

SOME APPROXIMATION PROBLEMS IN SEMI-ALGEBRAIC GEOMETRY

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Abstract. In this paper we deal with a best approximation of a vector with respect to a closed semi-algebraic set C in the space \mathbb{R}^n endowed with a semi-algebraic norm ν . Under additional assumptions on ν we prove semi-algebraicity of the set of points of unique approximation and other sets associated with the distance to C . For C irreducible algebraic we study the critical point correspondence and introduce the ν -distance degree, generalizing the notion developed by other authors for the Euclidean norm. We discuss separately the case of the ℓ^p norm ($p > 1$).

1. Introduction. Let $\nu : \mathbb{R}^n \rightarrow [0, \infty)$ be a norm on \mathbb{R}^n . In many applications one needs to approximate a given vector $\mathbf{x} \in \mathbb{R}^n$ by a point \mathbf{y} in a given closed subset $C \subset \mathbb{R}^n$. Usually the approximation to \mathbf{x} given by \mathbf{y} is measured by $\nu(\mathbf{x} - \mathbf{y})$. Then the distance of \mathbf{x} to C with respect to the norm ν is $\text{dist}_\nu(\mathbf{x}, C) := \min\{\nu(\mathbf{x} - \mathbf{y}) : \mathbf{y} \in C\}$. A point $\mathbf{y}^* \in C$ is called a *best ν -approximation* of \mathbf{x} if $\nu(\mathbf{x} - \mathbf{y}^*) = \text{dist}_\nu(\mathbf{x}, C)$. Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^n and let $\text{dist}_{\|\cdot\|}(\mathbf{x}, C)$ denote the distance $\text{dist}_{\|\cdot\|}(\mathbf{x}, C)$. We call a best $\|\cdot\|$ -approximation a *best C -approximation*, or briefly a *best approximation*. Of course the Euclidean norm plays a special role, but it is not the only norm important in practical applications. For example, in compressed sensing one needs to minimize $\|\mathbf{x}\|_1$ subject to linear conditions $A\mathbf{x} = \mathbf{y}$. This motivates us to study approximation problems when the norm ν is a semi-algebraic function on \mathbb{R}^n and the set C is a closed semi-algebraic subset of \mathbb{R}^n .

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Our paper is organized as follows: In Section 2 we recall the well-known relation between the differentiability of $\text{dist}_\nu(\cdot, C)$ at a point $\mathbf{x} \in \mathbb{R}^n$ and the uniqueness of ν -approximation of \mathbf{x} (see [21] for many results of this type, and the references therein). In Section 3 we collect some fundamental properties of semi-algebraic sets in \mathbb{R}^n and prove that (under suitable assumptions on the norm ν) the set of all points $\mathbf{x} \in \mathbb{R}^n$ at which the ν -approximation of \mathbf{x} to a (fixed) closed semi-algebraic set C is not unique is a nowhere dense semi-algebraic set. In subsequent sections we study critical points of the function $\text{dist}(\mathbf{x}; C)$ given \mathbf{x} and C . The interest in critical points is again motivated by applications: In general, the numerical methods for finding best approximation to \mathbf{x} in C aim at finding a local minimum of the function $\nu(\mathbf{x} - \mathbf{y})$, $\mathbf{y} \in C$, but most of these methods will converge at most to a critical point of $f_{\mathbf{x}, \nu}(\mathbf{y}) := \nu(\mathbf{x} - \mathbf{y})$, $\mathbf{y} \in C$. In Section 4 we assume that C is an irreducible algebraic set, but ν is still a quite arbitrary semi-algebraic norm. We then study the critical point correspondence $\Sigma(C)$, i.e., the closure of the semi-algebraic set $\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times C : \mathbf{y} \text{ is a critical point of } f_{\mathbf{x}, \nu}\} \setminus (\mathbb{R}^n \times \text{Sing } C)$. Using some dominating maps associated with stratification of $\Sigma(C)$ we introduce the notion of the ν -distance degree of C . The case of ℓ^p norms ($p > 1$) is studied in Section 5, while Section 6 contains more details about the critical point correspondence. Our results do not require advanced material in semi-algebraic or algebraic geometry, so anyone with basic knowledge of these areas should be able to reconstruct the arguments. At any rate, we include the necessary prerequisites.

Some relations with results already present in literature can be observed, but of course there are differences. First of all, the problem of approximating vectors in \mathbb{R}^n by points in a definable closed set $C \subset \mathbb{R}^n$ was studied in [6] with respect to the Euclidean norm. The author of that paper proved, among other things, that the approximation by points in C is unique outside some nowhere dense definable set. He also obtained an analogous result in the case when C is subanalytic. Both classes considered in [6] substantially generalize the class of semi-algebraic sets. However, here we present a straightforward approach to the semi-algebraic case while working with a wide class of semi-algebraic norms. Second, the notions of the critical point correspondence and the Euclidean distance degree of an algebraic variety C were introduced and studied in [7]. That paper focused on methods of computation of this degree when C has a parametric representation or when the complexification of C is an affine cone in \mathbb{C}^n (which allows one to work in \mathbb{P}^{n-1}). Our definition of the distance degree (Definition 4.5) generalizes that of [7] and works for a quite arbitrary semi-algebraic norm. Finally, in our case the critical point correspondence is a semi-algebraic (rather than algebraic) set and our distance degree agrees with the degree of this semi-algebraic set understood in the sense of [19]. We hope that some specialists will take an interest in the natural occurrence of the notion of the degree of a semi-algebraic set in this context. While we do not present any explicit computations, we should note that our methods have applications in approximation of matrices and tensors. The relevant results can be found in [10] and [11].

2. Uniqueness of a best ν -approximation. Let ν be a norm on \mathbb{R}^n . Let

$$B_\nu := \{\mathbf{x} \in \mathbb{R}^n : \nu(\mathbf{x}) \leq 1\}, \quad S_\nu := \{\mathbf{x} \in \mathbb{R}^n : \nu(\mathbf{x}) = 1\}, \tag{2.1}$$

denote respectively the unit ball and the unit sphere with respect to ν . It is well known that all norms on \mathbb{R}^n are equivalent, i.e.,

$$\kappa_1(\nu)\|\mathbf{x}\| \leq \nu(\mathbf{x}) \leq \kappa_2(\nu)\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbb{R}^n, \text{ where } 0 < \kappa_1(\nu) \leq \kappa_2(\nu). \tag{2.2}$$

Recall that the dual norm ν^* is defined as $\nu^*(\mathbf{x}) = \max_{\mathbf{y} \in S_\nu} \mathbf{y}^\top \mathbf{x}$. Since ν is a convex function on \mathbb{R}^n it follows that the hyperplane $\mathbf{y}^\top \mathbf{z} = 1, \mathbf{z} \in \mathbb{R}^n$, is a supporting hyperplane of B_ν at $\mathbf{x} \in S_\nu$ if and only if $\mathbf{y}^\top \mathbf{x} = 1$ and $\mathbf{y} \in S_{\nu^*}$. The subdifferential of ν at $\mathbf{x} \neq \mathbf{0}$ is given by

$$\partial\nu(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \nu^*(\mathbf{y}) = 1, \mathbf{y}^\top \mathbf{x} = \nu(\mathbf{x})\}. \tag{2.3}$$

In particular, ν is differentiable at $\mathbf{x} \neq \mathbf{0}$ if and only if the supporting hyperplane of B_ν at $\frac{1}{\nu(\mathbf{x})}\mathbf{x}$ is unique [18]. Assume that ν is differentiable at $\mathbf{x} \neq \mathbf{0}$. Then the differential of ν at \mathbf{x} , viewed as a row vector in \mathbb{R}^n , is the only vector in $\partial\nu(\mathbf{x})$. In this case we will denote the differential also by $\partial\nu(\mathbf{x})$. So the directional derivative of ν at \mathbf{x} in the direction $\mathbf{u} \in \mathbb{R}^n$ is given as $\partial\nu(\mathbf{x})\mathbf{u}$. We call ν a *differentiable norm* if ν is differentiable at each $\mathbf{x} \neq \mathbf{0}$, i.e., $\nu \in \mathcal{C}^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$ [18]. If ν is a differentiable norm then we denote $\partial\nu(\mathbf{x})$ by $\nabla\nu(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$.

Recall that ν is strictly convex if for each pair of points $\mathbf{x}, \mathbf{y} \in S_\nu, \mathbf{x} \neq \mathbf{y}$, the point $t\mathbf{x} + (1-t)\mathbf{y}$ lies in the interior of B_ν for $t \in (0, 1)$, i.e. $\nu(t\mathbf{x} + (1-t)\mathbf{y}) < 1$ for $t \in (0, 1)$. It is well known that ν is strictly convex if and only if each supporting hyperplane of B_ν at $\mathbf{x} \in S_\nu$ intersects B_ν only in \mathbf{x} . That is, ν is strictly convex if and only if for each two distinct point $\mathbf{x}_1, \mathbf{x}_2 \in S_\nu$ one has the equality $\partial\nu(\mathbf{x}_1) \cap \partial\nu(\mathbf{x}_2) = \emptyset$.

It is easy to construct a norm in \mathbb{R}^2 which is strictly convex and not differentiable. It is well known that if ν is differentiable then ν^* is strictly convex and if ν is strictly convex then ν^* is differentiable.

Note that the ℓ_p -norm on $\mathbb{R}^n, \|(x_1, \dots, x_n)^\top\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, is differentiable and strictly convex if and only if $p \in (1, \infty)$.

The following result is known in many variants, dating back at least to 1938 ([12]). We state and prove here a version best suited to our purposes. For related results, see [21], [10, §6].

THEOREM 2.1. *Suppose that ν is strictly convex. Then at each point $\mathbf{x} \notin C$ where $\text{dist}_\nu(\mathbf{x}, C)$ is differentiable, best ν -approximation is unique.*

Proof. The subdifferential of ν at \mathbf{a} is

$$\partial\nu(\mathbf{a}) = \{\mathbf{x}^* \in (\mathbb{R}^n)^* : \forall \mathbf{y} \nu(\mathbf{y}) \geq \nu(\mathbf{a}) + \mathbf{x}^*(\mathbf{y} - \mathbf{a})\}.$$

Let $f(\cdot) := \text{dist}_\nu(\cdot, C)$. Assume that f is differentiable at \mathbf{x} and let us view the differential of f at \mathbf{x} as a linear functional on \mathbb{R}^n . Let \mathbf{y}^0 be a best approximation to \mathbf{x} in C . Then $\partial f(\mathbf{x}) \in \partial\nu(\mathbf{x} - \mathbf{y}^0)$.

If there exists a $\mathbf{z}^0 \in C \setminus \{\mathbf{y}^0\}$ such that $\nu(\mathbf{x} - \mathbf{y}^0) = \nu(\mathbf{x} - \mathbf{z}^0) = f(\mathbf{x})$, then $\frac{1}{\nu(\mathbf{x} - \mathbf{y}^0)}\partial f(\mathbf{x}) \in \partial\nu(\frac{1}{\nu(\mathbf{x} - \mathbf{y}^0)}(\mathbf{x} - \mathbf{y}^0)) \cap \partial\nu(\frac{1}{\nu(\mathbf{x} - \mathbf{z}^0)}(\mathbf{x} - \mathbf{z}^0))$. But this is a contradiction with strict convexity of ν . ■

The following example, due to Sinan Güntürk (private communication), shows that we cannot drop the condition that ν is strictly convex in Theorem 2.1.

EXAMPLE 2.2. Let $C \subset \mathbb{R}^2$ be the line $\{(t, t) : t \in \mathbb{R}\}$ and $\nu = \|\cdot\|_1$. Then for $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$ it is straightforward to show that $\text{dist}_{\|\cdot\|_1}(\mathbf{x}, C) = |x_1 - x_2|$. For $\mathbf{x} \in \mathbb{R}^2 \setminus C$ a best approximation is an arbitrary point of the segment between the two points on C : $(x_1, x_1)^\top, (x_2, x_2)^\top$. That is, for each $\mathbf{x} \in \mathbb{R}^2 \setminus C$, $\text{dist}_{\|\cdot\|_1}(\mathbf{x}, C)$ is differentiable and \mathbf{x} does not have a unique best approximation.

We will need the following lemma:

LEMMA 2.3. *Let ν be an arbitrary norm. Let $a > 1$ and let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as $g := \nu^a$. Then*

$$g(\mathbf{x}) > 0 \text{ for } \mathbf{x} \neq \mathbf{0}, \quad g(\mathbf{0}) = 0, \quad Dg(\mathbf{0}) = \mathbf{0}, \quad \lim_{\|\mathbf{x}\| \rightarrow \infty} g(\mathbf{x}) = \infty, \quad (2.4)$$

where $Dg(\mathbf{0})$ is the differential of g at $\mathbf{0}$. Assume that $C \subset \mathbb{R}^n$ is a closed set. Then $\text{dist}_\nu(\mathbf{x}, C)^a$ is differentiable at each $\mathbf{y} \in C$, and its gradient is $\mathbf{0}$.

Proof. Use (2.2) to deduce (2.4). Suppose that $\mathbf{y} \in C$. Then $\text{dist}_\nu(\mathbf{y}, C) = 0$. Clearly

$$0 \leq \text{dist}_\nu(\mathbf{x}, C)^a - \text{dist}_\nu(\mathbf{y}, C)^a \leq \nu(\mathbf{x} - \mathbf{y})^a \leq (\kappa_2(\nu)\|\mathbf{x} - \mathbf{y}\|)^a.$$

The above inequalities yield directly that $\text{dist}_\nu(\mathbf{x}, C)^a$ is differentiable at each $\mathbf{y} \in C$, and its gradient is $\mathbf{0}$. ■

Note that in general the function $\text{dist}(\cdot, C)$ given by a norm ν may be not differentiable at $\mathbf{y} \in C$. A trivial example is $C = \{\mathbf{0}\} \subset \mathbb{R}^n$.

COROLLARY 2.4. *Let the assumptions of Theorem 2.1 hold. Assume that $a > 1$. Then at each point \mathbf{x} where $\text{dist}_\nu(\mathbf{x}, C)^a$ is differentiable, best ν -approximation is unique.*

3. Semi-algebraic sets. The background material for this section can be found e.g. in [1] or [5]. Recall that a set $S \subset \mathbb{R}^n$ is called semi-algebraic if it is a finite union of sets of the form $\{\mathbf{x} \in \mathbb{R}^n : P_i(\mathbf{x}) > 0, Q(\mathbf{x}) = 0, i \in \{1, \dots, \lambda\}\}$, where P_i, Q are polynomials on \mathbb{R}^n with real coefficients. The fundamental result about semi-algebraic sets (Tarski–Seidenberg theorem) says that a semi-algebraic set $S \subset \mathbb{R}^n$ can be described by a quantifier-free first order formula (with parameters in \mathbb{R} considered as a real closed field). It follows that the projection of a semi-algebraic set is semi-algebraic. The class of semi-algebraic sets is closed under finite unions, finite intersections and complements.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *semi-algebraic* if its graph $G(f) = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^n\}$ is semi-algebraic. The definition and properties of semi-algebraic sets immediately yield the following result:

LEMMA 3.1. *Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be semi-algebraic. Then for each rational $a = \frac{b}{c}$, where b, c are positive integers, the function f^a is semi-algebraic.*

The following property is useful:

PROPOSITION 3.2 ([20], Theorem I.2.9.13; [5], Exercise 2.10). *Given a semi-algebraic C^1 function f on \mathbb{R}^n , the partial derivatives $\partial f / \partial x_1, \dots, \partial f / \partial x_n$ are semi-algebraic.*

From now on we will only consider semi-algebraic norms in \mathbb{R}^n . We call a norm ν *semi-algebraic* if the function $\nu(\cdot)$ is semi-algebraic.

EXAMPLE 3.3. The norm $\|(x_1, \dots, x_n)^\top\|_a := (\sum_{i=1}^n |x_i|^a)^{1/a}$, $a \geq 1$, is semi-algebraic if a is rational. Indeed, assume that $a = \frac{b}{c}$, where $b \geq c \geq 1$ are coprime integers. Then

$$G(\|\cdot\|_a) = \left\{ (x_1, \dots, x_n, t)^\top : x_i = \pm y_i^c, y_i \geq 0, i = 1, \dots, n, \right. \\ \left. t = s^c, s \geq 0, \sum_{i=1}^n y_i^b - s^b = 0 \right\}.$$

The following result is well known in the case when ν is the Euclidean norm [5, §1.1]. We will sketch the proof in the general case.

LEMMA 3.4. *Let $C \subset \mathbb{R}^n$ be a nonempty closed semi-algebraic set and let ν be a semi-algebraic norm. Then the function $f(\cdot) := \text{dist}_\nu(\cdot, C)^a$, where $a = \frac{b}{c}$ with b, c positive integers, is semi-algebraic.*

Proof. Assume first that $a = b = 1$. Then the graph of f can be written as

$$\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : t \geq 0, \forall \mathbf{y} \in C t \leq \nu(\mathbf{x} - \mathbf{y}), \forall \varepsilon > 0 \exists \mathbf{y}^* \in C t + \varepsilon > \nu(\mathbf{x} - \mathbf{y}^*)\}.$$

This is a finite intersection of semi-algebraic sets. For example, the set $\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : \forall \mathbf{y} \in C t \leq \nu(\mathbf{x} - \mathbf{y})\}$ is the complement in \mathbb{R}^{n+1} of the set which is the projection onto \mathbb{R}^{n+1} of the set $B \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$,

$$B = (\mathbb{R}^{n+1} \times C) \cap u^{-1}(-\infty, 0),$$

where the function $u : \mathbb{R}^{n+1} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $u(\mathbf{x}, t, \mathbf{y}) = \nu(\mathbf{x} - \mathbf{y}) - t$ is semi-algebraic. Since preimages of semi-algebraic sets by semi-algebraic maps are semi-algebraic, B is semi-algebraic, and so is its projection. A similar argument applies to other sets in the intersection characterizing the graph of f . By Lemma 2.3 and Lemma 3.1, $\text{dist}_\nu(\cdot, C)^a$ is semi-algebraic if $a = \frac{b}{c}$ and b, c are positive integers. ■

For our next theorem, we will need the following results, proved in [8] as parts of Theorem 3.3 (Fact 1 and Fact 2). Recall that a semi-algebraic set is called smooth if it is an open subset of the set of smooth points of some irreducible algebraic set. Every semi-algebraic set has smooth semi-algebraic Whitney stratification, see e.g. [19].

PROPOSITION 3.5. *Let $X \subset \mathbb{R}^n$ be a semi-algebraic set and let $f : X \rightarrow \mathbb{R}^p$ be a semi-algebraic map. Then there is a smooth Whitney semi-algebraic stratification $X = \bigcup_{i=1}^L \Delta_i$ such that the graph $f|_{\Delta_i}$ is a smooth semi-algebraic set for each i .*

PROPOSITION 3.6. *Let $X \subset \mathbb{R}^n$ be a smooth semi-algebraic set and let $f : X \rightarrow \mathbb{R}^p$ be a map whose graph is a smooth semi-algebraic set. Then the set of points in X where f is not differentiable is contained in a closed semi-algebraic set of dimension less than the dimension of X .*

Now we can prove an approximation result.

THEOREM 3.7. *Let $C \subset \mathbb{R}^n$ be a closed semi-algebraic set. Assume that ν is a semi-algebraic norm such that ν and ν^* are differentiable. Then the set of all points $\mathbf{x} \in \mathbb{R}^n$ at which the ν -approximation to \mathbf{x} in C is not unique, denoted by $S(C)$, is a nowhere dense semi-algebraic set. In particular $S(C)$ is contained in some hypersurface $H \subset \mathbb{R}^n$.*

Proof. Let $f(\mathbf{x}) = \text{dist}_\nu(\mathbf{x}, C)$. Since f is semi-algebraic, the graph of $G(f)$ is a semi-algebraic set. Hence

$$G(f) \times C := \{(\mathbf{x}^\top, t, \mathbf{y}^\top)^\top \in \mathbb{R}^{2n+1} : \mathbf{x} \in \mathbb{R}^n, t = \text{dist}_\nu(\mathbf{x}, C), \mathbf{y} \in C\} \quad (3.1)$$

is semi-algebraic. Let

$$\begin{aligned} T(f) &:= \{(\mathbf{x}^\top, t, \mathbf{y}^\top, \mathbf{x}^\top, t, \mathbf{z}^\top)^\top \in \mathbb{R}^{2(2n+1)} : \\ &(\mathbf{x}^\top, t, \mathbf{y}^\top)^\top, (\mathbf{x}^\top, t, \mathbf{z}^\top)^\top \in G(f) \times C, \nu(\mathbf{x} - \mathbf{y}) = \nu(\mathbf{x} - \mathbf{z}) = t, \|\mathbf{y} - \mathbf{z}\|^2 > 0\}. \end{aligned} \quad (3.2)$$

Clearly, $T(f)$ is semi-algebraic. It is straightforward to see that $S(C)$ is the projection of $T(f)$ on the first n coordinates, so $S(C)$ is semi-algebraic. Theorem 2.1 along with Propositions 3.5 and 3.6 yields that $S(C)$ does not contain an open set. Therefore $S(C)$ is contained in a finite union of hypersurfaces, which is a hypersurface H . ■

The proof of Theorem 3.7 yields

COROLLARY 3.8. *Let the assumption of Theorem 3.7 hold. Define*

$$\Omega_\nu(C) := \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in \mathbb{R}^n \setminus S(C), \mathbf{z} \in C, \text{dist}_\nu(\mathbf{x}, C) = \nu(\mathbf{x} - \mathbf{z})\}.$$

Then $\Omega_\nu(C)$ is a semi-algebraic set of dimension n .

REMARK 3.9. A similar result for the Euclidean norm and a definable or subanalytic set was proved in [6]. The proof uses strict convexity of the norm in an essential way. We want to highlight the fact that our approach implies easily the fact that $S(C)$ is semi-algebraic. In a general context (of metric geometry, not necessarily semi-algebraic) there are many ways of proving that $S(C)$ is nowhere dense, see e.g. [21]. Also, in [9] it is proved directly that the dimension of $S(C) \subset \mathbb{R}^n$ is at most $n - 1$ (the argument works for a few standard notions of dimension, e.g. the Hausdorff dimension). We thank the referee for the latter reference. From the definition of dimension of a semi-algebraic set it follows in a straightforward way that a semi-algebraic set S has an empty interior if and only if $\dim S < n$.

4. The case of an irreducible variety. We first recall some basic facts about varieties and polynomial and rational maps used in this paper, see for example [5, 15, 16, 17]. We will work only with real and complex algebraic varieties, so for the purpose of stating general results we let \mathbb{F} denote either the field of real numbers \mathbb{R} or complex numbers \mathbb{C} . Let $\mathbb{F}[\mathbb{F}^n]$ be the ring of polynomials in n variables $\mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{F}^n$. For $p \in \mathbb{F}[\mathbb{F}^n]$ denote by $Z(p) \subset \mathbb{F}^n$ the zero set of p . $V \subset \mathbb{F}^n$ is called a *variety* or an *algebraic set* if there exists a finite number of polynomials $p_1, \dots, p_m \in \mathbb{F}[\mathbb{F}^n]$ so that $V = \bigcap_{i=1}^m Z(p_i)$. Let $V \subset \mathbb{F}^n$ be a variety and let $I_V \subset \mathbb{F}[\mathbb{F}^n]$ be the ideal of all polynomials that vanish on V . Then by Hilbert's theorem I_V is finitely generated. We assume here that I_V is

generated by p_1, \dots, p_m , i.e. $I_V = \langle p_1, \dots, p_m \rangle$. A variety V is called *irreducible* if V is not a union of two proper subvarieties of V . Recall that V is irreducible if and only if I_V is prime. It is well known every variety V decomposes uniquely as a finite union of distinct irreducible varieties.

For two given varieties $X, Y \subset \mathbb{F}^n$, the set $X \setminus Y$ is called a *quasi-variety*. It is well known that the closure of a quasi-variety in a standard or Zariski topology is a variety. For $S \subset \mathbb{F}^n$ we denote by $\text{Closure}(S)$ the closure of S in the standard topology in \mathbb{F}^n .

Assume that $V \subset \mathbb{F}^n$ is a variety defined as above. Denote by

$$D(V)(\mathbf{x}) = \left(\left[\frac{\partial p_i}{\partial x_j}(\mathbf{x}) \right]_{i,j=1}^{m,n} \right)^\top \in \mathbb{F}^{n \times m}, \quad \mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{F}^n,$$

the Jacobian matrix corresponding to V . Then the dimension of V , denoted by $\dim V$, is the minimal possible nullity of $D(V)(\mathbf{y})$ for $\mathbf{y} \in V$. (Recall that for an $n \times m$ matrix A , the nullity of A is the dimension of the null space of A , i.e., of $\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$.) Assume that $V \subset \mathbb{F}^n$ is an irreducible variety of dimension d , $1 \leq d < n$. A point $\mathbf{y} \in V$ is called *smooth* if $\text{rank } D(V)(\mathbf{y}) = n - d$. Otherwise $\mathbf{y} \in V$ is called *singular*. The set of singular points of V is denoted by $\text{Sing } V$. Note that $\text{Sing } V$ is the set of all points of V where all $n - d$ minors of $D(V)(\mathbf{y})$ are zero, hence $\text{Sing } V$ is a strict subvariety of V . The quasi-variety $W := V \setminus \text{Sing } V$ is a nonempty manifold of dimension d . For $\mathbb{F} = \mathbb{C}$ it is connected. For $\mathbb{F} = \mathbb{R}$ it consists of a finite number of connected components. For each $\mathbf{y} \in V \setminus \text{Sing } V$ we denote by $\mathbf{U}(\mathbf{y}) \in \mathbb{F}^n$ the $(n - d)$ -dimensional subspace spanned by the columns of $D(V)(\mathbf{y})$.

More generally, assume that the quasi-variety $W \subset \mathbb{C}^n$ is a complex connected manifold of dimension d . Then its closure is an irreducible variety.

Suppose that $C = \bigcap_{i=1}^m Z(p_i) \subset \mathbb{R}^n$ is an irreducible variety of real dimension d , where $I_C = \langle p_1, \dots, p_m \rangle$. Then $C_{\mathbb{C}}$ is the zero locus of $p_1(\mathbf{z}) = \dots = p_m(\mathbf{z}) = 0$ in \mathbb{C}^n . It is well known that $C_{\mathbb{C}}$ is a complex irreducible variety of complex dimension d , see e.g. [19]. We denote $D(C_{\mathbb{C}})(\mathbf{x})$ by $D(C)(\mathbf{x})$ when no ambiguity would arise.

Let X, Y be affine irreducible algebraic varieties over \mathbb{C} . A map $f : X \rightarrow Y$ is a regular map if it is polynomial in affine coordinates on X, Y . A map $f : X \rightarrow Y$ is called *rational* if there exists a Zariski open set $X' \subset X$ such that the restriction $f : X' \rightarrow Y$ is given as a well defined rational map in affine coordinates on X, Y . Such a map f is called *dominant* if $f(X')$ is Zariski dense in Y .

Let X, Y be irreducible affine varieties of the same dimension and let $f : X \rightarrow Y$ be a regular dominant map. Then the degree of f , denoted by $\deg f$, is defined as the (necessarily finite) degree of the field extension $[\mathbb{C}(X) : f^*(\mathbb{C}(Y))]$. Furthermore, $\deg f$ is the cardinality of the set $f^{-1}(\mathbf{y})$ for a generic $\mathbf{y} \in Y$.

The following result, introducing the notion of a critical point, is well known, see for example Lemma 2.7 in [15] for a polynomial real-valued function g , and we leave its proof to the reader:

LEMMA 4.1. *Let $C \subset \mathbb{R}^n$ be an irreducible variety. Assume that $g \in \mathcal{C}^1(\mathbb{R}^n)$. Then $\mathbf{y} \in C$ is a critical point of $g|_C$ if one of the following equivalent conditions holds:*

1. *Either $\mathbf{y} \in \text{Sing } C$ or $\mathbf{y} \in C \setminus \text{Sing } C$ and $\nabla g(\mathbf{y})|_{T_{\mathbf{y}}C} \equiv 0$, where $T_{\mathbf{y}}C$ is the tangent space to C at \mathbf{y} .*
2. *Either $\mathbf{y} \in \text{Sing } C$ or $\mathbf{y} \in C \setminus \text{Sing } C$ and $\nabla g(\mathbf{y}) \in \mathbf{U}(\mathbf{y})$, where $\mathbf{U}(\mathbf{y})$ is the column space of $D(C)(\mathbf{y})$, i.e., the normal space to C at \mathbf{y} .*
3. *Either $\mathbf{y} \in \text{Sing } C$ or $\mathbf{y} \in C \setminus \text{Sing } C$ and $\text{rank}[D(C)(\mathbf{y}) \nabla g^\top] = n - d$.*
4. *$\text{rank}[D(C)(\mathbf{y}) \nabla g^\top] \leq n - d$.*
5. *$\mathbf{y} \in C_C$ is in the zero set of all $n - d + 1$ minors of $[D(C)(\mathbf{y}) \nabla g^\top]$.*

We will now study the properties of the set of critical points of g . As the singular points of C are always critical points of g by the definition, it is natural to consider only the smooth points of C which are critical points g [15].

NOTATION 1. Let $m > n$ be two positive integers and \mathbb{F} be the field of real \mathbb{R} or complex numbers \mathbb{C} . For the remaining part, we let $\pi : \mathbb{F}^m \rightarrow \mathbb{F}^n$ denote the projection on the n first coordinates of vectors in \mathbb{F}^m .

LEMMA 4.2. *Let $g \in \mathcal{C}^1(\mathbb{R}^n)$ be semi-algebraic. For each $\mathbf{x} \in \mathbb{R}^n$ denote by $g_{\mathbf{x}} : \mathbb{R}^n \rightarrow \mathbb{R}$ the function $g_{\mathbf{x}}(\mathbf{y}) = g(\mathbf{x} - \mathbf{y})$. Assume that $C \subset \mathbb{R}^n$ is an irreducible variety. Then the sets:*

$$\begin{aligned} \Sigma_{g,0}(C) &= \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times C : \mathbf{y} \text{ is a critical point of } g_{\mathbf{x}}|_C\}, \\ \Sigma_{g,1}(C) &:= \Sigma_{g,0}(C) \setminus (\mathbb{R}^n \times \text{Sing } C), \\ \Sigma_g(C) &:= \text{Closure}(\Sigma_{g,1}(C)) \end{aligned}$$

are semi-algebraic. Then $\pi(\Sigma_{g,1}(C))$ is semi-algebraic.

Proof. $\Sigma_{g,0}(C)$ is the complement of the projection (onto the product of the first two factors) of the semi-algebraic set

$$B = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \mathbf{y} \in C \setminus \text{Sing } C, \mathbf{z} \in T_{\mathbf{y}}C, \nabla g_{\mathbf{x}}(\mathbf{y})(\mathbf{z}) \neq 0\}.$$

Hence $\Sigma_{g,0}(C)$ is semi-algebraic. Clearly, $\mathbb{R}^n \times \text{Sing } C$ is an algebraic set. Hence $\Sigma_{g,1}(C)$ is semi-algebraic, and its closure $\Sigma_g(C)$ is semi-algebraic. ■

Suppose that g is a polynomial. This will be the case when e.g. $g = \|x\|_q^q$ with $q \geq 2$ even. Then we can define the sets $\Sigma_{g,0}(C_{\mathbb{C}}), \Sigma_{g,1}(C_{\mathbb{C}}), \Sigma_g(C_{\mathbb{C}})$ over \mathbb{C} as in Lemma 4.2, where $\mathbb{R}^n, C, \text{Sing } C$ are replaced by $\mathbb{C}^n, C_{\mathbb{C}}, \text{Sing } C_{\mathbb{C}}$ respectively. The arguments of the proof of Lemma 4.1 yield that $\Sigma_{g,0}(C_{\mathbb{C}})$ is a complex subvariety. Hence $\Sigma_{g,1}(C_{\mathbb{C}})$ is a quasi-variety and $\Sigma_g(C_{\mathbb{C}})$ is a variety. See [11] for more details.

LEMMA 4.3. *Suppose that ν is a semi-algebraic norm such that ν and ν^* are differentiable. Assume that $g = \nu^a$, where $a = \frac{b}{c}$ and $b > c \geq 1$ are integers. Let $C \subset \mathbb{R}^n$ be an irreducible variety and assume that $\Omega_{\nu}(C)$ is defined as in Corollary 3.8. Then $\Omega_{\nu}(C) \subseteq \Sigma_{g,0}(C)$. Furthermore, $\Omega_{\nu}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \Omega_{\nu}(C) \cap \Sigma_{g,1}(C)$, and $\Omega_{\nu}(C) \setminus (\mathbb{R}^n \times \text{Sing } C), \pi(\Omega_{\nu}(C) \setminus (\mathbb{R}^n \times \text{Sing } C))$ are semi-algebraic sets of dimension n .*

Proof. Suppose first that $\mathbf{x} \in C$. Then \mathbf{x} is a unique best ν -approximation of \mathbf{x} and $(\mathbf{x}, \mathbf{x}) \in \Omega_\nu(C)$. As $\nabla \text{dist}(\cdot, C)^a$ is $\mathbf{0}$ at \mathbf{x} , we deduce that $(\mathbf{x}, \mathbf{x}) \in \Sigma_{g,0}(C)$. Let $\mathbf{x} \in \mathbb{R}^n \setminus (S(C) \cup C)$. So there is a unique \mathbf{z} such that $\text{dist}_\nu(\mathbf{x}, C) = \nu(\mathbf{x} - \mathbf{z}) > 0$, i.e., $(\mathbf{x}, \mathbf{z}) \in \Omega_\nu(C)$. If $\mathbf{z} \in \text{Sing } C$ then $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^n \times \text{Sing } C \subset \Sigma_{g,0}(C)$. Assume now that $\mathbf{z} \in C \setminus \text{Sing } C$. Since $\text{dist}_\nu(\mathbf{x}, C) = \nu(\mathbf{x} - \mathbf{z}) > 0$ it follows that $\nabla g_{\mathbf{x}}(\mathbf{z}) = 0$. Hence $(\mathbf{x}, \mathbf{z}) \in \Sigma_{g,0}(C)$ and $\Omega_\nu(C) \subseteq \Sigma_{g,0}(C)$. Furthermore, $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \Omega_\nu(C) \cap \Sigma_{g,1}(C)$. As $\Omega_\nu(C)$ is semi-algebraic, it follows that $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)$ and $\pi(\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C))$ are semi-algebraic.

Let $\mathbf{y} \in C \setminus \text{Sing } C$. Then $\min_{\mathbf{w} \in \text{Sing } C} \nu(\mathbf{y} - \mathbf{w}) = \phi(\mathbf{y}) > 0$. Let $O(\mathbf{y}) := \{\mathbf{x} \in \mathbb{R}^n : \nu(\mathbf{x} - \mathbf{y}) < \frac{\phi(\mathbf{y})}{2}\}$. Note that $O(\mathbf{y})$ is an open semi-algebraic set, hence its dimension equals n . Assume that $\mathbf{x} \in O(\mathbf{y})$. Then $\text{dist}_\nu(\mathbf{x}, C) = \nu(\mathbf{x} - \mathbf{z}) \leq \nu(\mathbf{x} - \mathbf{y}) < \frac{\phi(\mathbf{y})}{2}$. Suppose that $\mathbf{w} \in \text{Sing } C$. Then

$$\nu(\mathbf{x} - \mathbf{w}) = \nu(\mathbf{y} - \mathbf{w} + \mathbf{x} - \mathbf{y}) \geq \nu(\mathbf{y} - \mathbf{w}) - \nu(\mathbf{x} - \mathbf{y}) > \phi(\mathbf{y}) - \frac{\phi(\mathbf{y})}{2} = \frac{\phi(\mathbf{y})}{2}.$$

This shows that $\mathbf{z} \notin \text{Sing } C$. Thus for each $\mathbf{x} \in O(\mathbf{y}) \setminus S(C)$ we have $(\mathbf{x}, \mathbf{z}) \in \Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)$. Hence $\dim \Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \dim \Omega_\nu(C) = n$. Clearly, $\pi(\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)) \supset O(\mathbf{y}) \setminus S(C)$. Hence $\pi(\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C))$ is a semi-algebraic set of dimension n . ■

Assume that $\nu(\cdot) = \|\cdot\|$ and $g(\mathbf{x}) = \|\mathbf{x}\|^2 = \sum_{i=1}^n x_i^2$. Then $\Sigma_g(C_{\mathbb{C}})$ is an irreducible variety of dimension n [7, 11]. (A short justification that $\Sigma_g(C_{\mathbb{C}})$ is irreducible is given in the proof of Lemma 5.1.) $\Sigma_g(C_{\mathbb{C}})$ is called the *critical point correspondence* in [7]. Consider the projection $\pi : \Sigma_g(C_{\mathbb{C}}) \rightarrow \mathbb{C}^n$. Clearly, π is a polynomial map. It is a dominating map [7, 11]. This is a simple consequence of Lemma 4.3. Indeed, since $\pi(\Omega_\nu(C) \cap \Sigma_{g,1}(C))$ is a semi-algebraic set of real dimension $2n$ it follows that the algebraic set $\text{Closure}(\pi(\Sigma_g(C_{\mathbb{C}})))$ must be \mathbb{C}^n . As the complex dimension $\dim \Sigma_g(C_{\mathbb{C}}) = n$, it follows that π is a dominating map.

Let $\delta := \delta_{\|\cdot\|}(C)$ be the degree of $\pi : \Sigma_g(C_{\mathbb{C}}) \rightarrow \mathbb{C}^n$. That is, for a generic point $\mathbf{x} \in \mathbb{C}^n$ the set $\pi^{-1}(\mathbf{x}) \subset \Sigma_g(C_{\mathbb{C}})$ has δ distinct points:

$$\pi^{-1}(\mathbf{x}) = \{(\mathbf{x}, \mathbf{z}_1), \dots, (\mathbf{x}, \mathbf{z}_\delta)\}.$$

The number δ is called the *Euclidean distance degree* of C in [7]. It gives an upper bound for the number of critical smooth points for the function $g_{\mathbf{x}}|_C$ for a generic $\mathbf{x} \in \mathbb{R}^n$.

THEOREM 4.4. *Let the assumptions of Lemma 4.3 hold. Assume that $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \bigcup_{i=1}^N \Theta_i$ is a smooth semi-algebraic Whitney stratification. Suppose that each Θ_i is an open semi-algebraic subset of smooth points of an irreducible variety $V_i \subset \mathbb{R}^n \times \mathbb{R}^n$ of dimension n for $i = 1, \dots, M$. Furthermore, for $i > M$ each Θ_i has dimension less than n . Denote by $V_{i,\mathbb{C}} \subset \mathbb{C}^n \times \mathbb{C}^n$ the complexification of V_i . Then the projection $\pi : V_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ is dominating for $i = 1, \dots, M$.*

Proof. Clearly $\Omega_\nu(C)$ is a univalent graph and

$$\Theta_i = \{(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in \pi(\Theta_i), \mathbf{z} \in C \setminus \text{Sing } C \text{ is the unique best approximation of } \mathbf{x}\}.$$

Hence $\dim \Theta_i = \dim \pi(\Theta_i)$. Assume that $i \in \{1, \dots, M\}$. Then

$$n = \dim V_{i,\mathbb{C}} \geq \dim \pi(\text{Closure}(\pi(V_{i,\mathbb{C}}))).$$

As $\pi(\Theta_i) \subset \pi(V_{i,\mathbb{C}})$ and $\dim \pi(\Theta_i) = n$ it follows that $\pi : V_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ is dominating. ■

We now give a definition of the ν -distance degree of C :

DEFINITION 4.5. Let the assumptions of Theorem 4.4 hold. Let δ_i be the degree of the dominating map $\pi : V_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ for $i = 1, \dots, M$. Assume that V_1, \dots, V_k , where $k \leq M$, are k distinct irreducible varieties, while $V_i \in \{V_1, \dots, V_k\}$ for each $i > k$. Then $\delta_\nu(C) := \sum_{i=1}^k \delta_i$ is called the ν -distance degree of C .

REMARK 4.6. The union $\bigcup_i V_{i,\mathbb{C}}$ is the smallest complex algebraic variety containing $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)$. Therefore the ν -distance degree agrees with the degree of the semialgebraic set $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)$ as introduced in [19]. Note also that for $\nu(\cdot) = \|\cdot\|$ our definition of $\delta_{\|\cdot\|}(C)$ coincides with the Euclidean distance degree of C in [7]. There are many tools of algebraic geometry (including computational ones) that can be used to find the degree of the dominating map $\pi : V_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ [7, 11, 2, 3, 4].

In the last section we interpret the degree of $\pi : V_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ as a degree of critical point correspondence of an algebraic function induced by one stratum of dimension n given by Proposition 3.5 for the graph of the norm ν considered as a subset of $\mathbb{R}^n \times \mathbb{R}$.

5. Critical point correspondence for ℓ_p norms. Let $\mathbf{F}_{2m-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ($m \in \mathbb{N}$) denote the map $(z_1, \dots, z_n) \mapsto (z_1^{2m-1}, \dots, z_n^{2m-1})$. Note that \mathbf{F}_{2m-1} is a proper map of degree $(2m-1)^n$. For each irreducible variety $W \subset \mathbb{C}^n$ of dimension t , $\mathbf{F}^{-1}(W)$ is a union of at most $(2m-1)^n$ irreducible varieties, each of dimension t . Let $X \subset \mathbb{C}^n$ be an irreducible variety of dimension d . Then $\mathbf{F}_{2l+1}(X)$ is also an irreducible variety of dimension d .

Assume that $p = \frac{2m}{2l+1}$, where $m > l \geq 0$ are integers. Then

$$\|\mathbf{x}\|_p = \left(\sum_{j=1}^n x_j^{2m/(2l+1)} \right)^{(2l+1)/(2m)}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Note that $\|\mathbf{x}\|_p$ is a C^∞ -smooth function on $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Let $C \subset \mathbb{R}^n$ be an irreducible variety of dimension d , $1 \leq d < n$, as in §4. In this section we show that, as in the case of ℓ_2 norm, $\delta_{\|\cdot\|_p}(C)$ in Definition 4.5 is the degree of $\pi : V_{1,\mathbb{C}} \rightarrow \mathbb{C}^n$, and $V_{1,\mathbb{C}}$ can be considered as the critical point correspondence. (That is, in Definition 4.5 we take $k = 1$.)

Let $g(\mathbf{x}) := \|\mathbf{x}\|_p^p$. Assume that $\mathbf{y} \in C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}$. Denote by $DC(\mathbf{y}) \in \mathbb{C}^{n \times m}$ the complex Jacobian as defined in §4. Let $\mathbf{U}(\mathbf{y}) \subset \mathbb{C}^n$ be the subspace spanned by the columns of $DC(\mathbf{y})$. Then $\dim \mathbf{U}(\mathbf{y}) = n - d$. Let

$$\Sigma_{2m,2l+1,1}(C) = \{(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^n \times (C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}) : \mathbf{F}_{2m-2l-1}(\mathbf{z} - \mathbf{y}) \in \mathbf{F}_{2l+1}(\mathbf{U}(\mathbf{y}))\}.$$

That is,

$$\Sigma_{2m,2l+1,1}(C) = \{(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^n \times (C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}) : \mathbf{z} \in \mathbf{y} + \mathbf{F}_{2m-2l-1}^{-1}(\mathbf{F}_{2l+1}\mathbf{U}(\mathbf{y}))\}.$$

Let further $\Sigma_{2m,2l+1}(C) = \text{Closure}(\Sigma_{2m,2l+1,1}(C))$. Lemma 4.1 yields that any critical point of \mathbf{y}_0 of $g_{\mathbf{x}}(\mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_p^p$ in $C \setminus \text{Sing } C$ satisfies $(\mathbf{x}, \mathbf{y}_0) \in \Sigma_{2m,2l+1,1}(C)$.

LEMMA 5.1. *Let $m > l \geq 0$ be integers. Then $\Sigma_{2m,2l+1}(C) \subset \mathbb{C}^n \times \mathbb{C}^n$ is a closed algebraic variety, and each of its irreducible components is of dimension n . Furthermore, there exists exactly one irreducible component $\Sigma_{2m,2l+1}^1(C)$ of $\Sigma_{2m,2l+1}(C)$ containing all real points in $\Sigma_{2m,2l+1}(C)$.*

Proof. Consider first the case when $m = 1, l = 0$. Then $\Sigma_{2,1}(C) = \Sigma_{\|\cdot\|^2}(C)_{\mathbb{C}}$ is the critical correspondence variety studied in [7, 11], and discussed in §4. Observe first that

$$\Phi(C) := \{(\mathbf{u}, \mathbf{y}) : \mathbf{y} \in C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}, \mathbf{u} \in \mathbf{U}(\mathbf{y})\}$$

is a quasi-algebraic variety (see Lemma 4.1). Furthermore, it is isomorphic to an $(n - d)$ -dimensional vector bundle over $C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}$. Hence it is a connected complex manifold of dimension n . Therefore $\text{Closure}(\Phi(C))$ is an irreducible complex variety of dimension n . (It is usually called the *Nash modification* of $C_{\mathbb{C}}$.) Consider the linear automorphism A of $\mathbb{C}^n \times \mathbb{C}^n$: $(\mathbf{z}, \mathbf{w}) \mapsto (\mathbf{z} + \mathbf{w}, \mathbf{w})$. Then

$$\begin{aligned} A(\Sigma_{2,1}(C)) &= \Phi(C) \\ \tilde{\Sigma}_{2,1}(C) &:= \text{Closure}(\Phi(C)) = A(\Sigma_{2,1}(C)). \end{aligned}$$

Hence $\tilde{\Sigma}_{2,1}(C)$ and $\Sigma_{2,1}(C)$ are n -dimensional irreducible varieties.

Let $\tilde{\mathbf{F}}_{2m-1} : \mathbb{C}^n \times \mathbb{C}^n$ be given by $(\mathbf{z}, \mathbf{w}) \mapsto (\mathbf{F}_{2m-1}(\mathbf{z}), \mathbf{w})$. Clearly, $\tilde{\mathbf{F}}_{2m-1}$ is a proper polynomial map of degree $(2m - 1)^n$. Hence, for each irreducible variety in $W \subset \mathbb{C}^n \times \mathbb{C}^n$, $\tilde{\mathbf{F}}_{2m-1}^{-1}(W)$ is a union of at most $(2m - 1)^n$ irreducible varieties, each of dimension $\dim W$. Furthermore, $\tilde{\mathbf{F}}_{2m-1}(W)$ is an irreducible variety of dimension n . Let

$$\tilde{\Sigma}_{2m,2l+1}(C) = \tilde{\mathbf{F}}_{2m-2l-1}^{-1}(\tilde{\mathbf{F}}_{2l+1}(\tilde{\Sigma}_{2,1}(C))) = \bigcup_{i=1}^{N(2m)} \tilde{\Sigma}_{2m,2l+1}^i(C). \tag{5.1}$$

Here $\tilde{\Sigma}_{2m,2l+1}^i(C)$ are distinct irreducible components of $\tilde{\Sigma}_{2m,2l+1}(C)$, $i = 1, \dots, N(2m)$. It is straightforward to show

$$\begin{aligned} \Sigma_{2m,2l+1}(C) &= A^{-1}(\tilde{\Sigma}_{2m,2l+1}(C)) = \bigcup_{i=1}^{N(2m)} \Sigma_{2m,2l+1}^i(C) \\ \Sigma_{2m,2l+1}^i(C) &:= A^{-1}(\tilde{\Sigma}_{2m,2l+1}^i(C)), \quad i = 1, \dots, N(2m). \end{aligned} \tag{5.2}$$

Here $\Sigma_{2m,2l+1}^i(C)$ are distinct irreducible components of $\Sigma_{2m,2l+1}(C)$ of dimension n for $i = 1, \dots, N(2m)$.

Assume now that (\mathbf{x}, \mathbf{y}) is a real point in $\tilde{\Sigma}_{2m,2l+1}(C)$. So $\mathbf{u} := \mathbf{F}_{2m-2l-1}(\mathbf{x}) \in \mathbf{F}_{2l+1}(\mathbf{U}(\mathbf{y})) \cap \mathbb{R}^n$. Clearly, $\mathbf{F}_{2m-2l-1}^{-1}(\mathbf{u})$ contains exactly one real point \mathbf{x} . Hence

$$\tilde{\Sigma}_{2m,2l+1}(C) \cap (\mathbb{R}^n \times C) = \tilde{\Sigma}_{2m,2l+1}^1(C) \cap (\mathbb{R}^n \times C).$$

Thus $\Sigma_{2m,2l+1}^1(C)$ of $\Sigma_{2m,2l+1}(C)$ contains all real points in $\Sigma_{2m,2l+1}(C)$. ■

THEOREM 5.2. *Let $m > l \geq 0$ be integers and assume that $\Sigma_{2m,2l+1}(C)$ is defined as above. Let $\Sigma_{2m,2l+1}^1(C)$ be the irreducible component of dimension n of $\Sigma_{2m,2l+1}(C)$ which contains all the real points of $\Sigma_{2m,2l+1}(C)$. Let $p = \frac{2m}{2l+1}$ and $g(\mathbf{x}) = \|\mathbf{x}\|_p^p$. Let $\Omega_{\|\cdot\|_p}(C)$ and $\Sigma_{g,1}(C)$ be defined as in Corollary 3.8 and Lemma 4.2, respectively. Then*

$\Omega_{\|\cdot\|_p}(C) \cap \Sigma_{g,1}(C) \subset \Sigma_{2m,2l+1}^1(C)$. Hence $\delta_{\|\cdot\|_p}(C)$ is the degree of the dominating map $\pi : \Sigma_{2m,2l+1}^1(C) \rightarrow \mathbb{C}^n$. That is, $\Sigma_{2m,2l+1}^1(C)$ is the critical point correspondence for the ℓ_p norm.

Proof. As in the proof of Lemma 4.3, we have

$$\Omega_{\|\cdot\|_p}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \Omega_{\|\cdot\|_p}(C) \cap \Sigma_{g,1}(C) \subset \Sigma_{2m,2l+1,1}(C) \subset \Sigma_{2m,2l+1}(C).$$

As $\Omega_{\|\cdot\|_p}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) \subset \mathbb{R}^n$ it follows that $\Omega_{\|\cdot\|_p}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) \subset \Sigma_{2m,2l+1}^1(C)$. Let $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \bigcup_{i=1}^N \Phi_i$ be a smooth Whitney semi-algebraic stratification as in Theorem 4.4. Then each Φ_i of dimension n is an open semi-algebraic subset of $\Sigma_{2m,2l+1}^1(C)$. Theorem 4.4 yields that $\pi : \Sigma_{2m,2l+1}^1(C) \rightarrow \mathbb{C}^n$ is dominating. Definition 4.5 yields that $\delta_{\|\cdot\|_p}(C)$ is the degree of the dominating map $\pi : \Sigma_{2m,2l+1}^1(C) \rightarrow \mathbb{C}^n$. ■

6. Algebraic critical point correspondences. In this section we give more detailed information about the variety $V_{i,\mathbb{C}}$ appearing in Theorem 4.4. Let ν be a semi-algebraic norm such that ν and ν^* are differentiable. Consider the smooth Whitney stratification of $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$ given by Proposition 3.5. Assume that Δ_i is of dimension n . (Since ν is not differentiable at $\mathbf{0}$, we deduce that $\mathbf{0} \notin \Delta_i$.) Then the graph of $\nu|_{\Delta_i}$ is an open semi-algebraic set in an irreducible variety $W_i \subset \mathbb{R}^n \times \mathbb{R}$ of dimension n . Let $W_{i,\mathbb{C}} \subset \mathbb{C}^n \times \mathbb{C}$ be the complexification of W_i . Hence $W_{i,\mathbb{C}}$ is a hypersurface in $\mathbb{C}^n \times \mathbb{C}$. For simplicity of notation we let $W = W_{i,\mathbb{C}}$ when no ambiguity would arise. Thus

$$W = \{(\mathbf{z}, t) \in \mathbb{C}^{n+1} : G(\mathbf{z}, t) = 0\},$$

where

$$G(\mathbf{z}, t) = \sum_{j=0}^M a_j(\mathbf{z})t^j, \quad a_j(\mathbf{z}) \in \mathbb{C}[\mathbf{z}], \quad j = 0, \dots, M. \tag{6.1}$$

The arguments below show that $G(\mathbf{z}, t)$ is an irreducible polynomial in $\mathbb{C}[(\mathbf{z}, t)]$.

Let $\tau : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ be the projection onto the first component. Since $\tau(W_i) \supset \Delta_i$, it follows that $\tau : W_{i,\mathbb{C}} \rightarrow \mathbb{C}^n$ is a dominating polynomial map of degree m . Furthermore, $W_{i,\mathbb{C}}$ is the graph of a multivalued algebraic function f_i . For simplicity of notation we also let $f := f_i$ when no ambiguity would arise. That is, f satisfies a polynomial equation $G(\mathbf{z}, f) = 0$ where $G(\mathbf{z}, t)$ is given by (6.1). Let

$$W' := \left\{ (\mathbf{z}, t) \in W : \frac{\partial G}{\partial t} = 0 \right\}.$$

Clearly, W' is a strict subvariety of W . For $(\mathbf{z}_0, t_0) \in W \setminus W'$ the implicit function theorem yields that f is an analytic function of \mathbf{z} in a neighborhood of \mathbf{z}_0 with $f(\mathbf{z}_0) = t_0$. Let

$$\mathbf{F} := \left(\frac{\partial G}{\partial z_1}, \dots, \frac{\partial G}{\partial z_n} \right).$$

Note that $\mathbf{F} : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ is a polynomial map $(\mathbf{z}, t) \mapsto \mathbf{F}((\mathbf{z}, t))$ and that

$$\nabla f(\mathbf{z}) = \mathbf{H}((\mathbf{z}, t)) := -\frac{\mathbf{F}((\mathbf{z}, t))}{\frac{\partial G(\mathbf{z}, t)}{\partial t}}, \quad (\mathbf{z}, t) \in W \setminus W'. \tag{6.2}$$

For $\mathbf{y} \in C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}$ let $\mathbf{U}(\mathbf{y})$ be defined as in Lemma 4.1. (Recall that C is a d -dimensional variety in \mathbb{R}^n .) The critical set $\Sigma_{\nu,1}(C, W)$ is defined as

$$\Sigma_{\nu,1}(C, W) = \{(\mathbf{z}, t, \mathbf{y}) \in \mathbb{C}^n \times \mathbb{C} \times (C_{\mathbb{C}} \setminus \text{Sing } C_{\mathbb{C}}) : (\mathbf{z} - \mathbf{y}, t) \in W, \mathbf{F}(\mathbf{z} - \mathbf{y}, t) \in \mathbf{U}(\mathbf{y})\}.$$

The arguments for equivalence of the conditions in Lemma 4.1 yield that $\Sigma_{\nu,1}(C, W)$ is a quasi-variety. Hence $\Sigma_{\nu}(C, W) = \text{Closure}(\Sigma_{\nu,1}(C, W))$ is an algebraic set. As in the case of the ℓ_{2m} norm, $\Sigma_{\nu}(C, W)$ may contain several irreducible components.

THEOREM 6.1. *Assume that $\nu, \nu^* \in C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and ν is semi-algebraic. Consider the smooth Whitney stratification of $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$ given in Proposition 3.5. Assume that Δ_i is of dimension n . Let $W_i \subset \mathbb{R}^n \times \mathbb{R}$ be an n -dimensional irreducible variety that contains the graph of $\nu | \Delta_i$. Let $W_{i,\mathbb{C}}$ denote the complexification of W_i . Let $C \subset \mathbb{R}^n$ be an irreducible variety of dimension d . With $\Sigma_{\nu}(C, W_{i,\mathbb{C}})$ defined as above and $\Omega_{\nu}(C)$ as in Corollary 3.8, respectively, define*

$$\Omega_{\nu}(C, \Delta_i) := \{(\mathbf{x}, \mathbf{z}) \in \Omega_{\nu}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) : \mathbf{x} - \mathbf{z} \in \text{Closure}(\Delta_i) \setminus \{\mathbf{0}\}\}.$$

Then $\Omega_{\nu}(C, \Delta_i)$ is semi-algebraic. Furthermore,

1. Assume that $\pi(\Omega_{\nu}(C, \Delta_i))$ has dimension n . Then $\Sigma_{\nu}(C, W_{i,\mathbb{C}})$ contains a positive number of subvarieties $\tilde{V}_{i,1}, \dots, \tilde{V}_{i,l_i}$ such that $\dim \tilde{V}_{i,j} = n$ for $j = 1, \dots, l_i$.
2. Let $\omega : \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^n$ be the projection onto the product of the first and the last factor. Then for each $j \in \{1, \dots, l_i\}$ there exists a variety $V_{j'}$ in the set of k varieties $\{V_1, \dots, V_k\}$, as given in Definition 4.5, such that $\omega : \tilde{V}_{i,j} \rightarrow V_{j'}$ is dominating.

Moreover, for each $V_l \in \{V_1, \dots, V_k\}$, there exist i and j such that $\omega : \tilde{V}_{i,j} \rightarrow V_l$.

Proof. Since $\Omega_{\nu}(C)$ and Δ_i are semi-algebraic, we deduce immediately that $\Omega_{\nu}(C, \Delta_i)$ is semi-algebraic. Assume that the semi-algebraic set $\Delta'_i := \pi(\Omega_{\nu}(C, \Delta_i))$ has dimension n . Denote by $G_i(\mathbf{z}, t)$ the polynomial induced by $\nu | \Delta_i$. Let

$$\tilde{\Omega}_{\nu}(C, \Delta_i) := \{(\mathbf{x}, t, \mathbf{z}) : (\mathbf{x}, \mathbf{z}) \in \Omega_{\nu}(C, \Delta_i), t = \nu(\mathbf{x} - \mathbf{z})\}.$$

We claim that $\tilde{\Omega}_{\nu}(C, \Delta_i) \subset \Sigma_{\nu}(C, W_{i,\mathbb{C}})$. Since W_i is a closed set, it contains the graph of $\nu | \text{Closure}(\Delta_i)$. Assume that $(\mathbf{x}, t, \mathbf{z}) \in \tilde{\Omega}_{\nu}(C, \Delta_i)$. By the definition, $(\mathbf{x} - \mathbf{z}, t) \in W_i$. Furthermore, $\mathbf{z} \in C$ is a unique best ν -approximation to \mathbf{x} . By definition, $\mathbf{x} - \mathbf{z} \in \text{Closure}(\Delta_i) \setminus \{\mathbf{0}\}$. Hence $\mathbf{x} - \mathbf{z} \neq \mathbf{0}$. Therefore $\nabla \nu(\mathbf{x} - \mathbf{z}) \neq \mathbf{0}$. As $(\mathbf{x} - \mathbf{z}, \nu(\mathbf{x} - \mathbf{z})) \in W_i$ it follows that $\mathbf{F}(\mathbf{x} - \mathbf{z}, \nu(\mathbf{x} - \mathbf{z})) \in \mathbf{U}(\mathbf{z})$. Thus $(\mathbf{x}, \nu(\mathbf{x} - \mathbf{z}), \mathbf{z}) \in \Sigma_{\nu}(C, W_{i,\mathbb{C}})$.

Let $\Omega_{\nu}(C) \setminus (\mathbb{R}^n \times \text{Sing } C) = \bigcup_{l=1}^N \Theta_l$ be the smooth Whitney decomposition as in Theorem 4.4. Assume that $\dim \Theta_l = n$. Hence $\dim \pi(\Theta_l) = n$. Suppose furthermore that $\dim \pi(\Theta_l) \cap \Delta'_i = n$. Then the variety $Y_i = \text{Closure}(\omega(W_{i,\mathbb{C}}))$ contains $\Omega_{\nu}(C, \Delta_i)$. Let $V_{l,\mathbb{C}}$ be the irreducible variety defined in Theorem 4.4. Since V_l is the minimal variety containing Θ_l , it follows that $V_{l,\mathbb{C}} \subset Y_i$. Let

$$\tilde{Y}_{i,l} := \{(\mathbf{x}, t, \mathbf{z}) \in \Sigma_{\nu}(C, W_{i,\mathbb{C}}) : (\mathbf{x}, \mathbf{z}) \in V_{l,\mathbb{C}}\}.$$

Let $\tilde{V}_{i,l} \subset \tilde{Y}_{i,l}$ be the smallest subvariety of $\tilde{Y}_{i,l}$ which contains the points of the semi-algebraic set $\tilde{\Omega}_{\nu}(C, \Delta_i)$ of dimension n . Then $\dim \tilde{V}_{i,l} = n$ and $\omega : \tilde{V}_{i,l} \rightarrow V_{l,\mathbb{C}}$ is dominating.

It remains to show that each V_l appearing in Theorem 4.4 corresponds to some Θ_l of dimension n . Clearly, $\text{diag}(C) := \{(\mathbf{z}, \mathbf{z}) : \mathbf{z} \in C\}$ is an irreducible variety of dimension $d < n$. By refining the stratification if needed we can assume that in the decomposition of $\Omega_\nu(C) \setminus (\mathbb{R}^n \times \text{Sing } C)$ each Θ_i of dimension n does not intersect $\text{diag}(C)$, i.e. $\Theta_i \cap \text{diag}(C) = \emptyset$ for $i = 1, \dots, M$.

Assume that in the smooth Whitney stratification of $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$ given in Proposition 3.5 we have $\dim \Delta_i = n$ for $i = 1, \dots, L'$ and $\dim \Delta_i < n$ for $i > L'$. Hence $\mathbb{R}^n \setminus (\bigcup_{i=1}^{L'} \Delta_i)$ is a semi-algebraic set of dimension at most $n - 1$. Therefore

$$\mathbb{R}^n = \bigcup_{i=1}^M \text{Closure}(\Delta_i), \quad \mathbb{R}^n \setminus \{\mathbf{0}\} = \bigcup_{i=1}^M \text{Closure}(\Delta_i) \setminus \{\mathbf{0}\}.$$

Assume that $\dim \Theta_l = n$. So $\dim \pi(\Theta_l) = n$. As $\Theta_l \cap \text{diag}(C) = \emptyset$ it follows that $\mathbf{x} - \mathbf{z} \neq 0$ for each $(\mathbf{x}, \mathbf{z}) \in \Theta_l$. Hence

$$\Theta_l = \bigcup_{i=1}^{L'} (\Theta_l \cap \Omega_\nu(C, \Delta_i)), \quad \pi(\Theta_l) = \bigcup_{i=1}^{L'} (\pi(\Theta_l \cap \Omega_\nu(C, \Delta_i))).$$

Therefore there exists Δ_i of dimension n such that $\dim \pi(\Theta_l) \cap \pi(\Omega_\nu(C, \Delta_i)) = n$. ■

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