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ON A NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. Several sets of quaternionic functions are described and studied with respect to hyperholomorphy, addition and (non-commutative) multiplication, on open sets of $\mathbb{H}$, then Hamilton 4-manifolds analogous to Riemann surfaces, for $\mathbb{H}$ instead of $\mathbb{C}$, are defined, and so begin to describe a class of four-dimensional manifolds.

1. Introduction. We first recall the definition of the field $\mathbb{H}$ of quaternions using pairs of complex numbers and a modified Cauchy–Fueter operator (Section 2) that have been introduced by F. Colombo et al., [CLSSS07]. We will only use right multiplication. We will consider C^∞ $\mathbb{H}$-valued quaternionic functions defined on an open set U of $\mathbb{H}$ whose behavior mimics the behavior of holomorphic functions on an open set of $\mathbb{C}$. If such a function does not vanish identically, it has an (algebraic) inverse. Finally we describe properties of hyperholomorphic functions with respect to addition and multiplication.

In Section 3, we characterize the quaternionic functions which are, almost everywhere, hyperholomorphic and whose inverses are hyperholomorphic almost everywhere, on U, as the solutions of a system of two non-linear PDE. We find non-trivial examples of a solution showing that the considered space of functions is significant; we will call these functions *hypermeromorphic*.

At the moment, I am unable to get the general solution of the system of PDE. Same difficulty for subsequent occurring systems of PDE.

In Section 4, we describe a subspace of hyperholomorphic and hypermeromorphic functions defined almost everywhere on U, having "good properties for addition and multiplication"; we again obtain systems of non-linear PDE.

In Section 5 and the following, we consider globalization of the above notions, define Hamilton 4-manifolds analogous to Riemann surfaces, for $\mathbb{H}$ instead of $\mathbb{C}$, and give examples of such manifolds; our ultimate aim is to describe a class of 4-dimensional manifolds.

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2. Quaternions. $\mathbb{H}$-valued functions. Hyperholomorphic functions

See [CSSS04, CLSSS07, D13].

2.1. Quaternions. If $q \in \mathbb{H}$, then $q = z_1 + z_2\mathbf{j}$ where $z_1, z_2 \in \mathbb{C}$. We have $z_1\mathbf{j} = \mathbf{j}\overline{z}_1$, and note $|q| = |z_1|^2 + |z_2|^2$.

The conjugate of q is $\overline{q} = \overline{z}_1 - z_2\mathbf{j}$. Let us denote by $*$ the (right) multiplication in $\mathbb{H}$, then the right inverse of q is $q^{-1} = |q|^{-1}\overline{q}$.

2.2. Quaternionic functions. Let U be an open set of $\mathbb{H} \cong \mathbb{C}^2$ and $f \in C^\infty(U, \mathbb{H})$, then $f = f_1 + f_2\mathbf{j}$, where $f_1, f_2 \in C^\infty(U, \mathbb{C})$. The complex valued functions f_1, f_2 will be called the *components* of f.

DEFINITION 2.1. Let U be an open neighborhood of 0 in $\mathbb{H} \cong \mathbb{C}^2$.

(a) From now on, we will consider the quaternionic functions $f = f_1 + f_2\mathbf{j}$ having the following properties:

  (i) When f_1 and f_2 are not holomorphic, the set $Z(f_1) \cap Z(f_2)$ is discrete on U;

  (ii) for every $q \in Z(f_1) \cap Z(f_2)$, $J_q^\alpha(\cdot)$ denoting the *jet of order α at q* (see [M66]), let $m_i = \sup_{\alpha_i} J_q^{\alpha_i}(f_i) = 0$; m_i, $i = 1, 2$, is finite.

  $m_q = \inf m_i$ is the *order of the zeroes q of f*.

(b) We will also consider the quaternionic functions defined almost everywhere on U (i.e. outside a locally finite set of C^∞ hypersurfaces, namely $Z(f_1), Z(f_2)$).

2.3. Modified Cauchy–Fueter operator $\mathcal{D}$. Hyperholomorphic functions. See [CLSSS07, F39].

For $f \in C^\infty(U, \mathbb{H})$, with $f = f_1 + f_2\mathbf{j}$,

$$\mathcal{D}f(q) = \frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f(q).$$

A function $f \in C^\infty(U, \mathbb{H})$ is said *hyperholomorphic* if $\mathcal{D}f = 0$.

Characterization of the hyperholomorphic function f on U:

$$\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f_2}}{\partial z_2} = 0; \quad \frac{\partial f_1}{\partial \overline{z}_2} + \frac{\partial \overline{f_2}}{\partial z_1} = 0 \quad \text{on } U. \tag{1}$$

2.4. Several families of meromorphic functions. The conditions f_1 is holomorphic and f_2 is holomorphic are equivalent on U; the same is true for almost everywhere defined holomorphic functions on U.

By definition, *holomorphic* (*almost everywhere defined*) *functions of two complex variables* on U are such that $f_2 = 0$, and f_1 is (almost everywhere) holomorphic.

2.4.1. *Consider the almost everywhere defined hyperholomorphic functions on U whose components are real*

$$f = f_1 + f_2\mathbf{j}.$$

According to a remark of Guy Roos in March 2013, they are almost everywhere holomorphic [R13].

2.4.2. *The above considered almost everywhere holomorphic functions are meromorphic and constitute two $\mathbb{H}$-commutative algebras A_1, A_2, with common origin* 0. Let $f = a+b\mathbf{i}$, and $g = c+d\mathbf{j}$, with $a, b, c, d \in \mathbb{R}$ be two almost everywhere defined holomorphic functions, i.e. meromorphic functions on U.

A_1 is the set of the meromorphic functions $f = a+b\mathbf{i}$, and A_2 is the set of meromorphic functions $g = c + d\mathbf{j}$, with $a, b, c, d \in \mathbb{R}$.

The sums $f + g = a + c + d\mathbf{j} + b\mathbf{i}$ constitute the algebra $A_1 + A_2$ of meromorphic functions.

More generally, $A_{\alpha,\beta} = \alpha A_1 + \beta A_2$, with $\alpha, \beta \in \mathbb{R}$ is an algebra of meromorphic functions on U,

$$A_{\alpha,\beta} = \sum_{a,b,c,d\in\mathbb{R}} \alpha(a + b\mathbf{i}) + \beta(c + d\mathbf{j}).$$

2.4.3. We now begin to introduce multiplication for hyperholomorphic functions, addition and scalar multiplication being obvious.

2.5. Multiplication of almost everywhere defined hyperholomorphic functions

Proposition 2.2. *Let f', f'' be two almost everywhere defined hyperholomorphic functions. Then, their product $f' * f''$ satisfies*:

$$\mathcal{D}(f' * f'') = \mathcal{D}f' * \mathbf{j}f'' + \Big(f'\Big(\frac{\partial}{\partial \overline{z}_1}\Big) + \overline{f}'\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f''.$$

Proof. Let $f' = f_1' + f_2'\mathbf{j}$, $f'' = f_1'' + f_2''\mathbf{j}$ be two hyperholomorphic functions. We have

$$f' * f'' = (f_1' + f_2'\mathbf{j})(f_1'' + f_2''\mathbf{j}) = f_1'f_1'' - f_2'\overline{f}_2'' + (f_1'f_2'' + f_2'\overline{f}_1'')\mathbf{j}.$$

Compute

$$\frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)\big(f_1'f_1'' - f_2'\overline{f}_2'' + (f_1'f_2'' + f_2'\overline{f}_1'')\mathbf{j}\big).$$

By derivation of the first factors of the sum $f' * f''$, we get the first term:

$$\begin{aligned}\frac{1}{2}\Big(\frac{\partial f_1'}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f_1'}{\partial \overline{z}_2}\Big)(f_1'' + f_2''\mathbf{j}) + \frac{1}{2}\Big(\frac{\partial f_2'}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f_2'}{\partial \overline{z}_2}\Big)\mathbf{j}\mathbf{j}(\overline{f}_2'' - \overline{f}_1''\mathbf{j})\\ = \frac{1}{2}\Big(\frac{\partial f_1'}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f_1'}{\partial \overline{z}_2}\Big)(f_1'' + f_2''\mathbf{j}) + \frac{1}{2}\Big(\frac{\partial f_2'\mathbf{j}}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial f_2'\mathbf{j}}{\partial \overline{z}_2}\Big)\mathbf{j}(f_2''\mathbf{j} + f_1'') = \mathcal{D}f' * \mathbf{j}f''.\end{aligned}$$

By derivation in

$$\frac{1}{2}\Big(\frac{\partial}{\partial \overline{z}_1} + \mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)\big(f_1'f_1'' + f_2'\mathbf{j}f_2''\mathbf{j} + (f_1'f_2''\mathbf{j} + f_2'\mathbf{j}f_1'')\big)$$

of the second factors of the sum $f' * f''$, we get the second term (up to factor $\frac{1}{2}$):

$$\begin{aligned}f_1'\frac{\partial f_1''}{\partial \overline{z}_1} + \overline{f}_1'\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_2} + f_1'\frac{\partial f_2''}{\partial \overline{z}_1}\mathbf{j} + \overline{f}_1'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_2}\mathbf{j} + f_2'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_1}\mathbf{j} + \overline{f}_2'\mathbf{j}\frac{\partial f_2''}{\partial \overline{z}_2} + f_2'\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_1} + \overline{f}_2'\mathbf{j}\mathbf{j}\frac{\partial f_1''}{\partial \overline{z}_2}\\ = (f_1' + f_2'\mathbf{j})\Big(\frac{\partial}{\partial \overline{z}_1}\Big)(f_1'' + f_2''\mathbf{j}) + (\overline{f}_1' + \overline{f}_2'\mathbf{j})\mathbf{j}\frac{\partial}{\partial \overline{z}_2}(f_1'' + f_2''\mathbf{j})\\ = \Big((f_1' + f_2'\mathbf{j})\Big(\frac{\partial}{\partial \overline{z}_1}\Big) + (\overline{f}_1' + \overline{f}_2'\mathbf{j})\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)(f_1'' + f_2''\mathbf{j}) = \Big(f'\Big(\frac{\partial}{\partial \overline{z}_1}\Big) + \overline{f}'\mathbf{j}\frac{\partial}{\partial \overline{z}_2}\Big)f''. \;\blacksquare\end{aligned}$$

3. Almost everywhere hyperholomorphic functions whose inverses are almost everywhere hyperholomorphic

Definition 3.1. We call *inverse* of a quaternionic function $f : q \mapsto f(q)$, the function defined almost everywhere on U: $q \mapsto f(q)^{-1}$; then $f^{-1} = |f|^{-1}\overline{f}$, where $\overline{f}$ is the (quaternionic) conjugate of f, then $f^{-1} = |f|^{-1}(\overline{f_1} - f_2\mathbf{j})$.

Behavior of f^{-1} at $q \in Z(f)$. Let $n_1 = \sup J_q^\alpha(|f|^{-1}\overline{f_1})$; $n_2 = \sup J_q^\alpha(|f|^{-1}\overline{f_2})$.

Define $n_q = \sup n_i$, $i = 1, 2$, as the *order of the pole q of f^{-1}*.

3.1. Characterization

Proposition 3.2. *The following conditions are equivalent*:

(i) *the function f and its right inverse are hyperholomorphic, when they are defined*;
(ii) *we have the equations*:

$$(\overline{f_1} - f_1)\frac{\partial\overline{f_1}}{\partial z_1} - \overline{f_2}\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2} = 0$$

$$\overline{f_2}\frac{\partial f_1}{\partial z_1} + \frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1} - f_1) - f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} = 0.$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ be a hyperholomorphic function and $g = g_1 + g_2\mathbf{j} = |f|^{-1}(\overline{f_1} - f_2\mathbf{j})$ its inverse; so $g_1 = |f|^{-1}\overline{f_1}$; $g_2 = -|f|^{-1}f_2$, where $|f| = (f_1\overline{f_1} + f_2\overline{f_2})$.

$$\mathcal{D}g(q) = \frac{1}{2}\Big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\,\frac{\partial}{\partial\overline{z}_2}\Big)g(q) = \frac{1}{2}\Big(\frac{\partial g_1}{\partial\overline{z}_1} - \frac{\partial\overline{g}_2}{\partial z_2}\Big)(q) + \mathbf{j}\,\frac{1}{2}\Big(\frac{\partial g_1}{\partial\overline{z}_2} + \frac{\partial\overline{g}_2}{\partial z_1}\Big)(q)$$

$$\frac{\partial g_1}{\partial\overline{z}_1} = |f|^{-1}\frac{\partial\overline{f_1}}{\partial\overline{z}_1} - |f|^{-2}\overline{f_1}\Big(\frac{\partial f_1}{\partial\overline{z}_1}\overline{f_1} + f_1\frac{\partial\overline{f_1}}{\partial\overline{z}_1} + \frac{\partial f_2}{\partial\overline{z}_1}\overline{f_2} + f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_1}\Big)$$

$$-\frac{\partial\overline{g}_2}{\partial z_2} = |f|^{-1}\frac{\partial\overline{f_2}}{\partial z_2} - |f|^{-2}\overline{f_2}\Big(\frac{\partial f_1}{\partial z_2}\overline{f_1} + f_1\frac{\partial\overline{f_1}}{\partial z_2} + \frac{\partial f_2}{\partial z_2}\overline{f_2} + f_2\frac{\partial\overline{f_2}}{\partial z_2}\Big)$$

$$\frac{\partial g_1}{\partial\overline{z}_2} = |f|^{-1}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - |f|^{-2}\overline{f_1}\Big(\frac{\partial f_1}{\partial\overline{z}_2}\overline{f_1} + f_1\frac{\partial\overline{f_1}}{\partial\overline{z}_2} + \frac{\partial f_2}{\partial\overline{z}_2}\overline{f_2} + f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big)$$

$$\frac{\partial\overline{g}_2}{\partial z_1} = -|f|^{-1}\frac{\partial\overline{f_2}}{\partial z_1} + |f|^{-2}\overline{f_2}\Big(\frac{\partial f_1}{\partial z_1}\overline{f_1} + f_1\frac{\partial\overline{f_1}}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\overline{f_2} + f_2\frac{\partial\overline{f_2}}{\partial z_1}\Big)$$

$$\begin{aligned}
2|f|^2\mathcal{D}g = {} & (f_1\overline{f_1} + f_2\overline{f_2})\Big(\frac{\partial\overline{f_1}}{\partial\overline{z}_1} + \frac{\partial\overline{f_2}}{\partial z_2}\Big) - \overline{f_1}f_1\frac{\partial\overline{f_1}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_1}\frac{\partial f_1}{\partial\overline{z}_1} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_2}\frac{\partial f_2}{\partial\overline{z}_1} \\
& - \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_2} - f_1\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_2} - \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_2} - f_2\overline{f_2}\frac{\partial\overline{f_2}}{\partial z_2} \\
& + \mathbf{j}\Big((f_1\overline{f_1} + f_2\overline{f_2})\Big(\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \frac{\partial\overline{f_2}}{\partial z_1}\Big) - \overline{f_1}\,\overline{f_1}\frac{\partial f_1}{\partial\overline{z}_2} - \overline{f_1}f_1\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}\,\overline{f_2}\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} \\
& + \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\overline{f_2}\frac{\partial\overline{f_2}}{\partial z_1}\Big).
\end{aligned}$$

Use the fact that f is hyperholomorphic (equation (1)):

$$\begin{aligned}2|f|^2\mathcal{D}g = {}& f_1\overline{f_1}\frac{\partial\overline{f_2}}{\partial z_2} + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_1}\frac{\partial f_1}{\partial\overline{z}_1} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_2} - \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_2}\\ &+ \overline{f_2}\frac{\partial f_2}{\partial\overline{z}_1}(f_1 - \overline{f_1}) + \mathbf{j}\Big(+f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}\,\overline{f_2}\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\\ &+ \overline{f_1}\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f_1}) + \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1}\Big).\end{aligned}$$

f being hyperholomorphic, g hyperholomorphic is equivalent to the system of two equations:

$$f_1\overline{f_1}\frac{\partial\overline{f_2}}{\partial z_2} + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_1}\frac{\partial f_1}{\partial\overline{z}_1} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_1} - \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_2} - \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_2} + \overline{f_2}\frac{\partial f_2}{\partial\overline{z}_1}(f_1 - \overline{f_1}) = 0$$
$$f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}\,\overline{f_2}\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} + \overline{f_1}\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f_1}) + \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} = 0$$

f_1 and f_2 satisfy, by conjugation of the second equation:

$$f_2\overline{f_2}\frac{\partial f_1}{\partial z_1} - f_1f_1\frac{\partial\overline{f_1}}{\partial z_1} - f_1\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1} - f_1) + f_1\overline{f_1}\frac{\partial f_2}{\partial\overline{z}_2} - f_1f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - f_2f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} = 0$$
$$\overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}\,\overline{f_2}\frac{\partial f_2}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} + \overline{f_1}\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f_1}) = 0$$

Using (1), we get:

$$+ f_2\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1(\overline{f_1} - f_1)\frac{\partial\overline{f_1}}{\partial z_1} - f_1\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1} - f_1) - f_1f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - f_2f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} = 0$$
$$+ \overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + (f_1 - \overline{f_1})\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} + \overline{f_1}\frac{\partial f_1}{\partial\overline{z}_2}(f_1 - \overline{f_1}) + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} = 0.$$

Assume $f_1 \neq 0$, $f_2 \neq 0$. Then

$$\begin{aligned}&\overline{f_1}\Big(f_2\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1(\overline{f_1} - f_1)\frac{\partial\overline{f_1}}{\partial z_1} - f_1\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1} - f_1) - f_1f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - f_2f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big) = 0\\ &- f_2\Big(+\overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + (f_1 - \overline{f_1})\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1}\\ &\qquad\qquad - \overline{f_1}\frac{\partial\overline{f_2}}{\partial z_1}(f_1 - \overline{f_1}) + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big) = 0.\end{aligned}$$

By sum:

$$\begin{aligned}&\overline{f_1}\Big(f_1(\overline{f_1} - f_1)\frac{\partial\overline{f_1}}{\partial z_1} - f_1\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1} - f_1) - f_1f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2}\Big)\\ &\qquad - f_2\Big((f_1 - \overline{f_1})\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} - \overline{f_1}\frac{\partial\overline{f_2}}{\partial z_1}(f_1 - \overline{f_1}) + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2}\Big) = 0,\end{aligned}$$

i.e.

$$(\overline{f_1}f_1 + f_2\overline{f_2})\Big((\overline{f_1}-f_1)\frac{\partial\overline{f_1}}{\partial z_1} - \overline{f_2}\frac{\partial f_2}{\partial z_1} - f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2}\Big)=0$$

$$\overline{f_2}\Big(f_2\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_1(\overline{f_1}-f_1)\frac{\partial\overline{f_1}}{\partial z_1} - f_1\overline{f_2}\frac{\partial f_2}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1}-f_1) - f_1 f_2\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - f_2 f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big)=0$$

$$f_1\Big(\overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} + (f_1-\overline{f_1})\overline{f_2}\frac{\partial\overline{f_1}}{\partial z_1} + \overline{f_2}\,\overline{f_2}\frac{\partial f_2}{\partial z_1} - \overline{f_1}\frac{\partial\overline{f_2}}{\partial z_1}(f_1-\overline{f_1}) + f_2\overline{f_2}\frac{\partial\overline{f_1}}{\partial\overline{z}_2} - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big)=0$$

By sum

$$\overline{f_2}\Big(f_2\overline{f_2}\frac{\partial f_1}{\partial z_1} + f_2\frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1}-f_1) - f_2 f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big)$$
$$+ f_1\Big(\overline{f_1}\,\overline{f_2}\frac{\partial f_1}{\partial z_1} - \overline{f_1}\frac{\partial\overline{f_2}}{\partial z_1}(f_1-\overline{f_1}) - \overline{f_1}f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2}\Big)=0,$$

i.e.

$$\overline{f_2}\frac{\partial f_1}{\partial z_1} + \frac{\partial\overline{f_2}}{\partial z_1}(\overline{f_1}-f_1) - f_2\frac{\partial\overline{f_2}}{\partial\overline{z}_2} = 0. \ \blacksquare$$

Definition 3.3. We will call *w-hypermeromorphic function* (*w*- for *weak*) any almost everywhere defined hyperholomorphic function whose right inverse is hyperholomorphic almost everywhere.

4. On the spaces of hypermeromorphic functions

4.1. Sum of two w-hypermeromorphic functions

Proposition 4.1. *If f and g are two w-hypermeromorphic functions, then the following conditions are equivalent*:

(i) *the sum $h = f + g$ is w-hypermeromorphic*;
(ii) *h satisfies the following PDE*:

$$-\Big(\frac{\partial|h|}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial|h|}{\partial\overline{z}_2}\Big)(\overline{h}_1 - h_2\mathbf{j}) + |h|\Big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\Big)(\overline{h}_1 - h_2\mathbf{j}) = 0.$$

Proof. Exploit the condition

$$|h|^2\mathcal{D}(h^{-1}) = -\mathcal{D}(|h)|)(\overline{h}) + |h|\mathcal{D}(\overline{h}) = 0;$$

with $\overline{h} = \overline{h}_1 - h_2\mathbf{j}$

$$2\mathcal{D}\overline{h} = \Big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\Big)(\overline{h}_1 - h_2\mathbf{j}) = \frac{\partial\overline{h}_1}{\partial\overline{z}_1} + \frac{\partial\overline{h}_2}{\partial z_2} - \Big(\frac{\partial h_2}{\partial\overline{z}_1} - \frac{\partial h_1}{\partial z_2}\Big)\mathbf{j}$$

$$\mathcal{D}(|h|) = \mathcal{D}(h_1\overline{h}_1 + h_2\overline{h}_2) = \frac{1}{2}\Big(\frac{\partial}{\partial\overline{z}_1} + \mathbf{j}\frac{\partial}{\partial\overline{z}_2}\Big)(h_1\overline{h}_1 + h_2\overline{h}_2)$$
$$= \frac{1}{2}\Big(\overline{h}_1\frac{\partial h_1}{\partial\overline{z}_1} + \overline{h}_2\frac{\partial h_2}{\partial\overline{z}_1} + h_1\frac{\partial\overline{h}_1}{\partial\overline{z}_1} + h_2\frac{\partial\overline{h}_2}{\partial\overline{z}_1}\Big)$$
$$+ \frac{1}{2}\Big(\overline{h}_1\frac{\partial h_1}{\partial z_2} + \overline{h}_2\frac{\partial h_2}{\partial z_2} + h_1\frac{\partial\overline{h}_1}{\partial z_2} + h_2\frac{\partial\overline{h}_2}{\partial z_2}\Big)\mathbf{j} = 0. \ \blacksquare$$

4.2. Product of two w-hypermeromorphic functions

Proposition 4.2. *Let f, g be two w-hypermeromorphic functions on U, then the following conditions are equivalent*:

- (i) *the product $f * g$ is w-hypermeromorphic*;
- (ii) *f and g satisfy the system of PDE*:

$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_1}+\frac{\partial \overline{f_2}}{\partial z_2}\Big)+(f_1-\overline{f_1})\frac{\partial g_1}{\partial \overline{z}_1}+\overline{f_2}\,\frac{\partial g_1}{\partial z_2}-f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_1}=0,$$
$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_2}-\frac{\partial \overline{f_2}}{\partial z_1}\Big)+(f_1-\overline{f_1})\frac{\partial g_1}{\partial \overline{z}_2}-\overline{f_2}\,\frac{\partial g_1}{\partial z_1}-f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2}=0.$$

Proof. Let $f = f_1 + f_2\mathbf{j}$ and $g = g_1 + g_2\mathbf{j}$ be two hypermeromorphic functions and $f * g = f_1 g_1 - f_2\overline{g}_2 + (f_1 g_2 - f_2\overline{g}_1)\mathbf{j}$ their product, then

$$\frac{\partial f_1}{\partial \overline{z}_1}-\frac{\partial \overline{f_2}}{\partial z_2}=0,$$
$$\frac{\partial(f_1g_1-f_2\overline{g}_2)}{\partial \overline{z}_1}-\frac{\partial(\overline{f_1}\,\overline{g_2}-\overline{f_2}g_1)}{\partial z_2}$$
$$=g_1\Big(\frac{\partial f_1}{\partial \overline{z}_1}+\frac{\partial \overline{f_2}}{\partial z_2}\Big)-\overline{g_2}\Big(\frac{\partial \overline{f_1}}{\partial z_2}+\frac{\partial f_2}{\partial \overline{z}_1}\Big)+f_1\frac{\partial g_1}{\partial \overline{z}_1}-\overline{f_1}\,\frac{\partial \overline{g_2}}{\partial z_2}+\overline{f_2}\,\frac{\partial g_1}{\partial z_2}-f_2\frac{\partial \overline{g_2}}{\partial \overline{z}_1}=0,$$
$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_1}+\frac{\partial \overline{f_2}}{\partial z_2}\Big)+f_1\,\frac{\partial g_1}{\partial \overline{z}_1}-\overline{f_1}\,\frac{\partial \overline{g_2}}{\partial z_2}+\overline{f_2}\,\frac{\partial g_1}{\partial z_2}-f_2\frac{\partial \overline{g_2}}{\partial \overline{z}_1}=0,$$
$$\frac{\partial(f_1g_1-f_2\overline{g}_2)}{\partial \overline{z}_2}+\frac{\partial(\overline{f_1}\,\overline{g_2}-\overline{f_2}g_1)}{\partial z_1}$$
$$=g_1\Big(\frac{\partial f_1}{\partial \overline{z}_2}-\frac{\partial \overline{f_2}}{\partial z_1}\Big)+\overline{g_2}\Big(\frac{\partial \overline{f_1}}{\partial z_1}-\frac{\partial f_2}{\partial \overline{z}_2}\Big)+f_1\frac{\partial g_1}{\partial \overline{z}_2}-\overline{f_1}\,\frac{\partial \overline{g}_2}{\partial z_1}+\overline{f_2}\,\frac{\partial g_1}{\partial z_1}-f_2\frac{\partial \overline{g_2}}{\partial \overline{z}_2}=0,$$
$$g_1\Big(\frac{\partial f_1}{\partial \overline{z}_2}-\frac{\partial \overline{f_2}}{\partial z_1}\Big)+f_1\frac{\partial g_1}{\partial \overline{z}_2}+\overline{f_1}\,\frac{\partial \overline{g}_2}{\partial z_1}-\overline{f_2}\,\frac{\partial g_1}{\partial z_1}-f_2\frac{\partial \overline{g}_2}{\partial \overline{z}_2}=0. \ \blacksquare$$

Definition 4.3. We will call *hypermeromorphic* the w-hypermeromorphic functions whose sum and product are w-hypermeromorphic. Their space is nonempty, since it contains the space of the meromorphic functions.

5. Globalization. Hamilton 4-manifold

5.1. The hypermeromorphic functions on a relatively compact open set U of $\mathbb{H}$ play the part of the meromorphic functions on a relatively compact open set U of $\mathbb{C}$. We will call *pseudoholomorphic function on U* every hypermeromorphic function, without poles on U. We will call *smooth hypermeromorphic function* (*sha function*) *on U* every hypermeromorphic function, without zeroes and poles on U.

Lemma 5.1. *The quotient of two pseudoholomorphic functions on U, with the same zeroes and the same orders, is a sha function on U.*

5.2. Manifolds. The sha functions have been defined on open sets of $\mathbb{H} \cong \mathbb{C}^2$. Let X be a 4-dimensional manifold bearing an atlas $\mathcal{A}$ of charts (h_j, U_j) such as the transition functions $h_{i,j} : U_i \cap U_j \to \mathbb{H}$ are sha functions. $X = (X, \mathcal{A})$ will be called an $\mathcal{A}$-manifold analogous for $\mathbb{H}$ of a Riemann surface for $\mathbb{C}$. I also propose to call an $\mathcal{A}$-manifold a *Hamilton* 4*-manifold.*

5.3. Sheaves of pseudoholomorphic, hypermeromorphic functions

5.3.1. *Functions on an $\mathcal{A}$-manifold $X = (X, \mathcal{A})$.* A map $f : X \to \mathbb{H}$ is called a *pseudoholomorphic function* on X, if it is continuous and satisfies the following condition: for every chart (h, U) of X, $(f\,|\,U)h^{-1} : h(U) \to \mathbb{H})$ is pseudoholomorphic. In the same way, a map $f : X \to \mathbb{H}$ is called a *hypermeromorphic function* on X, if it is continuous and satisfies the following condition: for every chart (h, U) of X, $(f\,|\,U)h^{-1} : h(U) \to \mathbb{H})$ is hypermeromorphic.

5.3.2. *Examples of Hamilton* 4*-manifold.* The identity map of $\mathbb{H}$ is: $z_1 + z_2\mathbf{j} \mapsto z_1 + z_2\mathbf{j}$.

Example 1. $(id_{\mathbb{H}}, \mathbb{H})$ is the unique chart of the atlas defining $\mathbb{H}$ as an $\mathcal{A}$-manifold.

Proof. The identity map $(id_{\mathbb{H}}$ is $f_1 = z_1, f_2 = z_2$ is pseudoholomorphic. ∎

Example 2. Every open set V of X bears an induced structure of *Hamilton* 4*-manifold.*

Example 3 (Hamilton hypersphere $\mathbb{HP}$). In the space $\mathbb{H} \times \mathbb{H} \setminus \{0\}$, consider the equivalence relation $\rho_1 \mathcal{R} \rho_2$: there exists $\lambda \in \mathbb{H}^* = \mathbb{H} \setminus 0$ such that $\rho_2 = \rho_1 \lambda$ (right multiplication by λ). The elements of $\mathbb{H} \times \mathbb{H} \setminus \{0\}$ are the pairs $(q_1, q_2) \neq (0, 0)$. Let

$$\pi : \mathbb{H} \times \mathbb{H} \setminus \{0\} \to \big(\mathbb{H} \times \mathbb{H} \setminus \{0\}\big)/\mathcal{R}, \text{ denoted by } \mathbb{HP},$$
$$(q_1, q_2) \mapsto \text{class of } (q_1, q_2).$$

So, $\mathbb{HP}$ is the set of the quaternionic lines from the origin of $\mathbb{H}^2$.

Consider the pairs $(q_1, q_2) \in \mathbb{H}^2$, with $q_2 \neq 0$ we have $\pi(q_1, q_2) = \pi(q_1 q_2^{-1}, 1)$. Let $\zeta = q_1 q_2^{-1}$, $q_2 \neq 0$. In the same way, consider the pairs $(q_1, q_2) \in \mathbb{H}^2$, with $q_1 \neq 0$ we have $\pi(q_1, q_2) = \pi(1, q_2 q_1^{-1})$. Let $\zeta' = q_2 q_1^{-1}$, $q_1 \neq 0$. The charts ζ, ζ' have for domains U, U', two open sets of $\mathbb{HP}$, respectively homeomorphic to $\mathbb{H}$ forming an atlas of $\mathbb{HP}$. Remark that U covers the whole of $\mathbb{HP}$ except the point $\pi(q_1, 0)$ denoted ∞, and that U' covers the whole of $\mathbb{HP}$ except the point $\pi(0, q_2)$ denoted 0. $U' = \mathbb{HP} \setminus \{0\}$. Over $U \cap U'$, we have: $\zeta.\zeta' = 1$, i.e. $\zeta' = \zeta^{-1}$ and $\zeta = q_1 q_2^{-1}$.

5.3.3. *Pseudoholomorphic map or morphism.* Let X and Y be two Hamilton 4-manifolds, a map $f : X \to Y$ is said *pseudoholomorphic* if it is continuous and if, for every pair of pseudoholomorphic charts $(h, U), (k, V)$ such that $f(U) \subset V$,

$$k(f\,|\,U)h^{-1} : h(U) \to k(V) \text{ is pseudoholomorphic.}$$

5.3.4. *Sheaf of pseudoholomorphic functions.* Let U, V be two open sets of X such that $U \subset V$, then the restrictions to U of the pseudoholomorphic functions on V are pseudoholomorphic on U.

So we defined the *sheaf*, denoted by $\mathcal{P}$, of (non-commutative rings) *of pseudoholomorphic functions on X*. The pair (X,$\mathcal{P}$) is a *ringed space.*

In the same way, the *sheaf* of non-commutative rings, denoted by $\mathcal{M}$, *of hypermeromorphic functions is defined on X.*

5.3.5. *Hamiltonian submanifolds.* They are submanifolds whose function ring is pseudoholomorphic. We will implicitly use the following fact: If f is a pseudoholomorphic or hypermeromorphic function, the same is true for $a+f$, where a is any fixed quaternion.

The following examples are complex analytic submanifolds.

i) $\mathbb{H}$. Let a be a fixed quaternion, then $a+\mathbb{C}\subset\mathbb{H}$ is a *complex line from a embedded in* $\mathbb{H}$.

ii) $\mathbb{HP}$. *Complex projective line imbedded in* $\mathbb{HP}$. Let $i : z_1 \mapsto z_1+z_2\mathbf{j}$ and $j:\mathbb{CP}\mapsto\mathbb{HP}$,

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{C}\setminus 0 \arrow[r] \arrow[d, "i\times i"'] & \mathbb{H}\times\mathbb{H}\setminus\{0\} \arrow[d] \\
(\mathbb{C}\times\mathbb{C}\setminus 0)/\mathcal{R}' \arrow[r, "j"] & \big(\mathbb{H}\times\mathbb{H}\setminus\{0\}\big)/\mathcal{R}.
\end{tikzcd}$$

Let $p\in\mathbb{HP}$ be a fixed point. Then, $p+\mathbb{C}P^1$ is a *complex projective line* (or Riemann sphere) *from p, embedded in* $\mathbb{HP}$.

iii) Let S be a compact Riemann surface contained in $\mathbb{HP}$ as a Hamiltonian submanifold. Then $p+S$ is a *compact Riemann surface from p, embedded in* $\mathbb{HP}$.

5.3.6. *A family of complex submanifolds in a Hamilton 4-manifold.* We now use the properties and notions of Subsection 2.4.2. They are $a+A_{\alpha,\beta}$ and also for restrictions to an open set U of $\mathbb{H}$.

On a Hamilton 4-manifold X with an atlas $\mathcal{A}$ and every domain of chart U as above, we obtain:

Proposition 5.2. *Let $(X,\mathcal{P})$ be a Hamilton 4-manifold. There exists a family of complex analytic curves $C_{b,\gamma,\delta}$ of X. For every U domain of coordinates in $\mathcal{A}$ consider $A_{\gamma,\delta}$. By gluing, we get a complex analytic curve in $(X,\mathcal{P})$ from $b\in X$, and γ,δ are real parameters.*

Proof. Let $b\in X$, $\beta,\gamma\in\mathbb{R}$ be given, consider an atlas $\mathcal{A}$ whose domains of charts are either open sets U of X disjoint from $A_{\beta,\gamma}$, or $V_{\beta,\gamma}=U\cup(A_{\beta,\gamma}\cap U)$ where $A_{\beta,\gamma}\cap U$ is connected, not empty. The restrictions of the charts of $\mathcal{A}$ to $U\cup(A_{\beta,\gamma}\cap U)$ define an atlas of $C_{b,\gamma,\delta}$ as complex analytic subvariety of $(X,\mathcal{P})$, in the following way: assume $b\in V_{\beta,\gamma}\cap A_{\beta,\gamma}\cap U$ and consider the open sets analogous to $V_{\beta,\gamma}$ such that the various $V_{\beta,\gamma}$ be connected. Then the corresponding $A_{\beta,\gamma}\cap U$ constitute a covering of the unique complex analytic curve $C_{b,\gamma,\delta}$. ■

5.3.7. Let C be a complex analytic curve embedded into X and an atlas $\mathcal{A}$ such that every chart of domain U meeting C satisfies: $U\cap C$ is connected.

Theorem 5.3. *The set of complex analytic curves in X is the family $C_{b,\gamma,\delta}$.*

6. Hamilton 4-manifold of a hypermeromorphic function

6.1. Analytic continuation along a path [D90, p. 116].

Let X be a Hamilton 4-manifold, $\gamma:[0,1]\to X$ a continuous path from a to b, $\varphi\in\mathcal{P}_a$ a germ of pseudoholomorphic function at a.

Let $\tau \in [0,1]$ and $\varphi_\tau \in \mathcal{P}_{\gamma(\tau)}$, there exist an open neighborhood U_τ of $\gamma(\tau)$ in X and a pseudoholomorphic function $f_\tau \in \mathcal{P}(U_\tau)$ such that $\rho^{U_\tau}_{\gamma(\tau)} f_\tau = \varphi_\tau$. γ being continuous, there exists an open neighborhood W_τ of τ in [0,1] such that $\gamma(W_\tau) \subset U_\tau$.

Definition 6.1. A germ $\psi \in \mathcal{P}_b$ is said to be *the analytic continuation of* φ *along* γ if there exists a family $(\varphi_t)_{t\in[0,1]}$ such that:

1) $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.
2) for every $\tau \in [0,1]$, for every $t \in W_\tau$, we have: $\rho^{U_\tau}_{\gamma(t)} f_\tau = \varphi_t$.

Theorem 6.2 (Identity theorem). *Let X be a connected Hamilton 4-manifold and $f_1, f_2 : X \to Y$ be two morphisms which coincide in the neighborhood of a point $x_0 \in X$, then f_1, f_2 coincide on X.*

Proof as for Riemann surfaces, [D90, ch. 5].

Theorem 6.3 (cf. [D90, ch. 5, 4.1.5]). *Let X be a simply connected Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$ be a germ having an analytic continuation along every path from a. Then there exists a unique function $f \in \mathcal{P}(X)$ such that $\rho^X_a f = \varphi$.*

Let $p : Y \to X$ be a morphism of two Hamilton 4-manifolds; p is locally bi-pseudoholomorphic, then it defines, for every $y \in Y$, an isomorphism $p^*_y : \mathcal{P}_{x,p(y)} \to \mathcal{P}_{Y,y}$; this defines: $p_* = p_{*y} = (p^*_y)^{-1}$.

Definition 6.4. Let X be a Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$. A quadruple (Y, p, f, b) is called an *analytic continuation of* φ if:

(i) Y is a Hamilton 4-manifold, $p : Y \to X$ is a morphism;
(ii) f is a pseudoholomorphic function on Y;
(iii) $b \in p^{-1}(a) \subset Y$; $p_*(\rho^Y_b f) = \varphi$.

An analytic continuation is said to be *maximal* if it is a solution of the following universal map problem: for every analytic continuation (Z, q, g, c) of φ, there exists a fibered morphism $F : Z \to Y$ such that $F(c) = b$ and $F^*(f) = g$. Hence

If (Y, p, f, b) is a maximal analytic continuation of φ, it is unique up to an isomorphism. Y is called the Hamilton 4-manifold of φ.

Theorem 6.5. *Let X be a Hamilton 4-manifold, $a \in X$, $\varphi \in \mathcal{P}_a$. Then there exists a maximal analytic continuation of φ.*

Remark. Then, we will say that the above function f is *the unique maximal analytic continuation of the germ* φ. Moreover, the above definitions and the results of Section 2 are valid for the sheaf $\mathcal{M}$ of hypermeromorphic functions instead of the sheaf $\mathcal{P}$.

6.2. Main result

Theorem 6.6. *Let X be a Hamilton 4-manifold and $P(T) = T^n + c_1 T^{n-1} + \ldots + c_n \in \mathcal{M}(X)[T]$ be an irreducible polynomial of degree n. Then there exist a Hamilton 4-manifold Y, a ramified pseudoholomorphic covering (cf.* [D90, ch. 5] *for Riemann surfaces) with n leaves $\Pi : Y \to X$ and a hypermeromorphic function $F \in \mathcal{M}(Y)$ such that $(\Pi^* P)(F) = 0$.*

F is the unique maximal analytic continuation of every hypermeromorphic germ φ of X such that $P(\varphi)=0$; F is called the *hyperalgebraic function defined by the polynomial* P and Y is the *Hamilton* 4*-manifold of* F.

1) *X is compact connected.*
2) *Every pseudoholomorphic function on X is constant.*
3) *Every hypermeromorphic function f on X different from ∞ is rational.*
4) *In case $X=\mathbb{HP}$, in Theorem* 6.6*, c_j is rational.* Indeed, since c_j is hypermeromorphic, from 3), it is rational.

6.3. Proof of Theorem 6.6. In the notation of Example 3, ζ is a local coordinate on $X=\mathbb{HP}$. f has a finite set of poles $p_1,\dots,p_n$. Assume that ∞ is not a pole of f, then $p_1,\dots,p_n\in\mathbb{H}$. Let h_ν be the principal part of f at p_ν, then $f-h_\nu=a_\nu$, constant, from 2) and $h_\nu=\sum_{j=-k_\nu}^{-1}C_{\nu j}(\zeta-p_\nu^j)$ is a hypermeromorphic function, where $C_{\nu j}\in\mathbb{H}$.

6.3.1. *Elementary symmetric functions.* Let $\Pi: Y\to X$ be a non-ramified pseudoholomorphic covering with n leaves, and f be a hypermeromorphic function on Y. Every point $x\in X$ has an open neighborhood U such that $\Pi^{-1}(U)=\bigcup_{j=1}^n V_j$ where the V_j are disjoint and $\Pi\,|\,V_j: V_j\to U$ is bi-pseudoholomorphic $(j=1,\dots,n)$; let $\varphi_j: U\to V_j$ the reverse (i.e. set inverse) of $\Pi\,|\,V_j$ and $f_j=\varphi_j^*f=f.\varphi_j$. Then

$$\prod_{j=1}^{n}(T-f_j)=T^n+c_1T^{n-1}+\ldots+c_n;$$

$c_j=(-1)^j s_j(f_1,\dots,f_n)$, where s_j is the j-th elementary symmetric function in n variables. The c_j are hypermeromorphic, locally defined, but glue together into $c_1,\dots,c_n\in\mathcal{M}(X)$ and are called *the elementary symmetric functions of f with respect to* Π.

REMARK. The elementary symmetric functions of a hypermeromorphic function on Y are still defined when the covering Π is ramified.

6.3.2.

THEOREM 6.7. *Let* Π *be as in Theorem* 6.6*, with Y not necessarily connected, $A\subset X$ be a discrete closed subset containing all the critical values of* Π*, and $B=\Pi^{-1}(A)$.*

Let f be a pseudoholomorphic (*resp. hypermeromorphic*) *function on $Y\setminus B$ and*

$$c_1,\dots,c_n\in\mathcal{H}(X\setminus A)\quad (\textit{resp. } \mathcal{M}(X\setminus A))$$

the elementary symmetric functions of f. Then the following two conditions are equivalent:

(i) *f has a pseudoholomorphic* (*resp. hypermeromorphic*) *extension to Y*;
(ii) *for every $j=1,\dots,n$, c_j has a pseudoholomorphic* (*resp. hypermeromorphic*) *extension to X.*

6.3.3. *Existence of Y in Theorem* 6.6. Let $\Delta\in\mathcal{M}(X)$ be the discriminant of $P(T)$; $P(T)$ being irreducible, $\Delta\neq 0$: then there exists a discrete closed set $A\subset X$ such that, for every $x\in X'=X\setminus A$, $\Delta(x)\neq 0$, and all the functions c_j are pseudoholomorphic.

Let $Y'=\{\varphi\in\mathcal{H}_x, x\in X'; P(\varphi)=0\}\subset L\mathcal{H}$ be the étalé space defined by the sheaf $\mathcal{H}$, and $\Pi': Y'\to X$ $(\varphi\mapsto x)$.

It can be shown that, for every $x \in X'$, there exists an open neighborhood U of x in X' and functions $f_j \in \mathcal{H}(U)$, $j = 1, \ldots, n$, such that $P(T)\,|\,U = \Pi_{j=1}^{n}(T - f_j)$; then $\Pi'^{-1}(U) = \bigcup_{j=1}^{n}[U, f_j]$ where $[U, f_j] = \{f_{jy}, y \in U\}$ is an open set of $L\mathcal{H}$ and $\Pi'\,|\,[U, f_j] : [U, f_j] \to U$ is a homeomorphism; Y' is a Hamilton 2-manifold not necessarily connected, and a pseudoholomorphic, non-ramified covering of X'. It can be shown that Π' can be extended into a ramified pseudoholomorphic covering $\Pi : Y \to X$ of X for which $Y' = \Pi^{-1}(X')$.

The c_j are defined on the whole of X; from Theorem 6.7, f has an extension $F \in \mathcal{M}(X)$ such that

$$\Pi^* P(F) = F^n + (\Pi^* c_1)F^{n-1} + \ldots + \Pi^* c_n = 0.$$

It is easy to prove the connectedness of Y and the uniqueness of F. This ends the proof of Theorem 6.6.

7. The Hamilton 4-manifold Y of F when $X = \mathbb{HP}$

7.1. Recall the main properties of Y.

Y is of real dimension 4;
Y is connected;
Y is compact;
Y is C^∞.

Let m be the number of the critical values of Π and q_j these critical values; they define points of Y forming the 0-skeleton of a simplicial complex K carried by the manifold Y. K may be supposed to be C^∞ by parts. Cutting along the 3-faces of K defines a fundamental domain FD of the covering Π. FD is a 4-dimensional polytope in $\mathbb{HP}$ with an even number of 3-faces; gluing together the opposite 3-faces, we get a compact 4-dimensional polytope with homology of the Hamilton 4-manifold Y.

7.2. Homology of Y. $H^p(Y;\mathbb{Z})$, for $p = 0, \ldots, 4$, have to be evaluated, using the critical values q_j, and the Poincaré duality.

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Pierre Dolbeault passed away on June 12, 2015. Born near Paris on October 10, 1924, he entered the École Normale Supérieure in 1945 before spending a year in Princeton. Later he prepared a thèse d'état *under the supervision of Henri Cartan. Most of his career took place in France (Montpellier, Bordeaux, Poitiers and finally Paris). In December 1956, he married Simone Lemoine, who was also professor of mathematics successively in the universities of Bordeaux and Poitiers. Well connected to a large network of mathematicians, collaborators and friends, he often travelled to USA, Italy, Romania, Germany and many other countries. Poland was a special place for him. At the end of a long, fruitful scientific life, he had once more the opportunity to travel to Będlewo during the summer of 2014 at the occasion of the* Conference on Constructive Approximation of Functions of Several Complex Variables*. The participation to the meeting and the contribution to the proceedings have been his last scientific achievements. Mathematics have always been essential to him. During the last years he had to reduce his activities, but research in mathematics was the last one he was willing to abandon. At his own pace, he luckily managed to stay productive almost up to the end.*

Jean Dolbeault