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ON THE IMPLICIT FUNCTION THEOREM IN O-MINIMAL STRUCTURES

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Abstract. A local-global version of the implicit function theorem in o-minimal structures and a generalization of the theorem of Wilkie on covering open sets by open cells are proven.

Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].)

We will prove the following local-global version of the implicit function theorem

THEOREM 1. *Let $p \in \{1, 2, \dots\} \cup \{\infty\}$. Let Ω be a definable open subset of R^{n+m} and let $F = (F_1, \dots, F_m) : \Omega \longrightarrow R^m$ be a definable $\mathcal{C}^p$-mapping such that*

$$\frac{\partial(F_1, \dots, F_m)}{\partial(x_{n+1}, \dots, x_{n+m})} \neq 0 \quad \text{in } \Omega.$$

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Then there exists a finite family $f_i : C_i \longrightarrow R^m$ $(i \in \{1, \dots, s\})$ *of definable* $\mathcal{C}^p$*-mappings defined on definable open subsets* C_i *of* R^n *such that for each* i

$$F(x_1, \dots, x_n, f_i(x_1, \dots, x_n)) = 0 \quad \text{on} \quad C_i \quad \text{and} \quad \bigcup_{i=1}^{s} f_i = F^{-1}(0).$$

(*We adopt the convention to identify mappings with their graphs.*)

Remark 2. When Ω is bounded, it follows from a theorem of Wilkie [W] that in the above theorem each of C_i can be assumed to be an open definable cell in R^n.

By the classical implicit function theorem, the above theorem is an immediate corollary to the following

Proposition 3. *Let* M *be a definable* $\mathcal{C}^p$*-submanifold of* R^{n+m} *of dimension* n. *Let* $\pi : R^{n+m} = R^n \times R^m \longrightarrow R^n$ *be the natural projection. Assume that* $\pi|M$ *is a local diffeomorphism.*

Then there exists a finite family $A_1, \dots, A_s$ *of open definable subsets of* M *such that* $M = A_1 \cup \ldots \cup A_s$ *and, for each* $i \in \{1, \dots, s\}$, $\pi|A_i$ *is a* $\mathcal{C}^p$*-diffeomorphism* (*onto an open subset in* R^n).

Proposition 3 is a consequence of the following, much more general, elementary fact.

Proposition 4. *Let* E *be any definable subset of* R^{n+m}. *Let* $\pi : R^{n+m} = R^n \times R^m \longrightarrow R^n$ *be the natural projection. Assume that* $\pi|E$ *is locally injective.*

Then there exists a finite family $A_1, \dots, A_s$ *of open definable subsets of* E *such that* $E = A_1 \cup \ldots \cup A_s$ *and, for each* $i \in \{1, \dots, s\}$, $\pi|A_i$ *is injective.*

As an application of Proposition 3, we will prove the following generalization of the above mentioned theorem of Wilkie.

Proposition 5. *Any definable bounded* $\mathcal{C}^p$*-submanifold* M *of* R^{n+m} *of dimension* n *can be represented as a finite union* $M = C_1 \cup \ldots \cup C_s$, *where each of* C_i *is, after perhaps a permutation of coordinates in* R^{n+m}, *a definable* n*-dimensional cell in* R^{n+m}.

Proof of Proposition 4. We will prove by induction on $k \in \{0, \dots, n\}$ that if C is a definable subset of E of dimension k, then there exists a finite family $A_1, \dots, A_s$ of open definable subsets of E such that $C \subset A_1 \cup \ldots \cup A_s$ and each of $\pi|A_i$ $(i \in \{1, \dots, s\})$ is injective.

By using a cell decomposition we can assume without any loss of generality that C is a k-dimensional cell. Then $\pi|C$ is injective and $\pi(C)$ is a k-dimensional cell in R^n. After perhaps a permutation of coordinates in R^n, one can assume that

$$\pi(C) = \{(x_1, \dots, x_n) : (x_1, \dots, x_k) \in \Omega,\ x_j = \varphi_j(x_1, \dots, x_k)\ (j = k+1, \dots, n)\}$$

and

$$C = \{(x_1, \dots, x_{n+m}) : (x_1, \dots, x_k) \in \Omega,\ x_j = \varphi_j(x_1, \dots, x_k)\ (j = k+1, \dots, n+m)\},$$

where Ω is a definable open subset of R^k and $\varphi_j : \Omega \longrightarrow R$ $(j = k+1, \dots, n+m)$ are definable continuous functions.

For each $u = (u_1, \dots, u_k) \in \Omega$ and $\varepsilon > 0$, set

$$\Theta(u,\varepsilon) := \{(x_1,\dots,x_{n+m}) \in E : u = (x_1,\dots,x_k),\ |x_j - \varphi_j(u)| < \varepsilon\ (j = k+1,\dots,n+m)\}.$$

For each $u \in \Omega$, set $r(u) := \sup\{\varepsilon \in (0,1] : \pi|\Theta(u,\varepsilon) \text{ is injective}\}$. By the assumption of local injectivity r is well-defined and it is easy to check that r is definable. There exists a closed definable subset Z of Ω of dimension $< k$ such that $r|\Omega \setminus Z$ is continuous. It is clear that π is injective when restricted to the set

$$\bigcup_{u\in\Omega\setminus Z} \Theta(u, r(u)) = \{(x_1,\dots,x_{n+m}) \in E : u = (x_1,\dots,x_k) \in \Omega \setminus Z,\ |x_j - \varphi_j(u)| < r(u)\ (j = k+1,\dots,n+m)\},$$

which is an open definable neighborhood of $C|\Omega \setminus Z$ in E. Now, to finish the proof it suffices to apply the induction hypothesis to $C|Z$. ∎

Proof of Proposition 5.

Let $\tau : M \ni x \longmapsto T_xM \in \mathbb{G}_n(R^{n+m})$ be the Gauss map, $\{e_j\}$ $(j \in \{1,\dots,n+m\})$ — the canonical basis of R^{n+m} and let

$$V_\alpha := \Big\{L \in \mathbb{G}_n(R^{n+m}) : L \cap \Big(\sum_{i=1}^{m} Re_{\alpha_i}\Big) = \{0\}\Big\},$$

where $\alpha = (\alpha_1,\dots,\alpha_m)$, $1 \leqslant \alpha_1 < \dots < \alpha_m \leqslant n+m$. Since τ is definable, $\{\tau^{-1}(V_\alpha)\}$ is an open definable covering of M. This reduces the general case, after perhaps a permutation of coordinates, to that from Proposition 3. Hence, Proposition 5 follows from Proposition 3 and Remark 2. ∎

REMARK 6. The assumption in Proposition 5 that M is bounded is in fact superfluous.

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