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MAXIMALLY CONVERGENT RATIONAL APPROXIMANTS OF MEROMORPHIC FUNCTIONS

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Dedicated to Wiesław Pleśniak on the occasion of his retirement

Abstract. Let f be meromorphic on the compact set $E \subset \mathbb{C}$ with maximal Green domain of meromorphy $E_{\rho(f)}$, $\rho(f) < \infty$. We investigate rational approximants r_{n,m_n} of f on E with numerator degree $\le n$ and denominator degree $\le m_n$. We show that a geometric convergence rate of order $\rho(f)^{-n}$ on E implies uniform maximal convergence in m_1-measure inside $E_{\rho(f)}$ if $m_n = o(n/\log n)$ as $n \to \infty$. If $m_n = o(n)$, $n \to \infty$, then maximal convergence in capacity inside $E_{\rho(f)}$ can be proved at least for a subsequence $\Lambda \subset \mathbb{N}$. Moreover, an analogue of Walsh's estimate for the growth of polynomial approximants is proved for r_{n,m_n} outside $E_{\rho(f)}$.

1. Introduction. Let K be a compact subset of the complex plane $\mathbb{C}$ and let $\mathcal{B}(K)$ denote the collection of all probability measures with support in K. The logarithmic energy $I(\mu)$ of $\mu \in \mathcal{B}(K)$ is defined by

$$I(\mu) := \int\int \log \frac{1}{|z-t|}\, d\mu(z)\, d\mu(t)$$

and the energy V of K by

$$V := \inf\{I(\mu) : \mu \in \mathcal{B}(K)\}.$$

V is either finite or $+\infty$. The quantity

$$\operatorname{cap} K := e^{-V}$$

is called the logarithmic capacity or capacity of K. The capacity of any set $E \subset \mathbb{C}$ is defined by

$$\operatorname{cap} E := \sup\{\operatorname{cap} K : K \subset E,\ K \text{ compact}\}.$$

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In the following, E is a compact subset of $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ in the extended complex plane. The domain Ω is called *regular* if there exists a Green function $G(z) = G(z, \infty)$ on Ω with pole at ∞ satisfying $G(z) \to 0$ as $z \in \Omega$ tends to the boundary $\partial\Omega$ of Ω. Then $\operatorname{cap} E > 0$ and

$$\lim_{z\to\infty} (G(z) - \log|z|) = -\log \operatorname{cap} E.$$

For $\rho > 1$ we define the Green domains E_ρ by

$$E_\rho := \{z \in \Omega : G(z) < \log \rho\} \cup E$$

with boundaries $\Gamma_\rho := \partial E_\rho$.

For $B \subset \mathbb{C}$, we denote by $\mathcal{M}(B)$ the class of functions that are meromorphic in some open neighborhood of B. If $f \in \mathcal{M}(E)$, then there exists a maximal $\rho(f) > 1$ such that $f \in \mathcal{M}(E_{\rho(f)})$. $\rho(f) = \infty$ if and only if f is meromorphic on $\mathbb{C}$. We use $\|\cdot\|_B$ for the supremum norm on $B \subset \mathbb{C}$.

Given $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $\mathcal{R}_{n,m}$ be the collection of all rational functions,

$$\mathcal{R}_{n,m} := \{r = p/q : p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ q \not\equiv 0\}$$

where $\mathcal{P}_n$ (resp. $\mathcal{P}_m$) denotes the class of all algebraic polynomials of degree at most n (resp. m).

In [2] the starting point was a sequence of rational approximants $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ to $f \in \mathcal{M}(E)$ such that $r_{n,m_n} \in \mathcal{R}_{n,m_n}$ and

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \le \frac{1}{\rho(f)}. \tag{1.1}$$

Due to Walsh's theorem (cf. Walsh [11]), such type of approximants can be obtained for example by choosing $r_{n,m_n} = r^*_{n,m_n}$ as a best rational approximant of f on E, i.e.,

$$\|f - r^*_{n,m_n}\|_E = \inf_{r\in\mathcal{R}_{n,m_n}} \|f - r\|_E,$$

where $\lim_{n\to\infty} m_n = \infty$. In [2] the problem of overconvergence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ outside E was investigated. It turned out that convergence in m_1-measure is an efficient tool: Let B be a subset of $\mathbb{C}$ and define

$$m_1(B) := \inf(\Sigma|U_\nu|),$$

where the infimum is taken over all denumerable coverings $\{U_\nu\}$ of B by disks U_ν and $|U_\nu|$ denotes the radius of U_ν.

Let D be a domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to converge to φ *m_1-almost uniformly inside* D if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K_\varepsilon) < \varepsilon$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ on $K \setminus K_\varepsilon$ (cf. Gončar [3]).

M_1-UNIFORM CONVERGENCE ([2], Theorem 1). *Let $f \in \mathcal{M}(E)$, $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, where $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, be a sequence of rational approximants of f with* (1.1) *and*

$$m_n = o(n/\log n) \text{ as } n \to \infty, \quad \lim_{n\to\infty} m_n = \infty. \tag{1.2}$$

Then there exists an extension $\widetilde{f}$ *of* f *to* $E_{\rho(f)}$ *with the property: For any* $\varepsilon > 0$ *there exists a subset* $\Omega(\varepsilon) \in \mathbb{C}$ *with* $m_1(\Omega(\varepsilon)) < \varepsilon$ *such that* $\widetilde{f}$ *is continuous on* $E_\sigma \setminus \Omega(\varepsilon)$ *and*

$$\limsup_{n\to\infty} \|\widetilde{f} - r_{n,m_n}\|^{1/n}_{E_\sigma\setminus\Omega(\varepsilon)} \le \frac{\sigma}{\rho(f)} \tag{1.3}$$

for any σ *with* $1 < \sigma < \rho(f)$.

Hence, $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ converges m_1-almost uniformly to $\widetilde{f}$ inside $E_{\rho(f)}$. It remained open whether the continuous function $\widetilde{f}$ is always m_1-equivalent to f on $E_{\rho(f)}$. In [1] this problem was investigated by *convergence in capacity*, instead of using m_1-convergence.

Let D be again a domain in $\mathbb{C}$, φ a function in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to *converge in capacity inside* D to φ if for any compact set $K \subset D$ and any $\varepsilon > 0$

$$\operatorname{cap}\big(\{z \in K : |(\varphi - \varphi_n)(z)| \ge \varepsilon\}\big) \to 0 \text{ as } n \to \infty.$$

Moreover, $\{\varphi_n\}_{n\in\mathbb{N}}$ converges *uniformly in capacity inside* D to φ if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $\operatorname{cap} K_\varepsilon < \varepsilon$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ on $K \setminus K_\varepsilon$ (cf. Gončar [3]).

Convergence in capacity ([1]). *Let* E *be compact in* $\mathbb{C}$ *with regular connected complement,* $\{m_n\}_{n\in\mathbb{N}}$ *a sequence in* $\mathbb{N}$ *with*

$$m_n = o(n) \text{ as } n \to \infty, \ \lim_{n\to\infty} m_n = \infty, \tag{1.4}$$

and let $f \in \mathcal{M}(E)$ *and* $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, *a sequence of rational approximants of* f *such that* (1.1) *holds. Then* $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ *converges in capacity to* f *inside* $E_{\rho(f)}$. *Moreover, there exists a subsequence* $\Lambda \subset \mathbb{N}$ *such that the subsequence* $\{r_{n,m_n}\}_{n\in\Lambda}$ *converges uniformly in capacity to* f *inside* $E_{\rho(f)}$.

Now, the aim of the paper is to show first that in the above theorem on m_1-uniform convergence the function $\widetilde{f}$ is m_1-equivalent to f (Section 2). Second, we can sharpen the above result on uniform convergence in capacity to obtain maximal geometric convergence in capacity, but only for a subsequence $\Lambda \subset \mathbb{N}$. Moreover, we prove upper bounds for the growth of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ outside $E_{\rho(f)}$ of Walsh's type (Section 3).

2. Maximal m_1-convergence.

In the following, we always assume that E is a compact set in $\mathbb{C}$ with regular, connected complement and f is meromorphic on E with finite maximal parameter $\rho(f)$ of meromorphy. Moreover, we start with a sequence $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ of rational approximants of f with $r_{n,m_n} \in \mathcal{R}_{n,m_n}$ and

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\partial E} \le \frac{1}{\rho(f)}\,. \tag{2.1}$$

First, we want to show that in Theorem 1 of [2], i.e. in (1.3), we can identify $\widetilde{f}$ with f. As a consequence, we can apply the results in [2] for the growth and the zero distribution of rational approximants, satisfying (2.1) only. Hence, the number of applications—so far known for real rational approximants of real-valued functions f on an interval, and for Padé approximants resp. multipoint Padé approximants—is substantially enlarged.

Theorem 2.1. *Under the above conditions let*

$$m_n = o(n/\log n) \text{ as } n \to \infty, \quad \lim_{n\to\infty} m_n = \infty. \tag{2.2}$$

Then for any $\varepsilon > 0$ *there exists a subset* $\Omega(\varepsilon) \subset \mathbb{C}$ *with* $m_1(\Omega(\varepsilon)) < \varepsilon$ *such that*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{E_\sigma \setminus \Omega(\varepsilon)} \leq \frac{\sigma}{\rho(f)} \tag{2.3}$$

for any σ, $1 < \sigma < \rho(f)$.

The property (2.3) was called in [2] *maximal* m_1*-convergence.*

3. Maximal convergence in capacity. Concerning convergence in capacity we refer to

Theorem 3.1 ([1], Theorem 2.2). *Let* E *be compact in* $\mathbb{C}$ *with regular, connected complement* $\Omega = \overline{\mathbb{C}} \setminus E$, $f \in \mathcal{M}(E)$ *with* $\rho(f) < \infty$, $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ *a sequence with*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\partial E} \leq \frac{1}{\rho(f)}\,, \tag{3.1}$$

and let

$$m_n = o(n) \text{ as } n \to \infty, \quad \lim_{n\to\infty} m_n = \infty. \tag{3.2}$$

If $1 < \sigma < \rho(f)$, *and* $1 < \theta < \rho(f)/\sigma$, *then there exists a number* $n_0 = n_0(\sigma,\theta) \in \mathbb{N}$ *and compact sets* $\Omega_n(\sigma,\theta) \subset \overline{E}_\sigma$ *such that for all* $n \geq n_0(\sigma,\theta)$

$$\operatorname{cap} \Omega_n(\sigma,\theta) \leq d^{1/2}\Big(1 - \frac{\theta - 1}{1 + 3\theta}\Big)^{n/(2m_n)}$$

and

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma \setminus \Omega_n(\sigma,\theta)} \leq \Big(\frac{\theta\sigma}{\rho(f)}\Big)^n,$$

where d *is the diameter of* $E_{\rho(f)}$.

We remark that in [1] this theorem was stated with an arbitrary parameter d greater than the diameter of $E_{\rho(f)}$, originated by Nevanlinna's subadditivity property in [5] (cf. [6]). But any parameter $d \geq \operatorname{diameter}(E_{\rho(f)})$ is allowed for this subadditivity property (cf. Tsuji [8], Ransford [7]). Moreover, the additional condition $d > 1$ in the formulation of Theorem 2.2 in [1] was not used in the proof, but only to shorten the proof of the subsequent theorem on uniform convergence.

Theorem 3.2. *Let* $E, f, \{m_n\}_{n\in\mathbb{N}}$ *and* $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ *as in Theorem* 3.1. *Then there exists a subsequence* $\Lambda \subset \mathbb{N}$ *with the following property*:

For any σ, $1 < \sigma < \rho(f)$, *and any* $\varepsilon > 0$ *there exists a denumerable union of closed sets* $\Omega(\sigma,\varepsilon) \subset \overline{E}_\sigma$ *(a so-called* F_σ*-set) with* $\operatorname{cap}\Omega(\sigma,\varepsilon) < \varepsilon$ *such that*

$$\limsup_{n\in\Lambda, n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{E_\sigma \setminus \Omega(\sigma,\varepsilon)} \leq \frac{\sigma}{\rho(f)}\,. \tag{3.3}$$

In the following, we will use the standard notion of a F_σ-set as the denumerable union of closed sets; there is no connection with the parameter $1 < \sigma < \rho(f)$.

Note, that our construction of Λ in the proof will depend on $\{m_n\}_{n\in\mathbb{N}}$ and on the exceptional sets $\Omega_n(\sigma,\theta)$ of the function f. In contrast to this, Wallin [9] could prove pointwise convergence outside an exceptional set A for general Padé approximants with $\operatorname{cap} A = 0$ for a subsequence Λ constructed a priori depending only on $\{m_n\}_{n\in\mathbb{N}}$.

Analogously to maximal m_1-convergence we define the property of Theorem 3.2 as *maximal convergence in capacity.*

Definition 3.3. Let $f \in \mathcal{M}(E)$ with $\rho(f) < \infty$, $\Lambda \subset \mathbb{N}$. A sequence $\{r_n\}_{n\in\Lambda}$, $r_n \in \mathcal{R}_{n,n}$, is called *maximally convergent in capacity* to f on $E_{\rho(f)}$ if for any $\varepsilon > 0$ and any σ, $1 < \sigma < \rho(f)$, there exists a F_σ-set $\Omega(\sigma,\varepsilon) \subset \overline{E}_\sigma$ with $\operatorname{cap}\Omega(\sigma,\varepsilon) < \varepsilon$ such that

$$\limsup_{n\in\Lambda, n\to\infty} \|f - r_n\|^{1/n}_{\overline{E}_\sigma\setminus\Omega(\sigma,\varepsilon)} \le \frac{\sigma}{\rho(f)}\,.$$

Theorem 3.2 can be generalized slightly by starting directly from a subsequence $\widetilde{\Lambda} \subset \mathbb{N}$ instead of the whole natural numbers $\mathbb{N}$.

Corollary 3.4 (Maximal convergence in capacity). *Under the conditions for E, f, $\{m_n\}_{n\in\mathbb{N}}$ and $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ as in Theorem 3.1, let $\widetilde{\Lambda} \subset \mathbb{N}$ be some subsequence of $\mathbb{N}$. Then there exists a subsequence $\Lambda \subset \widetilde{\Lambda} \subset \mathbb{N}$ such that $\{r_{n,m_n}\}_{n\in\Lambda}$ is maximally convergent to f in capacity on $E_{\rho(f)}$.*

Comparing Theorem 2.1 with Theorem 3.2, we have maximal convergence in capacity only for a *subsequence*, but maximal m_1-convergence for the *whole* sequence, payed on the other hand by a stronger condition for the growth of $\{m_n\}_{n\in\mathbb{N}}$. In this context it is important to note that there exists a constant $\alpha > 0$ such that $m_1(B) \le \alpha \operatorname{cap} B$ for any Borel set $B \subset \mathbb{C}$ (Landkof [4], p. 203). Hence,

$$m_1(\Omega(\sigma,\varepsilon)) \le \alpha \operatorname{cap}\Omega(\sigma,\varepsilon) < \alpha\,\varepsilon,$$

and Theorem 3.2 implies maximal m_1-convergence for the subsequence $\Lambda \subset \mathbb{N}$.

The starting point in this section was the inequality (3.1). So it is quite natural to ask whether this inequality can be generalized to

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\Gamma_\sigma} \le \frac{\sigma}{\rho(f)}$$

for any σ, $1 < \sigma < \rho(f)$, analogous to polynomial approximation. Apparently, this cannot be true since the poles of r_{n,m_n} can destroy such an inequality on Γ_σ. Nevertheless, it is possible to obtain such a result for level lines Γ_{σ_n} with $\lim_{n\to\infty}\sigma_n = \sigma$.

Theorem 3.5. *In addition to the conditions of Theorem 3.1 let E be connected. Then for any σ, $1 < \sigma < \rho(f)$, there exists a sequence*

$$\{\sigma_n\}_{n=1}^{\infty},\ \sigma_n \le \sigma,\ \textit{with}\ \lim_{n\to\infty}\sigma_n = \sigma$$

such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\Gamma_{\sigma_n}} \le \frac{\sigma}{\rho(f)}\,. \tag{3.4}$$

Let us compare this theorem with a result in [2]: By Theorem 2.1 we obtain the m_1-maximal convergence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ to f on $E_{\rho(f)}$. If E is connected, then by Remark 4 in [2] there exists for any σ, $1 < \sigma < \rho(f)$, a sequence $\{\sigma_k\}_{k\in\mathbb{N}}$, $1 < \sigma_k < \rho(f)$, such that $\lim_{k\to\infty}\sigma_k = \sigma$ and

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\Gamma_{\sigma_k}} \le \frac{\sigma_k}{\rho(f)}\,,$$

where σ_k is independent of n. Hence this result is stronger than (3.4), but only under the condition (2.2) which is stronger than (3.2).

Finally, we can estimate the growth of $r_{n,m_n} = p_n/q_{m_n}$, resp. p_n, by normalizing r_{n,m_n} such that $\|q_{m_n}\|_E = 1$. We use a terminology of Walsh, namely the notion of *harmonic majorant.*

Definition 3.6 (cf. Walsh [10]). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of subharmonic functions in $\overline{\mathbb{C}} \setminus \overline{E}_\rho$ ($\rho > 1$). Then this sequence has the Green function

$$G_\rho(z) = G(z) - \log \rho$$

as a *harmonic majorant* in $\overline{\mathbb{C}} \setminus \overline{E}_\rho$ if for any compact set $S \subset \overline{\mathbb{C}} \setminus \overline{E}_\rho$

$$\limsup_{n\to\infty} \max_{z\in S} (f_n(z) - G_\rho(z)) \le 0.$$

Theorem 3.7. *Under the conditions of Theorem* 3.5 *the Green function*

$$G_{\rho(f)}(z) = G(z) - \log \rho(f) \quad \text{of } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$$

is a harmonic majorant for the sequence

$$\left\{ \frac{1}{n} \log |p_n(z)| \right\}_{n\in\mathbb{N}} \text{ in } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)},$$

where $r_{n,m_n} = p_n/q_{m_n}$ *is normalized by* $\|q_{m_n}\|_E = 1$.

4. Proof of Theorem 2.1. First, let us define for ε, $0 < \varepsilon < 1$, the exceptional set $\Omega(\varepsilon)$: We denote by

$$\eta_1, \eta_2, \ldots, \eta_s$$

the finite number of poles of f in E, notified according to their multiplicities. Since f is meromorphic in $E_{\rho(f)}$, the total number of poles of f in $E_{\rho(f)}$ is denumerable. Hence, we may arrange the poles η_i of f outside E, $i > s$, such that

$$G(\eta_i) \le G(\eta_{i+1}) \text{ for } i \ge s. \tag{4.1}$$

If the number of poles of f in $E_{\rho(f)}$ is finite, say $\widetilde{s}$, then we set $\eta_{\tilde{s}+j} = \eta, \ldots, j \ge 1$, where η is a fixed point on $\Gamma_{\rho(f)}$. Therefore, in any case we have defined an infinite sequence

$$\{\eta_i\}_{i=1}^\infty \text{ with } \lim_{i\to\infty} G(\eta_i) = \rho(f),$$

where all poles of f in $E_{\rho(f)}$ are listed and (4.1) is satisfied. Finally, let $\xi_{n,i}$ denote the poles of r_{n,m_n}, according to their multiplicities again. For $0 < \varepsilon < 1$ we define the open sets

$$\Omega_n(\varepsilon) := \bigcup_{\xi_{n,i}} \left\{ z \in \mathbb{C} : |z - \xi_{n,i}| < \frac{\varepsilon}{2n^3} \right\} \cup \left\{ z \in \mathbb{C} : |z - \eta_n| < \frac{\varepsilon}{2n^3} \right\}$$

and

$$\Omega(\varepsilon) := \bigcup_{n=1}^\infty \Omega_n(\varepsilon),$$

where we may assume, without loss of generality, that $m_n \le n - 1$ for all $n \in \mathbb{N}$. Then

$$m_1(\Omega(\varepsilon)) \le \sum_{n=1}^\infty m_1(\Omega_n(\varepsilon)) \le \frac{\varepsilon}{2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\varepsilon}{2} \frac{\pi^2}{6} < \varepsilon.$$

Now, let $r_{n,m_n} = p_n/q_{m_n}$ be normalized with respect to E_{τ^*}, i.e., we fix a parameter $\tau^* > \rho(f)$ such that $0 \in E_{\tau^*}$ and

$$q_{m_n}(z) = \prod_{\xi_{n,i} \in E_{\tau^*}} (z - \xi_{n,i}) \prod_{\xi_{n,i} \notin E_{\tau^*}} \Big(1 - \frac{z}{\xi_{n,i}}\Big). \tag{4.2}$$

Let us fix $1 < \sigma < \rho(f)$, $\delta > 0$ and τ such that

$$1 < \sigma < \rho(f) - \delta < \tau < \rho(f).$$

We denote by h^τ the monic polynomial whose zeros are the poles of f in $\overline{E}_\tau$, counted with their multiplicities. Then

$$(fh^\tau)(z) = f(z)h^\tau(z)$$

is holomorphic in $\overline{E}_\tau$. Let us denote by $p_n^\tau \in \mathcal{P}_n$ the best uniform polynomial approximation of fh^τ on E. Then there exists $n_1 = n_1(\sigma, \delta)$ such that for $n \geq n_1(\sigma, \delta)$ we have degree$(h^\tau) \leq m_n$ and

$$\|fh^\tau - r_{n,m_n} h^\tau\|_{\partial E} \leq \Big(\frac{1}{\rho(f) - \delta}\Big)^n, \tag{4.3}$$

$$\|fh^\tau - p_n^\tau\|_E \leq \Big(\frac{1}{\rho(f) - \delta}\Big)^n, \tag{4.4}$$

$$\|fh^\tau - p_n^\tau\|_{\overline{E}_\sigma} \leq \Big(\frac{\sigma}{\rho(f) - \delta}\Big)^n. \tag{4.5}$$

For (4.3) we have used (3.1); (4.4) and (4.5) follow from the theorem of Bernstein–Walsh (cf. [11]). Combining (4.3) and (4.4), for $n \geq n_1(\sigma, \delta)$

$$\|r_{n,m_n} h^\tau - p_n^\tau\|_{\partial E} \leq 2\Big(\frac{1}{\rho(f) - \delta}\Big)^n.$$

Multiplication by $q_{m_n}(z)$ implies, together with the normalization of q_{m_n} in (4.2), that there exists a constant $c_1 > 0$ such that

$$\|p_n h^\tau - p_n^\tau q_{m_n}\|_E \leq c_1^{m_n} \Big(\frac{1}{\rho(f) - \delta}\Big)^n \quad \text{for } n \geq n_1(\sigma, \delta).$$

Next, we apply the Lemma of Bernstein–Walsh [11] to the polynomial

$$w(z) = p_n h^\tau - p_n^\tau q_{m_n} \in \mathcal{P}_{n+m_n}$$

and obtain for $n \geq n_1(\sigma, \delta)$

$$|w(z)| \leq (c_1\sigma)^{m_n} \Big(\frac{\sigma}{\rho(f) - \delta}\Big)^n \quad \text{for } z \in \overline{E}_\sigma$$

or

$$|r_{n,m_n}(z) h^\tau(z) - p_n^\tau(z)| \leq (c_1\sigma)^{m_n} \Big(\frac{\sigma}{\rho(f) - \delta}\Big)^n \frac{1}{|q_{m_n}(z)|} \tag{4.6}$$

for $z \in \overline{E}_\sigma$ where $q_{m_n}(z) \neq 0$.

Since $\varepsilon < 1$ we can estimate $|q_{m_n}(z)|$ from below for $z \in \overline{E}_{\rho(f)} \setminus \Omega(\varepsilon)$ by

$$|q_{m_n}(z)| \geq \Big(\frac{\varepsilon}{2n^3}\Big)^{m_n} c_2^{m_n}, \tag{4.7}$$

where

$$c_2 := \min_{z \in \overline{E}_{\rho(f)}} \min_{\xi \in \overline{\mathbb{C}} \setminus E_{\tau^*}} \Big|1 - \frac{z}{\xi}\Big| > 0.$$

Next, we choose $n_0 = n_0(\tau)$ such that

$$G(\eta_n) > \log \tau \quad \text{for all } n \geq n_0(\tau).$$

Then $\operatorname{grad}(h^\tau) \leq n_0$ and for $z \in \mathbb{C} \setminus \Omega(\varepsilon)$ we obtain

$$|h^\tau(z)| \geq \Big(\frac{\varepsilon}{2n_0^3}\Big)^{n_0}. \tag{4.8}$$

Hence, for $z \in \overline{E}_\sigma \setminus \Omega(\varepsilon)$

$$|f(z) - r_{n,m_n}(z)| \leq \frac{1}{|h^\tau(z)|}\,|f(z)h^\tau(z) - p_n^\tau(z)| + \frac{1}{|h^\tau(z)q_{m_n}(z)|}\,|p_n(z)h^\tau(z) - p_n^\tau(z)q_{m_n}(z)|$$

and (4.5)–(4.8) imply for $z \in \overline{E}_\sigma \setminus \Omega(\varepsilon)$ and $n \geq n_1(\sigma, \delta)$

$$|f(z) - r_{n,m_n}(z)| \leq \Big(\frac{2n_0^3}{\varepsilon}\Big)^{n_0}\Big(\frac{\sigma}{\rho(f)-\delta}\Big)^n + \Big(\frac{2n_0^3}{\varepsilon}\Big)^{n_0}\Big(\frac{2c_1\sigma n^3}{c_2\varepsilon}\Big)^{m_n}\Big(\frac{\sigma}{\rho(f)-\delta}\Big)^n.$$

Hence

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\overline{E}_\sigma\setminus\Omega(\varepsilon)} \leq \frac{\sigma}{\rho(f)-\delta}\,.$$

Since $\delta > 0$ can be chosen arbitrary small, we get (2.3). ∎

5. Proofs of the theorems in Section 3. First, let us mention some estimates of capacity.

If $E \subset \mathbb{C}$ is compact and $P \subset \mathbb{R}$ is the orthogonal projection of E onto the real line then

$$\operatorname{cap} E \geq \operatorname{cap} P.$$

Let $T : E \to \mathbb{C}$ be a mapping satisfying

$$|T(z) - T(w)| \leq A|z - w| \text{ for } z, w \in E$$

then

$$\operatorname{cap} T(E) \leq A \operatorname{cap} E \quad \text{(contraction property)}.$$

Let $E = \bigcup_{k=1}^\infty E_k$ be a union of Borel sets in $\mathbb{C}$ and if $\operatorname{diameter}(E) \leq d$, then

$$1/\log\frac{d}{\operatorname{cap} E} \leq \sum_{k=1}^\infty 1/\log\frac{d}{\operatorname{cap} E_k}$$

(Nevanlinna [5], cf. Tsuji [8], Ransford [7]).

Let E be compact and connected with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ and let Φ denote the conformal mapping of Ω onto $\Delta = \{z \in \overline{\mathbb{C}} : |z| > 1\}$, normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.

Lemma 5.1. *Let $1 < \rho < \infty$ be fixed and let B be a compact set in $\mathbb{C} \setminus E_\rho$ where E_ρ is the Green domain to the parameter ρ. Then there exists a constant $c = c(\rho) > 0$ such that*

$$\operatorname{cap} \Phi(B) \leq c \operatorname{cap} B.$$

Proof. Let $z, w \in \mathbb{C} \setminus E$ and define

$$H(z,w) := \begin{cases} \dfrac{\Phi(z) - \Phi(w)}{z - w} & \text{for } z \neq w, \\ \Phi'(w) & \text{for } z = w. \end{cases}.$$

Then for fixed $z \in \mathbb{C} \setminus E$ the function $H(z,w)$ is holomorphic as a function of $w \in \mathbb{C} \setminus E$ since

$$\lim_{\substack{w \to z \\ w \neq z}} H(z,w) = \Phi'(z) = H(z,z).$$

Moreover, the function $H(z,w)$ is also holomorphic at the point $w = \infty$ for fixed $z \in \mathbb{C} \setminus E$ and

$$\lim_{w \to \infty} H(z,w) = \lim_{w \to \infty} \frac{\Phi(w)}{w - z} = \frac{1}{\operatorname{cap} E}.$$

Hence, we can apply the maximum principle for the function $H(z,w)$ on the domain $\overline{\mathbb{C}} \setminus E_\rho$ and obtain

$$\max_{w' \in \overline{\mathbb{C}} \setminus E_\rho} |H(z,w')| = \max_{w' \in \Gamma_\rho} |H(z,w')|,$$

i.e., for any $z \in \mathbb{C} \setminus E_\rho$ there exists a point $w_z \in \Gamma_\rho$ such that

$$|H(z,w)| \leq |H(z,w_z)|.$$

Therefore for any $(z,w) \in (\mathbb{C} \setminus E_\rho) \times (\mathbb{C} \setminus E_\rho)$

$$|H(z,w)| \leq |H(z,w_z)| = |H(w_z,z)| \leq |H(w_z,w_{w_z})|,$$

where $(w_z, w_{w_z}) \in \Gamma_\rho \times \Gamma_\rho$.

Now, let us consider the function

$$H(\xi,\eta) \text{ with } (\xi,\eta) \in \Gamma_\rho \times \Gamma_\rho.$$

$H(\xi,\eta)$ is continuous on $\Gamma_\rho \times \Gamma_\rho$ and attains its maximal modulus on $\Gamma_\rho \times \Gamma_\rho$. If we define

$$c = c(\rho) := \max_{(\xi,\eta) \in \Gamma_\rho \times \Gamma_\rho} |H(\xi,\eta)|,$$

then $c(\rho) > 0$ and we have

$$|H(z,w)| \leq c(\rho) \text{ for all } (z,w) \in (\mathbb{C} \setminus E_\rho) \times (\mathbb{C} \setminus E_\rho).$$

Consequently,

$$|\Phi(z) - \Phi(w)| \leq c(\rho)|z - w|$$

for any $z, w \in \mathbb{C} \setminus E_\rho$. Hence the contraction property of the logarithmic capacity yields

$$\operatorname{cap} \Phi(B) \leq c(\rho) \operatorname{cap} B$$

for any compact $B \subset \mathbb{C} \setminus E_\rho$. ∎

Proof of Theorem 3.2. Let us denote by

$$\{\sigma_k : k = 1, 2, 3, \dots\}$$

the denumerable set of rational numbers of the open interval $(1, \rho(f))$. Moreover, let $\{\theta_k\}$ be a strictly decreasing sequence with $\lim_{k \to \infty} \theta_k = 1$.

For $1 < \sigma < \rho(f)$ and $\theta > 1$, let us define as in Theorem 3.1 the set $\Omega_n(\sigma, \theta)$ and the number $n_0(\sigma, \theta)$, i.e., for $n \geq n_0(\sigma, \theta)$

$$|f(z) - r_{n,m_n}(z)| \leq \Big(\frac{\theta\sigma}{\rho(f)}\Big)^n \text{ for } z \in \overline{E}_\sigma \setminus \Omega_n(\sigma, \theta),$$

where $\Omega_n(\sigma, \theta) \subset \overline{E}_\sigma$ is compact and

$$\operatorname{cap} \Omega_n(\sigma, \theta) \leq d^{1/2} \gamma^{n/2m_n} \quad \text{with } \gamma = 1 - \frac{\theta - 1}{1 + 3\theta}\,. \tag{5.1}$$

Replacing (σ, θ) by (σ_1, θ_1), we can find, because of $m_n = o(n)$ as $n \to \infty$, a subsequence $\Lambda_1 = \{n_j^{(1)}\}_{j=1}^\infty$ of $\mathbb{N}$ such that $n_j^{(1)} \geq n_0(\sigma_1, \theta_1)$ for $j \geq 1$ and

$$\frac{m_{n_j^{(1)}}}{n_j^{(1)}} \leq \frac{1}{j^2} \log \frac{1}{\gamma_1} \quad \text{for } j = 1, 2, \ldots,$$

where

$$\gamma_1 = 1 - \frac{\theta_1 - 1}{1 + 3\theta_1}\,.$$

Recursively, we can define subsequences $\Lambda_k = \{n_j^{(k)}\}_{j=k}^\infty \subset \Lambda_{k-1}$ such that $n_j^{(k)} \geq n_0(\sigma_k, \theta_k)$ for $j \geq k$ and

$$\frac{m_{n_j^{(k)}}}{n_j^{(k)}} \leq \frac{1}{j^2} \frac{1}{k^2} \log \frac{1}{\gamma_k} \quad \text{for } j \geq k, \tag{5.2}$$

where

$$\gamma_k = 1 - \frac{\theta_k - 1}{1 + 3\theta_k}\,.$$

Next, we define

$$\Lambda := \{n_k^{(k)}\}_{k=1}^\infty,$$

and we want to show that Λ fulfils the assertions of Theorem 3.2.

Let us consider for $i \geq 1$ the F_σ-sets

$$\widetilde{\Omega}_i := \sum_{k=i}^\infty \sum_{j=i}^\infty \Omega_{n_j^{(k)}}(\sigma_k, \gamma_k),$$

then $\widetilde{\Omega}_1 \supset \widetilde{\Omega}_2 \supset \ldots$. For abbreviation, let

$$a_{k,j} := 1/\log \frac{d}{\operatorname{cap} \Omega_{n_j^{(k)}}(\sigma_k, \gamma_k)}$$

and we choose the parameter $d \geq \max(1, \operatorname{diameter}(E_{\rho(f)}))$. The subadditivity property of Nevanlinna yields

$$1/\log \frac{d}{\operatorname{cap} \widetilde{\Omega}_i} \leq \sum_{k=i}^\infty \sum_{j=i}^\infty a_{k,j} \tag{5.3}$$

for all $i \geq 1$. Moreover, by (5.1)

$$a_{k,j} \leq \Big(\log d - \frac{1}{2}\log d - \frac{n_j^{(k)}}{2m_{n_j^{(k)}}} \log \gamma_k\Big)^{-1}$$

and because of $d \geq 1$ and (5.2), for $j \geq k$

$$a_{k,j} \leq \Big(\frac{n_j^{(k)}}{2m_{n_j^{(k)}}} \log \frac{1}{\gamma_k}\Big)^{-1} \leq \frac{2}{k^2 j^2}\,.$$

Hence, the double series

$$S_1 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{k,j} \leq 2\Big(\sum_{k=1}^{\infty}\frac{1}{k^2}\Big)^2 = \frac{\pi^4}{18}$$

is convergent.

Now, let $0 < \varepsilon < d$ be fixed. Then we can define $i(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=i(\varepsilon)}^{\infty}\sum_{j=i(\varepsilon)}^{\infty} a_{k,j} < \Big(\log \frac{d}{\varepsilon}\Big)^{-1},$$

and by (5.3),

$$\operatorname{cap} \widetilde{\Omega}_{i(\varepsilon)} < \varepsilon. \tag{5.4}$$

Next, we fix $1 < \sigma < \rho(f)$ and $0 < \delta < \rho(f) - \sigma$. Then we can choose $k(\delta) \in \mathbb{N}$ such that

$$k(\delta) \geq i(\varepsilon),\ \ \sigma < \sigma_{k(\delta)} < \sigma + \delta \text{ and } 1 < \theta_{k(\delta)} < 1 + \delta. \tag{5.5}$$

Consequently, for $n = n_j^{(k(\delta))} \in \Lambda_{k(\delta)}$, $j \geq k(\delta)$, and all $z \in \overline{E}_{\sigma_{k(\delta)}} \setminus \Omega_n(\sigma_{k(\delta)}, \theta_{k(\delta)})$

$$|f(z) - r_{n,m_n}(z)| \leq \Big(\frac{\theta_{k(\delta)}\sigma_{k(\delta)}}{\rho(f)}\Big)^n \leq \Big(\frac{(1+\delta)(\sigma+\delta)}{\rho(f)}\Big)^n.$$

Hence, for all $n \in \Lambda_{k(\delta)}$

$$\|f - r_{n,m_n}\|^{1/n}_{\overline{E}_{\sigma_{k(\delta)}}\setminus\widetilde{\Omega}_{i(\varepsilon)}} \leq \frac{(1+\delta)(\sigma+\delta)}{\rho(f)}.$$

Let

$$\Omega(\sigma,\varepsilon) := \widetilde{\Omega}_{i(\varepsilon)} \operatorname{cap} \overline{E}_\sigma,$$

then $\Omega(\sigma,\varepsilon)$ is a F_σ-set in $\overline{E}_\sigma$ and by (5.4)

$$\operatorname{cap} \Omega(\sigma,\varepsilon) \leq \operatorname{cap} \widetilde{\Omega}_{i(\varepsilon)} < \varepsilon.$$

Moreover, for $n \in \Lambda_{k(\delta)}$ we have

$$\|f - r_{n,m_n}\|^{1/n}_{\overline{E}_\sigma\setminus\Omega(\sigma,\varepsilon)} \leq \frac{(1+\delta)(\sigma+\delta)}{\rho(f)}\,.$$

Let $\{\delta_i\}_{i=1}^\infty$ be a strictly decreasing sequence with $\delta_1 < \rho(f) - \sigma$ and $\lim_{i\to\infty}\delta_i = 0$, then we can choose $k(\delta_i)$ such that $k(\delta_i)$ satisfies (5.5), where $\delta = \delta_i$, and moreover, $k(\delta_{i+1}) > k(\delta_i)$. Hence, $\Lambda_{k(\delta_{i+1})} \subset \Lambda_{k(\delta_i)}$ and

$$\limsup_{n\in\Lambda, n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\overline{E}_\sigma\setminus\Omega(\sigma,\varepsilon)} \leq \lim_{i\to\infty} \limsup_{n\in\Lambda_{k(\delta_i)}, n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\overline{E}_\sigma\setminus\Omega(\sigma,\varepsilon)} \leq \frac{\sigma}{\rho(f)}\,. \ \blacksquare$$

The Corollary is proved along the same lines as in the preceding proof, the only difference is to choose Λ_1 as a subsequence of $\widetilde{\Lambda}$.

Proof of Theorem 3.5. By Theorem 3.1, for any σ, $1 < \sigma < \rho(f)$, and any θ, $1 < \theta < \rho(f)/\sigma$, there exists a number $n_0(\sigma,\theta)$ and compact set $\Omega_n(\sigma,\theta) \subset \overline{E}_\sigma$ such that for all $n \geq n_0(\sigma,\theta)$

$$\operatorname{cap}\Omega_n(\sigma,\theta) \leq d^{1/2}\Big(1 - \frac{\theta-1}{1+3\theta}\Big)^{n/(2m_n)}$$

and

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma\setminus\Omega_n(\sigma,\theta)} \leq \Big(\frac{\theta\sigma}{\rho(f)}\Big)^n.$$

Let Φ be the conformal mapping from Ω to the exterior of the unit disk, normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Let $\kappa := (1+\sigma)/2$, then, according to the transfer lemma for the logarithmic capacity, there exists a constant $c = c(\kappa)$ such that

$$\operatorname{cap}\Phi(B) \leq c \operatorname{cap} B$$

for any compact set $B \subset \mathbb{C} \setminus E_\kappa$.

Now, let us define two strictly decreasing sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\varepsilon_1 < (\sigma-1)/2$, and $\{\theta_k\}_{k\in\mathbb{N}}$, $1 < \theta_k < \rho(f)/\sigma$, such that

$$\lim_{k\to\infty} \varepsilon_k = 0 \text{ and } \lim_{k\to\infty} \theta_k = 1.$$

Then we will define the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ by induction: First, starting with ε_1 and θ_1 and using Theorem 3.1, we obtain a natural number $n_0(\sigma,\theta_1)$ and compact sets $\Omega_n(\sigma,\theta_1)$ in $\overline{E}_\sigma$ such that for $n \geq n_0(\sigma,\theta_1)$

$$\operatorname{cap}\Omega_n(\sigma,\theta_1) \leq d^{1/2}\Big(1 - \frac{\theta_1-1}{1+3\theta_1}\Big)^{n/(2m_n)}$$

and

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma\setminus\Omega_n(\sigma,\theta_1)} \leq \Big(\frac{\theta_1\sigma}{\rho(f)}\Big)^n.$$

Define

$$\varepsilon_{n,1} := d^{1/2}\Big(1 - \frac{\theta_1-1}{1+3\theta_1}\Big)^{n/(2m_n)},$$

then we can find $n_1 = n_1(\sigma,\theta_1) > n_0(\sigma,\theta_1)$ such that with $c = c(\kappa)$ of the previous lemma

$$\varepsilon_{n,1} \leq \varepsilon_1/(8c) \quad \text{for all } n \geq n_1(\sigma,\theta_1).$$

Next, we use polar coordinates in $\mathbb{C}$ and we consider the annulus

$$R_1 := \{z \in \mathbb{C} : \sigma - \varepsilon_1 \leq |z| \leq \sigma\}$$

and the projection $p_1 : \mathbb{C} \to \mathbb{R}_+$, defined by $p_1(z) := |z|$. The contraction principle for the logarithmic capacity implies

$$\operatorname{cap} p_1(R_1) = \operatorname{cap}([\sigma - \varepsilon_1, \sigma]) = \varepsilon_1/4. \tag{5.6}$$

Then we use the compact subsets

$$\widetilde{\Omega}_n(\sigma,\theta_1) := \Omega_n(\sigma,\theta_1)\operatorname{cap}(\mathbb{C} \setminus E_\kappa)$$

in $\overline{E}_\sigma \setminus E_\kappa$. The transfer lemma yields

$$\operatorname{cap}\Phi(\widetilde{\Omega}_n(\sigma,\theta_1)) \leq c \operatorname{cap}\widetilde{\Omega}_n(\sigma,\theta_1).$$

Hence, for $n \geq n_1(\sigma, \theta_1)$

$$\operatorname{cap} p_1(\Phi(\widetilde{\Omega}_n(\sigma,\theta_1))) \leq \operatorname{cap}\Phi(\widetilde{\Omega}_n(\sigma,\theta_1)) \leq c \operatorname{cap}\widetilde{\Omega}_n(\sigma,\theta_1) \leq c \operatorname{cap}\Omega_n(\sigma,\theta_1) \leq c\,\varepsilon_{n,1} \leq \varepsilon_1/8. \tag{5.7}$$

Combining (5.6) and (5.7), we conclude that there exists a parameter $\sigma_{n,1}$ with

$$\sigma - \varepsilon_1 \leq \sigma_{n,1} \leq \sigma \tag{5.8}$$

such that the level line $\Gamma_{\sigma_{n,1}}$ of Green's function $G(z)$ satisfies for all $n \geq n_1(\sigma,\theta_1)$

$$\Gamma_{\sigma_{n,1}} \subset \overline{E}_\sigma \setminus \Omega_n(\sigma,\theta_1)$$

and therefore,

$$\|f - r_{n,m_n}\|_{\Gamma_{\sigma_{n,1}}} \leq \Big(\frac{\theta_1\sigma}{\rho(f)}\Big)^n.$$

Now, let us assume that we have already defined $n_k(\sigma,\theta_k)$ with

$$n_k(\sigma,\theta_k) > n_{k-1}(\sigma,\theta_{k-1})$$

and $\sigma_{n,k}$ for $n \geq n_k(\sigma,\theta_k)$ with $\sigma - \varepsilon_k \leq \sigma_{n,k} \leq \sigma$ such that for $n \geq n_k(\sigma,\theta_k)$

$$\|f - r_{n,m_n}\|_{\Gamma_{\sigma_{n,k}}} \leq \Big(\frac{\theta_k\sigma}{\rho(f)}\Big)^n.$$

Then as above, there exists $n_0(\sigma,\theta_{k+1}) > n_k(\sigma,\theta_k)$ and compact sets $\Omega_n(\sigma,\theta_{k+1}) \subset E_{\overline{\sigma}}$ for $n \geq n_0(\sigma,\theta_{k+1})$ such that

$$\operatorname{cap}\Omega_n(\sigma,\theta_{k+1}) \leq \varepsilon_{n,k+1} := d^{1/2}\Big(1 - \frac{\theta_{k+1}-1}{1+3\theta_{k+1}}\Big)^{n/(2m_n)}$$

for all $n \geq n_0(\sigma,\theta_{k+1})$ and

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma \setminus \Omega_n(\sigma,\theta_{k+1})} \leq \Big(\frac{\theta_{k+1}\sigma}{\rho(f)}\Big)^n.$$

Again, we can find $n_{k+1}(\sigma,\theta_{k+1}) \geq n_0(\sigma,\theta_{k+1})$ such that

$$\varepsilon_{n,k+1} \leq \varepsilon_{k+1}/(8c) \text{ for } n \geq n_{k+1}(\sigma,\theta_{k+1}),$$

where $c = c(\kappa)$. Let

$$R_{k+1} = \{z \in \mathbb{C} : \sigma - \varepsilon_{k+1} \leq |z| \leq \sigma\}$$

then

$$\operatorname{cap} p_1(R_{k+1}) = \frac{\varepsilon_{k+1}}{4}. \tag{5.9}$$

As above, let

$$\widetilde{\Omega}_n(\sigma,\theta_{k+1}) = \Omega_n(\sigma,\theta_{k+1}) \operatorname{cap}(\mathbb{C} \setminus E_\kappa),$$

then for $n \geq n_1(\sigma,\theta_{k+1})$

$$\operatorname{cap} p_1(\Phi(\widetilde{\Omega}_n(\sigma,\theta_{k+1})) \leq \operatorname{cap}\Phi(\widetilde{\Omega}_n(\sigma,\theta_{k+1})) \leq c\operatorname{cap}\widetilde{\Omega}_n(\sigma,\theta_{k+1}) \leq c\operatorname{cap}\Omega_n(\sigma,\theta_{k+1}) \leq c\,\varepsilon_{n,k+1} \leq \varepsilon_{k+1}/8. \tag{5.10}$$

(5.9) together with (5.10) implies that there exist parameters $\sigma_{n,k+1}$ with

$$\sigma - \varepsilon_{k+1} \leq \sigma_{n,k+1} \leq \sigma$$

such that

$$\Gamma_{\sigma_{n,k+1}} \subset \overline{E}_\sigma \setminus \Omega_n(\sigma, \theta_{k+1})$$

and therefore, for $n \geq n_{k+1}(\sigma, \theta_{k+1})$

$$\|f - r_{n,m_n}\|_{\Gamma_{\sigma_{n,k+1}}} \leq \Big(\frac{\theta_{k+1}\sigma}{\rho(f)}\Big)^n.$$

Finally, we define for all $n \geq n_1(\sigma, \theta_1)$ and $k \geq 1$

$$\sigma_n := \sigma_{n,k} \text{ for } n_k(\sigma, \theta_k) \leq n < n_{k+1}(\sigma, \theta_{k+1})$$

and obtain

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\Gamma_{\sigma_n}} \leq \frac{\sigma}{\rho(f)}\,.$$

In addition we have

$$\lim_{n\to\infty} \sigma_n = \sigma \text{ and } \sigma_n \leq \sigma \text{ for all } n \geq n_1(\sigma, \theta_1). \ \blacksquare$$

Proof of Theorem 3.7. Fix σ, $1 < \sigma < \rho(f)$. According to Theorem 3.5 there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$, $\sigma_n \leq \sigma$, with $\lim_{n\to\infty} \sigma_n = \sigma$, and a sequence $\{\theta_n\}_{n\in\mathbb{N}}$, $\theta_n > 1$, with $\lim_{n\to\infty} \theta_n = 1$ such that

$$\|f - r_{n,m_n}\|_{\Gamma_{\sigma_n}} \leq \Big(\frac{\theta_n \sigma}{\rho(f)}\Big)^n$$

for all sufficiently large $n \geq n_0 \in \mathbb{N}$. Let us denote by h^σ the monic polynomial whose zeros are the poles of f in $\overline{E}_\sigma$, counted with their multiplicities. Then fh^σ is holomorphic in $\overline{E}_\sigma$. Let $m = \text{degree}(h^\sigma)$ then

$$\|fh^\sigma q_{m_n} - p_n h^\sigma\|_{\Gamma_{\sigma_n}} \leq \Big(\frac{\theta_n\sigma}{\rho(f)}\Big)^n \|h^\sigma q_{m_n}\|_{\Gamma_\sigma}$$

and

$$\|p_n h^\sigma\|_{\Gamma_{\sigma_n}} \leq \Big(\frac{\theta_n\sigma}{\rho(f)}\Big)^n \|h^\sigma\|_{\Gamma_\sigma} \|q_{m_n}\|_{\Gamma_\sigma} + \|fh^\sigma\|_{\Gamma_\sigma}\|q_{m_n}\|_{\Gamma_\sigma}.$$

Since fh^σ is holomorphic in E_σ and $\|q_{m_n}\|_E = 1$,

$$\|p_n h^\sigma\|_{\Gamma_{\sigma_n}} \leq \Big(\frac{\theta_n\sigma}{\rho(f)}\Big)^n \sigma^{m_n}\|h^\sigma\|_{\Gamma_\sigma} + \sigma^{m_n}\|fh^\sigma\|_{\Gamma_\sigma}$$

and we can choose $n_1 \geq n_0 \in \mathbb{N}$ and a constant $c_1 > 0$ such that

$$\|p_n h^\sigma\|_{\Gamma_{\sigma_n}} \leq c_1 \sigma^{m_n} \text{ for } n \geq n_1.$$

Then the Bernstein–Walsh Lemma yields for $z \in \mathbb{C} \setminus \overline{E}_{\sigma_n}$

$$|p_n(z)h^\sigma(z)| \leq c_1\, \sigma^{m_n} e^{(n+m)G_{\sigma_n}(z)} \tag{5.11}$$

for $n \geq n_1$.

Let S be compact in $\overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$, then we can choose τ such that

$$\tau > \rho(f) \text{ and } S \subset \overline{\mathbb{C}} \setminus E_\tau$$

and we get

$$\alpha := \min_{z\in\Gamma_\tau} \min_{\xi\in\overline{E}_{\rho(f)}} |z - \xi| > 0$$

and

$$\min_{z\in\Gamma_\tau} |h^\sigma(z)| \geq \alpha^m.$$

Now, (5.11) implies for $z \in \Gamma_\tau$ and $n \geq n_1$

$$\frac{1}{n}\log|p_n(z)| \leq \frac{1}{n}\log c_1 + \frac{m_n}{n}\log\sigma + \frac{n+m}{n}G_{\sigma_n}(z) + \frac{m}{n}\log\frac{1}{\alpha}$$

and, together with $m_n = o(n)$,

$$\limsup_{n\to\infty}\max_{z\in\Gamma_\tau}\Big(\frac{1}{n}\log|p_n(z)| - G_{\sigma_n}(z)\Big) \leq 0.$$

Since the Green functions $G_{\sigma_n}(z)$ converge uniformly to $G_\sigma(z)$ on Γ_τ as $n \to \infty$, we have got finally

$$\limsup_{n\to\infty}\max_{z\in\Gamma_\tau}\Big(\frac{1}{n}\log|p_n(z)| - G_{\sigma}(z)\Big) \leq 0.$$

The function

$$\frac{1}{n}\log|p_n(z)| - G_\sigma(z)$$

is subharmonic on $\overline{\mathbb{C}} \setminus \overline{E}_\sigma$. Hence the maximum principle implies

$$\max_{z\in S}\Big(\frac{1}{n}\log|p_n(z)| - G_{\sigma}(z)\Big) \leq \max_{z\in\Gamma_\tau}\Big(\frac{1}{n}\log|p_n(z)| - G_{\sigma}(z)\Big).$$

Consequently,

$$\limsup_{n\to\infty}\max_{z\in S}\Big(\frac{1}{n}\log|p_n(z)| - G_{\sigma}(z)\Big) \leq 0.$$

But this holds for any $\sigma < \rho(f)$. The uniform convergence of $G_\sigma(z)$ on S for $\sigma \to \rho(f)$ leads to

$$\limsup_{n\to\infty}\max_{z\in S}\Big(\frac{1}{n}\log|p_n(z)| - G_{\rho(f)}(z)\Big) \leq 0.$$

This inequality holds for any compact S in $\overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$. Hence, by definition the sequence

$$\Big\{\frac{1}{n}\log|p_n(z)|\Big\}_{n\in\mathbb{N}}$$

has $G_{\rho(f)}(z)$ as a harmonic majorant. ∎

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