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POLYNOMIAL INEQUALITIES IN BANACH SPACES

MIROSŁAW BARAN
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland
E-mail: miroslaw.baran.tarnow@gmail.com

Abstract. We point out relations between the injective complexification of a real Banach space and polynomial inequalities. In particular we prove a generalization of a classical Szegő inequality to the case of polynomial mappings between Banach spaces. As an application we observe a complex version of known Bernstein–Szegő type inequalities.

Introduction. The main goal of this paper is to prove a generalization of the classical Bernstein–Markov and Szegő inequalities for one variable polynomials

$$|Q'(t)| \le (\deg Q)(1-t^2)^{-1/2}\|Q\|_{[-1,1]}, \quad t \in (-1,1),$$
$$|Q'(t)| \le (\deg Q)(1-t^2)^{-1/2}(\|Q\|_{[-1,1]}^2 - Q^2(t))^{1/2}, \quad t \in (-1,1).$$

The first inequality holds for arbitrary polynomials while the second one is true for polynomials with real coefficients.

Some generalizations of the above inequalities for many variables or for Banach spaces have been found (see [B1]–[B5], [HA1, HA2], [KE], [M-S], [SAR]). Let us mention here the beautiful Harris inequality for a Hilbert space X:

$$|d_xQ(v)| \le (\deg Q)\Big(\|v\|^2 + \frac{(x\cdot v)^2}{1-\|x\|^2}\Big)^{1/2}\|Q\|, \quad x \in \operatorname{int}(B),\ v \in X,$$

where B is the unit closed ball in X and $x \cdot v$ denotes the scalar product of x and v in X.

One of the main results of this paper is a generalization of this inequality in the case X is an arbitrary real Banach space. The scalar product in such a situation is replaced by some *pseudo inner product* that is defined by using notion of *the injective norm*. This norm is a norm for so called injective complexification of real Banach space. This is one of main tools in this paper. A little surprising by-product is obtaining a tool for investigations of geometry of Banach spaces. Let us mention that complexifications of real Banach spaces were considered in [M-S-T] but in different context.

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1. Geometry of injective complexification of real Banach spaces. Let $X = (X, \|\cdot\|_X)$ be a real Banach space (we shall usually write $\|x\|$ instead of $\|x\|_X$ if a Banach space is fixed). Let $B = B_X$ and $S = S_X$ denote the closed unit ball and the unit sphere in X, respectively. Similarly, let $B^* = B_{X^*}$ and $S^* = S_{X^*}$ be the closed unit ball and the unit sphere in the dual space $X^* = (X^*, \|\cdot\|_{X^*})$ (we shall usually write $\|\ell\|^*$ instead of $\|\ell\|_{X^*}$). Let $\tilde{X} = X \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of X. $\tilde{X}$ is a complex Banach space with a norm equivalent to the projective norm $\|\cdot\|_\wedge$ from $\hat{X} = X \hat{\otimes} \mathbb{C}$ or (and) to the injective norm $\|\cdot\|_\vee$ from $\check{X} = X \check{\otimes}_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathcal{L}(X^*, \mathbb{C})$. Let us recall the definition of projective and injective norms in a situation interesting for us:

$$\|z\|_\wedge = \inf\Big\{\sum_{j=1}^{k} |\alpha_j|\,\|x_j\|_X : z = \sum_{j=1}^{k} \alpha_j x_j,\ \alpha_j \in \mathbb{C},\ x_j \in X\Big\}, \tag{1.1}$$

$$\|z\|_\vee = \sup_{\ell \in S^*} |\ell(z)|. \tag{1.2}$$

One can easily prove that

$$\|z\|_\vee = \sup_{|\theta| \le \pi} \|\cos\theta x - \sin\theta y\|, \quad z = x + iy, \tag{1.3}$$

$$\|z\|_\wedge \le \inf_{|\theta| \le \pi} \{\|\cos\theta x - \sin\theta y\| + \|\sin\theta x + \cos\theta y\|\} \tag{1.4}$$

and

$$\|z\|_\wedge \ge \sup_{|\theta| \le \pi} \big\{\|\cos\theta x - \sin\theta y\|^2 + \|\sin\theta x + \cos\theta y\|^2\big\}^{1/2}. \tag{1.4$'$}$$

These two inequalities are sharp because if X is a Hilbert space then we have equality in (1.4). The equality in (1.4′) holds if $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max(|x_1|, |x_2|)$. By $\hat{B}, \hat{S}$ and $\check{B}, \check{S}$ we shall denote the closed unit ball and the unit sphere in $\hat{X}$ and $\check{X}$, respectively. We shall treat X as a subset of $\tilde{X}$ such that $\tilde{X} = X + iX$.

Example 1.1. If $X = H$ is a real Hilbert space, $z = x + iy$ then (Drużkowski [D, D-P], in the case of $\mathbb{R}^n$ a formula for the injective norm was found by A. Turowicz while the projective norm is equal to the well-known Lie norm)

$$\|z\|_\vee = \Big(\frac{\|z\|^2 + |z^2|}{2}\Big)^{1/2}, \tag{1.5}$$

$$\|z\|_\wedge = \Big(\frac{\|z\|^2 + |z^2|}{2}\Big)^{1/2} + \Big(\frac{\|z\|^2 - |z^2|}{2}\Big)^{1/2}. \tag{1.6}$$

Moreover, if $x \cdot y = 0$, then

$$\|z\|_\vee = \max(\|x\|, \|y\|),\ \|z\|_\wedge = \|x\| + \|y\|. \tag{1.7}$$

(Here $\|x\|^2 = x^2 = x \cdot x$, $\|z\|^2 = x^2 + y^2$, $z^2 = x^2 - y^2 + 2ix \cdot y$, "$\cdot$" is the inner product in X.) Conditions (1.7) determine both norms because, as it has been proved by Drużkowski in [D], if $z = x + iy \in \tilde{X}$ is fixed, then there exists $\zeta \in \mathbb{T}$ such that

$$z = \zeta(x' + iy') \quad \text{and} \quad x' \cdot y' = 0. \tag{1.8}$$

It was calculated by the author that then

$$\min(\|x'\|,\|y'\|)=\lambda_1(z)=\Big(\frac{\|z\|^2-|z^2|}{2}\Big)^{1/2}, \tag{1.9}$$

$$\max(\|x'\|,\|y'\|)=\lambda_2(z)=\Big(\frac{\|z\|^2+|z^2|}{2}\Big)^{1/2}, \tag{1.10}$$

and $\lambda_1(z),\lambda_2(z)$ are eigenvalues of the square root of the Gram matrix $G(x,y)$.

Now we give a short proof of (1.7). Without loss of generality we may assume that x and y are linearly independent and put $w_1=\frac{x}{\|x\|}$, $w_2=\frac{y}{\|y\|}$. If $a\in S$, then there exist $w_3\in B$, $w_3\cdot x=w_3\cdot y=0$ and $\alpha\in\mathbb{R}^3$ such that $\alpha_1^2+\alpha_2^2+\alpha_3^2=1$, $a=\alpha_1w_1+\alpha_2w_2+\alpha_3w_3$. Let now $\ell\in S^*$. Then, by Riesz's theorem, $\ell(w)=w\cdot a$ for some $a\in S$ and we have

$$|\ell(z)|=\big|\alpha_1\|x\|+i\alpha_2\|y\|\big|\le\max(\|x\|,\|y\|),$$

whence $\|z\|_\vee=\max(\|x\|,\|y\|)$. Put

$$u=\frac{1}{\sqrt{2}}\,(w_1-w_2),\qquad v=\frac{1}{\sqrt{2}}\,(w_1+w_2)$$

and define $\ell(w)=w\cdot(u+iv)$. Then $\|x\|+\|y\|=|\ell(z)|$. Moreover, for an arbitrary $a\in S$, we have

$$|\ell(a)|=\Big|\frac{1}{\sqrt{2}}\,(\alpha_1-\alpha_2)+i\,\frac{1}{\sqrt{2}}\,(\alpha_1+\alpha_2)\Big|\le 1.$$

If now $z=\sum\beta_jx_j$, $\beta_j\in\mathbb{C}$, $x_j\in X$, then

$$|\ell(z)|=\Big|\sum\beta_j\ell\Big(\frac{x_j}{\|x_j\|}\Big)\,\|x_j\|\Big|\le\sum|\beta_j|\,\|x_j\|,$$

which gives the inequality $\|x\|+\|y\|\le\|z\|_\wedge$ and the equality follows.

Example 1.2. If $X=\ell^\infty_{\mathbb{R}}$ then $\check{X}=\ell^\infty_{\mathbb{C}}$. If $X=\ell^1_{\mathbb{R}}$, then $\hat{X}=\ell^1_{\mathbb{C}}$.

We can define in X the following *pseudo inner product*:

$$\mathcal{S}(x,y):=\big[\|x+iy\|_\vee^2-\|x\|^2\big]^{1/2}\big[\|x+iy\|_\vee^2-\|y\|^2\big]^{1/2}$$

for $x,y\in X$. We shall say x and y are *strongly orthogonal* iff $\mathcal{S}(x,y)=0$ and $x\perp_s y$ iff $\|x+iy\|_\vee=\|x\|$. Some properties of the above notion are contained in the following.

Proposition 1.3. *The function $\mathcal{S}$ has the following properties*:

1) $\mathcal{S}(x,y)=\mathcal{S}(y,x)$, $\mathcal{S}(\lambda x,\lambda y)=\lambda^2\mathcal{S}(x,y)$, $\lambda\in\mathbb{R}$.
2) $\|x\|=\sqrt{\mathcal{S}(x,x)}$.
3) *If* $\ell\in\mathcal{L}(X)$ *is a surjection then* $\ell\in\operatorname{Aut}(X)$ *iff* $\mathcal{S}(\ell x,\ell y)=\mathcal{S}(x,y)$.
4) $\mathcal{S}(x,y)\le\|x\|\|y\|$. *If x and y are linearly dependent then* $\mathcal{S}(x,y)=\|x\|\|y\|$. *The inverse implication is, in general, not true. However, if X is strictly convex then the equality* $\mathcal{S}(x,y)=\|x\|\|y\|$ *implies linear dependence of x and y.*
5) *If* $\mathcal{S}(x,y)=0$ *then* $x\perp y$ *or* $y\perp x$. *Here* $\perp$ *stands for the Birkhoff orthogonality.*
6) *If* $x\perp_s y$, *then* $x\perp_s\lambda y=0$ *for* $\lambda\in[-1,1]$.
7) *If X is a Hilbert space then* $\mathcal{S}(x,y)=|x\cdot y|$.

Proof. An easy proof of 1)–3) and 6) is left to the reader. A nontrivial property 7) is contained in Example 1.1. Let now $\|x+iy\|_\vee=\|x\|$ and let $\ell\in S^*$ be a support functional at $x/\|x\|$, so we have $\ell(x) = \|x\|$. Then

$$\|x\|^2 + \ell^2(y) = |\ell(x+iy)|^2 \le \|x\|^2$$

which gives $\ell(y) = 0$ and this is equivalent to the fact that $x \perp y$ in Birkhoff's sense: $\|x\| \le \|x + \lambda y\|$ for an arbitrary $\lambda \in \mathbb{R}$. The first part of 4) an easy exercise to the reader, we shall only prove that equality $\mathcal{S}(x,y) = \|x\|\|y\|$ implies linear dependence of x and y under assumption that X is strictly convex. To do this we need the following.

Claim. *If* $\mathcal{S}(x,y) = \|x\|\|y\|$ *then there exists* $\ell_0 \in S^*$ *such that*

$$\ell_0^2(x) + \ell_0^2(y) = \|x\|^2 + \|y\|^2.$$

Proof. It is easy to check that the equality $\mathcal{S}(x,y) = \|x\|\|y\|$ implies $\|x+iy\|_\vee^2 = \|x\|^2 + \|y\|^2$. Then there is $\theta_0 \in [-\pi,\pi]$ that satisfies

$$\|\cos\theta_0 x - \sin\theta_0 y\|^2 = \|x\|^2 + \|y\|^2.$$

Now we can choose an $\ell_0 \in S^*$ such that

$$\|\cos\theta_0 x - \sin\theta_0 y\| = \ell_0(\cos\theta_0 x - \sin\theta_0 y)$$

and we have

$$\cos 2\theta_0 \frac{1}{2}\left(\ell_0^2(x) - \ell_0^2(y)\right) - \sin 2\theta_0 \ell_0(x)\ell_0(y) + \frac{1}{2}\left(\ell_0^2(x) + \ell_0^2(y)\right) = \|x\|^2 + \|y\|^2.$$

Applying the Cauchy–Schwarz inequality we obtain

$$\ell_0^2(x)+\ell_0^2(y) \le \|x\|^2+\|y\|^2 \le \frac{1}{2}\left(\ell_0^2(x)+\ell_0^2(y)\right)+\left[\frac{1}{4}\left(\ell_0^2(x) - \ell_0^2(y)\right)^2 + \ell_0^2(x)\ell_0^2(y)\right]^{1/2}.$$

Hence $\|x\|^2 + \|y\|^2 = \ell_0^2(x) + \ell_0^2(y)$ and the claim is proved. ∎

Assuming that $\mathcal{S}(x,y) = \|x\|\,\|y\|$ with $x, y \neq 0$ we obtain, by Claim,

$$\|x\|^2 + \|y\|^2 = \ell_0^2(x) + \ell_0^2(y).$$

Since $\ell_0 \in S^*$ it has to be

$$|\ell_0(x)| = \|x\|, \ |\ell_0(y)| = \|y\|.$$

Put $x' = x/\|x\|$, $y' = y/\|y\|$. Then either $|\ell_0(\frac{1}{2}(x'+y'))| = 1$ or $|\ell_0(\frac{1}{2}(x'-y'))| = 1$, which implies $\|\frac{1}{2}(x'+y')\| = 1$ or $\|\frac{1}{2}(x'-y')\| = 1$. But X is strictly convex so we obtain $x' = y'$ or $x' = -y'$. This completes the proof. ∎

An important property of strong orthogonality has an easy proof. It is, in a sense, a generalization of Drużkowski's lemma (1.8) for Hilbert spaces (see [D]).

Lemma 1.4. *For arbitrary* $x, y \in X$ *there exists* $\zeta \in \mathbb{T}$ *such that* $\zeta(x+iy) = x' + iy'$ *and* $x' \perp_s y'$. *In particular,* $x' \perp y'$.

Proof. Since $\|x+iy\|_\vee = \sup_\theta \|\cos\theta x - \sin\theta y\|$, one can choose $\zeta = e^{i\theta_0}$ such that

$$\|x+iy\|_\vee = \|\cos\theta_0 x - \sin\theta_0 y\| = \|x'\|.$$

Then

$$\|x+iy\|_\vee = \|\zeta(x+iy)\|_\vee = \|x'+iy'\|_\vee = \|x'\|.$$

Hence $x' \perp_s y'$ and the proof is completed. ∎

Now define

$$x^{\perp_s} := \{y \in X : x \perp_s y\},$$
$$\mathcal{D}(f) = \{x \in X : f \text{ is Gâteaux differentiable at } x\}.$$

(Here and later $f(x) = \|x\|_X$.) Note that $\|y\| \le \|x\|$ for $y \in x^{\perp_s}$.

THEOREM 1.5. *We have the following inclusions*

$$\bigcup_{x \in \operatorname{extr}(B)} \mathbb{T}\{x + i\operatorname{extr}(x^{\perp_s})\} \subset \operatorname{extr}(\check{B}) \subset \bigcup_{x \in S} \mathbb{T}\{x + i\operatorname{extr}(x^{\perp_s})\}.$$

Proof. The right-hand side inclusion easily follows by Lemma 1.4. Let now $x \in \operatorname{extr}(B)$ and $y \in \operatorname{extr}(x^{\perp_s})$. Then $x + iy \in \check{S}$. Assume that

$$x + iy = (1-\alpha)(x_1 + iy_1) + \alpha(x_2 + iy_2),\ \ \alpha \in (0,1),$$

where $x_1 + iy_1$, $x_2 + iy_2 \in \check{S}$. Then $x_1, x_2 \in B$, whence $x_1 = x_2 = x$. Thus we get

$$x + iy = (1-\alpha)(x + iy_1) + \alpha(x + iy_2)$$

and therefore $x \perp_s y_1$, $x \perp_s y_2$ which gives $y_1 = y_2 = y$. The proof is completed. ■

COROLLARY 1.6. *For an arbitrary Banach space X one has*

$$\check{S} = \bigcup_{x \in S} \mathbb{T}\{x + ix^{\perp_s}\}.$$

COROLLARY 1.7. *If X is a strictly convex Banach space then*

$$\operatorname{extr}(\check{S}) = \bigcup_{x \in S} \mathbb{T}\{x + i\operatorname{extr}(x^{\perp_s})\}.$$

PROPOSITION 1.8. *If $x \in S$ and $y \in x^{\perp_s}$ then for an arbitrary $t \in \mathbb{R}_+$ one has*

$$0 \le \|x \pm ty\| - 1 \le \frac{t^2}{1 + \sqrt{1 + t^2}} \le \frac{1}{2}t^2.$$

Proof. If $y \in x^{\perp_s}$ then $x \perp y$, whence $\|x \pm ty\| \ge 1$ for $t \ge 0$. On the other hand if $t = \tan\theta$, then $|\cos\theta| = \frac{1}{\sqrt{1+t^2}}$ which gives

$$\frac{1}{\sqrt{1+t^2}}\|x \pm ty\| = \|\cos\theta x - \sin\theta y\| \le 1$$

and the second inequality easily follows. ■

COROLLARY 1.9. *If $x \in S$, $y \in x^{\perp_s} \setminus \{0\}$ then*

$$\lim_{\theta \to 0} \frac{f(x + \theta y) - f(x)}{\theta} = D_y f(x) = 0.$$

COROLLARY 1.10. *If $\operatorname{codim}(x^{\perp_s}) = 1$, then $x \in \mathcal{D}(f)$.*

REMARK 1.11. Corollary 1.10 is not true if we replace $x^{\perp_s}$ by $x^{\perp}$.

Fix an $x \in X$ with $\|x\| < 1$. Define $\Lambda_x : X \to \mathbb{R}_+$ by

$$\Lambda_x(v) = \inf\{\lambda > 0 : \|v + i\lambda x\|_{\vee} \le \lambda\}.$$

It follows from the proposition below that Λ_x is a norm in X.

PROPOSITION 1.12. *If $x \in X$, $\|x\| < 1$, is fixed then for an arbitrary $v \in X$*

$$\Lambda_x(v) = \sup_{\ell \in S^*} \frac{|\ell(v)|}{[1 - \ell^2(x)]^{1/2}}\,.$$

Proof. For $v = 0$ the equality trivially holds. Denote its right-hand side by Λ. We have

$$\ell^2(v) + \ell^2(\Lambda(v)x) \le \Lambda^2(v)$$

for every $\ell \in S^*$, whence we get the inequality $\Lambda_x(v) \le \Lambda(v)$. Let now $\lambda \ge \Lambda_x(v) + \epsilon$. Then for an arbitrary $\ell \in S^*$

$$\ell^2(v) + \ell^2(\lambda x) \le \lambda^2$$

and therefore

$$\lambda \ge \sup_{\ell \in S^*} \frac{|\ell(v)|}{[1 - \ell^2(x)]^{1/2}} = \Lambda(v).$$

Letting $\epsilon \to 0+$ we get the inequality $\Lambda_x(v) \ge \Lambda(v)$, which completes the proof. ∎

COROLLARY 1.13. *If $x \in X$, $\|x\| < 1$, is fixed then Λ_x is a norm in X equivalent to the norm $\|\cdot\|_X$:*

$$\|v\|_X \le \Lambda_x(v) \le (1 - \|x\|_X^2)^{-1/2}\|v\|_X.$$

Both inequalities are sharp because $\Lambda_0(v) = \|v\|$ and $\Lambda_x(x) = \|x\|(1 - \|x\|^2)^{-1/2}$.

COROLLARY 1.14. *The norm Λ_x is determined by the equalities*

$$\|v + i\Lambda_x(v)x\|_\vee = \Lambda_x(v),$$

$$\Lambda_x^2(v)(\Lambda_x^2(v) - \|v\|^2) = \frac{\mathcal{S}^2(v, \Lambda_x(v)x)}{1 - \|x\|^2}\,.$$

Let B_x and S_x be the unit ball and the unit sphere for Λ_x, respectively. Then

$$S_x = \{v \in X : \mathcal{S}(v, x) = (1 - \|v\|^2)^{1/2}(1 - \|x\|^2)^{1/2}\},$$

$$B_x = \{v \in X : \mathcal{S}(v, x) \le (1 - \|v\|^2)^{1/2}(1 - \|x\|^2)^{1/2}\}.$$

COROLLARY 1.15. *If $X = H$ is a Hilbert space then*

$$\Lambda_x(v) = \Big(\|v\|^2 + \frac{(x \cdot v)^2}{1 - \|x\|^2}\Big)^{1/2}.$$

COROLLARY 1.16. *If $x, v \in X$, $\|x\| < 1$, $\|v\| = 1$ then $v \in \mathcal{S}_x$ iff $\mathcal{S}(v, x) = 0$.*

REMARK 1.17. We shall call the dual norms Λ_x^*, $x \in \operatorname{int}(B)$, *equilibrium norms.* In the case X is a Hilbert space we have the equality

$$\Lambda_x^*(\ell) = (\|\ell\|^2 - \ell^2(x))^{1/2},\ \ell \in X^*.$$

One can also calculate that for $X = \ell_\infty$

$$\Lambda_x^*(s) = \sum_{j=1}^{\infty} \sqrt{1 - x_j^2}\,|s_j|,\ s \in \ell_1 \subset \ell_\infty^*.$$

If $X = X_1 \oplus_\infty X_2$, $\|(x_1, x_2)\|_X = \max(\|x_1\|_{X_1}, \|x_2\|_{X_2})$, then for $(x_1, x_2) \in \operatorname{int}(B_X)$

$$(X, \Lambda_{(x_1,x_2)}) = (X_1, \Lambda_{x_1}) \oplus_\infty (X_2, \Lambda_{x_2})$$

and

$$(X^*, \Lambda^*_{(x_1,x_2)}) = (X_1^*, \Lambda^*_{x_1}) \oplus_1 (X_2^*, \Lambda^*_{x_2}).$$

REMARK 1.18. The function $\mathcal{S}(x,y)$ is homogeneous with respect to each variable in the case X is a Hilbert space. It seems this is no longer true in other cases. So we may define two homogenizations of $\mathcal{S}(x,y)$:

$$\mathcal{S}_-(x,y) := \inf_{\lambda>0} \frac{1}{\lambda}\,\mathcal{S}(\lambda x,y),\ \mathcal{S}_+(x,y) := \sup_{\lambda>0} \frac{1}{\lambda}\,\mathcal{S}(\lambda x,y),\ x,y\in X.$$

Both functions $\mathcal{S}_-(x,y)$ and $\mathcal{S}_+(x,y)$ are homogeneous and satisfy the inequalities

$$\mathcal{S}_-(x,y) \le \mathcal{S}(x,y) \le \mathcal{S}_+(x,y) \le \|x\|\cdot\|y\|,\ x,y\in X.$$

Hence we easily obtain the following generalization of Corollary 1.15.

COROLLARY 1.19. *If X is a Banach space then for arbitrary $x,v\in X$ with $\|x\|<1$ we have the estimates*

$$\Big(\|v\|^2+\frac{\mathcal{S}_-(x,v)^2}{1-\|x\|^2}\Big)^{1/2} \le \Lambda_x(v) \le \Big(\|v\|^2+\frac{\mathcal{S}_+(x,v)^2}{1-\|x\|^2}\Big)^{1/2}.$$

2. The Joukowski function and its inverse. The *Joukowski function* g is defined by the following simple formula

$$g(\zeta)=\frac{1}{2}\,(\zeta+\zeta^{-1}),\ \zeta\in\mathbb{C}_*=\mathbb{C}\setminus\{0\}.$$

The basic properties of Joukowski's function are contained in the following

$$g(\zeta)=g(\zeta^{-1}),\ \zeta\in\mathbb{C}_*, \tag{2.1}$$

$$|g(\zeta)+1|+|g(\zeta)-1|=2g(|\zeta|),\ \zeta\in\mathbb{C}_*. \tag{2.2}$$

An immediate consequence of (2.2) is the fact that g gives biholomorphism of $\mathbb{D}^*:=\mathbb{C}\setminus\overline{\mathbb{D}}$ ($\mathbb{D}$ is the open unit disc in $\mathbb{C}$, by $\mathbb{T}$ we shall denote the unit circle in $\mathbb{C}$) onto $\mathbb{C}\setminus[-1,1]$. The image of the circle $g(\{|\zeta|=R\})$, $R>1$, is the ellipse

$$\mathcal{E}_R=\big\{u+iv : (u/g(R))^2+(v/(g^2(R)-1)^{1/2})^2=1\big\}.$$

The inverse function h has a form $h(w)=w+\sqrt{w^2-1}$ if we choose an appropriate branch of the square root function and the modulus of h extends continuously to the whole plane $\mathbb{C}$. Applying (2.2) we can easily check that

$$|h(\zeta)|=h\Big(\frac{1}{2}\,|\zeta+1|+\frac{1}{2}\,|\zeta-1|\Big),\ \zeta\in\mathbb{C}, \tag{2.3}$$

where $h(t)=t+\sqrt{t^2-1}$, $t\ge1$, with the usual arithmetic square root. Applying (2.3) one can prove that

$$\text{If } \alpha\in(-1,1),\ \beta\in\mathbb{R},\ \text{then } \log|h(\alpha+i\beta)|\le|\beta|(1-\alpha^2)^{-1/2}. \tag{2.3.1}$$

$$\text{If } \alpha\in(-1,1),\ 0<\varepsilon<1,\ \beta\in\mathbb{R},\ \text{and } |\beta|\le\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}\,(1-|\alpha|),$$

$$\text{then } (1-\varepsilon)|\beta|(1-\alpha^2)^{-1/2}\le\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|. \tag{2.3.2}$$

In particular (see [B8]), we have

$$\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|=|\beta|(1-\alpha^2)^{-1/2}\ \text{for } \alpha\in(-1,1),\ \beta\in\mathbb{R}. \tag{2.4}$$

The following connection between functions g and h and Chebyshev's polynomials T_k is known:

$$T_k(\zeta) = g(h^k(\zeta)), \tag{2.5}$$

$$\limsup_{k\to\infty} |T_k(\zeta)|^{1/k} = |h(\zeta)|, \ \zeta \in \mathbb{C}. \tag{2.6}$$

(If $\zeta \notin (-1,1)$ we can replace lim sup by lim.)

We shall need the following facts (we omit their easy proof).

Proposition 2.1. *For an arbitrary $\zeta \in \mathbb{C}$ we have*

$$\sup_{|\theta|\le\pi} |\cos\theta\zeta + i\sin\theta| = \frac{1}{2}|\zeta+1| + \frac{1}{2}|\zeta-1|. \tag{2.7}$$

If $\zeta \in \mathbb{D}^*$, $w \in \mathbb{C}$, then we have equivalences

$$|\bar{\zeta}w - \zeta^{-1}\bar{w}| \le \frac{1}{2}\left(|\zeta|^2 - |\zeta|^{-2}\right) \quad \textit{iff } |h(w)| \le |\zeta|. \tag{2.8}$$

$$|\bar{\zeta}w - \zeta^{-1}\bar{w}| = \frac{1}{2}(|\zeta|^2 - |\zeta|^{-2}) \quad \textit{iff } \ |h(w)| = |\zeta|. \tag{2.8$'$}$$

Corollary 2.2. *For an arbitrary $\zeta \in \mathbb{C}_*$*

$$\sup_{|\theta|\le\pi} |\cos\theta g(\zeta) + i\sin\theta| = g(|\zeta|). \tag{2.9}$$

3. Polynomial extremal functions for Banach spaces. Let us recall that a homogeneous polynomial P of degree k on X (on $\tilde{X}$) is the mapping of the form $P(x) = L(x,\ldots,x)$, where L is a continuous k-linear form on X (on $\tilde{X}$). We shall denote the vector space of all homogeneous polynomials of degree k by $\mathcal{H}_k(X)$ ($\mathcal{H}_k(\tilde{X})$) and we put $\mathcal{H}(X) = \bigcup_{k=0}^{\infty} \mathcal{H}_k(X)$, $\mathcal{H}(\tilde{X}) = \bigcup_{k=0}^{\infty} \mathcal{H}_k(\tilde{X})$. A polynomial Q is the mapping that is equal to a finite sum of homogeneous polynomials, so the rings of polynomials $\mathcal{P}(X)$ on X and $\mathcal{P}(\tilde{X})$ on $\tilde{X}$ have natural gradations

$$\mathcal{P}(X) = \bigotimes_{k=0}^{\infty} \mathcal{H}_k(X), \quad \mathcal{P}(\tilde{X}) = \bigotimes_{k=0}^{\infty} \mathcal{H}_k(\tilde{X}).$$

Now we define two polynomial extremal functions in the same way as it was made by Siciak in $\mathbb{C}^n$

$$\Phi_B(z) = \sup\{|Q(z)|^{1/\deg Q} : Q \in \mathcal{P}(\tilde{X}),\ \deg Q \ge 1,\ \|Q\|_B \le 1\}, \quad z \in \tilde{X},$$
$$\Psi_S(z) = \sup\{|P(z)|^{1/\deg P} : P \in \mathcal{P}(\tilde{X}),\ \deg P \ge 1,\ \|P\|_S \le 1\}, \quad z \in \tilde{X}.$$

Let now $Y = (X \oplus \mathbb{R})_2 = (X \times \mathbb{R}, q)$, where the norm q is given by

$$q(x,t) = (\|x\|_X^2 + t^2)^{1/2},$$

and let B_1, S_1 be the unit ball and the unit sphere in Y and let S_1^* be the unit sphere in $Y^* = (Y^*, q^*)$. It is not difficult to check that for an arbitrary $\ell \in X^*$ we have

$$q^*(\ell \oplus \lambda\,\mathrm{id}_{\mathbb{R}}) = \left([\|\ell\|^*]^2 + \lambda^2\right)^{1/2}.$$

Put $\check{Y} = Y \check{\otimes} \mathbb{C}$. We shall denote the injective norm in $\check{Y}$ also by $\|\cdot\|_\vee$. In fact, we have the equality

$$\|(z,0)\|_\vee = \|z\|_\vee.$$

Proposition 3.1. *The mapping*

$$\chi : \overline{\mathbb{D}^*} \times \check{S} \ni (\zeta, c) \mapsto \frac{1}{2}\,(\zeta c + \zeta^{-1}\bar{c}) \in \tilde{X}$$

is a surjection, the restriction of χ *to* $\mathbb{D}^* \times \check{S}$ *is a surjection onto* $X \setminus B$ *and the restriction of* χ *to* $\mathbb{T} \times \check{S}$ *is a surjection onto* B. *In particular, if* $\zeta \in \mathbb{T}$ *and* $\|c\|_\vee \le 1$, *then* $\frac{1}{2}(\zeta c + \zeta^{-1}\bar{c}) \in B$.

Proof. Let $z = x + iy \in \tilde{X}$ and assume $y \neq 0$. Let $R > 1$ be fixed and put

$$a = a(R) = \frac{1}{g(R)}\,x,\ b = b(R) = (g^2(R) - 1)^{-1/2}y,\ c = c(R) = a + ib.$$

Since

$$\max(\|a\|, \|b\|) \le \|c\|_\vee \le \|c\|_\wedge \le \|a\| + \|b\|,$$

we see that $\|c\|_\vee \to 0$ as $R \to \infty$ and $\|c\|_\vee \to \infty$ as $R \to 1+$. Therefore we can find $R > 1$ such that $\|c\|_\vee = 1$ and we get $z = \chi(R, c)$. If $z = x \neq 0$, then for $iz = ix$ we have the previous case. At the end, $0 = \chi(1, ia)$ for arbitrary $a \in S \subset \check{S}$. Let now $\zeta \in \mathbb{D}^*$, so $R = |\zeta| > 1$. It suffices to consider the case $\chi(R, c)$. Let $c = a + ib \in \check{S}$. If $\chi(R, c) \in B$, then $b = 0$ and $\|a\| = 1$. But $\|\chi(R, c)\| = g(R) > 1$, which is impossible. This contradiction completes the proof. ■

Proposition 3.2. *Let* $\ell \in S^*$ *be fixed. Then for all* $z \in \tilde{X}$ *we have inequality*

$$|h(\ell(z))| \le \Phi_B(z).$$

Proof. Put $Q_k(z) = T_k(\ell(z))$. Then $Q_k \in \mathcal{P}(\tilde{X})$, $\deg Q_k = k$ and $\|Q_k\|_B \le 1$, so we have

$$|Q_k(z)|^{1/k} \le \Phi_B(z),\ z \in \tilde{X}.$$

Letting $k \to \infty$ and applying (2.6) we get the required inequality. ■

Corollary 3.3. *For an arbitrary* $z \in \tilde{X}$ *we have the inequality*

$$\sup_{\ell \in S^*} |h(\ell(z))| \le \Phi_B(z).$$

Lemma 3.4. *If* $c \in \check{S}$ *is fixed, then for arbitrary* $\zeta \in \overline{\mathbb{D}^*}$

$$\Phi_B(\chi(\zeta, c)) \le |\zeta|.$$

In particular, $\Phi_B = 1$ *on* B.

Proof. Fix $Q \in \mathcal{P}(\tilde{X})$, $\deg Q \ge 1$, $\|Q\|_B \le 1$. Then the function

$$u(\zeta) := \frac{1}{\deg Q}\log|Q(\chi(\zeta, c))| - \log|\zeta|$$

is subharmonic and bounded from above in $\mathbb{D}^*$. Moreover, $u(\zeta) \le 0$ on $\mathbb{T}$. Hence, by the maximum principle for subharmonic functions, we get the inequality

$$|Q(\chi(\zeta, c))|^{1/\deg Q} \le |\zeta|,$$

which implies the inequality for the extremal function. ■

Lemma 3.5. *For an arbitrary $z \in \tilde{X}$,*

$$\sup_{\ell \in S^*} \Big(\frac{1}{2}\,|\ell(z)+1| + \frac{1}{2}\,|\ell(z)-1|\Big) = \|(z,i)\|_\vee$$

and

$$\sup_{\ell \in S^*} |h(\ell(z))| = h(\|(z,i)\|_\vee).$$

Proof. We have

$$\begin{aligned}\|(z,i)\|_\vee &= \sup_{\ell_1 \in S_1^*} |\ell_1(z,i)| = \sup\{|\ell(z)+\lambda i| : \ell \in B^*,\ \lambda \in \mathbb{R},\ [\|\ell\|^*]^2 + \lambda^2 = 1\} \\ &= \sup\{|\cos\theta \ell(z) + \sin\theta i| : \ell \in S^*,\ |\theta| \le \pi\} = \sup_{\ell \in S^*} \sup_{|\theta| \le \pi} |\cos\theta \ell(z) + \sin\theta i|\,,\end{aligned}$$

whence, by applying (2.7) and (2.3), the lemma follows. ■

Now we can formulate the main result of this section. In the case of $X = \mathbb{R}^n$ a formula for Φ_B was proved by Lundin [LU] and, a more precise version, by Baran [B1], [B2]. There are some reasons that Lundin's method cannot be applied in the infinite dimensional case while a method from [B1] works so well.

Theorem 3.6. *For an arbitrary $z \in \tilde{X}$ we have*

$$\Phi_B(z) = h(\|(z,i)\|_\vee)$$

and for arbitrary $c \in \check{S}$

$$\Phi_B(\chi(\zeta,c)) = |\zeta|.$$

Proof. It suffices to consider the case $z \in \tilde{X} \setminus B$. Then

$$R = h(\|(z,i)\|_\vee) = \sup_{\ell \in S^*} |h(\ell(z))| > 1.$$

Define

$$c = \Big(\frac{1}{2}\,(R^2 - R^{-2})\Big)^{-1} (Rz - R^{-2}\bar{z}).$$

Then $\chi(R,c) = z$ and applying (2.8) and (2.8′) we easily check that $c \in \check{S}$. Hence, by Lemma 3.4, we obtain

$$\Phi_B(z) = \Phi_B(\chi(R,c)) \le R = h(\|(z,i)\|_\vee).$$

Applying now Corollary 3.3 we get the equality. The second part of the theorem easily follows by the first one and its proof. ■

Corollary 3.7. *For an arbitrary $(z,\zeta) \in \check{Y}$, $\zeta \ne 0$ we have*

$$\|(z,\zeta)\|_\vee = |\zeta| g(\Phi_B(i\zeta^{-1}z)),$$

and for each $z \in \check{X}$ we have

$$\|z\|_\vee = \lim_{t \to 0+} tg(\Phi_B(it^{-1}z)) = \frac{1}{2} \lim_{t \to 0+} t\Phi_B\Big(\frac{1}{t}\,z\Big).$$

Corollary 3.8. *For an arbitrary $z \in \tilde{X}$ we have*

$$\|z\|_\wedge \le \Psi_S(z) \le 2\|z\|_\vee.$$

Proof. We have

$$\|z\|_\wedge = \sup_{\ell\in\hat{S}^*} |\ell(z)|,$$

and since

$$\|\ell\|_\wedge^* = \sup\{|\ell(z)| : z \in \hat{S}\} \geq \|\ell\|_S,$$

we get inequality $\|z\|_\wedge \leq \Psi_S(z)$. On the other hand $\Psi_S(z) \leq \Phi_B(z)$, and, since Ψ_S is homogeneous, we have

$$\Psi_S(z) \leq \lim_{r\to 0+} r\Phi_B\Big(\frac{z}{r}\Big) = 2\|z\|_\vee. \ \blacksquare$$

Let $X = H$ be a real Hilbert space with an inner product "$\cdot$". Then in the space $\tilde{H}$ there is a natural complex Hilbert structure. Let $\|\cdot\|$ be the Hilbertian norm on $\tilde{H}$ and let $z^2 = z \cdot \bar{z}$. Let B be the unit ball in H. The next corollary in the finite-dimensional case was firstly proved by Lundin [LU] (in an equivalent form) and a simpler proof was given in [B1] (see also [KL]). A proof presented here is quite new.

Corollary 3.9. *For an arbitrary $z \in \tilde{H}$*

$$\Phi_B(z) = \big(h(\|z\|^2 + |z^2 - 1|)\big)^{1/2}.$$

Proof. Put $Y = (H \oplus \mathbb{R})_2$. Then Y is again a real Hilbert space. Let $\check{Y} = Y \check{\otimes} \mathbb{C}$. Applying this result to the space Y we easily obtain, by Theorem 3.6, the required formula (because $h(2t^2 - 1) = h^2(t)$, $t \geq 1$). $\blacksquare$

Remark 3.10. If $H = \mathbb{R}^2$, then (1.5) reduces to the following simple formula

$$\|(z_1, z_2)\|_\vee = \frac{1}{2}\,|z_1 + iz_2| + \frac{1}{2}\,|z_1 - iz_2|. \tag{3.1}$$

In fact, this formula can also be obtained from Corollary 3.9, because $|h| = \Phi_{[-1,1]}$. Later we shall give an interesting application of (3.1).

Now define a function $\gamma : \tilde{X} \times \mathbb{C} \to \mathbb{R}_+$ by

$$\gamma(z,\zeta) = \|(z,\zeta)\|_\vee + (\|(z,\zeta)\|_\vee^2 - |\zeta|^2)^{1/2}.$$

It is easily seen that

$$\gamma(z,\zeta) = |\zeta|\Phi_B(\zeta^{-1}iz),\ \zeta \in \mathbb{C}_*,$$

in particular,

$$\Phi_B(z) = \gamma(z, i).$$

It follows by homogeneity of Ψ_B that

$$\Psi_B(z) \leq \inf_{\zeta\in\mathbb{C}_*} |\zeta|\Phi_B(\zeta^{-1}z) = \inf_{\zeta\in\mathbb{C}_*} \gamma(z,\zeta). \tag{3.2}$$

In the case $X = H$ is a Hilbert space we have

$$\gamma(z,\zeta) = \Big[\frac{1}{2}\,(\|z\|^2 + |\zeta|^2) + \frac{1}{2}\,|z^2 + \zeta^2|\Big]^{1/2} + \Big[\frac{1}{2}\,(\|z\|^2 - |\zeta|^2) + \frac{1}{2}\,|z^2 + \zeta^2|\Big]^{1/2},$$

which implies

$$\gamma(z, \pm i\sqrt{z^2}) = \|z\|_\wedge. \tag{3.3}$$

It follows from (3.2), (3.3) and Corollary 3.8 that we have proved the following result (Siciak [SI1, SI2] and Drużkowski [D]).

PROPOSITION 3.11. *If H is a Hilbert space then for an arbitrary $z \in \tilde{H}$*

$$\Psi_B(z) = \|z\|_\wedge.$$

REMARK 3.12. One can also prove the above proposition in the following way, which gives a little more. Put $\hat{g}(\zeta) = \frac{1}{2}(\zeta - \zeta^{-1})$, $\zeta \in \mathbb{C}_*$. Then for all $a, b \in H$, $a \cdot b = 0$ and for all $\zeta \in \mathbb{C}_*$, we have

$$\|g(\zeta)a \pm i\hat{g}(\zeta)b\|_\wedge = \big|g(\zeta)\|a\| + \hat{g}(\zeta)\|b\|\big|. \tag{3.4}$$

One can check that logarithm of the right-hand side in (3.4) is a harmonic function, which gives the following

COROLLARY 3.13. *If $X = H$ is a real Hilbert space then* $\log \Psi_S$ *is harmonic on each $\chi(\zeta, c)$, $c \in \check{S}$. In particular, this implies $\Psi_S(z) = \|z\|_\wedge$, $z \in \tilde{X}$.*

REMARK 3.14. One can suppose that the equality $\Psi_B(z) = \|z\|_\wedge$, $z \in \tilde{X}$, holds for an arbitrary Banach space X. However it is not true, see [B6]. We conjecture that $\Psi_B(z) = \inf_{\zeta \in \mathbb{C}_*} \gamma(z, \zeta)$ for an arbitrary Banach space X.

DEFINITION 3.15. Let S be the unit sphere in a real Hilbert space H. Put

$$\mathbb{S} = \{z = x + iy \in \tilde{H} : z^2 = z \cdot \bar{z} = x \cdot x - y \cdot y + 2ix \cdot y = 1\}.$$

In the case $S = S^{n-1} \subset \mathbb{R}^n$ we have $\mathbb{S} = \{z \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 1\}$. It is clear that $S \subset \mathbb{S}$, so $\mathbb{S}$ is a complexification of S. Define also

$$\mathbb{V}_2(H) = \{(a, b) \in S \times S : a \cdot b = 0\}.$$

If $\dim H < \infty$ then $\mathbb{V}_2(H)$ is a special case of so called Stiefel's manifold. It is known ([B5]) that $\mathbb{V}_2(H) = \operatorname{extr} \check{B}$.

It is easy to check the following important

PROPOSITION 3.16. *The mapping*

$$\chi : (\mathbb{C} \setminus \mathbb{D}) \times \mathbb{V}_2(H) \ni (\zeta, (a, b)) \mapsto \frac{1}{2}(\zeta + \zeta^{-1})a + i\,\frac{1}{2}(\zeta - \zeta^{-1})b \in \tilde{H}$$

is a surjection onto $\mathbb{S}$. Its restriction to $\mathbb{T} \times \mathbb{V}_2(H)$ is a surjection onto S.

Now we shall prove the following interesting and important fact.

THEOREM 3.17. *If S is the unit sphere in a real Hilbert space H then for all $z = x + iy \in \mathbb{S}$ we have the equality*

$$\Phi_S(z) = \Phi_B(z) = \Psi_S(z) = \|x\| + \|y\|.$$

Proof. For arbitrary $z \in \tilde{H}$ we have the inequalities

$$\Psi_S(z) = \Psi_B(z) \le \Phi_B(z) \le \Phi_S(z).$$

By the maximum principle we get the inequality

$$\Phi_S(\chi(\zeta, c)) \le |\zeta|,\ \zeta \in \mathbb{C} \setminus \mathbb{D},\ c \in \mathbb{V}_2(H).$$

We obtain by (3.3)

$$\|z\|_\wedge = \gamma(z,i) = \left[\frac{1}{2}\left(\|z\|^2+1\right)\right]^{1/2} + \left[\frac{1}{2}\left(\|z\|^2-1\right)\right]^{1/2}, \ z\in\mathbb{S}.$$

Since, for $z=\chi(\zeta,c)$, $\|z\|^2=\frac{1}{2}\left(|\zeta|^2+|\zeta|^{-2}\right)$, it yields $\|z\|_\wedge=|\zeta|$, which completes the proof. ■

Example 3.18. We present some formulas for homogeneous extremal function. For details we refer to [B7].

Let S be the unit ball with respect to a norm q in $\mathbb{R}^2$. If $u(t)=\log q(1,t)$ then

$$\Psi_S(z_1,z_2)=|z_1|\exp\mathcal{P}u(z_2/z_1),$$

with

$$\mathcal{P}u(\zeta)=(\Im\zeta)\frac{1}{\pi}\int_{-\infty}^{\infty}|\zeta-t|^{-2}u(t)\,dt=\frac{1}{\pi}\int_{-\infty}^{\infty}u(ty+x)\,\frac{dt}{1+t^2},$$

where $\zeta=x+iy$, $y\ge 0$.

If $q_n(x)=(x_1^{2n}+x_2^{2n})^{1/(2n)}$ and $S_n=\{x\in\mathbb{R}^2 : q_n(x)=1\}$, then for all $z\in\mathbb{C}^2$,

$$\Psi_{S_n}(z)=\left[\prod_{j=1}^{n}\left(|z_1|^2-2\alpha_j\Re(z_1\overline{z_2})+|z_2|^2+2|\beta_j||\Im(z_1\overline{z_2})|\right)^{1/2}\right]^{1/n},$$

where $\zeta_j=\alpha_j+i\beta_j\in\sqrt[2n]{-1}$, $j=1,\dots,n$, with $\zeta_j\ne\overline{\zeta_k}$ for $j\ne k$.

If $q_\infty(x)=\max(|x_1|,|x_2|)$ and $S_\infty=\{x\in\mathbb{R}^2 : q_\infty(x)=1\}$, then for all $z\in\mathbb{C}^2$,

$$\Psi_{S_\infty}(z)=\exp\left[\int_0^{2\pi}\log\left(|z_1|^2-2\cos\theta\Re(z_1\overline{z_2})+|z_2|^2+2\left|\sin\theta\Im(z_1\overline{z_2})\right|\right)^{1/2}\frac{d\theta}{2\pi}\right].$$

Let ν be a fixed norm $\widetilde{X}$ that is equivalent to the projective norm $\|z\|_\wedge$ of a given norm $\|\cdot\|$ in X. If $A\subset\widetilde{X}^*$ is a bounded convex set, then $\sup\{\nu^*(l-k) : l,k\in A\}$ is the diameter $\operatorname{diam}(A)=\operatorname{diam}_{\nu^*}(A)$ of A with respect to the norm ν^*. Following recent results in the finite dimensional case (see [B-BC1, B-BC2]) we define a capacity $C_\nu(B)$ for the unit ball B in $(X,\|\cdot\|)$ with respect to ν by

$$C_\nu(B)=\liminf_{r\to\infty}\frac{r}{\varphi_\nu(B,r)},$$

where

$$\varphi_\nu(B,r)=\sup\{\Phi_B(x+z) : x\in B,\ \nu(z)\le r\},\ r\ge 0.$$

Proposition 3.19. *Under the definitions above we have the equalities*

$$\varphi_\nu(B,r)=h\Big(1+\frac{1}{2}\operatorname{diam}_{\nu^*}(B^*)r\Big),\ C_\nu(B)=\frac{1}{\operatorname{diam}_{\nu^*}(B^*)}.$$

Proof. Let $x \in B$, $z = u + iv$, $\nu(u+iv) = \rho \le r$. We can write, applying Proposition 2.1, Theorem 3.6 and formulas related to them,

$$\begin{aligned}
&\sup\{\Phi_B(x+z) : x \in B, \nu(z) \le r\}\\
&\quad = h\Big(\sup_{l\in S^*}\sup_{0\le\rho\le r}\sup_{\nu(u+iv)=\rho}\sup_{x\in B}\sup_{|\theta|\le\pi}\sup_{|t|\le\pi} |\cos\theta(l(x) + e^{it}l(u+iv)) + i\sin\theta|\Big)\\
&\quad = h\Big(\sup_{l\in S^*}\sup_{0\le\rho\le r}\sup_{\nu(u+iv)=\rho}\sup_{x\in B}\sup_{|\theta|\le\pi} (|\cos\theta l(x) + i\sin\theta| + |l(u+iv)|\ |\cos\theta|)\Big)\\
&\qquad = h\Big(\sup_{l\in S^*}\sup_{0\le\rho\le r}\sup_{\nu(u+iv)=\rho} (1 + |l(u+iv)|)\Big) = h\big(\sup_{l\in S^*}(1+\nu^*(l)r)\big)\\
&\qquad\qquad = h\Big(1 + \frac{1}{2}\operatorname{diam}_{\nu^*}(B^*)r\Big).
\end{aligned}$$

The second formula is a consequence of the equality $\lim_{r\to\infty}\frac{r}{h(1+\alpha r)} = \frac{1}{2a}$. ■

4. Inequalities for the derivative of polynomials. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Banach spaces. By $\mathcal{P}_k(X,Y)$ we shall denote the set of polynomial mappings $P : X \to Y$ of degree $\le k$. It means that $P = P_0 + \ldots + P_m$, $m \le k$, where $P_j(x) = F_j(x,\ldots,x)$ and F_j is a continuous j-linear form. If $Q \in \mathcal{P}_k(X,Y)$ then we define

$$\|Q\|_{B_X} = \sup\{\|Q(x)\|_Y : x \in B_X\}.$$

The main result of this section is the following Szegő type inequality.

THEOREM 4.1. *Let $Q \in \mathcal{P}_k(X,Y)$ with $\|Q\|_{B_X} \le 1$. If $\|x\|_X < 1$ and $\|Q(x)\|_Y < 1$, then for an arbitrary $v \in X$*

$$\Lambda_{Q(x)}(d_xQ(v)) \le k\Lambda_x(v). \tag{4.1}$$

This means that $d_xQ : (X,\Lambda_x) \to (Y,\Lambda_{Q(x)})$ has the norm $\|d_xQ\| \le \deg Q$.

COROLLARY 4.2. *If $Q \in \mathcal{P}_k(X)$, $x \in \operatorname{int}(B)$ and $v \in X$ then*

$$|d_xQ(v)| \le k\Lambda_x(v)(\|Q\|^2 - Q^2(x))^{1/2}. \tag{4.2}$$

In particular, if $\|v\| = 1$ and $\mathcal{S}(v,x) = 0$ then

$$|d_xQ(v)| \le k(\|Q\|^2 - Q^2(x))^{1/2}.$$

For an arbitrary Banach space X we have for $x \in \operatorname{int}(B)$, $v \in X$ the following generalization of the Harris inequality

$$|d_xQ(v)| \le k\Big(\|v\|^2 + \frac{\mathcal{S}_+(x,v)^2}{1-\|x\|^2}\Big)^{1/2}(\|Q\|^2 - Q^2(x))^{1/2}.$$

COROLLARY 4.3. *If $Q : X \to \mathbb{C}$ is a polynomial of degree $\le k$ then*

$$\frac{1}{2}\Big|\Lambda_x(v)Q(x) + i\,\frac{1}{k}\,d_xQ(v)\Big| + \frac{1}{2}\Big|\Lambda_x(v)Q(x) - i\,\frac{1}{k}\,d_xQ(v)\Big| \le \Lambda_x(v)\|Q\|. \tag{4.3}$$

REMARK 4.4. It is an easy exercise that (4.1) implies (4.2). It is not difficult to prove that (4.2) implies also (4.1). To see this fix an $\ell \in Y^*$ and consider a polynomial $Q_\ell(x) = \ell(Q(x)) \in \mathcal{P}_k(X)$ where $Q \in \mathcal{P}_k(X,Y)$, $\|Q\|_{B_X} = 1$ and $\|Q(x)\|_Y < 1$. Then, by (4.2), we get

$$\Lambda_x^2(v)\ell(Q(x))^2 + \frac{1}{k^2}\,\ell(d_xQ(v))^2 \le \Lambda_x^2(v)\|\ell\circ Q\|^2,$$

which implies, by taking the supremum over $\ell \in B_Y^*$, that

$$\Big\| \frac{1}{k}\, d_x Q(v) + i\Lambda_x(v) Q(x) \Big\|_{\vee} \le \Lambda_x(v),$$

which gives (4.1).

For a proof of (4.2) we shall need the following facts.

Proposition 4.5. *For an arbitrary $v \in X$ and for all $x \in \operatorname{int}(B)$ we have*

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \log \Phi_B(x + i\varepsilon v) = \Lambda_x(v). \tag{4.4}$$

If $Q \in \mathcal{P}(X)$, $\|Q\|_B \le 1$ then for all $z \in \tilde{X}$

$$|h(Q(z))|^{1/\deg Q} \le \Phi_B(z). \tag{4.5}$$

Proof. The first part of Proposition 4.5 is a consequence of Theorem 3.6, Lemma 3.5 and inequalities (2.3.1) and (2.3.2). The second part is an easy corollary to (2.5) and (2.6) because

$$|T_k(Q(z))|^{1/\deg Q} \le \Phi_B(z). \ \blacksquare$$

Proof of (4.2). Let $Q \in \mathcal{P}(X)$, $\|Q\|_B \le 1$. Fix an $x \in \operatorname{int}(B)$ and an $\alpha \in (0,1)$ and put $Q_\alpha = \alpha Q$. Then, for a fixed $v \in X$,

$$Q_\alpha(x + i\varepsilon v) = Q_\alpha(x) + i\varepsilon d_x Q_\alpha(v) + O(\epsilon^2)$$

as $\epsilon \to 0+$. Since, by (4.5),

$$\frac{1}{\deg Q} \frac{1}{\varepsilon} \log |h(Q_\alpha(z))| \le \frac{1}{\varepsilon} \log \Phi_B(z) \tag{4.6}$$

and $\|x\| < 1$ and $Q_\alpha(x) < 1$ we can apply (4.4) and (2.3.1) and (2.3.2) to the right-hand side and to the left-hand side of (4.6), respectively. This gives

$$\alpha |d_x Q(v)| \le (\deg Q) \Lambda_x(v) (1 - \alpha^2 Q^2(x))^{1/2}.$$

Letting $\alpha \to 1-$ and replacing Q by $\frac{1}{\|Q\|} Q$ we obtain (4.2). The proof is complete. $\blacksquare$

Note also the following inequalities.

Corollary 4.6. *If $x \in \operatorname{int}(B)$ is fixed then for all $\ell \in X^*$*

$$(1 - \|x\|^2)^{1/2} \|\ell\| \le \Lambda_x^*(\ell) \le (\|\ell\|^2 - \ell^2(x))^{1/2}.$$

Corollary 4.7. *If $Q \in \mathcal{P}(\tilde{X})$ and $x \in \operatorname{int}(B)$ and $v \in X$ are fixed then*

$$|d_x Q(v)| \le (\deg Q) \|v\| (1 - \|x\|^2)^{-1/2} \|Q\|_B.$$

Now we can formulate a generalization of classical Markov's inequality. A proposition below was proved by Sarantopoulos [SAR] and, in the important case $X = \mathbb{R}^n$, it was independently proved by Baran [B3] (by applying a method presented here).

Proposition 4.8. *If $Q \in \mathcal{P}(\tilde{X})$ and $x \in B$ is fixed then $d_x Q : X \to \mathbb{C}$ has the norm $\|d_x Q\| \le (\deg Q)^2 \|Q\|_B$.*

Corollary 4.9. *If $Q \in \mathcal{P}(X, \tilde{Y})$ and $x \in B_X$ is fixed then $d_x Q : X \to \check{Y}$ has the norm $\|d_x Q\| \le (\deg Q)^2 \|Q\|_B$.*

Proof of Proposition 4.8. Fix a $v \in X \setminus \{0\}$. Then

$$q_v : X \ni x \mapsto d_x Q(v) \in \mathbb{C}$$

is a polynomial of degree $\deg q_v \leq \deg Q - 1$. If now $x \in B$ is fixed then $q_v(\lambda x)$ is a polynomial of degree $\leq \deg Q - 1$, which satisfies the inequality

$$|q_v(\lambda x)| \leq (\deg Q)\|v\|(1-\lambda^2)^{-1/2}\|Q\|_B, \ \lambda \in (-1,1),$$

whence, by Schur's lemma,

$$|q_v(\lambda x)| \leq (\deg Q)^2\|v\|\|Q\|_B, \ \lambda \in [-1,1],$$

which completes the proof. ■

Proof of Corollary 4.9. If $Q \in \mathcal{P}(X, \tilde{Y})$ then we consider a polynomial $Q_\ell := \ell \circ Q \in \mathcal{P}(X, \mathbb{C})$, where $\ell \in Y^*$ is fixed. By Proposition 4.8 we have

$$|d_x Q_\ell(v)| = |\ell(d_x Q(v))| \leq (\deg Q)^2\|v\|\|Q_\ell\|_B,$$

whence, taking the supremum over $\ell \in S_Y^*$ we obtain

$$\|d_x Q(v)\|_\vee \leq (\deg Q)^2\|v\|\|Q\|_B.$$

This inequality finishes the proof. ■

Applying similar methods we shall prove Harris' generalization of the Szegő inequality for trigonometric polynomials.

THEOREM 4.10. *Let H be a real Hilbert space. If $(x,v) \in \mathbb{V}_2(H)$ then for an arbitrary $Q \in \mathcal{P}(H)$ we have*

$$d_x Q(v)| = |\nabla Q(x) \cdot v| \leq (\deg Q)(\|Q\|_S^2 - Q^2(x))^{1/2}.$$

Proof. It suffices to prove that

$$|d_x Q(v)| \leq (\deg Q)(1 - Q^2(x))^{1/2}$$

for $\|Q\|_S < 1$. In such a situation, by Theorem 3.17, we can write the inequality

$$\frac{1}{\deg Q}\log|h(Q(z))| \leq \log \Phi_S(z) = \log\|z\|_\wedge, \ z \in \mathbb{S}.$$

Take an $\epsilon > 0$ and put $z = \sqrt{1+\epsilon^2}\,x + i\epsilon v$. Then $z \in \mathbb{S}$, $\|z\|_\wedge = \epsilon + \sqrt{1+\epsilon^2}$ and

$$Q(z) = Q\big(\sqrt{1+\epsilon^2}\,x\big) + i\epsilon\nabla Q\big(\sqrt{1+\epsilon^2}\,x\big)\cdot v + O(\epsilon^2).$$

Since

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\log\big|h\big(Q(\sqrt{1+\varepsilon^2}\,x + i\varepsilon v)\big)\big| = \frac{|\nabla Q(x)\cdot v|}{(1-Q^2(x))^{1/2}},$$

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\log\big(\varepsilon + \sqrt{1+\varepsilon^2}\big) = 1,$$

the result follows. ■

COROLLARY 4.11 (A generalization of Bernstein's inequality). *If $(x,v) \in \mathbb{V}_2(H)$ then for an arbitrary $Q \in \mathcal{P}(H,\mathbb{C})$*

$$|d_x Q(v)| \leq (\deg Q)\|Q\|_S.$$

Finally, one can prove the following inequality that contains inequalities for derivatives of real and complex polynomials. A proof is proposed as an exercise for the reader.

THEOREM 4.12. *If* $(x,v)\in\mathbb{V}_2(H)$ *then for all* $Q\in\mathcal{P}_k(H,\mathbb{C})$

$$\frac{1}{2}\Big|Q(x)+i\,\frac{1}{k}\,d_xQ(v)\Big|+\frac{1}{2}\Big|Q(x)-i\,\frac{1}{k}\,d_xQ(v)\Big|\le\|Q\|_S.$$

REMARK 4.13. Let $\widetilde{H}=H\oplus\mathbb{R}$, so that $\langle(x,t),(x',t')\rangle=\langle x,x'\rangle+tt'$. If $x\in B_H$ then $(x,\sqrt{1-x^2})\in S_{\widetilde{H}}$. We have

$$\big\langle(x,\sqrt{1-x^2}),(v,u)\big\rangle=0\quad\Leftrightarrow\quad\langle x,v\rangle+\sqrt{1-x^2}\,u=0.$$

If now $\|x\|<1$ then

$$u=-\frac{\langle x,v\rangle}{\sqrt{1-x^2}}\,,\ \ \|(v,u)\|=\Big(v^2+\frac{\langle x,v\rangle^2}{1-x^2}\Big)^{1/2}.$$

If $Q(x,t)=P(x)$ then $d_{(x,t)}Q=d_xP$ and applying Corollary 4.11 we get Bernstein type inequality for the ball B_H

$$|d_xP(v)|\le(\deg P)\Big(v^2+\frac{\langle x,v\rangle^2}{1-x^2}\Big)^{1/2}\|P\|_{B_H},\ \ x\in B_H$$

and

$$\|d_xP\|\le\deg P(1-x^2)^{-1/2}\|P\|_{B_H}.$$

REMARK 4.14. Fix an $x_0\in H\setminus\{0\}$, $||x_0||<1$ and $v_0\in S$. Put $t_0=\sqrt{1-x_0^2}$,

$$u_0=-\frac{\langle x_0,v_0\rangle}{\sqrt{1-x_0^2}}\,,\qquad\widetilde{v_0}=\frac{\sqrt{1-x_0^2}}{\sqrt{1-G(x_0,v_0)}}\,v_0,$$
$$\widetilde{u_0}=\frac{\sqrt{1-x_0^2}}{\sqrt{1-G(x_0,v_0)}}\,u_0=-\frac{\langle x_0,v_0\rangle}{\sqrt{1-G(x_0,v_0)}}\,.$$

If $Q(x,t)=(\langle x,\widetilde{v_0}\rangle+t\widetilde{u_0})P(x)$ then

$$d_{(x,t)}Q(v,u)=(\langle v,\widetilde{v_0}\rangle+u\widetilde{u_0})P(x)+(\langle x,\widetilde{v_0}\rangle+t\widetilde{u_0})d_xP(v).$$

Now for $((x,t),(v,u))\in\mathbb{V}_2(\widetilde{H})$ we obtain

$$|d_{(x,t)}Q(v,u)|\le(\deg P+1)\sup_{x\in B_H}|(\langle x,\widetilde{v_0}\rangle+t\widetilde{u_0})P(x)|.$$

In particular, taking $x=x_0$, $v=\widetilde{v_0}$, $u=\widetilde{u_0}$, we have

$$|P(x_0)|\le(\deg P+1)\sup_{x\in B_H}\big|\big(\langle x,\widetilde{v_0}\rangle+\sqrt{1-x^2}\,\widetilde{u_0}\big)P(x)\big|$$
$$=(\deg P+1)\sup_{x\in B_H}\big|\big(\langle x,v_0\rangle\sqrt{1-x_0^2}-\langle x_0,v_0\rangle\sqrt{1-x^2}\,\widetilde{u_0}\big)P(x)\big|(1-G(x_0,v_0))^{-1/2}.$$

For $x_0\in S_H$ and $v_0=x_0$ we get

$$|P(x_0)|\le(\deg P+1)\max_{x\in B_H}\sqrt{1-x^2}\,|P(x)|$$

that is a special case of Schur's inequality for $x_0\in S_H$. To get a full version of Schur's inequality we apply a method from [B6] and [G-M] (based, as it seems, on paper [SAR]).

We replace a polynomial P by $P_r(x) = P(rx)$, where $r \in (0,1]$. We shall have for $x_0 \in S_H$, $r \in (0,1]$

$$\begin{aligned}|P(rx_0)| = |P_r(x_0)| &\le (\deg P + 1) \max_{x \in B_H} \sqrt{1-x^2}\,|P_r(x)| \\ &\le (\deg P+1) \max_{x\in B_H} \sqrt{1-r^2x^2}\,|P(rx)| \le (\deg P+1) \max_{x\in B_H} \sqrt{1-x^2}\,|P(x)|\end{aligned}$$

so we get

$$\|P\|_{B_H} \le (\deg P+1)\big\|\sqrt{1-x^2}\,P(x)\big\|_{B_H}.$$

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