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STOCHASTIC CHARACTERIZATION OF PLURISUBHARMONICITY AND CONVEXITY OF FUNCTIONS

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Dedicated to Professor Wiesław Pleśniak on his 70th birthday

Abstract. It is described how both plurisubharmonicity and convexity of functions can be characterized in terms of simple to work with classes of holomorphic martingales, namely a class of driftless Itô processes satisfying a skew-symmetry property and a family of linear modifications of Brownian motion parametrized by a compact set.

1. Introduction. For many decades it has been known that the classical potential theory can be reformulated and developed within a probabilistic framework (see e.g. [D] or [PS]). In the multidimensional complex case the situation is rather different. Despite that over the years many significant and beautiful results have been obtained on the border between pluricomplex analysis and the theory of stochastic processes, it would be fair to say that no probabilistic counterpart of pluripotential theory exists as such. One of the main reasons might be simply the fact that most complex analysts do not display much interest in probabilistic methods. The purpose of this short note is to show that even fairly elementary use of stochastic approach is sufficient to characterize the two classes of subharmonic functions of primary importance in complex analysis: plurisubharmonic functions and the convex functions. To this end we will use two simple to handle families of stochastic processes. One of the families consists of driftless Itô processes whose diffusion coefficients satisfy a skew-symmetry condition. The other is a much smaller parametric collection of linear transformations of Brownian motion with a compact parameter set.

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The organization of the paper is as follows. First, notational conventions are listed alongside some background comments. Then a stochastic characterization of plurisubharmonicity of functions is presented in both global and local form. As a consequence a similar characterization of convexity of functions is obtained, as well a multivariate version of a theorem about compositions of plurisubharmonic functions and convex functions.

2. Notation and preliminaries.

In what follows $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the space $\mathbb{K}^n$ we will use either the Euclidean norm $\|\cdot\|$ or the maximum norm $|\cdot|$, with the corresponding open balls denoted by $B(a,r)$ or $P(a,r)$ respectively, where $a \in \mathbb{K}^n$ and $r > 0$. The set of all $m \times n$ matrices with entries from $\mathbb{K}$ will be denoted by $\mathbb{K}^{m\times n}$, and — when convenient — will be identified with the space of $\mathbb{K}$-linear mappings $\mathcal{L}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^m)$ from $\mathbb{K}^n$ to $\mathbb{K}^m$.

Because of the singular value decomposition theorem all linear isomorphisms of $\mathbb{K}^n$ can be seen as ellipsoidal modifications of the underlying Euclidean space. We define an *ellipsoid* (open or closed) as the image of an Euclidean ball in $\mathbb{K}^n$ via a $\mathbb{K}$-linear isomorphism of $\mathbb{K}^n$. If $v = (v_1, \dots, v_n) \in \mathbb{K}^n$, let $D[v] \in \mathbb{K}^{n\times n}$ denote the diagonal matrix with the entries of v on the diagonal. We will define the family of *ellipsoidal modifiers* as

$$\mathcal{E}(n) = \{UD[v] : v \in \partial P(0,1) \cap \mathbb{R}^n_+,\ U \in \mathcal{I}(n)\},$$

where $\mathcal{I}(n)$ denotes the orthogonal group $\mathcal{O}(n)$ if $\mathbb{K} = \mathbb{R}$ and the unitary group $\mathcal{U}(n)$ if $\mathbb{K} = \mathbb{C}$. Obviously the set $\mathcal{E}(n) \subset \mathcal{L}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K}^n)$ is compact. In fact the use of $\partial P(0,1)$ in the definition of $\mathcal{E}(n)$ is somewhat arbitrary — any compact subset S of the positive orthant $\mathbb{R}^n_+$, such that $(1,1,\dots,1) \in S$ and $\mathbb{R}_+ S = \mathbb{R}^n_+$, would do equally well. Obviously every linear isomorphism of $\mathbb{K}^n$ is a composition of a ellipsoidal modifier with an isometry.

Let $D \subset \mathbb{C}^n$ be open. It is well-known (see e.g. [K] (Thm 2.9.12) or [L] (Thm 1, p. 18)) that $u \in \mathrm{PSH}(D)$ if and only if $u \circ L \in \mathrm{SH}(L^{-1}(D))$ for any $\mathbb{C}$-linear isomorphism $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Similarly, if $D \subset \mathbb{R}^n$ is open and $n \geq 2$, then $u \in \mathrm{CVX}(D)$ if and only if $u \circ L \in \mathrm{SH}(L^{-1}(D))$ for any $\mathbb{R}$-linear isomorphism $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. At the same time, if D is open in $\mathbb{K}^n$, then $u \in \mathrm{SH}(D)$ if and only if $u \circ L \in \mathrm{SH}(L^{-1}(D))$ for any L in $\mathcal{U}(n)$ if $\mathbb{K} = \mathbb{C}$ or for any L in $\mathcal{O}(n)$ if $\mathbb{K} = \mathbb{R}$. Note also that subharmonicity is invariant with respect to a linear transformation given by a diagonal matrix in $\mathbb{K}^{n\times n}$ only if the transformation is a homothety. In view of these comments and the singular value decomposition theorem in order to assert plurisubharmonicity or convexity of u it suffices to verify subharmonicity of the compositions $u \circ L$ only for all non-singular ellipsoidal modifiers $L \in \mathcal{E}(n)$.

By an $\mathcal{E}(n)$-*Brownian motion* starting at a point $a \in \mathbb{K}^n$ we will mean the process $L(B_t) + a$, where $L \in \mathcal{E}(n)$ and B_t is a standard Brownian motion in $\mathbb{K}^n$.

A *holomorphic martingale* is defined to be a stochastic process $Z_t = (Z_t^1, \dots, Z_t^n)$ with values in a domain D in $\mathbb{C}^n$ such that all processes Z^j and $Z^j Z^k$ are continuous local martingales (see [F2]). The concept was originally introduced in 1972 by Getoor and Sharpe [GS] for $n = 1$, under the name *conformal martingale*. The name was used also in higher dimensions (e.g. by Schwartz, Fukushima, Okada, Kaneko, Ubøe, Kazamaki, Taniguchi) but in 1987 Fukushima [F1] proposed to replace *conformal* by *holomorphic* arguing that conformality and holomorphy are not synonymous in dimensions higher than one.

The key relationship between holomorphic martingales and plurisubharmonic functions is expressed in the following classic result due to Schwartz [S]:

THEOREM 2.1. *The composition of a bounded plurisubharmonic function with a holomorphic martingale is a submartingale.*

In what follows, it will be assumed that the coefficients of the considered Itô processes are adapted, "progressively" square integrable, and regular enough to guarantee the uniqueness, existence and path-wise continuity of (strong) solutions of the underlying stochastic differential equations. Let Z_t is a $\mathbb{C}^m$-valued Itô process driven by a complex n-dimensional Brownian motion W_t, that is

$$dZ_t = \alpha(t, Z_t)\,dt + \beta(t, Z_t)\,dW_t + \gamma(t, Z_t)\,d\bar{W}_t, \tag{1}$$

where α, β, γ are measurable functions defined on $\mathbb{R}_+ \times \mathbb{C}^m$ (with values in $\mathbb{C}^{m\times 1}, \mathbb{C}^{m\times n}$, and $\mathbb{C}^{m\times n}$, respectively) and $Z_0 \in \mathbb{C}^m$. We will say that the process Z_t is an *$\mathcal{OM}$-process* if it is driftless (i.e. $\alpha = 0$) and the matrix $\beta\gamma^T$ is skew-symmetric at any value of $(t, z) \in \mathbb{R}_+ \times \mathbb{C}^m$. In fact, it was shown by Ubøe [U] that this skew-symmetry condition is equivalent to Z_t being a (global) holomorphic martingale. This is the motivation behind the symbol $\mathcal{OM}$ as the letter $\mathcal{O}$ is often used to denote the property of being holomorphic, whereas $\mathcal{M}$ stands for the word "martingale". We will derive the holomorphic martingale property of $\mathcal{OM}$-processes as a consequence of Theorem 3.1 in the next section.

Elementary calculations, based on the sequential and bloc identification of complex and real vectors and matrices, can be used to derive from the real Itô formula its complex counterpart. Alternatively, like in [U], one can first prove directly a complex version of Itô's formula in which the outcome is expressed in terms of the components of dZ_t and their conjugates, and then perform conversion to the form stated below. The *complex version of Itô's formula* can be stated as follows: *if u is a real-valued $\mathcal{C}^2$ function of m complex variables, then*

$$du(Z) = \mathcal{L}u\,dt + 2\,\mathrm{Re}\{u_z(\beta\,dW + \gamma\,d\bar{W})\},$$

where $u_z = (\partial u/\partial z_1, \ldots, \partial u/\partial z_m)$ and

$$\mathcal{L}u = 2\,\mathrm{Re}\Big\{\sum_{j=1}^{m} \alpha_j \frac{\partial u}{\partial z_j} + \sum_{j,k=1}^{m} (\beta\gamma^T + \gamma\beta^T)_{jk} \frac{\partial^2 u}{\partial z_j \partial z_k}\Big\} + 2 \sum_{j,k=1}^{m} (\beta\beta^* + \gamma\gamma^*)_{jk} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}.$$

The asterisk stands here for the complex adjoint.

3. Stochastic criteria of plurisubharmonicity and convexity. In what follows, if Z_t is a stochastic process starting at a point in a domain $G \subset \mathbb{K}^n$, the symbol $\tau_{Z,G}$ will denote the first exit time of the process Z from G.

The implication ①⇒② in the next theorem is a global counterpart of the result of Schwartz stated above.

Theorem 3.1. *Let* $u : \mathbb{C}^n \longrightarrow \mathbb{R}$ *be an upper semicontinuous function of at most quadratic growth*:

$$\sup_{z\in\mathbb{C}^n} \frac{|u(z)|}{1+\|z\|^2} < \infty.$$

Then the following conditions are equivalent.

① $u \in \mathrm{PSH}(\mathbb{C}^n)$.

② *If* Z_t *is an* $\mathcal{OM}$*-process in* $\mathbb{C}^n$ *driven by an* n*-dimensional complex Brownian motion* W_t*, then the process* $t \mapsto u(Z_t)$ *is a submartingale with respect to the filtration generated by* W_t.

③ *If* W_t *is an* $\mathcal{E}(n)$*-Brownian motion in* $\mathbb{C}^n$*, then the process* $t \mapsto u(W_t)$ *is a submartingale with respect to its own filtration.*

Proof. We start with the implication ①⇒②. Let $\chi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ be a smoothing kernel, that is the function $\chi_\varepsilon(z) = \varepsilon^{-2n}\chi(z/\varepsilon)$, where $\chi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ depends on $\|z\|$ only, $\chi \ge 0$, $\operatorname{supp}\chi = \bar{B}(0,1)$ and $\|\chi\|_{L^1} = 1$. Then the convolutions $u * \chi_\epsilon$ have at most quadratic growth whereas their complex gradients $(u*\chi_\epsilon)_z$ have at most linear growth. The latter implies that the integrals with respect to Brownian motion, which we obtain by applying the complex version of Itô formula to the processes $t \mapsto (u*\chi_\epsilon)(Z_t)$ are progressively square integrable and hence are martingales. Consequently the complex version of Itô formula and basic properties of Itô integrals imply that all processes $t \mapsto (u*\chi_\epsilon)(Z_t)$ are submartingales as the functions $u*\chi_\epsilon$ are plurisubharmonic. Hence also $t \mapsto u(Z_t)$ is a submartingale by the monotone convergence theorem.

The implication ②⇒③ is trivial. In view of our earlier comments, in order to show ③⇒①, it suffices to assert subharmonicity of compositions $u_L = u\circ L$, where L is a linear isomorphism, knowing that $u_L(W_t)$ is a submartingale for any Brownian motion W_t with arbitrarily chosen starting point. To this end one can use a standard argument. Take an open ball G and a harmonic function $h \in H(G)\cap\mathcal{C}(\bar{G})$ such that $u_L \le h$ on ∂G. Let $z \in G$. Let W_t be a Brownian motion starting at z. It follows from the submartingale property and the optimal sampling theorem, that $u_L(z) \le \mathrm{E}[u_L(W_{\tau_{W,G}})] \le \mathrm{E}[h(W_{\tau_{W,G}})] = h(z)$. ■

By applying the theorem to $u(z_1,\ldots,z_n) = \pm\mathrm{Re}(z_j z_k)$ and $u(z_1,\ldots,z_n) = \pm\mathrm{Im}(z_j z_k)$ we get the following statement.

Corollary 3.2 (Ubøe [U]). *If* $Z_t = (Z_t^1,\ldots,Z_t^n)$ *is an* $\mathcal{OM}$*-process in* $\mathbb{C}^n$*, then the processes* $Z_t^j Z_t^k$ *are martingales for all* $j,k \in \{1,\ldots,n\}$*, and so* Z_t *is a holomorphic martingale.*

In view of Theorem 3.1 and Corollary 3.2, the implication ①⇒② in the following theorem is a special case of Schwartz' Theorem, but the argument we use is rather different from that used in [S].

Theorem 3.3. *Let* $D \subset \mathbb{C}^n$ *be a domain. Let* $u : D \longrightarrow \mathbb{R}$ *be upper semicontinuous and locally bounded. Then the following conditions are equivalent*:

① $u \in \mathrm{PSH}(D)$.

② *If* $G \Subset \mathbb{C}^n$ *such that* $\widehat{G} \subset D$ *and if* Z_t *is an* $\mathcal{OM}$*-process with* $Z_0 \in G$*, driven by an* n*-dimensional complex Brownian motion* W_t*, then the process* $t \mapsto u(Z_{t\wedge\tau_{Z,G}})$

is a submartingale with respect to the filtration generated by W_t. *Here* $\widehat{G}$ *denotes the polynomially convex hull of* $\bar{G}$.

③ *If* $G \Subset \mathbb{C}^n$ *such that* $\widehat{G} \subset D$, *then the process* $t \mapsto u(W_{t\wedge\tau_{W,G}})$ *is a submartingale with respect to the filtration generated by any* $\mathcal{E}(n)$*-Brownian motion* W_t *starting at a point in* G.

④ *For any open ellipsoid* $G \Subset D$ *the function* u *is submeanvalued in the sense that for any* $\mathcal{E}(n)$*-Brownian motion* W_t *starting at* $z \in G$

$$u(z) \le \mathrm{E}[u(W_{\tau_{W,G}})].$$

Proof. To show that ①⇒② let $G \Subset \mathbb{C}^n$ be such that $\widehat{G} \subset D$ and let $u \in \mathrm{PSH}(D)$. Let $K = \bar{G}$ and let $K_\delta = \{z \in \mathbb{C}^n : \operatorname{dist}(z, K) < \delta\}$ denote the δ-dilation of K for $\delta > 0$. Since $\widehat{G} \subset D$, if we take a sufficiently small δ, then $\widehat{K_\delta} \subset D$. Let V_{K_δ} be the pluricomplex Green function of K_δ. Then $V_{K_\delta} \in \mathcal{C}(\mathbb{C}^n) \cap \mathbb{L}$, where $\mathbb{L}$ denotes the Lelong class

$$\mathbb{L} = \bigl\{v \in \mathrm{PSH}(\mathbb{C}^n) : \sup_{z\in\mathbb{C}^n} [v(z) - \log(1 + \|z\|)] < \infty\bigr\}.$$

Furthermore $V_{K_\delta} \ge 0$ and $V_{K_\delta}^{-1}(0) = \widehat{K_\delta}$. (See e.g. [K] for background in pluripotential theory.) Choose positive constants γ and M so that $\omega = \{z \in \mathbb{C}^n : V_{K_\delta}(z) < \gamma\} \Subset D$ and $M > \sup\{|u(z)| : z \in \bar{\omega}\}$. Let

$$V = \frac{2M}{\gamma} V_{K_\delta} - M.$$

Then the function

$$U = \begin{cases} \max\{u, V\} & \text{in } \omega, \\ V & \text{outside } \omega, \end{cases}$$

is plurisubharmonic in $\mathbb{C}^n$ and equal to u in $\bar{G}$. Moreover — being a positive multiple of an element of $\mathbb{L}$ — it has less than quadratic growth. Let Z_t be an $\mathcal{OM}$-process starting at a point in G and driven by an n-dimensional complex Brownian motion W_t. Then, in view of Theorem 3.1, the process $U \circ Z$ is a submartingale with respect to the filtration $\mathcal{F}_t$ generated by W_t. This leads to the localized conclusion of ②. Indeed, in view of the Optional Sampling Theorem (see e.g. [IW]) the process $t \mapsto U(Z_{t\wedge\tau_{Z,G}})$ is a submartingale with respect to the filtration $\mathcal{F}_{t\wedge\tau_{Z,G}}$. It also is a submartingale with respect to the filtration $\mathcal{F}_t$. This yields ②.

The implications ②⇒③⇒④ are obvious. To show ④⇒① fix a complex linear isomorphism $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and a point $a \in L^{-1}(D)$. Because our choice of L is arbitrary, it suffices to show that $u \circ L \in \mathrm{SH}(L^{-1}(D))$. Let $r > 0$ be such that $\bar{\mathbb{B}}(a, r) \subset L^{-1}(D)$ and let $h \in \mathcal{C}(\bar{\mathbb{B}}(a, r)) \cap \mathrm{H}(\mathbb{B}(a, r))$. Assume that $u \circ L \le h$ on $\partial\mathbb{B}(a, r)$. We have to show that the inequality is true inside the ball. Let $z \in \mathbb{B}(a, r)$ and let W_t be a Brownian motion starting at z. Then $\tilde{W}_t = L(W_t)$ is an $\mathcal{E}(n)$-Brownian motion starting at $L(z)$. Let us denote by τ the common exit time of the processes W_t and $\tilde{W}_t$ from $\mathbb{B}(a, r)$ and $L(\mathbb{B}(a, r))$ respectively. Then

$$u(L(z)) \le \mathrm{E}[u(\tilde{B}_\tau)] = \mathrm{E}[(u \circ L)(B_\tau)] \le \mathrm{E}[h(B_\tau)] = h(z),$$

which concludes the argument. ■

The next two corollaries are counterparts of the properties described in the above theorems but for convex functions.

Corollary 3.4. *Let $n \geq 2$ and let $u \in \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ be such that*

$$\sup_{x \in \mathbb{R}^n} \frac{|u(x)|}{1 + \|x\|^2} < \infty.$$

Then the following conditions are equivalent.

① *$u \in \mathrm{CVX}(\mathbb{R}^n)$.*
② *If Z_t is an $\mathbb{R}^n$-values square integrable martingale, then the process $t \mapsto u(Z_t)$ is a submartingale with respect to the same filtration.*
③ *If W_t is an $\mathcal{E}(n)$-Brownian motion in $\mathbb{R}^n$, then the process $t \mapsto u(W_t)$ is a submartingale with respect to its own filtration.*

Proof. The implication ①⇒② follows from Jensen's inequality for conditional expectations. Because of formal similarities between plurisubharmonic and convex functions, the rest of the proof can proceed along the same lines as the proofs of the above theorems. Alternatively one can utilize Theorem 3.1 and the fact that u is convex if and only if $z \mapsto u(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$ is plurisubharmonic. ∎

Corollary 3.5. *If $n \geq 2$, $D \subset \mathbb{R}^n$ and $u \in \mathcal{C}(D)$, then the following conditions are equivalent*:

① *$u \in \mathrm{CVX}(D)$.*
② *If $G \Subset D$ is a convex domain and Z_t is a $\bar{G}$-valued martingale starting at a point in G, then the process $t \mapsto u(Z_t)$ is a submartingale.*
③ *If $G \Subset D$ is a convex domain, then the process $t \mapsto u(W_{t \wedge \tau_{W,G}})$ is a submartingale with respect to the filtration generated by any $\mathcal{E}(n)$-Brownian motion W_t starting at a point in G.*
④ *For any open ellipsoid $G \Subset D$ the function u is submeanvalued in the sense that*

$$u(x) \leq \mathrm{E}[u(W_{t \wedge \tau_{W,G}})]$$

for any $\mathcal{E}(n)$-Brownian motion W_t starting at $x \in G$.

Proof. The arguments outlined in the proof of the previous corollary apply here as well. ∎

The next property is a consequence of the multidimensional version of Jensen's inequality for conditional expectations of random vectors.

Corollary 3.6. *Let $f : D \longrightarrow \mathbb{R}$ be a convex function defined on a convex set $D \subset \mathbb{R}^n$. Let $-\infty \leq a_j < b_j < \infty$, for $j = 1, \ldots, n$, be such that $(a_1, b_1) \times \ldots \times (a_n, b_n) \subset D$. Let $G \subset \mathbb{C}^n$ be a domain and let $u = (u_1, \ldots, u_n) : G \longrightarrow [a_1, b_1) \times \ldots \times [a_n, b_n)$ be a mapping such that for each $j \in \{1, \ldots, n\}$ either u_j is pluriharmonic or, alternatively, u_j is plurisubharmonic and f is non-decreasing with respect to the j-th variable. Then f extends to the range of u* (*by taking suitable infima*) *and $f \circ u \in \mathrm{PSH}(G)$.*

Proof. Any non-pluriharmonic component u_j which is not locally bounded can be replaced by $\max\{u_j, M_j\}$ with a suitable constant M_j. Therefore our characterization of plurisubharmonicity, combined with the multidimensional Jensen's inequality for conditional expectations, yields the desired conclusion for the modified functions. By letting $M_j \to -\infty$ we get the general case. ■

Corollary 3.6 can be also obtained without using probabilistic methods in the same way as a number of results from [R] (see also [G], [GK] and [H]) concerning compositions of multivariate convex functions with various types of subharmonic functions and their generalizations.

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