CONSTRUCTIVE APPROXIMATION OF FUNCTIONS
BANACH CENTER PUBLICATIONS, VOLUME 107
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2015

TRANSFINITE DIAMETER ON CURVES, HANKEL DETERMINANTS AND THE MOMENT PROBLEM

LEN BOS

Department of Computer Science, University of Verona
strada Le Grazie 15, Verona, Italy
E-mail: leonardpeter.bos@univr.it

SIONE MA'U

Department of Mathematics, University of Auckland
Auckland, New Zealand
E-mail: s.mau@auckland.ac.nz

Abstract. We discuss problems on Hankel determinants and the classical moment problem related to and inspired by certain Vandermonde determinants for polynomial interpolation on (quadratic) algebraic curves in $\mathbb{C}^2$.

1. Introduction. Suppose that $K \subset \mathbb{C}^d$ is compact and that $\mathcal{P}_n(K)$ is the space of algebraic polynomials of degree at most n restricted to K. For a basis $\mathcal{B}_n := \{P_1, \ldots, P_N\} \subset \mathcal{P}_n(K)$ with $N := \dim(\mathcal{P}_n(K))$, and N points $X_n := \{z_1, \ldots, z_N\} \subset K$, we may form the so-called Vandermonde determinant

$$\mathrm{vdm}(\mathcal{B}_n; X_n) := \det([P_i(z_j)]_{1\le i,j\le N}).$$

Write

$$V(\mathcal{B}_n; K) := \max\{|\mathrm{vdm}(\mathcal{B}_n, X_n)| : X_n \subseteq K\};$$

a set of points X_n at which the maximum is attained is called a set of *Fekete points*, and

$$\lim_{n\to\infty} V(\mathcal{B}_n; K)^{1/\ell_n}$$

(if it exists) is called the *transfinite diameter of K*. The exponent ℓ_n is the degree of homogeneity of the determinant, whose exact value is not yet important.

2010 *Mathematics Subject Classification*: Primary 31A15; Secondary 44A60.
Key words and phrases: quadratic complex curve, Vandermonde determinant, Hankel determinant, Fekete points, Transfinite diameter.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc107-0-6

Fekete points and transfinite diameter (capacity) are topics of classical complex analysis. Of particular importance are the recent papers of Berman and Boucksom (e.g. [1, 2]) where they give a very general treatment. Their theory does apply when the underlying space is an algebraic complex variety but their formulas require the normalization by the known value of the transfinite diameter of an example set. Recently, the problem of transfinite diameter on algebraic varieties has been studied from the algebraic point of view in [4].

This current work began with the attempt to compute some explicit examples of transfinite diameter on algebraic varieties. Surprisingly, already the case of the simplest such variety, i.e., a quadratic curve, leads directly and intimately into some classical problems of the asymptotics of Hankel determinants and the Moment Problem, and here we report on some of these investigations.

For a non-degenerate quadratic curve in $\mathbb{C}^2$

$$V := \big\{(x,y) \in \mathbb{C}^2 : Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\big\}$$

we may assume without loss of generality that $C \neq 0$. In this case it is easy to verify that

$$\mathcal{B}_n := \{1, x, x^2, \ldots, x^n\} \cup y\{1, x, x^2, \ldots, x^{n-1}\},$$

of dimension $2n+1$, is a basis for $\mathcal{P}_n(V)$, and $\ell_n = n(n+1)$. It is known that $\lim_{n\to\infty} V(\mathcal{B}_n, K)^{1/\ell_n}$ exists, where $K \subset V$ is a compact set (see [4, Theorem 6.7]).

For an explicit example, it is reasonable to consider the case of a set in a variety that is the graph above the unit circle, i.e., a set K of the form $K = \{(x,y) \in \mathbb{C}^2 : |x| = 1, y = A(x)\}$. We first give a formula for the associated Vandermonde determinant evaluated at points over the roots of unity.

Proposition 1.1. *Suppose that* $A(z) = \sum_{j=0}^{\infty} a_j z^j$ *has radius of convergence strictly greater than* 1*. Consider the basis of bivariate polynomials*

$$\mathcal{B}_n := \{1, x, x^2, \ldots, x^n\} \cup y\{1, x, x^2, \ldots, x^{n-1}\} = \{b_j(x,y) : 0 \le j \le 2n\} \quad (say)$$

and the $2n+1$ *points*

$$X_n := \{(\omega^k, A(\omega^k)) : 0 \le k \le 2n\} = \{(x_i, y_i) : 0 \le i \le 2n\} \quad (say)$$

where $\omega := \exp(2\pi i/(2n+1))$ *is the* $(2n+1)$*-th principal root of unity. Further, let*

$$Q_{2n}(z) := \sum_{k=0}^{2n} q_k z^k$$

be the univariate polynomial of degree at most $2n$ *that interpolates* $A(z)$ *at the points of* X_n*. Consider the associated (bivariate) Vandermonde determinant*

$$\mathrm{vdm}(\mathcal{B}_n; X_n) := \det[b_j(x_i, y_i)]_{0 \le i,j \le 2n}.$$

Then

$$\mathrm{vdm}(\mathcal{B}_n; X_n) = \pm\mathrm{vdm}(\{1, x, \ldots, x^{2n}\}; \{1, \omega, \ldots, \omega^{2n}\}) \times \det \begin{bmatrix} q_2 & q_3 & \cdot & \cdot & \cdot & q_{n+1} \\ q_3 & q_4 & \cdot & \cdot & \cdot & q_{n+2} \\ q_4 & q_5 & \cdot & \cdot & \cdot & q_{n+3} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ q_{n+1} & q_{n+2} & \cdot & \cdot & \cdot & q_{2n} \end{bmatrix}.$$

Proof. First note that since $Q_{2n}(z)$ is the interpolant of $A(z)$ at $\{1, \omega, \omega^2, \ldots, \omega^{2n}\}$,

$$X_n = \{(\omega^k, A(\omega^k)) : 0 \le k \le 2n\} = \{(\omega^k, Q_{2n}(\omega^k)) : 0 \le k \le 2n\} =: \widetilde{X}_n, \quad \text{say.}$$

Hence

$$\mathrm{vdm}(\mathcal{B}_n; X_n) = \mathrm{vdm}(\mathcal{B}_n; \widetilde{X}_n).$$

Further, note that at the points of $\widetilde{X}_n$ the values of the basis functions are those of the functions

$$\widetilde{\mathcal{B}}_n := \{1, x, \ldots x^n\} \cup Q_{2n}(x)\{1, x, \ldots, x^{n-1}\}.$$

Hence

$$\mathrm{vdm}(\mathcal{B}_n; X_n) = \mathrm{vdm}(\widetilde{\mathcal{B}}_n; \widetilde{X}_n).$$

Now consider an element of $\widetilde{\mathcal{B}}_n$ of the form

$$x^r Q_{2n}(x) = x^r \sum_{k=0}^{2n} q_k z^k = \sum_{k=0}^{2n} q_k z^{k+r}, \quad 0 \le r \le n-1.$$

Hence, at one of the roots of unity, ω^s, we have

$$x^r Q_{2n}(x)|_{x=\omega^s} = \sum_{k=0}^{2n} q_k (\omega^s)^{k+r}.$$

Then extending the definition of the coefficients q_k cyclically, i.e., setting $q_m = q_k$ for $m \equiv k \bmod (2n+1)$, and using the fact that ω^s is a $(2n+1)$-th root of unity, we obtain

$$x^r Q_{2n}(x)|_{x=\omega^s} = \sum_{k=0}^{2n} q_{k-r} (\omega^s)^k.$$

In other words, $x^r Q_{2n}(x)$ agrees with

$$Q_{2n}^{(r)}(x) := \sum_{k=0}^{2n} \overrightarrow{q}_k^{(r)} x^k$$

where $\overrightarrow{q}^{(r)} := (q_r, \ldots, q_{2n}, q_0, \ldots, q_{r-1})$ is the vector $(q_0, q_1, \ldots, q_{2n})$ of coefficients of $Q_{2n}(x)$ permuted cyclically r times to the right.

It follows that

$$\mathrm{vdm}(\mathcal{B}_n; X_n) = \mathrm{vdm}(\mathcal{B}'_n; X'_n)$$

where

$$\mathcal{B}'_n := \{1, x, x^2, \ldots, x^n, Q_{2n}^{(0)}(x), Q_{2n}^{(1)}(x), \ldots, Q_{2n}^{(n-1)}\}$$

and

$$X_n' := \{1, \omega, \omega^2, \ldots, \omega^{2n}\}.$$

Now, the transition matrix, in block form, from $\{1, x, x^2, \ldots, x^{2n}\}$ (the standard basis of $\mathcal{P}_{2n}(\mathbb{C})$) to the basis $\mathcal{B}_n'$ is

$$T_n := \left[\begin{array}{c|cccc} I_{n+1} & & 0 & & \\ \hline & q_{n+1} & q_{n+2} & \cdots & q_{2n} \\ & q_n & q_{n+1} & \cdots & q_{2n-1} \\ * & q_{n-1} & q_n & \cdots & q_{2n-2} \\ & \vdots & \vdots & & \vdots \\ & q_2 & q_3 & \cdots & q_{n+1} \end{array}\right] \in \mathbb{R}^{(2n+1)\times(2n+1)}$$

where $I_{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}$ is the identity matrix. We easily see that

$$\det(T_n) = \det(I_{n+1}) \times \det \begin{bmatrix} q_{n+1} & q_{n+2} & \cdots & q_{2n} \\ q_n & q_{n+1} & \cdots & q_{2n-1} \\ q_{n-1} & q_n & \cdots & q_{2n-2} \\ \vdots & \vdots & & \vdots \\ q_2 & q_3 & \cdots & q_{n+1} \end{bmatrix}$$

(the Toeplitz form). Inverting the rows we get

$$\det(T_n) = \pm \det \begin{bmatrix} q_2 & q_3 & \cdots & q_{n+1} \\ q_3 & q_4 & \cdots & q_{n+2} \\ q_4 & q_5 & \cdots & q_{n+3} \\ \vdots & \vdots & & \vdots \\ q_{n+1} & q_{n+2} & \cdots & q_{2n} \end{bmatrix},$$

the Hankel form.

Note further that

$$\mathrm{vdm}\big(\{1, x, \ldots, x^{2n}\}; \{1, \omega, \ldots, \omega^{2n}\}\big)$$

is the classical Vandermonde determinant at the roots of unity. In this particular case, the Vandermonde matrix is just $[\omega^{(i-1)(j-1)}]_{1\le i,j\le 2n+1}$, i.e., the classical Fourier matrix $\mathcal{F}_{2n+1}$. Since, as is well known, $\mathcal{F}_{2n+1}^*\mathcal{F}_{2n+1} = (2n+1)I_{2n+1}$,

$$|\det(\mathcal{F}_{2n+1})| = (2n+1)^{(2n+1)/2}$$

and we may actually conclude that

$$|\mathrm{vdm}(\mathcal{B}_n; X_n)| = (2n+1)^{(2n+1)/2} \left| \det \begin{bmatrix} q_2 & q_3 & \cdots & q_{n+1} \\ q_3 & q_4 & \cdots & q_{n+2} \\ q_4 & q_5 & \cdots & q_{n+3} \\ \vdots & \vdots & & \vdots \\ q_{n+1} & q_{n+2} & \cdots & q_{2n} \end{bmatrix} \right|. \quad \blacksquare$$

The following explicit formula for an interpolating polynomial of an analytic function at the roots of unity will be of use to us.

Lemma 1.2. *Suppose that $A(z) = \sum_{j=0}^{\infty} a_j z^j$ has radius of convergence strictly greater than 1. Then the polynomial $p_m(z)$ of degree at most $m-1$ that interpolates $A(z)$ at the m-th roots of unity, ω^r, $r = 0, 1, \ldots, (m-1)$, $\omega := \exp(2\pi i/m)$, is given by*

$$p_m(z) = \sum_{k=0}^{m-1} q_k z^k, \quad \textit{where} \quad q_k := \sum_{j=0}^{\infty} a_{k+mj}.$$

Proof. We calculate, for $0 \le r \le m-1$,

$$A(\omega^r) = \sum_{i=0}^{\infty} a_i \omega^{ri}.$$

Now, writing $i = mj + k$ for $j \ge 0$ and $0 \le k \le m-1$, we obtain

$$\begin{aligned} A(\omega^r) &= \sum_{j=0}^{\infty}\Big[\sum_{k=0}^{m-1}\big\{a_{mj+k}\omega^{r(mj+k)}\big\}\Big] = \sum_{j=0}^{\infty}\Big[\sum_{k=0}^{m-1}\big\{a_{mj+k}(\omega^m)^{rj}(\omega^r)^k\big\}\Big] \\ &= \sum_{j=0}^{\infty}\Big[\sum_{k=0}^{m-1}\big\{a_{mj+k} \times 1 \times (\omega^r)^k\big\}\Big] = \sum_{k=0}^{m-1}\Big[\sum_{j=0}^{\infty} a_{mj+k}\Big](\omega^r)^k \\ &= \sum_{k=0}^{m-1} q_k(\omega^r)^k = p_m(\omega^r), \end{aligned}$$

i.e., $p_m(z)$ and $A(z)$ have the same values at the roots of unity. ∎

Definition 1.3. A formal power series $F(z) = \sum_{k=0}^{\infty} a_k z^k$ is said to be the *generating function* of the Hankel matrices

$$H_n := [a_{i+j-1}]_{1 \le i,j \le n} \in \mathbb{C}^{n \times n}, \quad n = 1, 2, \ldots.$$

Example 1. Consider $F(z) = \sum_{k=1}^{\infty} z^k/k = -\log(1-z)$, $|z| < 1$. Then $H_n = [1/(i+j-1)] \in \mathbb{R}^{n \times n}$ is the classical Hilbert matrix. It is known that

$$\det(H_n) = \frac{\left(\prod_{j=1}^{n-1} j!\right)^4}{\prod_{j=1}^{2n} j!}$$

from which one may easily conclude that

$$\lim_{n\to\infty} (\det(H_n))^{1/n^2} = \frac{1}{4}\,.$$

One might notice that the limit 1/4 is the transfinite diameter (also the logarithmic capacity) of the interval $[0,1]$. This is not by coincidence. Indeed, for the Hilbert matrix,

$$(H_n)_{ij} = \frac{1}{i+j-1} = \int_0^1 x^{i-1}x^{j-1}\,dx$$

so that $H_n = G_n(\{1, x, x^2, \ldots, x^{n-1}\}; dx)$, the Gram matrix of the basis

$$\{1, x, x^2, \ldots, x^{n-1}\}$$

with respect to Lebesgue measure on $[0,1]$. Recall that for a basis $\{e_1, \ldots, e_n\}$ of $\mathcal{P}_{n-1}(\mathbb{C})$ and finite Borel measure μ on $\mathbb{C}$, $G_n = G_n(\{e_1, \ldots, e_n\}, \mu)$ is the $n \times n$ matrix whose (i,j)-th entry is $\int_{\mathbb{C}} e_i \bar{e}_j\, d\mu$.

It is a basic fact that, for a measure μ with support $S(\mu)=K\subset\mathbb{C}$ sufficiently regular,

$$\lim_{n\to\infty}\det(G_n)^{1/n^2}=\mathrm{cap}(K).$$

Indeed, this is the topic of the recent paper by Bloom [3]. We give a summary of the important facts.

First, let $p_0,p_1,\ldots$ be the *monic* orthogonal polynomials (with $\deg(p_n)=n$) with respect to the measure μ. Stahl and Totik [10] say that the measure is *regular* (class **Reg**) if

$$\lim_{n\to\infty}\left(\|p_n\|_{L^2(\mu)}\right)^{1/n}=\mathrm{cap}(K).$$

Bloom then proves (his Theorem 1.2) that μ is regular in this sense if and only if

$$\lim_{n\to\infty}\det(G_n)^{1/n^2}=\mathrm{cap}(K).$$

For the example of Lebesgue measure on $[0,1]$ the orthogonal polynomials are $p_n(x)=C_nP_n(2x-1)$ where $P_n(x)$ are the classical Legendre polynomials and C_n is the appropriate normalization constant. Indeed, it is easy to check that $C_n=\frac{(n!)^2}{(2n)!}$ so that

$$\|p_n\|_{L^2(dx)}=\frac{1}{2n+1}\,\frac{(n!)^2}{(2n)!}$$

from which it easily follows that

$$\lim_{n\to\infty}\left(\|p_n\|_{L^2(\mu)}\right)^{1/n}=1/4=\mathrm{cap}([0,1])$$

and we have Stahl–Totik regularity. From Bloom's theorem we then have

$$\lim_{n\to\infty}\det(H_n)^{1/n^2}=\lim_{n\to\infty}\det(G_n)^{1/n^2}=\mathrm{cap}([0,1])=\frac{1}{4}\,,$$

as we saw by direct calculation.

It is worth noting that Stahl and Totik [10, Theorem 3.2.3] show that, when K is regular for the exterior Dirichlet problem, μ is of class **Reg** iff the uniform norm of a polynomial on K is subexponentially equivalent to its $L^2(\mu)$ norm. Specifically, iff

$$\lim_{n\to\infty}\Bigg(\sup_{\deg(P)\le n,\,P\not\equiv 0}\frac{\|P\|_K}{\|P\|_{L^2(\mu)}}\Bigg)^{1/n}=1. \tag{1}$$

Sometimes this latter condition is referred to as the Bernstein–Markov property.

Stahl and Totik [10, Theorem 4.1.1] give the following sufficient condition for a measure to be in **Reg**, generalizing that given by Erdős and Turán for the interval $[-1,1]$.

PROPOSITION 1.4. *Suppose that $K\subset\mathbb{C}$ is compact and regular for the exterior Dirichlet problem. If $d\mu$ is a measure such that*

$$d\mu(z)=w(z)\,d\mu_K(z)\quad \textit{with } w(x)>0 \textit{ a.e.}\quad d\mu_K,$$

*where $d\mu_K$ is the equilibrium measure for K, then $d\mu\in$ **Reg**.*

2. The Goluzin–Pólya and Heine integral formulas. Goluzin [7, p. 307] (cf. Edrei [6, (2,29)]) gives a remarkable integral formula for Hankel determinants. Specifically, suppose that $E\subset\mathbb{C}$ is compact and that B denotes the component of the complement of E that contains ∞. Suppose further that $f(z)$ is regular in B and has the expansion

$f(z) = \sum_{k=1}^{\infty} a_k z^{-k}$, convergent in a neighbourhood of ∞. Then for $H_n \in \mathbb{C}^{n\times n}$ the Hankel matrix generated by $F(z) := f(1/z)$,

$$n!\det(H_n) = \frac{1}{(2\pi i)^n}\int_\gamma\int_\gamma\cdots\int_\gamma f(z_1)f(z_2)\cdots f(z_n)\mathrm{vdm}^2(z_1,\dots,z_n)\,dz_1\dots dz_n. \quad (2)$$

Here $\mathrm{vdm}(z_1,\dots,z_n) = \prod_{1\le i<j\le n}(z_j - z_i)$ denotes the classical univariate Vandermonde determinant and $\gamma \subset B$ is a closed Jordan curve that encloses E.

As a consequence of this formula we have the following, first proved by Pólya in 1928.

THEOREM 2.1 (Pólya [9] but see also [7, Theorem 3]). *Under the above assumptions on E and $f(z)$,*

$$\limsup_{n\to\infty} |\det(H_n)|^{1/n^2} \le \mathrm{cap}(E).$$

We are interested in when the limit exists (and is *equal* to $\mathrm{cap}(E)$). Suppose then that $F(z)$ is the (shifted) generating function of the moments of a measure μ supported on a real interval $E = [a,b]$, i.e.,

$$F(z) = \sum_{k=1}^{\infty}\Big(\int_a^b x^{k-1}\,d\mu(x)\Big)z^k$$

so that

$$\begin{aligned} F(z) &= \sum_{k=1}^{\infty}\Big(\int_a^b x^{k-1}\,d\mu(x)\Big)z^k = \int_a^b z\Big(\sum_{k=1}^{\infty}(xz)^{k-1}\Big)\,d\mu(x) \\ &= \int_a^b z\frac{1}{1-xz}\,d\mu(x) = \int_a^b \frac{z}{1-xz}\,d\mu(x) \end{aligned} \quad (3)$$

and

$$f(z) = F(1/z) = \int_a^b \frac{1}{z-x}\,d\mu(x) \quad (4)$$

which, by analytic continuation, are valid for $z \in B$. In other words, $f(z)$ is the Cauchy transform of the measure μ. Inserting this formula for $f(z)$ in the Goluzin–Pólya formula (2), we obtain

$$\begin{aligned} n!\det(H_n) &= \frac{1}{(2\pi i)^n}\int_\gamma\int_\gamma\cdots\int_\gamma\Big\{\prod_{k=1}^{n}\int_a^b \frac{1}{z_k - x_k}\,d\mu(x_k)\Big\} \\ &\qquad\times \mathrm{vdm}^2(z_1,\dots,z_n)dz_1\cdots dz_n \\ &= \int_a^b\cdots\int_a^b\Big[\frac{1}{(2\pi i)^n}\int_\gamma\cdots\int_\gamma\prod_{k=1}^{n}\frac{1}{z_k - x_k} \\ &\qquad\times \mathrm{vdm}^2(z_1,\dots,z_n)dz_1\cdots dz_n\Big]\,d\mu(x_1)\cdots d\mu(x_n) \\ &= \int_a^b\cdots\int_a^b \mathrm{vdm}^2(x_1,\dots,x_n)\,d\mu(x_1)\cdots d\mu(x_n). \end{aligned}$$

This latter formula for a Gram determinant is given in Szegő [11, p. 27] where he attributes it to Heine.

It follows from the results of Bloom [3], mentioned above, that in this case, we have

Proposition 2.2. *Suppose that $E = [a, b]$ and that $f(z)$ (or $F(z)$) is the moment generating function of a measure μ supported on E. Then*

$$\lim_{n\to\infty} |\det(H_n)|^{1/n^2} = \mathrm{cap}(E)$$

if and only if μ is in the Stahl–Totik class **Reg**.

The problem of telling whether or not a generating function is indeed the moment generating function of a measure is known as the Classical Hausdorff Moment problem and has been much studied.

Definition 2.3. Let $c = (c_j)_{j\geq 0}$ be a real sequence. Then c is said to be a *moment sequence* of a measure $d\mu$ on the interval $[a, b]$ if

$$c_j = \int_a^b x^j \, d\mu(x).$$

Of special interest to us is the following:

Theorem 2.4 (Liu and Pego [8, Corollary 1]). *c is the moment sequence of a measure $d\mu$ on $[0, b]$ if and only if the (shifted) generating function $F(z) := \sum_{k=1}^{\infty} c_{k-1} z^k$ is a Pick function, analytic on $(-\infty, 1/b)$.*

Definition 2.5. A function $f(z)$ is said to be a *Pick function*, analytic on the open interval (a, b), if

1. $f(z)$ is analytic in the open upper half-plane, $\mathrm{Im}(z) > 0$, and leaves it invariant, i.e., $\mathrm{Im}(f(z)) \geq 0$ for all $\mathrm{Im}(z) > 0$.
2. $f(z)$ takes real values on (a, b) and admits an analytic continuation by reflection from the upper half-plane across (a, b).

For more on Pick functions see the monograph by Donoghue [5].

Example 1 continued. As noted, here $F(z) = -\log(1 - z)$. It is elementary to check that $F(z)$ is indeed a Pick function, analytic on $(-\infty, 1)$, confirming the fact that its coefficient sequence $(1, 1/2, 1/3, \dots)$ is the moment sequence for a (probability) measure supported on $[0, 1]$. Which precise measure μ is given by the following recipe (cf. [8, equation (9)]). For an interval $(\alpha, \beta) \subset [0, 1]$ (in this case),

$$\mu(\alpha, \beta) = \lim_{y\to 0} -\frac{1}{\pi} \int_\alpha^\beta \mathrm{Im}(f(x + iy)) \, dx. \tag{5}$$

For this example

$$-f(z) = \log(1 - 1/z) = \log\Big(1 - \frac{x - iy}{x^2 + y^2}\Big) = \log\Big(\frac{x^2 + y^2 - x + iy}{x^2 + y^2}\Big).$$

Hence

$$\lim_{y\to 0} -f(z) = \log\Big(\frac{x^2 - x}{x^2}\Big) = \log\Big(\frac{x - 1}{x}\Big).$$

Since $x \in (\alpha, \beta) \subset [0, 1]$, $(x - 1)/x < 0$

$$\lim_{y\to 0} -\mathrm{Im}(f(z)) = \pi$$

and we see that $\mu(\alpha, \beta) = \beta - \alpha$ and so $d\mu$ is just Lebesgue measure on $[0, 1]$.

EXAMPLE 2. Consider the generating function $F(z) = e^z - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} z^k$ and the corresponding Hankel matrices $H_n \in \mathbb{R}^{n\times n}$ with entries

$$(H_n)_{ij} = 1/(i+j-1)!.$$

Then $f(z) = F(1/z) = e^{1/z} - 1$ is regular outside the (polar) set $E = \{0\}$, with $\mathrm{cap}(E) = 0$. Hence, by the Pólya Theorem 2.1, $\limsup_{n\to\infty} |\det(H_n)|^{1/n^2} \le 0$ and thus

$$\lim_{n\to\infty} |\det(H_n)|^{1/n^2} = 0.$$

It is interesting to note that there is also a simple closed formula for the determinant.

LEMMA 2.6. *For the above Hankel matrix,*

$$\det(H_n) = \pm \frac{\prod_{j=1}^{n-1} j!}{\prod_{j=n}^{2n-1} j!}\,.$$

Proof. Let $p_j(x) := x(x-1)\cdots(x-(n-1-j))$, $1 \le j \le n$, with $p_1(x) := 1$. Note that $\deg(p_j) = n-j$. Then

$$\begin{aligned}(H_n)_{ij} &= \frac{1}{(i+j-1)!} = \frac{1}{(n-1+i)!}\frac{(n+i-1)!}{(j+i-1)!}\\ &= \frac{1}{(n-1+i)!}(n+i-1)(n+i-2)\cdots(j+1+i-1)\\ &= \frac{1}{(n-1+i)!}(n+i-1)((n+i-1)-1)\cdots((n+i-1)-(n-1-j))\\ &= \frac{1}{(n-1+i)!}\,p_j(n+i-1).\end{aligned}$$

Hence

$$\begin{aligned}\det(H_n) &= \left(\prod_{i=1}^{n} \frac{1}{(n-1+i)!}\right)\det\big([p_j(n+i-1)]_{1\le i,j\le n}\big)\\ &= \left(\prod_{j=n}^{2n-1}\frac{1}{j!}\right)\mathrm{vdm}\big(\{p_1,p_2,\ldots,p_n\};(n,n+1,\ldots,2n-1)\big)\end{aligned}$$

the Vandermonde determinant with basis $\{p_1, p_2, \ldots, p_n\}$ and points $(n, n+1, \ldots, 2n-1)$, in this precise order. Since the polynomials p_j are monic, we may replace the basis $\{p_1, p_2, \ldots, p_n\}$ by $\{x^{n-1}, x^{n-2} \ldots, 1\}$ to obtain

$$\begin{aligned}\det(H_n) &= \pm\left(\prod_{j=n}^{2n-1}\frac{1}{j!}\right)\left(\prod_{1\le i<j\le n}((n+j-1)-(n+i-1))\right)\\ &= \pm\left(\prod_{j=n}^{2n-1}\frac{1}{j!}\right)\left(\prod_{1\le i<j\le n}(j-i)\right) = \pm\left(\prod_{j=n}^{2n-1}\frac{1}{j!}\right)\left(\prod_{j=1}^{n-1} j!\right).\end{aligned}$$

In fact, the sign can easily determined to be $(-1)^{\lfloor n/2\rfloor}$. ∎

From this formula it can be directly verified that indeed $\lim_{n\to\infty} |\det(H_n)|^{1/n^2} = 0$. But one may also calculate that

$$\lim_{n\to\infty} |\det(H_n)|^{1/(n^2\log(n))} = e^{-1}.$$

Of relevance is the following theorem of Wilson in 1935, [12].

THEOREM 2.7. *If $f(z)$ has only poles and a single essential singularity of order ρ, then*

$$\limsup_{n\to\infty} |\det(H_n)|^{1/(n^2\log(n))} \le e^{-1/\rho}.$$

EXAMPLE 3. A simple way of generating examples is to choose a priori a measure μ and then calculate the generating function $F(z)$ from (3). For example, consider

$$d\mu := \frac{2}{\pi}\sqrt{\frac{1-x}{x}}\,dx$$

supported on $[0,1]$. This measure is a special case of a Jacobi measure and hence is (easily) verified to be in the Stahl–Totik class **Reg**.

By elementary means we may calculate

$$F(z) = \int_0^1 \frac{z}{1-xz}\,d\mu(x) = \frac{2}{\pi}\int_0^1 \frac{z}{1-xz}\sqrt{\frac{1-x}{x}}\,dx = 2(1-\sqrt{1-z})$$

where the branch of the square root is chosen to be positive on the positive real axis.

As it must be, $F(z)$ is a Pick function, analytic on $(-\infty,1)$, and since the measure is in the class **Reg**, by Proposition 2.2,

$$\lim_{n\to\infty} |\det(H_n)|^{1/n^2} = \mathrm{cap}([0,1]) = 1/4.$$

Interestingly, also in this case one can give an explicit simple formula for $\det(H_n)$.

LEMMA 2.8. *For*

$$F(z) = 2(1-\sqrt{1-z}) = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\,2^{-2k}\binom{2k}{k} z^k$$

we have

$$\det(H_n) = 4^{-n(n-1)}.$$

Proof. First note that if we set $D\in\mathbb{R}^{n\times n}$ to be the diagonal matrix with $D_{ii} := 4^{i-1}$, $1\le i\le n$, it suffices to show that $A_n := DH_nD$ has determinant

$$\det(A_n) = 4^{n(n-1)/2}4^{-n(n-1)}4^{n(n-1)/2} = 1.$$

We note that

$$\begin{aligned}(A_n)_{ij} &= 4^{i-1}(H_n)_{ij}4^{j-1} = 2\times 4^{i+j-2}\,\frac{1}{2(i+j-1)-1}\,4^{-(i+j-1)}\binom{2(i+j-1)}{i+j-1}\\ &= \frac{1}{2}\,\frac{1}{2(i+j)-3}\binom{2(i+j-1)}{i+j-1}.\end{aligned}$$

Now consider the *lower triangular matrix* $L\in\mathbb{R}^{n\times n}$ given by

$$L_{ij} := (-1)^{i+j}\,\frac{(i+j-2)!}{(i-j)!(2j-2)!},\quad 1\le j\le i\le n.$$

Note that the diagonal elements of L are

$$L_{ii} = (-1)^{2i}\,\frac{(2i-2)!}{0!(2i-2)!} = 1,\quad 1\le i\le n,$$

and hence $\det(L) = 1$.

We claim that $LA_n = U$ where $U \in \mathbb{R}^{n\times n}$ is an *upper triangular* matrix with diagonal $U_{ii} = 1$, $1 \le i \le n$. From this it follows immediately that $\det(A_n) = 1$. To show this we must prove that

(a) $(LA_n)_{ij} = 0$ for $j < i$, and
(b) $(LA_n)_{ii} = 1$, $1 \le i \le n$.

To see (a) we calculate, for $1 \le j < i \le n$,

$$\begin{aligned}
(LA_n)_{ij} &= \sum_{k=1}^{i} L_{ik}(A_n)_{kj} \\
&= \sum_{k=1}^{i} (-1)^{i+k} \frac{(i+k-2)!}{(i-k)!(2k-2)!}\,\frac{1}{2}\,\frac{1}{2(k+j)-3}\binom{2(k+j-1)}{k+j-1} \\
&= (-1)^i\,\frac{1}{2}\sum_{k=1}^{i}(-1)^k\,\frac{(i+k-2)!}{(i-k)!(2k-2)!}\,\frac{1}{2(k+j)-3}\binom{2(k+j-1)}{k+j-1} \\
&= (-1)^{i+1}\,\frac{1}{2}\sum_{k'=0}^{i-1}(-1)^{k'}\,\frac{(i+k'-1)!}{(i-1-k')!(2k')!}\,\frac{1}{2(k'+j)-1}\binom{2(k'+j)}{k'+j} \\
&\qquad \text{(having set } k' = k-1) \\
&= (-1)^{i+1}\frac{1}{2}\,\frac{1}{(i-1)!}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\frac{k!(i+k-1)!(2(k+j))!}{(2k)!(2(k+j)-1)((k+j)!)^2}\,.
\end{aligned}$$

Now observe that

$$\frac{k!(i+k-1)!(2(k+j))!}{(2k)!(2(k+j)-1)((k+j)!)^2} =: p_{ij}(k)$$

is a polynomial of degree at most $i-2$ in k. Indeed,

$$\begin{aligned}
\frac{(2(k+j))!}{(2k)!(2(k+j)-1)} &= \frac{(2k+2j)(2k+2j-1)(2k+2j-2)\cdots(2k+1)}{2(k+j)-1} \\
&= (2k+2j)(2k+2j-2)(2k+2j-3)\cdots(2k+1), \\
\frac{(i+k-1)!}{(k+j)!} &= (k+i-1)(k+i-2)\cdots(k+j+1) \qquad (\text{since } j \le i-1)
\end{aligned}$$

and

$$\frac{k!}{(k+j)!} = \frac{1}{(k+j)(k+j-1)\cdots(k+1)}\,.$$

Hence,

$$\begin{aligned}
p_{ij}(k) &= (k+i-1)(k+i-2)\cdots(k+j+1) \\
&\qquad \times \frac{(2k+2j)(2k+2j-2)(2k+2j-3)\cdots(2k+1)}{(k+j)(k+j-1)\cdots(k+1)}\,.
\end{aligned}$$

But the denominator divides the numerator as for each factor of the denominator, twice that factor is present in the numerator. It follows that $p_{ij}(k)$ is indeed a polynomial in k with degree

$$\deg(p_{ij}) = [(i-1)-(j+1)-1] + [(2j-1)-j] = (i-1-j)+(j-1) = i-2,$$

as claimed.

With this notation, we have, for $1 \le j < i \le n$,

$$\begin{aligned}(LA_n)_{ij} &= (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}p_{ij}(k)\\ &= (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\,(i-1)!\,(-1)^{i-1}p_{ij}[0,1,2,\ldots,(i-1)]\end{aligned}$$

where $p_{ij}[0,1,2,\ldots,(i-1)]$ denotes the divided difference of order $i-1$ of p_{ij} at the nodes $0,1,\ldots,(i-1)$. This is zero as $\deg(p_{ij}) < i-1$ and we have shown (a).

For the diagonal case (b), as before, we may write

$$(LA_n)_{ii} = (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}p_i(k)$$

with

$$p_i(k) := \frac{k!\,(i+k-1)!\,(2(k+i))!}{(2k)!\,(2(k+i)-1)((k+i)!)^2}\,.$$

However, $p_i(k)$ is *no* longer a polynomial in k. In fact, in this case,

$$\begin{aligned}\frac{(i+k-1)!}{(k+i)!} &= \frac{1}{k+i}\,,\\ \frac{(2(k+i))!}{(2k)!\,(2(k+i)-1)} &= \frac{(2k+2i)(2k+2i-1)(2k+2i-2)\cdots(2k+1)}{2k+2i-1}\\ &= (2k+2i)(2k+2i-2)(2k+2i-3)\cdots(2k+1),\end{aligned}$$

and

$$\frac{k!}{(k+i)!} = \frac{1}{(k+i)(k+i-1)\cdots(k+1)}\,,$$

so that

$$p_i(k) = \frac{1}{k+i}\left\{\frac{(2k+2i)(2k+2i-1)(2k+2i-2)\cdots(2k+1)}{(k+i)(k+i-1)\cdots(k+1)}\right\}.$$

Again, for the expression in parentheses, the denominator divides the numerator so that $p_i(k)$ is a rational function of the form

$$p_i(k) = \frac{\widetilde{p}_i(k)}{k+i}\,,$$

where $\widetilde{p}_i(k)$ is a polynomial of degree $i-1$ in k.

Dividing, we get, as $k=-i$ is the root of the denominator,

$$p_i(k) = q_i(k) + \frac{\widetilde{p}_i(-i)}{k+i}$$

with $q_i(k)$ a polynomial of degree $i-2$ in k (identically zero for $i=1$).

Hence, we have

$$\begin{aligned}
(LA_n)_{ii} &= (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}p_i(k)\\
&= (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\Big\{q_i(k)+\frac{\widetilde{p}_i(-i)}{k+i}\Big\}\\
&= (-1)^{i+1}\,\frac{1}{2}\,\frac{1}{(i-1)!}\,(i-1)!(-1)^{i-1}\Big\{q_i(k)+\frac{\widetilde{p}_i(-i)}{k+i}\Big\}[0,1,\dots,(i-1)]\\
&= \frac{1}{2}\,\widetilde{p}_i(-i)\cdot\frac{1}{k+i}[0,1,2,\dots,(i-1)]
\end{aligned}$$

as $\deg(q_i)\le i-2$.

We may calculate

$$\begin{aligned}
\widetilde{p}_i(-i) &= 2^i(-3)(-5)\cdots(-(2i-1))\\
&= (-1)^{i-1}2^i\times 3\times 5\times 7\times\cdots\times(2i-1).
\end{aligned}$$

It remains to evaluate the divided difference

$$\frac{1}{k+i}[0,1,2,\dots,(i-1)].$$

To do this, note that this divided difference is the leading coefficient of the polynomial (of degree $i-1$) that interpolates the function $1/(x+i)$ at the nodes $x=0,1,2,\dots,(i-1)$. Let $P_{i-1}(x)$ denote this polynomial. We must have

$$\frac{1}{x+i}-P_{i-1}(x)=\frac{C}{x+i}\prod_{j=0}^{i-1}(x-j)$$

for some constant C. Multiplying by $x+i$ we obtain

$$1-(x+i)P_{i-1}(x)=C\prod_{j=0}^{i-1}(x-j).$$

Evaluating at $x=-i$ we get

$$C=\frac{1}{\prod_{j=0}^{i-1}(-i-j)}=(-1)^i\frac{1}{\prod_{j=0}^{i-1}(j+i)}\,.$$

But, the leading coefficient of $P_{i-1}(x)$ is $-C$ and hence

$$\frac{1}{k+i}[0,1,2,\dots,(i-1)]=(-1)^{i-1}\frac{1}{\prod_{j=0}^{i-1}(j+i)}\,.$$

Consequently,

$$\begin{aligned}
(LA_n)_{ii} &= \frac{1}{2}\,(-1)^{i-1}2^i(3)(5)\cdots(2i-1)(-1)^{i-1}\,\frac{1}{\prod_{j=0}^{i-1}(j+i)}\\
&= \frac{2^{i-1}\cdot 3\cdot 5\cdot 7\cdots(2i-1)}{\prod_{j=0}^{i-1}(j+i)}\\
&= \frac{2^{i-1}\cdot 3\cdot 5\cdot 7\cdots(2i-1)}{(2i-1)(2i-2)\cdots i}\,\frac{(i-1)!}{(i-1)!}=\frac{(2i-1)!}{(2i-1)!}=1.\ \blacksquare
\end{aligned}$$

3. Some simple transformation rules. Recall that for a generating function $F(z) = \sum_{k=1}^{\infty} a_k z^k$ we associate $f(z) := F(1/z) = \sum_{k=1}^{\infty} a_k z^{-k}$. We will use the notation $H_n(f)$ to denote the $n \times n$ Hankel matrix generated by f (or F),

$$(H_n(f))_{ij} := a_{i+j-1}.$$

For $r \in \mathbb{Z}_+$ we may consider the shifted series

$$F_r(z) := \sum_{k=1}^{\infty} a_{k+r} z^k$$

and associated shifted Hankel matrix $H_n(f_r) \in \mathbb{C}^{n \times n}$ with entries

$$(H_n(f_r))_{ij} = a_{i+j+r-1}.$$

Edrei [6, p. 31] gives the following (among others) basic transformation rules.

PROPOSITION 3.1. *We have*

1. *For* $\beta \in \mathbb{C}$, $\det(H_n(f(z+\beta))) = \det(H_n(f(z)))$.
2. *For* $\gamma \in \mathbb{C}$, $\det(H_n(f(\gamma z)) = \gamma^{-n^2} \det(H_n(f(z)))$.
3. *For* $\beta \in \mathbb{C}$ *let* $g(z) = 1/(z - \beta - f(z)) = \sum_{k=1}^{\infty} b_k z^{-k}$. *Then*
$$\det(H_{n+1}(g(z))) = \det(H_n(f(z))), \quad n \geq 1.$$
4. *For* $m \in \mathbb{Z}_+$, $\det(H_n(f(z^m))) = \begin{cases} 0 & \text{if } n \not\equiv 0 \bmod m \\ \pm(\det(H_{n/m}(f(z))))^m & \text{if } n \equiv 0 \bmod m. \end{cases}$

In particular, item 4. shows that, in general, $\lim_{n\to\infty} |\det(H_n(f))|^{1/n^2}$ need *not* exist.

EXAMPLE 4. In Example 3 we considered $F(z) = 2(1 - \sqrt{1-z})$. For $\gamma > 0$, let

$$G(z) := F(z/\gamma) = 2(1 - \sqrt{1 - z/\gamma}) = \frac{2}{\sqrt{\gamma}} (\sqrt{\gamma} - \sqrt{\gamma - z}).$$

By item 2. above, we have

$$\lim_{n\to\infty} |\det(H_n(G))|^{1/n^2} = \gamma^{-1} \lim_{n\to\infty} |\det(H_n(F))|^{1/n^2} = \gamma^{-1}/4 = \frac{1}{4\gamma}.$$

There is another simple transformation rule that will be of some use to us.

LEMMA 3.2. *Let* $g(z) = zf(z^2)$. *Then*

$$\det(H_n(g(z))) = \begin{cases} \det(H_k(f)) \det(H_k(f_1)) & n = 2k \\ \det(H_k(f)) \det(H_{k-1}(f_1)) & n = 2k-1. \end{cases}$$

Here $f_1(z)$ *denotes the shifted generating function defined above.*

Proof. Now

$$g(z) = zf(z^2) = \sum_{k=1}^{\infty} a_k z^{-(2k-1)} = \frac{a_1}{z} + \frac{0}{z^2} + \frac{a_2}{z^3} + \frac{0}{z^4} + \frac{a_3}{z^5} + \dots .$$

Hence $H_n(g)$ is a "checkerboard" matrix with alternating entries 0. For $n = 2k$, it has the form

$$H_n(g) = \begin{bmatrix} a_1 & 0 & a_2 & 0 & \dots & a_k & 0 \\ 0 & a_2 & 0 & a_3 & \dots & 0 & a_{k+1} \\ a_2 & 0 & a_3 & 0 & \dots & a_{k+1} & 0 \\ 0 & a_3 & 0 & a_4 & \dots & 0 & a_{k+2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{k+1} & 0 & a_{k+2} & \dots & 0 & a_{2k} \end{bmatrix}.$$

By interchanging rows and columns it may be put in the form

$$\begin{bmatrix} a_1 & a_2 & \dots & a_k & 0 & \dots & \dots & 0 \\ a_2 & a_3 & \dots & a_{k+1} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ a_k & a_{k+1} & \dots & a_{2k-1} & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & a_2 & a_3 & \dots & a_{k+1} \\ 0 & 0 & \dots & 0 & a_3 & a_4 & \dots & a_{k+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & a_{k+1} & a_{k+2} & \dots & a_{2k} \end{bmatrix}$$

from which the result follows immediately. The computation for $n = 2k-1$, odd, is similar and so we do not include it. ∎

In this context the shifted Hankel matrices $H(f_r)$ arise naturally. Edrei has shown

PROPOSITION 3.3 (Edrei [6, Theorem I]). *For each $r \in \mathbb{Z}_+$,*

$$\limsup_{n\to\infty} |\det(H_n(f_r))|^{1/n^2} = \limsup_{n\to\infty} |\det(H_n(f))|^{1/n^2},$$

i.e., the lim sup*s are the same for all $r \in \mathbb{Z}_+$.*

Note that it is necessary to use a lim sup as the limit, as noted above, may not exist. However, in the case when $F(z)$ is the generating function for a measure $\mu \in \mathbf{Reg}$ supported on an interval, we may prove the same result for limits.

The proof is a straightforward application of Bloom's result, Proposition 2.2. First, note that if

$$f(z) = \int_0^1 \frac{1}{z-x}\, d\mu(x)$$

then

$$\begin{aligned} f_1(z) = zf(z) - a_1 &= \int_0^1 \frac{z}{z-x}\, d\mu(x) - a_1 = \int_0^1 \Big(1 + \frac{x}{z-x}\Big)\, d\mu(x) - a_1 \\ &= \int_0^1 d\mu(x) + \int_0^1 \frac{1}{z-x}\, x\, d\mu(x) - a_1 = a_1 + \int_0^1 \frac{1}{z-x}\, x\, d\mu(x) - a_1 \\ &= \int_0^1 \frac{1}{z-x}\, x\, d\mu(x). \end{aligned}$$

In other words, $f_1(z)$ is the moment generating function of the measure $x\, d\mu(x)$ and, more generally, $f_r(z)$ is the moment generating function of the measure $x^r\, d\mu(x)$.

Lemma 3.4. *Suppose that the measure $d\mu(x) \in$ **Reg** with support $[0,1]$. Then for $r \in \mathbb{Z}_+$, the measure $x^r d\mu(x)$ is also in* **Reg**.

Proof. Clearly it suffices to prove this for $r = 1$. To this end, let

$$A_n := \sup_{\deg(P)\le n,\, P\not\equiv 0} \frac{\|P\|_K}{\|P\|_{L^2(x\mu)}} \quad \text{and} \quad B_n := \sup_{\deg(P)\le n,\, P\not\equiv 0} \frac{\|P\|_K}{\|P\|_{L^2(\mu)}}.$$

We will verify the Bernstein–Markov condition (1) for the measure $x\,d\mu(x)$, i.e., that $\lim_{n\to\infty} A_n^{1/n} = 1$, using the hypothesis that $\lim_{n\to\infty} B_n^{1/n} = 1$.

Note first that $\|P\|^2_{L^2(x\mu)} = \int_0^1 xP^2(x)\,d\mu \le \|P\|_K^2 \mu(K)$ so that $A_n \ge \sqrt{\mu(K)}$.

For an upper bound, let

$$C_n := \sup_{\deg(P)\le n,\, P\not\equiv 0} \frac{\|P\|_K}{\|P\|_{L^2(x^2dx)}}.$$

Since (by the Erdős–Turán condition) the measure $x^2\,dx$ is in **Reg**,

$$\lim_{n\to\infty} C_n^{1/n} = 1.$$

Now, for $\deg(P) \le n$,

$$\begin{aligned} \frac{\|P\|_K}{\|P\|_{L^2(x\mu)}} &\le \frac{\|P\|_K}{\|P\|_{L^2(x^2\mu)}} = \frac{\|P\|_K}{\left[\int_0^1 (xP(x))^2\,d\mu\right]^{1/2}} \\ &= \frac{\|P\|_K}{\|xP(x)\|_K} \times \frac{\|xP(x)\|_K}{\|xP(x)\|_{L^2(\mu)}} \le \frac{\|P\|_K}{\|xP(x)\|_K} \times B_{n+1}, \end{aligned}$$

as $\deg(xP(x)) \le n+1$.

But,

$$\begin{aligned} \|P\|_K \le C_n \|P\|_{L^2(x^2dx)} &= C_n \left[\int_0^1 x^2 P^2(x)\,dx\right]^{1/2} \\ &\le \left[\max_{0\le x\le 1} (xP(x))^2\right]^{1/2} = C_n \|xP(x)\|_K. \end{aligned}$$

Hence,

$$\frac{\|P\|_K}{\|P\|_{L^2(x\mu)}} \le C_n \times B_{n+1}$$

and, taking suprema, $A_n \le C_n B_{n+1}$. In summary, we have shown that

$$\sqrt{\mu(K)} \le A_n \le C_n B_{n+1}$$

from which it follows easily that $\lim_{n\to\infty} A_n^{1/n} = 1$. ∎

Hence by Proposition 2.2, we have

Proposition 3.5. *Suppose that $\mu \in$ **Reg**, with support $E = [0,1]$, and that $f(z)$ is the moment generating function of μ. Then, for $r \in Z_+$,*

$$\lim_{n\to\infty} |\det(H_n(f_r))|^{1/n^2} = \mathrm{cap}(E) = 1/4.$$

Combined with Lemma 3.2 we immediately conclude that

PROPOSITION 3.6. *Suppose that $\mu \in \mathbf{Reg}$, with support $E = [0,1]$, and that $f(z) = F(1/z)$ is the moment generating function of μ. Let $g(z) := zf(z^2)$. Then*

$$\begin{aligned}\lim_{n\to\infty} |\det(H_n(g))|^{1/n^2} &= \lim_{k\to\infty} |\det(H_k(f))|^{1/(4k^2)} \cdot \lim_{k\to\infty} \left|\det(H_k(f_1))\right|^{1/(4k^2)} \\ &= (\mathrm{cap}(E))^{1/4}(\mathrm{cap}(E))^{1/4} = \sqrt{\mathrm{cap}(E)} = \sqrt{1/4} = 1/2.\end{aligned}$$

EXAMPLE 5. As in Example 3, let $F(z) = 2(1 - \sqrt{1-z})$ and $f(z) = F(1/z)$. We have shown that $F(z)$ is the generating function of a measure in the Stahl–Totik class **Reg**, supported on $[0,1]$ and hence that

$$\lim_{n\to\infty} |\det(H_n(F))|^{1/n^2} = 1/4.$$

Further, let $g(z) = zf(z^2)$ and

$$G(z) = g(1/z) = \frac{2}{z}\,(1 - \sqrt{1-z^2}).$$

The function $\sqrt{1-z^2}$ has an expansion (about 0) of the form

$$\sqrt{1-z^2} = 1 + 0z + a_2 z^2 + a_3 z^3 + \dots$$

with odd coefficients $a_{2j-1} = 0$, $j = 1, 2, \dots$. Hence

$$G(z) = -\frac{2}{z}\,(a_2 z^2 + a_3 z^3 + \cdots) = -2(a_2 z + a_3 z^2 + a_4 z^3 + \dots).$$

It follows from Proposition 3.6 that

$$\begin{aligned}&\lim_{n\to\infty}\left|\det\begin{bmatrix} a_2 & a_3 & \dots & a_{n+1} \\ a_3 & a_4 & \dots & a_{n+2} \\ a_4 & a_5 & \dots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n}\end{bmatrix}\right|^{1/n^2} \\ &\qquad = \lim_{n\to\infty} 2^{-1/n} \times \lim_{n\to\infty} |\det(H_n(G))|^{1/n^2} = 1/2.\end{aligned}$$

We may also use the second Edrei transformation rule and consider, for $R > 0$,

$$G(z/R) = 2\,\frac{R}{z}\left(1 - \sqrt{1 - z^2/R^2}\right) = 2\,\frac{R}{z} - \frac{2}{z}\,\sqrt{R^2 - z^2}.$$

If we then expand

$$\sqrt{R^2 - z^2} = R + 0z + b_2 z^2 + b_3 z^3 + \dots$$

with $b_{2j-1} = 0$, $j = 1, 2, \dots$, we obtain

$$G(z/R) = -2(b_2 z + b_3 z^2 + b_4 z^3 + \dots).$$

It follows that

$$\lim_{n\to\infty}\left|\det\begin{bmatrix} b_2 & b_3 & \cdots & b_{n+1}\\ b_3 & b_4 & \cdots & b_{n+2}\\ b_4 & b_5 & \cdots & b_{n+3}\\ \vdots & \vdots & & \vdots\\ b_{n+1} & b_{n+2} & \cdots & b_{2n}\end{bmatrix}\right|^{1/n^2}$$
$$= \lim_{n\to\infty} 2^{-1/n}\times\lim_{n\to\infty}|\det(H_n(G(z/R)))|^{1/n^2} = \frac{1}{2R}\,.$$

We note that the determinants of this example have the same form as in Proposition 1.1.

4. Asymptotics of Vandermonde determinants for points on a quadratic curve. Consider a general non-degenerate quadratic curve in $\mathbb{C}^2$. After a rotation of coordinates (if necessary), its defining equation may be put in the form

$$Ax^2+By^2+Cx+Dy+E=0.$$

After a further *complex* rescaling and translation it may be reduced to one of two forms:

1. $y^2+x=R$
2. $y^2+x^2=R^2$.

We first consider Case 1: $y^2+x=R$ where we take $R>1$ and let $K=\{(x,y)\in\mathbb{C}^2 : |x|=1,\ y=\sqrt{R-x}\}$, where the branch of the square root is taken to be the one positive on the positive real half-line. K is just the compact set on the "upper" sheet of the curve lying above the unit circle.

Setting $A(z):=\sqrt{R-z}=\sum_{k=0}^\infty a_kz^k$, as in Proposition 1.1, let $Q_{2n}(z)=\sum_{k=0}^{2n}q_kz^k$ be the polynomial of degree $2n$ which interpolates $A(z)$ at the $(2n+1)$-th roots of unity.

Recall that in Proposition 1.1 (with the same notation) we established that for the Vandermonde determinant with bivariate basis $\mathcal{B}_n$ at the $2n+1$ points X_n,

$$|\mathrm{vdm}(\mathcal{B}_n;X_n)| = (2n+1)^{(2n+1)/2}\left|\det\begin{bmatrix} q_2 & q_3 & \cdots & q_{n+1}\\ q_3 & q_4 & \cdots & q_{n+2}\\ q_4 & q_5 & \cdots & q_{n+3}\\ \vdots & \vdots & & \vdots\\ q_{n+1} & q_{n+2} & \cdots & q_{2n}\end{bmatrix}\right|.$$

Hence,

$$\lim_{n\to\infty}|\mathrm{vdm}(\mathcal{B}_n;X_n)|^{1/n^2} = \lim_{n\to\infty}\left|\det\begin{bmatrix} q_2 & q_3 & \cdots & q_{n+1}\\ q_3 & q_4 & \cdots & q_{n+2}\\ q_4 & q_5 & \cdots & q_{n+3}\\ \vdots & \vdots & & \vdots\\ q_{n+1} & q_{n+2} & \cdots & q_{2n}\end{bmatrix}\right|^{1/n^2}$$

(provided the limits exist). But we claim

Proposition 4.1. *Suppose that $A(z) = \sqrt{R-z} = \sum_{k=0}^{\infty} a_k z^k$ with $R > 1$, and that $Q_{2n}(z) = \sum_{k=0}^{2n} q_k z^k$ is the polynomial of degree $2n$ that interpolates $A(z)$ at the $(2n+1)$-th roots of unity. Then*

$$\lim_{n\to\infty} \left| \det \begin{bmatrix} q_2 & q_3 & \cdots & q_{n+1} \\ q_3 & q_4 & \cdots & q_{n+2} \\ q_4 & q_5 & \cdots & q_{n+3} \\ \vdots & \vdots & & \vdots \\ q_{n+1} & q_{n+2} & \cdots & q_{2n} \end{bmatrix} \right|^{1/n^2}$$

$$= \lim_{n\to\infty} \left| \det \begin{bmatrix} a_2 & a_3 & \cdots & a_{n+1} \\ a_3 & a_4 & \cdots & a_{n+2} \\ a_4 & a_5 & \cdots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{bmatrix} \right|^{1/n^2} = \frac{1}{4R}.$$

Proof. First recall, from Example 3, that

$$2(1-\sqrt{1-z}) = \sum_{j=1}^{\infty} m_j z^j$$

where

$$m_j = \int_0^1 x^{j-1}\, d\mu(x)$$

are the moments of the measure

$$d\mu(x) := \frac{2}{\pi}\sqrt{\frac{1-x}{x}}\, dx \in \mathbf{Reg}.$$

By Proposition 3.5 we know that

$$\lim_{n\to\infty} \left| \det \begin{bmatrix} m_2 & m_3 & \cdots & m_{n+1} \\ m_3 & m_4 & \cdots & m_{n+2} \\ m_4 & m_5 & \cdots & m_{n+3} \\ \vdots & \vdots & & \vdots \\ m_{n+1} & m_{n+2} & \cdots & m_{2n} \end{bmatrix} \right|^{1/n^2} = \operatorname{cap}([0,1]) = \frac{1}{4}.$$

Now,

$$\sqrt{R-z} = \sqrt{R}\sqrt{1-z/R} = \sqrt{R}\left(1 - \frac{1}{2}\sum_{k=1}^{\infty} \frac{m_k}{R^k}\, z^k\right)$$

and so

$$\det \begin{bmatrix} a_2 & a_3 & \cdots & a_{n+1} \\ a_3 & a_4 & \cdots & a_{n+2} \\ a_4 & a_5 & \cdots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{bmatrix}$$

$$= \Big(-\frac{\sqrt{R}}{2}\Big)^n \det \begin{bmatrix} m_2/R^2 & m_3/R^3 & \dots & m_{n+1}/R^{n+1} \\ m_3/R^3 & m_4/R^4 & \dots & m_{n+2}/R^{n+2} \\ m_4/R^4 & m_5/R^5 & \dots & m_{n+3}/R^{n+3} \\ \vdots & \vdots & & \vdots \\ m_{n+1}/R^{n+1} & m_{n+2}/R^{n+2} & \dots & m_{2n}/R^{2n} \end{bmatrix}$$

$$= \Big(-\frac{\sqrt{R}}{2}\Big)^n R^{-n(n+1)} \det \begin{bmatrix} m_2 & m_3 & \dots & m_{n+1} \\ m_3 & m_4 & \dots & m_{n+2} \\ m_4 & m_5 & \dots & m_{n+3} \\ \vdots & \vdots & & \vdots \\ m_{n+1} & m_{n+2} & \dots & m_{2n} \end{bmatrix}.$$

It follows that

$$\lim_{n\to\infty} \left|\det \begin{bmatrix} a_2 & a_3 & \dots & a_{n+1} \\ a_3 & a_4 & \dots & a_{n+2} \\ a_4 & a_5 & \dots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n} \end{bmatrix}\right|^{1/n^2} = R^{-1} \times \frac{1}{4} = \frac{1}{4R}.$$

Now consider the analogous determinants with the a_k substituted by the q_k. By Lemma 1.2, with $m = 2n+1$,

$$\begin{aligned} q_k &= \sum_{j=0}^\infty a_{k+(2n+1)j} = -\frac{\sqrt{R}}{2}\sum_{j=0}^\infty R^{-(k+(2n+1)j)} m_{k+(2n+1)j} \\ &= -\frac{\sqrt{R}}{2}\sum_{j=0}^\infty R^{-(k+(2n+1)j)} \int_0^1 x^{k+(2n+1)j-1}\, d\mu(x) \\ &= -\frac{\sqrt{R}}{2}\int_0^1 R^{-k}x^{k-1}\Big(\sum_{j=0}^\infty (x/R)^{(2n+1)j}\Big)\, d\mu(x) \\ &= -\frac{\sqrt{R}}{2}R^{-k}\int_0^1 \frac{x^{k-1}}{1-(x/R)^{2n+1}}\, d\mu(x) = -\frac{\sqrt{R}}{2}R^{-k}\int_0^1 x^{k-1}\, d\mu_n(x) \end{aligned}$$

where

$$d\mu_n(x) := \frac{1}{1-(x/R)^{2n+1}}\, d\mu(x).$$

In other words,

$$q_k = -\frac{\sqrt{R}}{2}R^{-k}M_k$$

where M_k is the moment with respect to the measure $d\mu_n(x)$.

Note that

$$d\mu(x) \le d\mu_n(x) \le \frac{R^{2n+1}}{R^{2n+1}-1}\, d\mu(x) \le \frac{R}{R-1}\, d\mu(x)$$

so that the $L^2(\mu)$ and $L^2(\mu_n)$ norms are comparable up to a constant *independent* of n. Consequently the Bernstein–Markov constants for $d\mu$ and $d\mu_n$ are similarly comparable and we also have

$$\lim_{n\to\infty}\left|\det\begin{bmatrix} M_2 & M_3 & \dots & M_{n+1} \\ M_3 & M_4 & \dots & M_{n+2} \\ M_4 & M_5 & \dots & M_{n+3} \\ \vdots & \vdots & & \vdots \\ M_{n+1} & M_{n+2} & \dots & M_{2n} \end{bmatrix}\right|^{1/n^2} = \mathrm{cap}([0,1]) = \frac{1}{4}.$$

Then, by the same calculations as we did for the a_k,

$$\det\begin{bmatrix} q_2 & q_3 & \dots & q_{n+1} \\ q_3 & q_4 & \dots & q_{n+2} \\ q_4 & q_5 & \dots & q_{n+3} \\ \vdots & \vdots & & \vdots \\ q_{n+1} & q_{n+2} & \dots & q_{2n} \end{bmatrix} = \Big(-\frac{\sqrt{R}}{2}\Big)^n R^{-n(n+1)} \det\begin{bmatrix} M_2 & M_3 & \dots & M_{n+1} \\ M_3 & M_4 & \dots & M_{n+2} \\ M_4 & M_5 & \dots & M_{n+3} \\ \vdots & \vdots & & \vdots \\ M_{n+1} & M_{n+2} & \dots & M_{2n} \end{bmatrix}$$

and the result follows. ∎

An immediate corollary is

COROLLARY 4.2. *With the above notation*

$$\lim_{n\to\infty} |n\,\mathrm{vdm}(\mathcal{B}_n; X_n)|^{1/n^2} = \frac{1}{4R}.$$

Note that by definition, this gives a lower bound for the transfinite diameter of K on the curve.

For Case 2, $x^2+y^2 = R^2$, $R > 1$, we set $A(z) = \sqrt{R^2 - z^2} = \sum_{k=0}^{\infty} a_k z^k$ and, as before, let $K = \{(x,y) \in \mathbb{C}^2 : |x| = 1,\ y = A(z)\}$ and $Q_{2n}(z) = \sum_{k=0} q_k z^k$ the polynomial of degree $2n$ that interpolates $A(z)$ at the $(2n+1)$-th roots of unity. We conjecture that also in this case

$$\lim_{n\to\infty}\left|n\det\begin{bmatrix} q_2 & q_3 & \dots & q_{n+1} \\ q_3 & q_4 & \dots & q_{n+2} \\ q_4 & q_5 & \dots & q_{n+3} \\ \vdots & \vdots & & \vdots \\ q_{n+1} & q_{n+2} & \dots & q_{2n} \end{bmatrix}\right|^{1/n^2} = \lim_{n\to\infty}\left|\det\begin{bmatrix} a_2 & a_3 & \dots & a_{n+1} \\ a_3 & a_4 & \dots & a_{n+2} \\ a_4 & a_5 & \dots & a_{n+3} \\ \vdots & \vdots & & \vdots \\ a_{n+1} & a_{n+2} & \dots & a_{2n} \end{bmatrix}\right|^{1/n^2}$$

$(= 1/(2R))$, and expect to provide a proof in a forthcoming paper.

References

[1] R. Berman, S. Boucksom, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. 181 (2010), 337–394.

[2] R. Berman, S. Boucksom, D. Witt Nystrom, *Fekete points and convergence towards equilibrium measures on complex manifolds*, Acta Math. 207 (2011), 1–27.

[3] T. Bloom, *Asymptotics of integrals of Vandermonde determinants*, preprint.

[4] D. Cox, S. Ma'u, *Transfinite diameter on complex algebraic varieties*, preprint, arXiv: 1410.6962

[5] W. Donoghue, *Monotone Matrix Functions and Analytic Continuation*, Grundlehren Math. Wiss. 207, Springer, New York 1974.

[6] A. Edrei, *Sur les déterminants récurrents e les singularités d'une function donnée per son développement de Taylor*, Compositio Math. 7 (1940), 20–88.

[7] G. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Transl. Math. Monogr. 26, Amer. Math. Soc., Providence, RI 1969.

[8] J.-G. Liu, R. Pego, *On generating functions of Hausdorff moment sequences*, preprint, arXiv: 1401.8052.

[9] G. Pólya, *Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe*, Math. Ann. 99 (1928), 687–706.

[10] H. Stahl, V. Totik, *General Orthogonal Polynomials*, Encyclopedia Math. Appl. 43, Cambridge Univ. Press, Cambridge 1992.

[11] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., New York 1939.

[12] R. Wilson, *An extension of the Hadamard–Polya determinantal criteria for uniform functions*, Proc. London Math. Soc. (2) 39 (1935), 363–371.