aligned} R_{S,i}f &= R_{S,i}f_e + R_{S,i}f_{o,i} \\ &= \sum_{n\in I_1}\langle f_e,\Phi_n\rangle R_i\Phi_n - \sum_{n\in I_2}\langle f_{o,i},\Phi_n\rangle R_i^*\Phi_n \\ &= R_i(f_e) - R_i^*(f_{o,i}). \end{aligned}$$

Finally, the conclusion follows using Remark 5.1 and the fact that $f_e = (f + f\circ\sigma_i)/2$ and $f_{o,i} = (f - f\circ\sigma_i)/2$. ∎

6. Proof of Proposition 4.1. We need

LEMMA 6.1. *For every $f \in E$, $g(f) \in L^p(\mathcal{X}, d\mu)$ for $1 < p < \infty$.*

Proof. Since $\lambda_0 = 0$, we have $g(\varphi_0) = 0$, and for $\lambda_n > 0$,

$$g(\varphi_n)(x) = \Big(\int_0^\infty te^{-2\sqrt{\lambda_n}\,t}\,dt\Big)^{1/2} |(\sqrt{\lambda_n}\,\varphi_n(x), \delta_1\varphi_n(x), \ldots, \delta_d\varphi_n(x))| \in L^p(\mathcal{X}, d\mu), \quad \text{by } \mathbf{H2}(\mathrm{a}).$$

The result follows from the sublinearity of g. ∎

Proof of Proposition 4.1. We only need to prove inequality (4.1) since (4.2) and the boundedness of $\widetilde{g}_i$ on $L^p(\mathcal{X}, d\mu)$ follow from a generalization of [23, Theorem 10 p. 111, Corollary 2 p. 120] given in [4]. The generalization only assumes conditions I and II of [23, Theorem 10, Corollary 2]. In our context those conditions correspond to **H3**(a).

CASE $1 < p \le 2$. Let $f \in E$ and $\epsilon > 0$. Applying Lemma 7.2 to $F(t,x) = P^t f(x)$, setting $F_\epsilon = (F^2 + \epsilon^2)^{1/2}$ and taking into account that $(p-1)F^2 + \epsilon^2 \ge (p-1)(F^2 + \epsilon^2)$ we have

$$\begin{aligned}
[g(f)(x)]^2 &= \int_0^\infty t|\nabla P^t f(x)|^2\,dt \\
&\le \frac{1}{p(p-1)} \int_0^\infty tF_\epsilon^{2-p} \mathcal{L} F_\epsilon^p\,dt \\
&\le \frac{1}{p(p-1)} [P^* f]_\epsilon^{2-p} \int_0^\infty t\mathcal{L} F_\epsilon^p\,dt,
\end{aligned}$$

where $\mathcal{L} = \partial_t^2 - L$ and $P^* f(x) = \sup_{t>0} |P^t f(x)|$. Observe that from the formula given in Lemma 7.2, $\mathcal{L} F_\epsilon^p \ge 0$. Then, by using Hölder's inequality with $q = 2/(2-p)$ and Lemma 7.1 applied to $F(t,x) = F_\epsilon^p$ (the verification of the hypotheses of Lemma 7.1 is given below), we get

$$\begin{aligned}
\|g(f)\|_{L^p(d\mu)}^p &\le C_p \int_{\mathcal{X}} [P^* f(x)]_\epsilon^{p(1-p/2)} \Big(\int_0^\infty t\mathcal{L} F_\epsilon^p\,dt\Big)^{p/2} d\mu(x) \\
&\le C_p \|[P^* f]_\epsilon\|_{L^p(d\mu)}^{p(1-p/2)} \Big(\int_{\mathcal{X}} \int_0^\infty t\mathcal{L} F_\epsilon^p\,dt\,d\mu(x)\Big)^{p/2} \\
&\le C_p \|[P^* f]_\epsilon\|_{L^p(d\mu)}^{p(1-p/2)} \Big(\int_{\mathcal{X}} ((f(x))^2 + \epsilon^2)^{p/2}\,d\mu(x)\Big)^{p/2}.
\end{aligned}$$

By applying Lebesgue's theorem on a set of finite measure as $\epsilon \to 0^+$ we have

$$\|g(f)\|_{L^p(d\mu)}^p \le C_p \|P^* f\|_{L^p(d\mu)}^{p(1-p/2)} \|f\|_{L^p(d\mu)}^{p^2/2} \le C_p \|f\|_{L^p(d\mu)}^p,$$

where we have used the L^p-boundedness of P^* that follows from [23, Maximal Theorem, p. 73]. Hypothesis I of the Maximal Theorem is **H3**(a), and hypothesis II follows from the spectral decomposition of the operator P^t.

Let us now check the hypotheses of Lemma 7.1 for $F(t,x) = F_\epsilon^p = ((P^t f(x))^2 + \epsilon^2)^{p/2}$ with $f = \sum_{|n|\le N} c_n \varphi_n$ and

$$P^t f(x) = \sum_{|n|\le N} e^{-\sqrt{\lambda_n}\,t} c_n \varphi_n(x).$$

Since $\sup_{t>0} |F(t,x)| \le (\sum_{|n|\le N} |c_n|\,|\varphi_n(x)| + \epsilon)^p$, hypothesis (1) of Lemma 7.1 is fulfilled from **H2**(a) and the finiteness of μ.

Taking into account that $\lambda_0 = 0$, we have

$$t|\partial_t F(t,x)| \le p\beta t e^{-\alpha t}\Big(\sum_{|n|\le N} |c_n|\,|\varphi_n(x)| + \epsilon\Big)^p =: S(t,x)$$

with $\alpha = \inf\{\sqrt{\lambda_n} : 0 < |n| \le N\} > 0$ and $\beta = \sup\{\sqrt{\lambda_n} : |n| \le N\}$. This function $S(t,x)$ satisfies hypothesis (4) of Lemma 7.1.

Since $\varphi_0^i(x_i) = c$ for all $x_i \in X$, we have $(\varphi_0^i)' = 0$ and so

$$\begin{aligned}
t|\partial_{x_i} F(t,x)| &\le pt\Big(\sum_{|n|\le N} |c_n|\,|\varphi_n(x)| + \epsilon\Big)^{p-1} |\partial_{x_i} P^t f| \\
&\le pt\Big(\sum_{|n|\le N} |c_n|\,|\varphi_n(x)| + \epsilon\Big)^{p-1} \sum_{\{|m|\le N: m_i\ge 1\}} e^{-\sqrt{\lambda_m}\,t} |c_m|\,|\partial_{x_i}\varphi_m| \\
&\le pte^{-\alpha t}\Big(\sum_{|n|\le N} |c_n|\,|\varphi_n(x)| + \epsilon\Big)^{p-1} \sum_{|m|\le N} |c_m|\,|\partial_{x_i}\varphi_m| \\
&=: \psi(t,x),
\end{aligned}$$

By applying Hölder's inequality with $s = p/(p-1)$ and $s' = p$ and taking into account **H2**(a) we obtain

$$\big\||\varphi_n|^{p-1}\partial_{x_i}\varphi_m\big\|_{L^1(\mathcal{X},d\mu)} \le \|\varphi_n\|^{p-1}_{L^p(\mathcal{X},d\mu)} \|\partial_{x_i}\varphi_m\|_{L^p(\mathcal{X},d\mu)}.$$

Thus $\psi(t,x)$ satisfies hypothesis (3) of Lemma 7.1.

Finally, hypothesis (2) of Lemma 7.1 corresponds to **H1**(a).

Case $p > 2$. By the semigroup properties it is easy to prove that for $f \in E$ we have, for all $i = 1, \dots, d$,

$$\partial_t P^t f(x) = 2P^{t/2}(\partial_t P^{t/2} f)(x), \tag{6.1}$$

$$\delta_i P^t f(x) = \tilde{P}_i^{t/2}(\delta_i P^{t/2} f)(x), \tag{6.2}$$

for all $x \in \mathcal{X}$ and $t > 0$. Now, by Corollary 7.4, for every nonnegative $h \in C_c^2(\mathcal{X})$,

$$\tilde{P}_i^t h(x) \le P^t h(x). \tag{6.3}$$

Let $f \in E$ and $h \in C_c^2(\mathcal{X})$ be nonnegative. Then

$$f = c_0\varphi_0 + f_1$$

with

$$\varphi_0 = 1, \qquad f_1 = \sum_{0<|n|\le N} c_n\varphi_n, \qquad c_0 = \int_{\mathcal{X}} f\,d\mu.$$

Thus, $\|c_0\varphi_0\|_p \le \|f\|_p$ and $\|f_1\|_p \le 2\|f\|_p$. Moreover for all $t > 0$ and $x \in \mathcal{X}$, $\nabla P^t f(x) = \nabla P^t f_1(x)$.

By (6.1), (6.2), **H3**(b), the symmetry of P^t and $\tilde{P}_i^t$, (6.3), a change of variables and Lemma 7.2 for $p = 2$ we get

$$\begin{aligned}
\int_{\mathcal{X}} |g(f)(x)|^2 h(x)\,d\mu(x) &= \int_{\mathcal{X}} |g(f_1)(x)|^2 h(x)\,d\mu(x)\\
&= \int_{\mathcal{X}}\int_0^\infty t|\nabla P^t f_1(x)|^2\,dt\,h(x)\,d\mu(x)\\
&= \int_{\mathcal{X}}\int_0^\infty t\Big[4|P^{t/2}(\partial_t P^{t/2} f_1)(x)|^2 + \sum_{i=1}^d |\tilde{P}_i^{t/2}(\delta_i P^{t/2} f_1)(x)|^2\Big]\,dt\,h(x)\,d\mu\\
&\le 4\int_{\mathcal{X}}\int_0^\infty t\Big[P^{t/2}([\partial_t P^{t/2} f_1]^2)(x) + \sum_{i=1}^d \tilde{P}_i^{t/2}([\delta_i P^{t/2} f_1]^2)(x)\Big]\,dt\,h(x)\,d\mu\\
&= 4\int_{\mathcal{X}}\int_0^\infty t\Big[[\partial_t P^{t/2} f_1(x)]^2 P^{t/2}h(x) + \sum_{i=1}^d [\delta_i P^{t/2} f_1(x)]^2 \tilde{P}_i^{t/2}h(x)\Big]\,dt\,d\mu\\
&\le 4\int_0^\infty t\int_{\mathcal{X}} |\nabla P^{t/2} f_1(x)|^2 P^{t/2}h(x)\,d\mu(x)\,dt\\
&= 16\int_0^\infty t\int_{\mathcal{X}} |\nabla P^t f_1(x)|^2 P^t h(x)\,d\mu(x)\,dt\\
&= 8\int_0^\infty t\int_{\mathcal{X}} \mathcal{L}[(P^t f_1(x))^2]P^t h(x)\,d\mu(x)\,dt.
\end{aligned}$$

Let us remark that for $F, G : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}$ smooth enough,

$$\mathcal{L}(FG) = \mathcal{L}(F)G + F\mathcal{L}(G) + 2\nabla F\cdot\nabla G. \tag{6.4}$$

Taking into account (6.4) and $\mathcal{L}P^t h(x) = 0$ we can write the above double integral as

$$\int_0^\infty t \int_{\mathcal{X}} \mathcal{L}[(P^t f_1(x))^2 P^t h(x)]\, d\mu(x)\, dt$$
$$- 2 \int_0^\infty t \int_{\mathcal{X}} \nabla (P^t f_1(x))^2 \cdot \nabla P^t h(x)\, d\mu(x)\, dt$$
$$= \int_0^\infty t \int_{\mathcal{X}} \mathcal{L}[(P^t f_1(x))^2 P^t h(x)]\, d\mu(x)\, dt$$
$$- 4 \int_0^\infty t \int_{\mathcal{X}} P^t f_1(x) \nabla P^t f_1(x) \cdot \nabla P^t h(x)\, d\mu(x)\, dt =: I - II.$$

Applying Schwarz's inequality to the second term we get

$$|II| \leq 4 \int_{\mathcal{X}} P^* f_1(x) \int_0^\infty t |\nabla P^t f(x)|\, |\nabla P^t h(x)|\, dt\, d\mu(x)$$
$$\leq 4 \int_{\mathcal{X}} P^* f_1(x)\, g(f)(x)\, g(h)(x)\, d\mu(x).$$

Thus, the integrand of II belongs to $L^1(\mathbb{R}_+ \times \mathcal{X}, dt d\mu)$ by using Hölder's inequality, the L^p-boundedness of P^*, Lemma 6.1 and the boundedness of g for $p < 2$. Lemma 7.2 for $p = 2$ gives

$$\int_0^\infty t \int_{\mathcal{X}} \mathcal{L}[(P^t f_1(x))^2] P^t h(x)\, d\mu(x)\, dt \leq 2\|h\|_\infty \|g(f)\|_2^2 < \infty \tag{6.5}$$

by Lemma 6.1 for $p = 2$. As a consequence, the integrand of I is also in $L^1(\mathbb{R}_+ \times \mathcal{X}, dt d\mu)$ and applying Lemma 7.1 to $F(t,x) = (P^t f_1)^2 P^t h$ (the verification of the remaining hypotheses of Lemma 7.1 is given at the end of this proof), we get

$$I \leq \int_{\mathcal{X}} (f_1(x))^2 h(x)\, d\mu(x).$$

Thus,

$$\int_{\mathcal{X}} |g(f)(x)|^2 h(x)\, d\mu(x) \leq 8 \int_{\mathcal{X}} (f_1(x))^2 h(x)\, d\mu(x)$$
$$+ 32 \int_{\mathcal{X}} P^* f_1(x) g(f)(x) g(h)(x)\, d\mu(x).$$

Assume that $p \geq 4$, $2/p + 1/q = 1$ and $\|h\|_q \leq 1$. By Hölder's inequality, L^p-boundedness of P^* and the current theorem for $q \leq 2$, we obtain

$$\|g(f)\|_p^2 \leq C_q(\|f_1\|_p^2 + \|g(f)\|_p \|f_1\|_p)$$
$$\leq 4C_q(\|f\|_p^2 + \|g(f)\|_p \|f\|_p).$$

Thus, taking into account that $\|g(f)\|_p < \infty$ (see Lemma 6.1), we get the corresponding estimate for $p \geq 4$. For the remaining p's we use Marcinkiewicz's Interpolation Theorem.

Let us check the remaining hypotheses of Lemma 7.1 when $F(t,x) = (P^t f_1(x))^2 P^t h(x)$.

For $f_1 = \sum_{0<|n|\leq N} c_n \varphi_n$, we have $P^t f_1(x) = \sum_{0<|n|\leq N} e^{-\sqrt{\lambda_n}\,t} c_n \varphi_n(x)$.

Taking into account the spectral decomposition of P^t and **H3**(a), we get

$$\sup_{t>0} |F(t,x)| \leq \Big(\sum_{|n|\leq N} |c_n|\,|\varphi_n(x)|\Big)^2 \|h\|_\infty;$$

thus hypothesis (1) of Lemma 7.1 is satisfied due to assumption **H2**.

From the inequalities

$$t|\partial_t (P^t f_1(x))^2| P^t h(x) \leq 2\beta e^{-\alpha t} \Big(\sum_{|n|\leq N} |c_n|\,|\varphi_n(x)|\Big)^2 \; \|h\|_\infty$$

with $\alpha = \inf\{\sqrt{\lambda_n} : 0 < |n| \leq N\} > 0$ and $\beta = \sup\{\sqrt{\lambda_n} : |n| \leq N\}$, and

$$t(P^t f_1(x))^2 |\partial_t P^t h(x)| \leq e^{-2\alpha t} \Big(\sum_{|n|\leq N} |c_n|\,|\varphi_n(x)|\Big)^2 t|\partial_t P^t h(x)|,$$

we obtain $t|\partial_t F(t,x)| \leq S(t,x)$ with

$$S(t,x) = e^{-\alpha t}(2\beta t\|h\|_\infty + t|\partial_t P^t h(x)|) \Big(\sum_{|n|\leq N} |c_n|\,|\varphi_n(x)|\Big)^2.$$

This S satisfies hypothesis (4) of Lemma 7.1 due to Lemma 7.5 (see (7.3)).

By similar computations for $\partial_{x_i} F$, but now using Lemma 7.5 (see (7.2)), we find that $t|\partial_{x_i} F(t,x)| \leq \psi(t,x)$ with

$$\psi(t,x) = 2te^{-\alpha t} \Big(\sum_{|n|\leq N} |c_n|\,|\varphi_n|\Big) \Big(\sum_{|n|\leq N} |c_n|\,|\partial_{x_i}\varphi_n|\Big) + \|\partial_{y_i} h\|_\infty,$$

which belongs to $L^1(\mathbb{R}_+ \times \mathcal{X})$. ∎

7. Appendix. In the following lemmas we assume that hypotheses **H1** through **H3** hold.

LEMMA 7.1. *Let $\mathcal{L} = \partial_t^2 - (L + h(x))$ with h a nonnegative continuous function on $\mathcal{X}$. Let $F : \mathbb{R}_+ \times \bar{\mathcal{X}} \to [0,\infty)$ be a $C^2(\mathbb{R}_+ \times \mathcal{X}) \cap C^1(\mathbb{R}_+ \times \bar{\mathcal{X}})$ function such that $\mathcal{L}F \geq 0$ on $\mathbb{R}_+ \times \mathcal{X}$ or $\int_0^\infty \int_{\mathcal{X}} t|\mathcal{L}F(t,x)|\,d\mu(x)\,dt < \infty$ and the limit $\lim_{t\to 0^+} F(t,x)$ exists for every $x \in \mathcal{X}$. For every $t > 0$ sufficiently large, define*

$$\mathcal{X}_t = X_t \times \cdots \times X_t \quad \text{with} \quad X_t = [(b + e^{-t}) \vee (-t), (c - e^{-t}) \wedge t] =: [b_t, c_t].$$

Assume that:

(1) $\sup_{t>0} |F(t,x)| \in L^1(\mathcal{X}, d\mu)$.
(2) *For every* $i = 1, \dots, d$, *either* $p_i^2(b_t)w_i(b_t)$ *and* $p_i^2(c_t)w_i(c_t) \to 0$ *as* $t \to \infty$; *or* $\lim_{t\to\infty} p_i^2(b_t)w_i(b_t) = \lim_{t\to\infty} p_i^2(c_t)w_i(c_t) \neq 0$, $F_{x_i}(t,b,x^i) = F_{x_i}(t,c,x^i)$ *for all* $x^i \in X^{d-1}$ *whenever* b *and* c *are finite; or* $t|F_{x_i}(t,x_i,x^i)| \le \psi_1(x_i)\psi_2(t,x^i)$ *with* $\psi_1(x_i) \to 0$ *as* $x_i \to b, c$ *and* $\psi_2 \in L^1(\mathbb{R}_+ \times \mathcal{X}^i, dt d\mu^i(x^i))$.
(3) *For every* $i = 1, \dots, d$, $t|F_{x_i}(t,x)| \lesssim \psi(t,x)$ *with* $\psi \in L^1(\mathbb{R}_+ \times \mathcal{X}, dt d\mu(x))$.
(4) $t|F_t(t,x)| \lesssim S(t,x)$ *for all* $t > 0$ *and* $x \in \mathcal{X}$ *where the function* S *is nonnegative, continuous, vanishes as* $t \to 0^+$ *and* $t \to \infty$, *and belongs to* $L^1(\mathbb{R}_+ \times \mathcal{X}, dt d\mu)$.

Then

$$\int_0^\infty \int_{\mathcal{X}} t\mathcal{L}F(t,x)\,d\mu(x)\,dt \le \int_{\mathcal{X}} F(0,x)\,d\mu(x), \tag{7.1}$$

where $F(0,x) := \lim_{t\to 0^+} F(t,x)$.

Proof. For $T > 0$ and sufficiently large, we will apply the divergence theorem on the domain $\mathcal{D}_T = \Lambda_T \times \mathcal{X}_T$ with $\Lambda_T = [e^{-T}, T]$. Observe that

$$t\mathcal{L}Fw = -F_t w + \operatorname{div}_{(t,x)} \mathbf{G} - th(x)Fw$$

with

$$\mathbf{G}(t,x) = (tF_t, tp_1^2(x_1)F_{x_1}, \dots, tp_d^2(x_d)F_{x_d})w(x).$$

Taking into account that $h(x) \ge 0$ on $\mathcal{X}$ we have

$$\begin{aligned}
&\iint_{\mathcal{D}_T} t\mathcal{L}F(t,x)\,d\mu(x)\,dt \\
&\qquad \le -\iint_{\mathcal{D}_T} \partial_t F(t,x)\,d\mu(x)\,dt + \iint_{\mathcal{D}_T} \operatorname{div}_{(t,x)} \mathbf{G}\,dx\,dt \\
&\qquad = \int_{\mathcal{X}_T} F(e^{-T},x)\,d\mu(x) - \int_{\mathcal{X}_T} F(T,x)\,d\mu(x) + \int_{\partial\mathcal{D}_T} \mathbf{G}\cdot\eta\,d\sigma \\
&\qquad \le \int_{\mathcal{X}} F(e^{-T},x)\,d\mu(x) + \int_{\partial\mathcal{D}_T} \mathbf{G}\cdot\eta\,d\sigma.
\end{aligned}$$

Either by Beppo-Levi's theorem or by Lebesgue's theorem, as $T \to \infty$, the left hand side above tends to the left hand side of (7.1), and by Lebesgue's theorem together with (1) the first term of the right hand side tends to the right hand side of (7.1). We should prove that $\int_{\partial\mathcal{D}_T} \mathbf{G}\cdot\eta\,d\sigma \to 0$ as $T \to \infty$. The two sides of the parallelepiped corresponding to time would be $t = e^{-T}$ and $t = T$, whose corresponding normal vectors are $\eta = (-1, \mathbf{0})$ and $(1, \mathbf{0})$ respectively. The two sides of the parallelepiped corresponding to the ith coordinate are $x_i = b_T$ and $x_i = c_T$, whose corresponding normal vectors are $\eta = (0, -\mathbf{e}_i)$ and $(0, \mathbf{e}_i)$ respectively. The integration on these sides leads to integrals

$$-e^{-T} \int_{\mathcal{X}_T} F_t(e^{-T}, x)\, d\mu(x) \quad \text{and} \quad T \int_{\mathcal{X}_T} F_t(T, x)\, d\mu(x),$$

which from (4) and Lebesgue's theorem tend to 0 as $T \to \infty$. On the other hand, the ith spatial integral is the difference

$$p_i^2(c_T) w_i(c_T) \int_{e^{-T}}^{T} \int_{\mathcal{X}_T^i} t F_{x_i}(t, c_T, x^i)\, d\mu^i(x^i)\, dt \\ - p_i^2(b_T) w_i(b_T) \int_{e^{-T}}^{T} \int_{\mathcal{X}_T^i} t F_{x_i}(t, b_T, x^i)\, d\mu^i(x^i)\, dt,$$

where the superscript i means that we consider all the variables but i. From (2) and (3), the difference of these integrals also tends to 0 as $T \to \infty$. ∎

Lemma 7.2. *Let $F : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}$ be a C^2 function such that $\mathcal{L}F = 0$ with $\mathcal{L} = \partial_t^2 - L$. For every $\epsilon > 0$, let $F_\epsilon = (F^2 + \epsilon^2)^{1/2}$. Then for every $1 \le p < \infty$,*

$$\mathcal{L}F_\epsilon^p = pF_\epsilon^{p-2} \frac{(p-1)F^2 + \epsilon^2}{F^2 + \epsilon^2} |\nabla F|^2,$$

where $\nabla = (\partial_t, \delta_1, \ldots, \delta_d)$. For $p = 2$,

$$\mathcal{L}F^2 = \mathcal{L}F_\epsilon^2 = 2|\nabla F|^2, \qquad \forall \epsilon > 0.$$

Proof. This follows by simple calculations. ∎

Lemma 7.3. *Assume that $[\delta_i, \delta_i^*] \ge 0$ on X. Let $f \in C_c(\mathcal{X})$ be a nonnegative function. For $v_1(t, x) = T^t f(x)$ and $v_2(t, x) = \widetilde{T}_i^t f(x)$ we have*

$$v_2(t, x) \le v_1(t, x) \quad \text{for all } (t, x) \in (0, \infty) \times \mathcal{X}.$$

Proof. First, let $f(x) = \prod_{j=1}^d f_j(x_j)$ with f_j in D_j dense in the space $L^2(X, w_j(x)dx)$ and $f_j \ge 0$. In this case,

$$\widetilde{T}_i^t f(x) = \prod_{j \ne i} \sum_{n_j=0}^{\infty} e^{-\lambda_{n_j}^j t} \langle f_j, \varphi_{n_j}^j \rangle \varphi_{n_j}^j(x_j) \sum_{n_i=1}^{\infty} e^{-\lambda_{n_i}^i t} \frac{\langle f_i, \delta_i \varphi_{n_i}^i \rangle}{\|\delta_i \varphi_{n_i}^i\|_2^2} \delta_i \varphi_{n_i}^i(x_i) \\ = u_2(t, x_i) \prod_{j \ne i} \sum_{n_j=0}^{\infty} e^{-\lambda_{n_j}^j t} \langle f_j, \varphi_{n_j}^j \rangle \varphi_{n_j}^j(x_j),$$

and

$$T^t f(x) = \prod_{j\neq i} \sum_{n_j=0}^{\infty} e^{-\lambda^j_{n_j} t} \langle f_j, \varphi^j_{n_j}\rangle \varphi^j_{n_j}(x_j) \sum_{n_i=0}^{\infty} e^{-\lambda^i_{n_i} t} \langle f_i, \varphi^i_{n_i}\rangle \varphi^i_{n_i}(x_i)$$
$$= u_1(t, x_i) \prod_{j\neq i} \sum_{n_j=0}^{\infty} e^{-\lambda^j_{n_j} t} \langle f_j, \varphi^j_{n_j}\rangle \varphi^j_{n_j}(x_j),$$

where u_1 and u_2 were defined in **H3**(c). Now, we will apply [5, Theorem 1, p. 411] to the operator $\tilde{L} = -\partial_t - L_i = -\partial_t - \delta_i^*\delta_i$ and the function $u(t, x_i) = u_2(t, x_i) - u_1(t, x_i)$.

Let us check that the hypotheses of [5, Theorem 1, p. 411] are satisfied. General conditions I and II from [5] correspond to **H1**(b). Since $\tilde{L}u_1 = 0$ and $(\partial_t + \delta_i\delta_i^*)u_2 = 0$ on $X \times (0,\infty)$, we have $\tilde{L}u = [\delta_i, \delta_i^*]u_2 \geq 0$ on $(0,\infty) \times X$, by **H1**(c) and **H3**(a). On the other hand, $u(0, x_i) = 0$ for $x_i \in X$ because $\{\varphi^i_{n_j}\}_{n_j}$ and $\{\delta_i\varphi^i_{n_j}\}_{n_j}$ are bases in $L^2(X, w_i(x_i)dx_i)$; $u \leq 0$ on $(0, T) \times \partial X$ for all $T > 0$ by **H3**(c); and condition (2.5) of [5, Theorem 1, p. 411] follows since these semigroups are bounded by **H3**(a). Therefore by [5, Theorem 1, p. 411], $u(t, x_i) \leq 0$ on $(0, T) \times X$ for every $T > 0$. Thus $u_2(t, x_i) \leq u_1(t, x_i)$ for all $x_i \in X$ and $t > 0$. Since

$$\prod_{j\neq i} \sum_{n_j=0}^{\infty} e^{-\lambda^j_{n_j} t} \langle f_j, \varphi^j_{n_j}\rangle \varphi^j_{n_j}(x_j) \geq 0$$

as a consequence of **H3**(a) on X^{d-1} it follows that

$$v_2(t, x) \leq v_1(t, x) \quad \text{for all } (t, x) \in (0,\infty) \times \mathcal{X}$$

for $f(x) = \prod_{j=1}^d f_j(x_j)$ with f_j in D_j dense in $L^2(X, w_j(x)dx)$ and $f_j \geq 0$.

Now, let $f \in C_c(\mathcal{X})$ be nonnegative. Then there exists $(f_n) \subset D = \operatorname{span}\{f(x) = \prod_{j=1}^d f_j(x_j) \text{ with } f_j \in D_j\} \cap \{f \geq 0\}$ such that $\|f_n - f\|_2 \to 0$, $\|T^t f_n - T^t f\|_2 \to 0$ and $\|\widetilde{T}_i^t f_n - \widetilde{T}_i^t f\|_2 \to 0$ as $n \to \infty$, and $\widetilde{T}_i^t f_n(x) \leq T^t f_n(x)$ for all n. Hence there is a subsequence of $(T^t f_n)$ and $(\widetilde{T}_i^t f_n)$ that converges pointwise to $T^t f$ and $\widetilde{T}_i^t f$ respectively for almost every $x \in \mathcal{X}$. Thus $\widetilde{T}_i^t f(x) \leq T^t f(x)$ for almost every $x \in \mathcal{X}$. But since these semigroups are continuous on $\mathcal{X}$, the inequality follows for every $x \in \mathcal{X}$. ∎

COROLLARY 7.4. *Assume that* $[\delta_i, \delta_i^*] \geq 0$ *on* X. *Let* $f \in C_c(\mathcal{X})$ *be a nonnegative function. Then*

$$\widetilde{P}_i^t f(x) \leq P^t f(x) \quad \textit{for every } (x, t) \in \mathcal{X} \times (0,\infty).$$

Proof. This follows from the definitions of P^t, $\widetilde{P}_i^t$ and Lemma 7.3. ∎

LEMMA 7.5. *For any* $f \in C_c^2(\mathcal{X})$,

$$|\partial_{x_i} P^t f(x)| \leq C\|\partial_{y_i} f\|_\infty, \tag{7.2}$$

and

$$t|\partial_t P^t f(x)| \le \big\|\phi + s|\phi'|\big\|_{L^1((0,\infty),ds)} \|f\|_\infty, \tag{7.3}$$
$$\lim_{t\to 0^+} |t\partial_t P^t f(x)| = 0.$$

Proof. From **H2**(a), regarding the elliptic differential operator N_i it is easy to see that $\partial_{x_i} T^t \varphi_n = T_i^t \partial_{y_i}\varphi_n$. This is also true for every $f \in C_c^2(\mathcal{X})$ under the assumption **H2**(b). In fact,

$$\begin{aligned}
\partial_{x_i} T^t f(x) &= \partial_{x_i} \sum_{n\in\mathbb{N}_0^d} \langle f, \varphi_n\rangle_{L^2(\mathcal{X},d\mu)} T^t \varphi_n \\
&= \sum_{n\in\mathbb{N}_0^d} \langle f, \varphi_n\rangle_{L^2(\mathcal{X},d\mu)} \partial_{x_i} T^t \varphi_n \\
&= \sum_{n\in\mathbb{N}_0^d,\, n_i>0} \langle f, \varphi_n\rangle_{L^2(\mathcal{X},d\mu)} T_i^t(\partial_{x_i}\varphi_n) \\
&= \sum_{n\in\mathbb{N}_0^d,\, n_i>0} \frac{\langle \partial_{x_i} f, \partial_{x_i}\varphi_n\rangle_{L^2(\mathcal{X},d\nu_i)}}{\|\partial_{x_i}\varphi_n\|^2_{L^2(\mathcal{X},d\nu_i)}} T_i^t(\partial_{x_i}\varphi_n) = T_i^t(\partial_{x_i} f)(x).
\end{aligned}$$

Here we have used $\langle \partial_{x_i} f, \partial_{x_i}\varphi_n\rangle_{L^2(\mathcal{X},d\nu_i)} = \langle \delta_i f, \delta_i\varphi_n\rangle_\mu = \lambda^i_{n_i}\langle f, \varphi_n\rangle_\mu$, with $\|\partial_{x_i}\varphi_n\|^2_{L^2(\mathcal{X},d\nu_i)} = \|\delta_i\varphi_n\|^2_{L^2(\mathcal{X},d\mu)} = \lambda^i_{n_i}$.

Now, by definition of P^t,

$$\begin{aligned}
|\partial_{x_i} P^t f(x)| &= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \partial_{x_i} T^{t^2/4u} f(x)\, du \right| \\
&= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_i^{t^2/4u} (\partial_{y_i} f)(x)\, du \right| \\
&\le C \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, du\, \|\partial_{y_i} f\|_\infty = C\|\partial_{y_i} f\|_\infty.
\end{aligned}$$

This proves inequality (7.2).

On the other hand, (7.3) is derived from the formula

$$\begin{aligned}
t\partial_t P^t f(x) &= \int_0^\infty \left[\frac{1}{t^2}\phi(s/t^2) + \frac{1}{t^2}\,\frac{s}{t^2}\phi'(s/t^2)\right] T^s f(x)\, ds \\
&= \int_0^\infty (\phi(s) + s\phi'(s)) T^{st^2} f(x)\, ds,
\end{aligned}$$

which follows from the spectral decomposition of the Poisson semigroup P^t given in **H3**, and taking into account that $\phi + s\phi' \in L^1(\mathbb{R}_+, ds)$ and $\int_0^\infty (\phi(s) + s\phi'(s))\, ds = 0$ together with Lebesgue's theorem. ∎

Lemma 7.6. *Let μ be a positive Radon measure on X and suppose that there exists $\delta > 0$ such that*

$$\int_X e^{\delta|x|}\,d\mu(x) < \infty.$$

Then the set of polynomials restricted to X is dense in $L^p(X, d\mu)$ for $1 \le p < \infty$.

For the proof see [1, Theorem 6] where $\mathbb{R}$ has to be replaced by X and the reasoning follows the same lines.

8. Examples. In this section we apply the previous results to some classical polynomial settings.

Trigonometric polynomial expansions. For this setting,

$$\begin{aligned}
&X = (-\pi, \pi), && \mathcal{X} = (-\pi, \pi)^d,\\
&L_i = -\partial_{x_i}^2, && L = -\Delta,\\
&p_i(x_i) = 1, &&\\
&\delta_i = \partial_{x_i}, &&\\
&\delta_i^* = -\delta_i, &&\\
&w_i(x_i) = \tfrac{1}{2\pi}, && w(x) = \tfrac{1}{(2\pi)^d}.
\end{aligned}$$

We now verify the hypotheses.

H1:

(a) $p_i^2(-\pi)w_i(-\pi) = p_i^2(\pi)w_i(\pi) = 1$, but $\varphi_{n_i}^i(-\pi) = \varphi_{n_i}^i(\pi)$ and $(\varphi_{n_i}^i)'(-\pi) = (\varphi_{n_i}^i)'(\pi)$.
(c) Choose $\phi(\tau) = 1$ and thus $\Theta(\tau) = \tau$.
(d) $[\delta_i, \delta_i^*] = 0$.

H2:

(a) Eigenfunctions:

$$\varphi_n^i(x_i) = \begin{cases} 1, & n = 0,\\ \sqrt{2}\sin kx_i, & n = 2k-1,\\ \sqrt{2}\cos kx_i, & n = 2k, \end{cases} \qquad \varphi_n(x) = \prod_{i=1}^d \varphi_{n_i}^i(x_i),$$

$$\lambda_n^i = \lfloor (n+1)/2 \rfloor^2,\ n = 0, 1, \ldots, \qquad \lambda_n = \sum_{i=1}^d \lfloor (n_i+1)/2 \rfloor^2,$$
$$n = (n_1, \ldots, n_d).$$

Since

$$\delta_i\varphi_n = \begin{cases} \lfloor (n_i+1)/2 \rfloor \varphi_{n+e_i} & \text{for } n_i \text{ odd,} \\ \lfloor (n_i+1)/2 \rfloor \varphi_{n-e_i} & \text{for } n_i \text{ even,} \end{cases}$$

we have $F_i = G_i = E \subset L^p(\mathcal{X}, d\mu)$.

(b) can be found in [20].

H3:

(a) is a consequence of $T^t 1 = 1$ and Hölder's inequality.
(b) Use the Cauchy–Schwarz inequality.
(c) Since $N_i = L$, $T_i^t = T^t$ is bounded on L^∞.

In this setting, the results about the L^p-boundedness of the Riesz transforms for $1 < p < \infty$ are classical and can be found in [23].

Hermite polynomial expansions. For this setting,

$$\begin{aligned}
&X = \mathbb{R}, && \mathcal{X} = \mathbb{R}^d, \\
&L_i = -\partial_{x_i}^2 + 2x_i\partial_{x_i}, && L = -\varDelta + 2x\cdot\nabla, \\
&p_i(x_i) = 1, && \\
&\delta_i = \partial_{x_i}, && \\
&\delta_i^* = -\partial_{x_i} + 2x_i, && \\
&w_i(x_i) = \frac{e^{-x_i^2}}{\sqrt{\pi}}, && w(x) = \frac{e^{-|x|^2}}{(\sqrt{\pi})^d}.
\end{aligned}$$

H1:

(a) $p_i^2(b_t)w_i(b_t) = e^{-(-t)^2}$ and $p_i^2(c_t)w_i(c_t) = e^{-t^2}$ both tend to 0 as $t \to \infty$.
(b) Choose $\phi(\tau) = \sqrt{\tau}$ and thus $\varTheta(\tau) = 2\sqrt{\tau}$.
(c) $[\delta_i, \delta_i^*] = 2 > 0$.

H2:

(a) Eigenfunctions:

$$\begin{aligned}
&H_n(x_i) = (-1)^n e^{x_i^2}\frac{d^n}{dx_i^n}(e^{-x_i^2}), && \\
&\|H_n\|_2^2 = n!2^n, && \\
&\varphi_n^i(x_i) = \frac{H_n(x_i)}{\|H_n\|_2}, && \varphi_n(x) = \prod_{i=1}^d \varphi_{n_i}^i(x_i), \\
&\lambda_n^i = 2n,\ n = 0,1,\ldots, && \lambda_n = 2|n|,\ n = (n_1,\ldots,n_d), \\
& && |n| = n_1 + \cdots + n_d.
\end{aligned}$$

Since $\int_{-\infty}^{\infty} e^{\delta|x|}e^{-x^2}\,dx < \infty$ for all δ, by Lemma 7.6 the set of one-dimensional polynomials is dense in $L^p(X)$ for $1 < p < \infty$ and every polynomial is a linear combination of Hermite polynomials. Thus E is dense in $L^p(\mathcal{X}, d\mu)$. Moreover, since $\delta_i\varphi_n = -\sqrt{2n_i}\,\varphi_{n-e_i}$, we have $F_i = G_i = E \subset L^p(\mathcal{X}, d\mu)$.

(b) can be found in [20].

H3:

(a) is a consequence of $T^t 1 = 1$ and Hölder's inequality.
(b) Use the Cauchy–Schwarz inequality.
(c) is not necessary to check because $b = -\infty$ and $c = \infty$.
(d) Since $N_i = L + 2$, $T_i^t = e^{-2t}T^t$ is bounded on L^∞.

In this setting, the results about the L^p-boundedness, $1 < p < \infty$, for first and higher order Riesz transforms with respect to the Gaussian measure can be found e.g. in [12, 6, 21, 25, 7, 9].

Laguerre polynomial expansions

$X = (0,\infty) = \mathbb{R}_+,$ $\qquad \mathcal{X} = \mathbb{R}_+^d,$

$$L_i = L_i^{\alpha_i} = -x_i\partial_{x_i}^2 - (\alpha_i+1-x_i)\partial_{x_i}, \qquad L = L^\alpha = -\sum_{i=1}^d (x_i\partial_{x_i}^2 + (\alpha_i+1-x_i)\partial_{x_i}),$$

$\alpha = (\alpha_1, \dots, \alpha_d),$

$\alpha_i > -1,$

$p_i(x_i) = \sqrt{x_i},$

$\delta_i = \sqrt{x_i}\,\partial_{x_i},$

$$\delta_i^* = -\sqrt{x_i}\,\partial_{x_i} - \frac{\alpha_i+1/2}{\sqrt{x_i}} + \sqrt{x_i},$$

$$w_i(x_i) = \frac{x_i^{\alpha_i}e^{-x_i}}{\Gamma(\alpha_i+1)}, \qquad w(x) = \prod_{i=1}^d \frac{x_i^{\alpha_i}e^{-x_i}}{\Gamma(\alpha_i+1)}.$$

H1:

(a) $p_i^2(b_t)w_i(b_t) = (e^{-t})^{\alpha_i+1}e^{-e^{-t}}$ and $p_i^2(c_t)w_i(c_t) = t^{\alpha_i+1}e^{-t}$ both tend to 0 as $t \to \infty$.

(b) Choose $\phi(\tau) = \sqrt{\tau}$ and thus $\Theta(\tau) = 2\sqrt{\tau}$.

(c) $[\delta_i, \delta_i^*] = \frac{\alpha_i+1/2+x_i}{2x_i} \ge 0$ if and only if $\alpha_i \ge -1/2$.

H2:

(a) Eigenfunctions:

$$L_n^{\alpha_i}(x_i) = \frac{(-1)^n}{n!}x_i^{-\alpha_i}e^{x_i}\frac{d^n}{dx_i^n}(x_i^{\alpha_i+n}e^{-x_i}),$$

$$\|L_n^{\alpha_i}\|_2^2 = \frac{\Gamma(\alpha_i+n+1)}{\Gamma(\alpha_i+1)\Gamma(n+1)},$$

$$\varphi_n^i(x_i) = \varphi_n^{\alpha_i}(x_i) = \frac{L_n^{\alpha_i}(x_i)}{\|L_n^{\alpha_i}\|_2}, \qquad \varphi_n^{\alpha}(x) = \prod_{i=1}^d \varphi_{n_i}^{\alpha_i}(x_i),$$

$$\lambda_n^i = n,\ n = 0,1,\ldots, \qquad \lambda_n = |n|,\ n = (n_1,\ldots,n_d),$$

$$|n| = n_1 + \cdots + n_d.$$

Since $\int_0^\infty e^{\delta x} x^\alpha e^{-x}\,dx < \infty$ as long as $\delta < 1$, by applying Lemma 7.6, the set of one-dimensional polynomials is dense in $L^p(X)$ for $1 < p < \infty$ and every polynomial is a linear combination of Laguerre polynomials. Hence E is dense in $L^p(\mathcal{X}, d\mu)$. Since the derivative of a polynomial is another polynomial, we have $G_i = E$. On the other hand, observe that

$$\delta_i \varphi_n^\alpha(x) = -\sqrt{n_i}\sqrt{x_i}\,\varphi_{n-e_i}^{\alpha+e_i}(x) \in L^p(\mathcal{X}, d\mu),$$

thus $F_i \subset L^p(\mathcal{X}, d\mu)$.

(b) can be found in [20].

H3:

(a) is a consequence of $T^t 1 = 1$ and Hölder's inequality.

(b) Use the Cauchy–Schwarz inequality.

(c) We have to analyze $b = 0$ since $c = \infty$. In this case $u_2(t,0) = 0 \le u_1(t,0)$ for all $t > 0$.

(d) Since $N_i = L^{\alpha+e_i} + 1$, $T_i^t = e^{-t} T_{\alpha+e_i}^t$ is bounded on L^∞.

In the Laguerre setting, the L^p-boundedness of the Riesz transforms, $1 < p < \infty$, with respect to the Laguerre measure $d\mu_\alpha$, with $\alpha_i \ge -1/2$, is obtained in [16]. In our case, the restriction on the parameter α_i is a consequence of the nonnegativity of the associated commutator $[\delta_i, \delta_i^*]$.

Jacobi polynomial expansions. In this case, we have

$$X = (-1,1), \qquad \mathcal{X} = (-1,1)^d,$$

$$L_i = L_i^{\alpha_i,\beta_i} = -(1-x_i^2)\partial_{x_i}^2 - (\beta_i - \alpha_i - (\alpha_i+\beta_i+2)x_i)\partial_{x_i},$$

$$L = L^{\alpha,\beta} = -\sum_{i=1}^d ((1-x_i^2)\partial_{x_i}^2 + (\beta_i - \alpha_i - (\alpha_i+\beta_i+2)x_i)\partial_{x_i}),$$

$$\alpha_i, \beta_i > -1, \qquad \alpha = (\alpha_1,\ldots,\alpha_d),\ \beta = (\beta_i,\ldots,\beta_d),$$

$$p_i(x_i) = \sqrt{1-x_i^2},$$

$$\delta_i = \sqrt{1-x_i^2}\,\partial_{x_i},$$

$$\delta_i^* = -\sqrt{1-x_i^2}\,\partial_{x_i} + (\alpha_i+1/2)\sqrt{\frac{1+x_i}{1-x_i}} \\ -(\beta_i+1/2)\sqrt{\frac{1-x_i}{1+x_i}},$$

$$w_i(x_i) = \frac{(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}}{C_{\alpha_i,\beta_i}}, \qquad w(x) = \prod_{i=1}^d w_i(x_i)$$

with

$$C_{\alpha_i,\beta_i} := \frac{2^{\alpha_i+\beta_i+1}\Gamma(\alpha_i+1)\Gamma(\beta_i+1)}{\Gamma(\alpha_i+\beta_i+2)}.$$

H1:

(a) We can see that $p_i^2(b_t)w_i(b_t) = (2-e^{-t})^{\alpha_i+1}(e^{-t})^{\beta_i+1}$ and $p_i^2(c_t)w_i(c_t) = (e^{-t})^{\alpha_i+1}(2-e^{-t})^{\beta_i+1}$ both tend to 0 as $t\to\infty$.
(b) Choose $\phi(\tau)=\sqrt{\tau}$ and thus $\Theta(\tau)=2\sqrt{\tau}$.
(c) $[\delta_i,\delta_i^*] = \frac{\alpha_i+1/2}{1-x_i} + \frac{\beta_i+1/2}{1+x_i} \geq 0$ if and only if $\alpha_i,\beta_i \geq -1/2$.

H2:

(a) Eigenfunctions:

$$J_n^{\alpha_i,\beta_i}(x_i) = \frac{(-1)^n}{2^n n!}(1-x_i)^{-\alpha_i}(1+x_i)^{\beta_i} \\ \times\frac{d^n}{dx_i^n}((1-x_i)^{\alpha_i+n}(1+x_i)^{\beta_i+n}),$$

$$\|J_n^{\alpha_i,\beta_i}\|_2^2 = \frac{\Gamma(\alpha_i+\beta_i+2)}{\Gamma(\alpha_i+1)\Gamma(\beta_i+1)} \\ \times\frac{\Gamma(\alpha_i+n+1)\Gamma(\beta_i+n+1)}{(\alpha_1+\beta_i+2n+1)\Gamma(\alpha_1+\beta_i+n+1)\Gamma(n+1)},$$

$$\varphi_n^i(x_i) = \varphi_n^{\alpha_i,\beta_i}(x_i) = \frac{J_n^{\alpha_i,\beta_i}(x_i)}{\|J_n^{\alpha_i,\beta_i}\|_2}, \qquad \varphi_n^{\alpha,\beta}(x) = \prod_{i=1}^d \varphi_{n_i}^{\alpha_i}(x_i),$$

$$\lambda_n^i = n(n+\alpha_i+\beta_i+1), \qquad \lambda_n = \lambda_n^{\alpha,\beta} = \sum_{i=1}^d (n_i(n_i+\alpha_i+\beta_i+1)),$$

$$n=0,1,\ldots, \qquad n=(n_1,\ldots,n_d).$$

Since $\int_{-1}^1 e^{\delta|x|}(1-x)^\alpha(1+x)^\beta\,dx < \infty$ for all δ, by Lemma 7.6 the set of one-dimensional polynomials is dense in $L^p(X)$ for $1<p<\infty$ and every polynomial is a linear combination of Jacobi polynomials. Hence E is dense in $L^p(\mathcal{X},d\mu)$. Since the derivative of a polynomial is another polynomial, we have $G_i = E$. On the other hand, observe that

$$\delta_i \varphi_n^\alpha(x) = -\sqrt{\lambda_{n_i}^i}\sqrt{1-x_i^2}\,\varphi_{n-e_i}^{\alpha+e_i,\beta+e_i}(x) \in L^p(\mathcal{X}, d\mu),$$

thus $F_i \subset L^p(\mathcal{X}, d\mu)$.

(b) can be found in [20].

H3:

(a) is a consequence of $T^t 1 = 1$ and Hölder's inequality.

(b) Use the Cauchy–Schwarz inequality.

(c) We have to analyze $b = -1$ and $c = 1$. For $b = -1$, $u_2(t, -1) = 0 \leq u_1(t, -1)$ for all $t > 0$. As for $c = 1$, we have a similar inequality since $u_2(t, 1) = 0$.

(d) Since $N_i = L^{\alpha+e_i,\beta+e_i} + \alpha_i + \beta_i + 2$, $T_i^t = e^{-(\alpha_i+\beta_i+2)t} T_{\alpha+e_i,\beta+e_i}^t$ is bounded on L^∞.

For the Jacobi polynomial expansions, P. Sjögren and A. Nowak [17, 18] considered expansions based on multi-dimensional Jacobi polynomials and studied the L^p-boundedness of the Riesz transforms, $1 < p < \infty$, with respect to the measure $dw_{\alpha,\beta}(x)$ on $(-1,1)^d$ and $\alpha_i, \beta_i > -1$ for each i. In particular, under a slight restriction on the type parameters ($\alpha_i, \beta_i \geq -1/2$ for each i), they prove that these operators are bounded in L^p, $1 < p < \infty$, with constants independent of dimension. We find the same restrictions on the parameters and again they are a consequence of the nonnegativity of the commutator $[\delta_i, \delta_i^*]$. Recently, Langowski [10] followed the symmetrization procedure by considering the setting of symmetrized Jacobi expansions. In particular, he also obtained some new results in the original setting of classical Jacobi expansions.

Remark 8.1. On the other hand, Stempak and Wróbel [24] have also used the same technique in a context of function expansions. In particular, they have proved dimension free L^p-estimates for Riesz transforms associated with multi-dimensional Laguerre function expansions of Hermite type, but the range of the admissible Laguerre type multi-index in these estimates depends on p. For $1 < p \leq 2$ this range is almost optimal. This is a starting point to try to use a similar technique in order to analyze a general context of orthogonal function expansions.

Remark 8.2. Recently, Wróbel [26] derived a scheme to deduce the L^p-boundedness of certain d-dimensional Riesz transforms from the L^p-boundedness of appropriate one-dimensional Riesz transforms, by using an H^∞ joint functional calculus for strongly commuting operators. Since the L^p-bounds are all independent of the dimension, we see that when our hypotheses are satisfied we have the L^p-boundedness of d-dimensional Riesz transforms, with dimension free constants.

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