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ON PÓLYA'S THEOREM IN SEVERAL COMPLEX VARIABLES

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Abstract. Let K be a compact set in $\mathbb{C}$, f a function analytic in $\overline{\mathbb{C}} \setminus K$ vanishing at ∞. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{-k-1}$ be its Taylor expansion at ∞, and $H_s(f) = \det(a_{k+l})_{k,l=0}^{s}$ the sequence of Hankel determinants. The classical Pólya inequality says that

$$\limsup_{s\to\infty} |H_s(f)|^{1/s^2} \le d(K),$$

where $d(K)$ is the transfinite diameter of K. Goluzin has shown that for some class of compacta this inequality is sharp. We provide here a sharpness result for the multivariate analog of Pólya's inequality, considered by the second author in Math. USSR Sbornik 25 (1975), 350–364.

1. Introduction and preliminaries. We denote by $A(\mathbb{C}^n)^*$ the dual space to the space $A(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$, equipped with the locally convex topology of locally uniform convergence in $\mathbb{C}^n$. Following Hörmander ([12], Section 4.5), we call the elements of $A(\mathbb{C}^n)^*$ *analytic functionals*.

Let $\mathbb{Z}_+^n$ be the collection of all n-dimensional vectors with non-negative integer coordinates. For $k = (k_1, \dots, k_\nu, \dots, k_n) \in \mathbb{Z}_+^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $z^k = z_1^{k_1} \cdots z_n^{k_n}$ and $|k| := k_1 + \dots + k_n$ be the degree of the monomial z^k. We consider the enumeration $\{k(i)\}_{i\in\mathbb{N}}$ of the set $\mathbb{Z}_+^n$ such that $|k(i)| \le |k(i+1)|$ and on each set $\{|k(i)| = s\}$ the enumeration coincides with the lexicographic order relative to $k_1, \dots, k_n$. We will write $s(i) := |k(i)|$. The number of multiindices of degree at most s is $m_s := C_{s+n}^{s}$ and the number of those of degree exactly s is $N_s = m_s - m_{s-1} = C_{n+s-1}^{s}$, $s \ge 1$; $N_0 = 1$. Let $l_s := \sum_{q=0}^{s} q N_q$ for $s = 0, 1, 2, \dots$.

Consider Vandermondians:

$$V(\zeta_1, \dots, \zeta_i) := \det(e_\alpha(\zeta_\beta))_{\alpha,\beta=1}^{i}, \quad i \in \mathbb{N},$$

where $e_\alpha(z) := z^{k(\alpha)}$, $\alpha \in \mathbb{N}$, and $(\zeta_\beta) \in K^i$.

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For a compact set $K \subset \mathbb{C}^n$, define "maximal Vandermondians":

$$V_i := \sup\left\{|V(\zeta_1, \dots, \zeta_i)| : (\zeta_j) \in K^i\right\}, \quad i \in \mathbb{N}.$$

Set $d_s(K) := (V_{m_s})^{1/l_s}$. The transfinite diameter of K is the number

$$d(K) := \limsup_{s\to\infty} d_s(K). \tag{1.1}$$

In the one-dimensional case, this notion was introduced by Fekete [7] for $n = 1$, and by Leja [15] for $n \geq 2$. That, in fact, the usual limit can be taken in (1.1) was proved in [7] for $n = 1$ and in [26] for $n \geq 2$.

The *pluripotential Green function* of a compact set $K \subset \mathbb{C}^n$ is defined as follows

$$g_K(z) = \limsup_{\zeta\to z} \sup\{u(\zeta) : u|_K \leq 0,\ u \in \mathcal{L}(\mathbb{C}^n)\},$$

where $\mathcal{L}(\mathbb{C}^n)$ represents the Lelong class consisting of all functions $u \in Psh(\mathbb{C}^n)$ such that $u(\zeta) - \ln|\zeta|$ is bounded from above near infinity. Since $d(K) = d(\hat{K})$, with no loss of generality, we are going to consider polynomially convex compact sets. We will also consider the class of functions $\mathcal{L}^+(\mathbb{C}^n) := \{u \in \mathcal{L}(\mathbb{C}^n) : u(z) \geq \log^+|z| + C\}$. The function $g_K(z)$ is either plurisubharmonic in $\mathbb{C}^n$ or identically equal to $+\infty$. For more detail about the pluripotential Green function, we refer the reader to [13], [21] and [27].

The *Monge–Ampère energy* $\mathcal{E}(u, v)$ *of* u *relative to* v for $u, v \in \mathcal{L}^+(\mathbb{C}^n)$ is defined as follows ([3], Section 5):

$$\mathcal{E}(u, v) := \int_{\mathbb{C}^n} (u - v) \sum_{j=0}^{n} (dd^c u)^j \wedge (dd^c v)^{n-j}.$$

Let K be a compact set in $\mathbb{C}^n$. $A(K)$ represents the locally convex space of all germs of analytic functions on K, equipped with the countable inductive limit topology, i.e.,

$$A(K) = \liminf_{j\to\infty} A(D_j)$$

considered in regard to the inclusion of sets. D_j are open sets such that $D_{j+1} \Subset D_j$ for each $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} D_j$. Thus, in this setting, a sequence $\{u_j\}$ of germs converges to a germ u in this topology in case there exists an open neighborhood $V \supset K$ and functions $g_j, g \in A(V)$ being the representatives of the germs u_j, u respectively, such that g_j converges uniformly to g on any compact subset of V.

The Pólya Theorem (Theorem 2.1) and its multivariate analog (Theorem 2.3), considered by the second author in [26], are discussed in Section 2. The sharpness result of the generalized Pólya inequality (Section 4) is based on the comparison of the expression (2.6) for Hankel-like determinants from [26] with the expression (2.9) for the transfinite diameter from Bloom and Levenberg [4]. The main result of this article (Theorem 4.7) says that, for *real* compact sets, the equality *attains* in the generalized Pólya inequality (2.7) for some analytic functional $f^* \in A(K)^*$. This result seems to be new even in the one-dimensional case. Additionally, we introduce two sharpness properties for compact sets $K \subseteq \mathbb{C}^n$ and study the stability of these properties relative to the approximations from inside and outside (Propositions 4.3 and 4.4).

2. Pólya's theorem.

The following result is due to Pólya [20].

THEOREM 2.1. *Let K be a polynomially convex compact set in $\mathbb{C}$ and $f \in A(\overline{\mathbb{C}} \setminus K)$ have the following expansion in a neighborhood of ∞:*

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}} \,. \tag{2.1}$$

Let $A_s(f) := \det(a_{k+m})_{k,m=0}^{s-1}$, $s \in \mathbb{N}$, be a sequence of Hankel determinants composed from the coefficients of the expansion (2.1). *Then*

$$D(f) := \limsup_{s\to\infty} |A_s(f)|^{1/s^2} \leq d(K). \tag{2.2}$$

A direct multivariate analog of the inequality (2.2) makes no sense, since there are functions analytic on the complement of K but constants only. Schiffer and Siciak ([22]) proved some analog for the product of plane compact sets $K = K_1 \times K_2 \times \ldots \times K_n \subset \mathbb{C}^n$ and functions $f \in A((\overline{\mathbb{C}} - K_1) \times \ldots \times (\overline{\mathbb{C}} - K_n))$. Sheinov ([24], [25]) considered another analog of Pólya's inequality for a *linearly convex* compact set K, considering the Taylor expansion at the origin for functions analytic in the domain $D = K^*$ linearly convex adjoint (conjugate) to K (projective complement of K by Martineau [19]).

The case of an arbitrary compact set $K \subset \mathbb{C}^n$ was studied in [26]. It was suggested there, instead of analytic functions on some artificial "complement" of K, to deal with those analytic functionals in $\mathbb{C}^n$ that are extendible continuously onto the space $A(\widehat{K})$. We denote by $A_0(\{\infty^n\})$ the space of all analytic germs f' at $\infty^n = (\infty, \infty, \ldots, \infty) \in \overline{\mathbb{C}^n}$ with Taylor expansion of the form

$$f'(z) = \sum_{k\in\mathbb{Z}^n} \frac{a_k}{z^{k+I}} \,, \quad I = (1, 1, \ldots, 1), \tag{2.3}$$

converging in some neighborhood of ∞^n.

LEMMA 2.2. *There is an isomorphism,*

$$T : A(\mathbb{C}^n)^* \to A_0(\{\infty^n\}), \tag{2.4}$$

such that, for each f^* and $f' = Tf^*$, we have

$$f^*(\varphi) = \langle \varphi, f' \rangle := \Big(\frac{1}{2\pi i}\Big)^n \int_{\mathbb{T}_R^n} \varphi(\zeta) f'(\zeta)\, d\zeta, \quad \varphi \in A(\mathbb{C}^n),$$

where

$$\mathbb{T}_R^n := \big\{z = (z_\nu) \in \mathbb{C}^n : |z_\nu| = R,\ \nu = 1, \ldots, n\big\}, \quad R = R(f^*). \tag{2.5}$$

Proof. See, e.g., [6], Chapter 3. ∎

Let us define, for every analytic functional f^*, a related sequence of multivariate Hankel-like determinants constructed from the coefficients of the expansion (2.3):

$$H_i = H_i(f^*) := \det(a_{k(\alpha)+k(\beta)})_{\alpha,\beta=1}^{i}, \quad i \in \mathbb{N}, \tag{2.6}$$

with $a_{k(\alpha)} := f^*(e_\alpha) = \langle e_\alpha, f' \rangle$, $\alpha \in \mathbb{N}$, $f' = Tf^*$. Now we are ready to formulate the general form of multivariate Pólya's inequality.

Theorem 2.3. *Suppose that K is a compact set in $\mathbb{C}^n$, f^* is an analytic functional which has a continuous extension onto $A(K)$ and $f' = Tf^*$ is the corresponding analytic germ at ∞^n. Then for the determinants* (2.6), *the inequality holds*:

$$D(f') := \limsup_{i\to\infty} |H_i(f^*)|^{1/(2l_{s(i)})} \le d(K). \tag{2.7}$$

There has been proved in [26] a bit weaker result with the outer transfinite diameter $\widehat{d}(K)$ instead of $d(K)$, but later it was proved that $\widehat{d}(K) = d(K)$ (see Proposition 3.1 below).

We send the reader for the proof to [26], Theorem 3. However we cite here the following equality, which is crucial there and will be essentially used in Section 4:

$$i!\,|H_i(f^*)| = |f^*_{\zeta^{(i)}}(\dots f^*_{\zeta^{(j)}}(\dots f^*_{\zeta^{(1)}}([V(\zeta^{(1)},\zeta^{(2)},\dots,\zeta^{(i)})]^2)\dots)\dots)|, \quad i\in\mathbb{N}, \tag{2.8}$$

here the notation $f^*_{\zeta^{(j)}}$ means that the functional f^* is applied sequentially to a function of the variable $\zeta^{(j)}$ by keeping the other variables fixed.

Remark 2.4. The classical Pólya's Theorem (Theorem 2.1) is a particular case of Theorem 2.3 since, due to Grothendieck–Köthe–Silva duality (see [10], [11], [14], [23]), every $f \in A(\overline{\mathbb{C}} \setminus K)$ satisfying (2.1) in a neighborhood of ∞ represents a linear continuous functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C})^*$. Hereafter $\hookrightarrow$ denotes a linear continuous embedding.

Let $K \subset \mathbb{C}^n$ be a compact set, and μ be a bounded positive Borel measure on K. The pair (K,μ) is said to satisfy *Bernstein–Markov inequality* for holomorphic polynomials in $\mathbb{C}^n$ if, given $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ such that for all polynomials p_s of degree at most s

$$\|p_s\|_K \le M(1+\varepsilon)^s \|p_s\|_{L^2(\mu)}.$$

Theorem 2.5 (Bloom–Levenberg [4]). *Let $K \subset \mathbb{C}^n$ be a compact set, μ be a bounded positive Borel measure on K and let (K,μ) satisfy Bernstein–Markov inequality. Then*

$$\lim_{s\to\infty} Z_s(K,\mu)^{1/(2l_s(n))} = d(K),$$

where

$$Z_s(K,\mu) = \int_{K^{m_s(n)}} \left|V(\zeta^{(1)},\dots,\zeta^{(m_s(n))})\right|^2 d\mu(\zeta^{(1)})\, d\mu \dots d\mu(\zeta^{(m_s(n))}). \tag{2.9}$$

Remark 2.6. In [5] (Proposition 3.4 and Corollary 3.5), the same authors proved that for any compact set $K \subseteq \mathbb{C}^n$, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K,μ) satisfies Bernstein–Markov property.

3. Stability of transfinite diameter. The following proposition provides the stability of transfinite diameter of a compact set in $\mathbb{C}^n$ approximated from outside.

Proposition 3.1 (V. A. Znamenskii [30, 31], Levenberg [16]). *Let K be a compact set in $\mathbb{C}^n$ and $\{K_j\}$ a sequence of compact sets such that $K_{j+1} \subseteq K_j$ for all $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} K_j$. Then,*

$$\widehat{d}(K) := \lim_{j\to\infty} d(K_j) = d(K).$$

In this section, we prove a stability property of transfinite diameter relative to the approximation from inside. The following is an easy consequence of Lemma 6.5 in [2]:

LEMMA 3.2. *Suppose that K is a non-pluripolar compact set in $\mathbb{C}^n$, and $\{K_j\}$ is a sequence of non-pluripolar compact sets such that $K_j \subseteq K_{j+1} \subseteq K$, $j \in \mathbb{N}$, and for $L := \bigcup_{j=1}^{\infty} K_j$, we have*

$$\int_{K \setminus L} (dd^c g_K)^n = 0. \tag{3.1}$$

Then

$$\lim_{j \to \infty} g_{K_j}(z) = g_K(z), \quad z \in \mathbb{C}^n.$$

THEOREM 3.3. *Under the conditions of Lemma 3.2, we have*

$$\lim_{j \to \infty} d(K_j) = d(K).$$

Proof. We will use the unweighted energy version of Rumely's formula (see e.g., Theorem 5.1 of [17], or Section 9.1 of [3]). Since, by Lemma 3.2, $g_{K_j} \downarrow g_K$, applying the remark after Lemma 3.5 in [17], one obtains

$$-\ln d(K_j) = \frac{1}{n(2\pi)^n}\, \mathcal{E}(g_{K_j}, g_T) \uparrow \frac{1}{n(2\pi)^n}\, \mathcal{E}(g_K, g_T) = -\ln d(K), \quad \textit{as } j \to \infty,$$

where T is the unit torus in $\mathbb{C}^n$. ∎

4. Sharpness of Pólya's inequality.

The following theorem is proved by Goluzin in [8] (see also [9], Section 11).

THEOREM 4.1. *For functions which are analytic in an infinite domain B with boundary K consisting of a finite number of closed Jordan curves and having the expansion*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k}\,, \tag{4.1}$$

in a neighborhood of $z = \infty$, the inequality $D(f) = \limsup_{s \to \infty} |A_s(f)|^{1/s^2} \leq d(K)$ given by Theorem 2.1 is sharp.

Another way of expressing Theorem 4.1 is, for a compact set $K \subseteq \mathbb{C}$

$$d(K) = \sup\bigl\{D(f) : f \in A(\overline{\mathbb{C}} \setminus K)\bigr\}, \tag{4.2}$$

if the boundary ∂K consists of a finite number of closed Jordan curves.

DEFINITION 4.2. Let K be a polynomially convex compact set in $\mathbb{C}^n$. K is said to satisfy the *sharpness property* in the Pólya inequality, shortly denoted as $K \in (SP)$, if

$$d(K) = \sup\{D(f') : f' = T(f^*),\ f^* \in A(K)^*\}.$$

We say that K has a *strong sharpness property* in Pólya inequality, denoted by $K \in (SSP)$, if there exists a $f^* \in A(K)^*$ such that

$$D(f') = d(K)$$

for $f' = T(f^*)$, where T is defined as in Lemma 2.2.

If K is a pluripolar compact set in $\mathbb{C}^n$, then $K \in (SSP)$ by the result of Levenberg–Taylor ([18]) which says that $d(K) = 0$ if and only if K is pluripolar. From now on, we only consider non-pluripolar compact sets.

Proposition 4.3. *Let K be a compact set in $\mathbb{C}^n$, $\{K_i\}$ a sequence of compact sets with $K = \bigcap_{i=1}^{\infty} K_i$. Assume $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then there exists a sequence of analytic functionals $\{f_i^*\}$ such that $f_i^* \in A(K_i)^*$ for each $i \in \mathbb{N}$ and*

$$\lim_{i\to\infty} D(f_i') = d(K). \tag{4.3}$$

Proof. By Definition 4.2, for each $i \in \mathbb{N}$, there exists $f_i^* \in A(K_i)^*$ with $f_i' = T(f_i^*)$ such that $d(K_i) \leq D(f_i') + \frac{1}{i}$. Theorem 2.3 gives $D(f_i') \leq d(K_i)$. By using Proposition 3.1, we have

$$d(K) = \lim_{i\to\infty} d(K_i) \leq \lim_{i\to\infty} D(f_i') \leq \lim_{i\to\infty} d(K_i) = d(K),$$

which gives the limit (4.3). ■

As seen from Proposition 4.3, (SP) is not preserved under the approximation from outside, however, for an approximation from inside, we have the stability of the property (SP):

Proposition 4.4. *Let the conditions of Lemma* 3.2 *be given. Suppose further that $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then $K \in (SP)$.*

Proof. The proof is almost the same as the proof of Proposition 4.3 except we only use Theorem 3.3 instead of Proposition 3.1 in the end, hence we have the following:

$$d(K) \leq \lim_{i\to\infty} D(f_i') = \sup\{D(f_i') : i \in \mathbb{N}\} \leq d(K),$$

which concludes that $d(K) = \sup\{D(f_i') : i \in \mathbb{N}\}$ and so $K \in (SP)$ by Definition 4.3. ■

For an arbitrary compact set in $\mathbb{C}$, a following sharpness statement, which is weaker than (SP), is derived easily from Goluzin's result above.

Proposition 4.5. *Let K be a compact set in $\mathbb{C}$, $\{K_i\}$ a sequence of compact sets with the properties $K_{i+1} \Subset K_i$ for all $i \in \mathbb{N}$, $K = \bigcap_{i=1}^{\infty} K_i$. Then there exists a sequence of functions $f_i \in A(\overline{\mathbb{C}} \setminus K_i)$ such that*

$$\lim_{i\to\infty} D(f_i) = d(K). \tag{4.4}$$

Proof. For each $i \in \mathbb{N}$, we can find a compact set L_i whose boundary consists of a finite number of closed analytic Jordan curves so that $K_{i+1} \Subset L_i \Subset K_i$ holds. By the result of Goluzin, there exists $f_i \in A(\overline{\mathbb{C}} \setminus L_i)$ such that, $d(L_i) < D(f_i) + \frac{1}{i}$. Since $f_i \in A(\overline{\mathbb{C}} \setminus K_i)$ holds, we get by Theorem 2.1, $D(f_i) \leq d(K_i)$. Hence by using Proposition 3.1 we obtain the following

$$d(K) = \lim_{i\to\infty} d(L_i) \leq \lim_{i\to\infty} D(f_i) \leq \lim_{i\to\infty} d(K_i) = d(K),$$

which gives the desired limit (4.4). ■

Let K be a compact set in $\mathbb{C}^n$, and $J : A(K) \to C(K)$ the natural restriction homomorphism. $AC(K)$ is the Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ with respect to the uniform norm.

Lemma 4.6. *Let K be an infinite polynomially convex compact set in $\mathbb{C}^n$. Then, for each bounded Borel measure $\mu \in \mathcal{M}(K)$, there exists an analytic functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$ and a corresponding analytic germ $f' = Tf^*$ such that*

$$f^*(f) = \langle f, f' \rangle = \int_K f(\zeta)\, d\mu(\zeta), \tag{4.5}$$

for every $f \in A(\mathbb{C}^n)$.

Proof. The dense embedding $A(K) \hookrightarrow AC(K)$ implies, for the dual spaces, the following embedding: $AC(K)^* \hookrightarrow A(K)^*$. Since $AC(K)$ is a closed subspace of $C(K)$, every bounded Borel measure $\mu \in \mathcal{M}(K)$ defines a linear continuous functional $F^* \in AC(K)^*$ such that

$$F^*(f) = \int_K f(\zeta)\, d\mu(\zeta)$$

for every $f \in AC(K)$. Then, the restriction $f^* = F^*|_{A(K)}$ belongs to $A(K)^*$. By Lemma 2.2, since $A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$, there is $f' \in A_0(\{\infty^n\})$ such that

$$f^*(f) = \langle f, f' \rangle = \Big(\frac{1}{2\pi i}\Big)^n \int_{\mathbb{T}^n_R} f(\zeta) f'(\zeta)\, d\zeta, \quad f \in A(\mathbb{C}^n),$$

where $\mathbb{T}^n_R$ is defined as in (2.5), and R is sufficiently large. ■

Now we show that, for any real compact set in $\mathbb{C}^n$, the equality in the estimate (2.7) is attained at some $f^* \in A(K)^*$.

Theorem 4.7. *Let $K \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$ be a compact set. Then $K \in (SSP)$.*

Proof. By Theorem 2.5 and Remark 2.6, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies the Bernstein–Markov inequality. Let f^* be an analytic functional corresponding to μ by Lemma 4.6. Initially, we show that $Z_s(K, \mu) = m_s(n)!\, |H_{m_s(n)}(f^*)|$, where $Z_s(K, \mu)$ and $H_{m_s(n)}(f^*)$ are defined in Section 2. Indeed, considering the relation (2.8) gives:

$$m_s(n)!\, |H_{m_s(n)}(f^*)| = |f^*_{\zeta^{(m_s(n))}}(\dots(f^*_{\zeta^{(1)}}([V(\zeta^{(1)}, \dots, \zeta^{(m_s(n))})]^2)\dots)|, \tag{4.6}$$

Since K is a real subset and so $[V(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(m_s(n))})]^2$ is nonnegative, by iterating (4.5) $m_s(n)$ times, the right-hand side of (4.6) becomes:

$$\int_K \cdots \int_K \big|V(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(m_s(n))})\big|^2\, d\mu(\zeta^{(1)}) \dots d\mu(\zeta^{(m_s(n))}),$$

which is equal to $Z_s(K, \mu)$. Since $(m_s(n)!)^{1/(2l_s(n))} \to 1$ as $s \to \infty$, we have, by Theorem 2.5,

$$d(K) = \lim_{s\to\infty} Z_s(K, \mu)^{1/(2l_s(n))} = \lim_{s\to\infty} |H_{m_s(n)}(f^*)|^{1/(2l_s(n))} = D(f'). \ ■$$

Problem. Characterize compact sets in $\mathbb{C}^n$ such that either (SP) or (SSP) holds.

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