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ON WEIGHTED HARDY SPACES ON THE UNIT DISK

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Abstract. In this paper we completely characterize those weighted Hardy spaces that are Poletsky–Stessin Hardy spaces H^p_u. We also provide a reduction of H^∞ problems to H^p_u problems and demonstrate how such a reduction can be used to make shortcuts in the proofs of the interpolation theorem and corona problem.

1. Introduction. Let λ be the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Among many different definitions of weighted Hardy spaces the closest to our purpose is the definition in [7] and [2]. Let $\alpha \in L^1(\mathbb{T})$ be a non-negative function such that $\log \alpha \in L^1(\mathbb{T})$. Then $L^p_\alpha(\mathbb{T})$ is the space of all functions with the finite norm

$$\|\phi\|_{\alpha,p} = \Big(\int_0^{2\pi} |\phi(e^{i\theta})|^p \alpha(e^{i\theta})\, d\lambda\Big)^{1/p}$$

for $0 < p < \infty$ and $H^p_\alpha = N^+ \cap L^p_\alpha(\mathbb{T})$, where N^+ is the Smirnov class. If $\alpha \equiv 1$ then we will use the symbols H^p and $\|\cdot\|_p$.

Later for a plurisubharmonic exhaustion function u on a hyperconvex domain $D \subset \mathbb{C}^n$ Poletsky and Stessin introduced in [8] weighted Hardy spaces $H^p_u(D)$ to generalize the notion of the classical Hardy spaces and then studied the composition operators generated by the holomorphic mappings between such domains.

Recently, M. Alan and N. Göğüş in [1], S. Şahin in [9] and K. R. Shrestha in [10, 11] obtained the description of these spaces on the unit disk and called them Poletsky–Stessin Hardy spaces. In particular, they showed that the new spaces form a subclass of weighted Hardy spaces introduced at the beginning of this section.

In Section 3 we finish this description proving that H^p_α is a Poletsky–Stessin space if and only if the weight α is lower semicontinuous and greater than some $c > 0$ on $\mathbb{T}$.

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Although weighted Hardy spaces can be studied per se there is also an expectation that they can be useful for the classical theory. In Section 4 we prove the main result of this paper: if a closed convex set A intersects unit balls in all $H^p_u(\mathbb{D})$ for some $p > 1$ then it intersects the unit ball in H^∞. Thus to find bounded solutions to a linear problem it suffices to show that they exist at all $H^p_u(\mathbb{D})$ and their norms are uniformly bounded.

In the last two section we use this fact to demonstrate shortcuts in the proofs of the interpolation theorem and corona problem.

2. Duality. Let $a(z)$ be a holomorphic function such that $|a(e^{i\theta})| = \alpha(e^{i\theta})$ on $[0, 2\pi]$ a.e. and a never takes the zero value. Such a function does exist and belongs to H^1 because $\log\alpha$ is integrable on $\mathbb{T}$ so we can take a harmonic function

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \log\alpha(e^{i\theta})P(z, e^{i\theta})\, d\theta,$$

add a conjugate function g and write $a(z) = e^{h(z)+ig(z)}$.

In [2] for $f \in L^p_\alpha(\mathbb{T})$ the operator $A_p f = a^{1/p} f$ was introduced. Then

$$\|A_p f\|_p^p = \int_0^{2\pi} |f(e^{i\theta})|^p \alpha(e^{i\theta})\, d\lambda = \|f\|^p_{\alpha,p}.$$

Thus A_p is an isometrical imbedding of $L^p_\alpha(\mathbb{T})$ into $L^p(\mathbb{T})$.

We add other requirements on the weight α asking that $\alpha \geq c > 0$ on $\mathbb{T}$. Clearly, $A_p^{-1} f = \alpha^{-1/p} f$ is also an isometry and the inverse of A_p. Hence A_p is an isometric isomorphism of $L^p_\alpha(\mathbb{T}))$ onto $L^p(\mathbb{T})$. Moreover, A_p maps H^p_α isometrically onto H^p.

If $\phi \in L^p_\alpha(\mathbb{T})$ then $\operatorname{dist}(\phi, H^p_\alpha) = \operatorname{dist}(A_p\phi, H^p)$. By the classical result (see [6]) for $p > 1$

$$\operatorname{dist}(A_p\phi, H^p) = \Big|\sup_{g\in H^q, \|g\|_{H^q}=1} \int_0^{2\pi} a^{1/p}(e^{i\theta})\phi(e^{i\theta})e^{i\theta}g(e^{i\theta})\, d\lambda\Big|. \tag{1}$$

Since the H^q_α-norm of $\alpha^{-1/q}g(z)$ coincides with the H^q-norm of g we can get the following duality result:

Theorem 2.1. *If $\phi \in L^p_\alpha(\mathbb{T})$ then*

$$\operatorname{dist}(\phi, H^p_\alpha) = \Big|\sup_{g\in H^q_\alpha, \|g\|_{H^q_\alpha}=1} \int_{\mathbb{T}} \phi(e^{i\theta})a(e^{i\theta})e^{i\theta}g(e^{i\theta})\, d\lambda\Big|.$$

3. Poletsky–Stessin Hardy spaces. Let $\mathbb{D}$ be the unit disc $\{|z| < 1\}$ in $\mathbb{C}$. A continuous subharmonic function $u : \mathbb{D} \to [-\infty, 0)$ such that $u(z) \to 0$ as $|z| \to 1$ is called an *exhaustion function.* The class of such functions will be denoted by $\mathcal{E}$. Following [3] for $r < 0$ we set

$$B_{u,r} = \{z \in \mathbb{D} : u(z) < r\} \text{ and } S_{u,r} = \{z \in \mathbb{D} : u(z) = r\}.$$

As in [3] we let $u_r = \max\{u, r\}$ and define the measure

$$\mu_{u,r} = \Delta u_r - \chi_{\mathbb{D}\setminus B_r}\Delta u,$$

where Δ is the Laplace operator. Clearly $\mu_{u,r} \geq 0$ and is supported by $S_{u,r}$.

Let us denote by $\mathcal{E}_1$ the set of all continuous negative subharmonic exhaustion functions u on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \Delta u = 1.$$

In the same paper Demailly (see Theorems 1.7 and 3.1 there) proved the following result which we adapt to the case of $\mathbb{D}$.

THEOREM 3.1 (Lelong–Jensen formula). *Let ϕ be a subharmonic function on $\mathbb{D}$. Then ϕ is $\mu_{u,r}$-integrable for every $r < 0$ and*

$$\int_{S_{u,r}} \phi \, d\mu_{u,r} = \int_{B_{u,r}} \phi \, \Delta u + \int_{B_{u,r}} (r - u) \, \Delta \phi.$$

Moreover, if $u \in \mathcal{E}_1$ then the measures $\mu_{u,r}$ converge weak-$*$ in $C^*(\overline{\mathbb{D}})$ to a measure $\mu_u \geq 0$ supported by $\mathbb{T}$ as $r \to 0^-$.

As a consequence of this theorem we have the following corollary.

COROLLARY 3.2. *If ϕ is a non-negative subharmonic function, then the function*

$$r \mapsto \int_{S_{u,r}} \phi \, d\mu_{u,r}$$

is increasing on $(-\infty, 0)$.

Using the measures $\mu_{u,r}$ Poletsky and Stessin introduced in [8] the weighted Hardy spaces associated with an exhaustion $u \in \mathcal{E}$. Following [8] we define the space H_u^p, $0 < p < \infty$, consisting of all holomorphic functions $f(z)$ in $\mathbb{D}$ that satisfy

$$\|f\|_{u,p}^p = \overline{\lim_{r \to 0^-}} \int_{S_{u,r}} |f|^p \, d\mu_{u,r} < \infty.$$

By Corollary 3.2 we can replace the $\overline{\lim}$ in the above definition with lim. By Theorem 3.1 and the monotone convergence theorem it follows that

$$\|f\|_{u,p}^p = \int_{\mathbb{D}} |f|^p \, \Delta u - \int_{\mathbb{D}} u \, \Delta |f|^p. \tag{2}$$

The classical Hardy spaces H^p correspond to the exhaustion function $u(z) = \log |z|$ ([8, Section 4]). Hence the classical definition of the Hardy spaces is subsumed in this new definition.

It is proved in [8] that the spaces H_u^p are Banach when $p \geq 1$ and if $v, u \in \mathcal{E}$ and $v \leq cu$ in a neighborhood of $\mathbb{T}$ for some $c > 0$, then $H_v^p \subset H_u^p$ and if $f \in H_v^p$ then $\|f\|_{u,p}^p \leq c\|f\|_{v,p}^p$. Thus by Hopf's lemma the space $H_u^p(\mathbb{D})$ is contained in the classical Hardy space $H^p(\mathbb{D})$.

It has been established (see [1], [9] and [10]) that the boundary measure $\mu_u = \alpha_u \, d\lambda$ for some $\alpha_u \in L^1(\mathbb{T})$ and $f \in H_u^p$ if and only if $f \in H^p$ and $\|f\|_{p,\alpha_u} < \infty$. Moreover, $\|f\|_{u,p} = \|f\|_{\alpha_u,p}$. Hence

$$\int_{\mathbb{T}} |f|^p \, d\mu_u = \int_{\mathbb{D}} |f|^p \, \Delta u - \int_{\mathbb{D}} u \, \Delta |f|^p. \tag{3}$$

The weight α_u has the following properties:

1. $\alpha_u(e^{i\theta}) = \int_{\mathbb{D}} P(z, e^{i\theta})\,\Delta u(z)$, where $P(z, e^{i\theta})$ is the Poisson kernel;
2. $\|\alpha_u\|_{L^1} = 1$ if and only if $u \in \mathcal{E}_1$;
3. $\alpha_u(e^{i\theta})$ is lower semicontinuous and $\alpha_u(e^{i\theta}) \ge c$ on $\mathbb{T}$ for some $c > 0$.

The class of Poletsky–Stessin Hardy spaces is more narrow than weighted spaces discussed in Section 2 just because the weight function α_u must be lower semicontinuous and be greater than some $c > 0$ on $\mathbb{T}$. As the following result shows these are the only restrictions on weights.

THEOREM 3.3. *Let α be a measurable function on $\mathbb{T}$. Then $\alpha\,d\lambda = \mu_u$ for some $u \in \mathcal{E}_1$ if and only if α is lower semicontinuous, $\alpha(e^{i\theta}) \ge c > 0$ on $\mathbb{T}$ and*

$$\int_{\mathbb{T}} \alpha\,d\lambda = 1. \tag{4}$$

Proof. Let $\alpha \in C(\mathbb{T})$ be a function such that $\alpha \ge c > 0$ on $\mathbb{T}$. For $0 < r < 1$ define

$$\alpha_r(e^{i\theta}) = \int_{\mathbb{T}} P(re^{i\theta}, e^{i\varphi})\alpha(e^{i\varphi})\,d\lambda(\varphi).$$

Then $\alpha_r \to \alpha$ uniformly on $\mathbb{T}$ as $r \to 1$. Clearly $\alpha_r \in C^\infty(\mathbb{T})$.

Define

$$u_r(z) = \int \log\left|\frac{z - re^{i\varphi}}{1 - re^{-i\varphi}z}\right|\alpha(e^{i\varphi})\,d\lambda(\varphi).$$

Then u_r is a subharmonic exhaustion function on $\mathbb{D}$ and by the Riesz Decomposition Theorem its Laplacian Δu_r is supported by $\mathbb{T}(r) = \{z = re^{i\phi}\}$ and is equal to $\alpha(e^{i\varphi})\,d\lambda(\varphi)$. Hence

$$\int_{\mathbb{D}} \Delta u_r(z) = \int_{\mathbb{T}} \alpha(e^{i\varphi})\,d\lambda(\varphi).$$

The weight $\alpha_r(e^{i\theta})$ of u_r is equal to

$$\int_{\mathbb{T}} P(re^{i\varphi}, e^{i\theta})\alpha(e^{i\varphi})\,d\lambda(\varphi) = \alpha_r(e^{i\theta}).$$

Hence any $\alpha \in C(\mathbb{T})$ can be uniformly approximated by a function β_u such that $\beta_u\,d\lambda = \mu_u$ and $u \in \mathcal{E}$.

If α is any lower semicontinuous function satisfying (4) and such that $\alpha \ge c > 0$ on $\mathbb{T}$, then α is the pointwise limit of an increasing sequence of continuous functions α_j such that $\alpha_j \ge c/2 > 0$ on $\mathbb{T}$. Replacing α_j with the functions $\alpha_j - 2^{-j}$ we may assume that the function $\beta_j = \alpha_j - \alpha_{j-1} \ge 2^{-j-1}$ on $\mathbb{T}$. (Here we set $\alpha_0 = 0$.) By the argument above we can approximate the functions β_j by continuous functions γ_j such that $\gamma_j \ge 2^{-j-2}$ on $\mathbb{T}$, $\gamma_j\,d\lambda = \mu_{u_j}$ for some $u_j \in \mathcal{E}$ and

$$\sum_{j=1}^{\infty} \gamma_j = \alpha.$$

Let $v_j = \max\{u_j, -2^{-j}\}$. Since for a fixed j the weak-$*$ limits of $\mu_{u_j,r}$ and $\mu_{v_j,r}$ as $r \to 0^-$ coincide we see that $\alpha_{v_j} = \alpha_{u_j} = \gamma_j$. If $v = \sum v_j$ then v is a continuous exhaustion of $\mathbb{D}$ such that $\lim_{|z|\to 1} v(z) = 0$. Moreover,

$$\int_{\mathbb{D}} \Delta v = \sum_{j=1}^{\infty} \int_{\mathbb{D}} \Delta v_j = \sum_{j=1}^{\infty} \int_{\mathbb{T}} \gamma_j = \int_{\mathbb{T}} \alpha = 1.$$

Hence $v \in \mathcal{E}_1$.

Now

$$\int_{\mathbb{D}} P(z, e^{i\theta}) \Delta v(z) = \sum_{j=1}^{\infty} \int_{\mathbb{D}} P(z, e^{i\theta}) \Delta v_j(z) = \sum_{j=1}^{\infty} \gamma_j(e^{i\theta}) = \alpha(e^{i\theta}).$$

Thus $\mu_v = \alpha$.

The converse statements follows from the results [1], [9] and [10] mentioned above. ■

Among the advantages of these spaces comparatively to spaces studied in [2] we can list the following. First of all, one does not need the existence of boundary values or the notion of Smirnov class to introduce these spaces. This is especially attractive for the theory of functions in several variables on non-smooth domains.

Another advantage is the existence of Carleson measures. Given a weight α, a measure ν on the unit disk $\mathbb{D}$ is called *α-Carleson with the constant $C(\alpha)$* if

$$\int_{\mathbb{D}} |f|^p \, d\nu \le C(\alpha) \int_{\mathbb{T}} |f|^p \alpha \, d\lambda$$

for all $p > 1$ and all $f \in H^p_\alpha$. If $\alpha \equiv 1$ then such measures are called *Carleson measures.* In [7] one can find the characterization of α-Carleson measures for α satisfying Muckenhoupt's conditions similar to the classical characterization of Carleson measures by L. Carleson. In the case of Poletsky–Stessin Hardy spaces it follows immediately from (3) that the measure Δu is α_u-Carleson with the constant $C(\alpha_u) = 1$. By Theorem 3.3 we see that α-Carleson measures with constant 1 exist for all lower semicontinuous weights.

Thirdly, the formula (3) helps to obtain additional information. For example, one can get integrability of derivative. Since $\Delta |f|^p = \frac{p^2}{4} |f|^{p-2} |f'|^2$ for all $f \in H^p_u$, $p \ge 1$, we have the inequality

$$\int_{\mathbb{T}} |f|^p \, d\mu_u \ge \frac{p^2}{4} \int_{\mathbb{D}} |u| \, |f|^{p-2} |f'|^2 \, dx \, dy.$$

4. From H^p_u to H^∞. Let $u_1, \dots, u_k$ be exhaustion functions from $\mathcal{E}_1$ and let $u = (u_1, \dots, u_k)$. We say that $u \in \mathcal{E}_1^k$. Let H^p_u to be the direct product $H^p_{u_1} \times \ldots \times H^p_{u_k}$ with the norm

$$\|(f_1, \dots, f_k)\|_{u,p} = \sum_{j=1}^{k} \|f_j\|_{u_j,p}.$$

We will use the notation $(H^p)^k$ and $\|f\|_p$ when $\alpha_{u_1} = \ldots = \alpha_{u_k} = 1$. We denote by $B_{u,p}(r)$ the closed ball of radius r centered at the origin of H^p_u.

The norm on $(H^\infty)^k$ will be defined as

$$\|f\|_\infty = \sum_{j=1}^{k} \|f_j\|_\infty$$

and $B_\infty(r)$ is the closed ball of radius r centered at the origin of $(H^\infty)^k$. Then $B_\infty(r) \subset B_{u,p}(r)$.

THEOREM 4.1. *Let $A \subset (H^p)^k$, $p > 1$, be a closed convex set. Then $A \cap B_\infty(1) \neq \emptyset$ if and only if $A \cap B_{u,p}(1) \neq \emptyset$ for all exhaustion vector-functions $u = (u_1, \dots, u_k) \in \mathcal{E}_1^k$.*

Proof. The "only if" part of the theorem is obvious. The "if" part will be proved by contradiction. Let us take $0 < \varepsilon < 1$ and suppose that $A \cap B_\infty(r_0) = \emptyset$ for $r_0 = (1-\varepsilon)^{-1}$. By the Hahn–Banach theorem there exists $g = (g_1, \dots, g_k) \in (L^q(\mathbb{T}))^k$ such that

$$\sum_{j=1}^k \operatorname{Re} \int_{\mathbb{T}} f_j g_j \, d\lambda \geq 1$$

for all $f \in A$ and

$$\sum_{j=1}^k \operatorname{Re} \int_{\mathbb{T}} f_j g_j \, d\lambda \leq 1$$

for all $f \in B_\infty(r_0)$. Multiplying f_j by appropriate constants a_j with $|a_j| = 1$ we see that

$$\sum_{j=1}^k \Big| \int_{\mathbb{T}} f_j g_j \, d\lambda \Big| \leq r_0^{-1} = 1 - \varepsilon$$

for all $f \in B_\infty(1)$.

Let $\tilde{g}_j(z) = g_j(z)/z$. Then $\tilde{g}_j \in L^q(\mathbb{T}) \subset L^1(\mathbb{T})$ for all j. By a duality result (see [6, VII.2]) there exist $h_j \in H^1$ and $p_j \in H^\infty$ such that $\|p_j\|_\infty = 1$, $p_j(0) = 0$ and

$$(\tilde{g}_j - h_j)p_j = |\tilde{g}_j - h_j|$$

almost everywhere.

We take $f = (f_1, \dots, f_k) \in (H^\infty)^k$ such that $f_i \equiv 0$ when $i \neq j$ and $f_j(z) = p_j(z)/z$. Clearly, $f \in B_\infty(1)$. Therefore

$$1 - \varepsilon \geq \Big| \int_{\mathbb{T}} f_j g_j \, d\lambda \Big| = \Big| \int_{\mathbb{T}} (\tilde{g}_j - h_j) p_j \, d\lambda \Big| = \int_{\mathbb{T}} |\tilde{g}_j - h_j| \, d\lambda.$$

There is $\tilde{h}_j \in H^q$ so that $\|h_j - \tilde{h}_j\|_1 \leq \varepsilon/2$. Let $\phi_j = |\tilde{g}_j - \tilde{h}_j|$. Then

$$\int_{\mathbb{T}} \phi_j \, d\lambda \leq \int_{\mathbb{T}} \big(|\tilde{g}_j - h_j| + |h_j - \tilde{h}_j| \big) \, d\lambda \leq 1 - \varepsilon/2,$$

and for $f \in A$,

$$\sum_{j=1}^k \int_{\mathbb{T}} \phi_j |f_j| \, d\lambda = \sum_{j=1}^k \int_{\mathbb{T}} |(g_j - z\tilde{h}_j) f_j| \, d\lambda \geq \sum_{j=1}^k \Big| \int_{\mathbb{T}} (g_j - z\tilde{h}_j) f_j \, d\lambda \Big| \geq 1.$$

Let $\tilde{\phi}_j = \max\{\phi_j, \varepsilon/4\}$. Then $\|\tilde{\phi}_j\|_1 \leq \|\phi_j + \varepsilon/4\|_1 \leq 1 - \varepsilon/4$. Now for $f \in A$,

$$\sum_{j=1}^k \int_{\mathbb{T}} |f_j| \tilde{\phi}_j \, d\lambda \geq \sum_{j=1}^k \int_{\mathbb{T}} |f_j| \phi_j \, d\lambda \geq 1.$$

For any $\delta > 0$ and $1 \leq j \leq k$ there exists $\psi_j \in C(\mathbb{T})$ such that $\psi_j \geq \varepsilon/8$, $\|\psi_j\|_1 = \|\tilde{\phi}_j\|_1$ and $\|\psi_j - \tilde{\phi}_j\|_q < \delta$.

For $f \in A$,

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \psi_j \, d\lambda \geq \sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \tilde{\phi}_j \, d\lambda - \sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| \, |\psi_j - \tilde{\phi}_j| \, d\lambda \geq 1 - \delta \|f\|_p.$$

By Theorem 3.3 there are exhaustion functions u_j, $1 \leq j \leq k$, such that $\mu_{u_j} = a_j \psi_j$, where a_j is chosen so that $\|a_j \psi_j\|_1 = 1$. Let $u = (u_1, \ldots, u_k)$. Note that $a_j \geq (1-\varepsilon/4)^{-1}$.

If $f \in B_{u,p}(1)$ then

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| a_j \psi_j \, d\lambda \leq \sum_{j=1}^{k} \|f_j\|_{u,p} \|a_j \psi_j\|_1^{1/q} \leq 1$$

and

$$\|f\|_p = \sum_{j=1}^{k} \Big(\int_{\mathbb{T}} |f_j|^p \, d\lambda \Big)^{1/p} \leq \Big(\frac{\varepsilon}{8} \Big)^{-1/p} \sum_{j=1}^{k} \|f_j\|_{u_j,p} \leq \Big(\frac{\varepsilon}{8} \Big)^{-1/p} = c.$$

Thus if $f \in A$ and $\|f\|_p > c$ then $f \notin B_{u,p}(1)$. On the other hand, if $f \in A$ and $\|f\|_p \leq c$, then

$$\sum_{j=1}^{k} \int_{\mathbb{T}} |f_j| a_j \psi_j \, d\lambda \geq (1 - \varepsilon/4)^{-1} (1 - c\delta).$$

Taking $\delta > 0$ so small that $(1-\varepsilon/4)^{-1}(1-c\delta) > 1$ we see that $A \cap B_{u,p}(1) = \emptyset$. Hence $A \cap B_\infty(r_0) \neq \emptyset$ for all $r_0 > 1$.

Let $\{f_n\}$ be a sequence of functions such that $f_n \in A \cap B_\infty(1+1/n)$. We may assume that $\{f_n\}$ converges uniformly on compacta to a function $f \in B_\infty(1)$. This implies that $\{f_n\}$ converges to f weakly. Since any convex closed set is weakly closed we see that $f \in A$. ∎

As the following corollary shows it is possible to use the theorem above when all functions u_j are equal although the constants will change.

Corollary 4.2. *Let $A \subset (H^p)^k$, $p > 1$, be a closed convex set. Suppose $A \cap B_{\mathbf{u},p}(1) \neq \emptyset$ for all exhaustion vector-functions $\mathbf{u} = (u, \ldots, u) \in \mathcal{E}_1^k$. Then $A \cap B_\infty(k) \neq \emptyset$. Conversely, if $A \cap B_\infty(1) \neq \emptyset$ then $A \cap B_{\mathbf{u},p}(1) \neq \emptyset$.*

Proof. Let $v = (v_1, \ldots, v_k) \in \mathcal{E}_1^k$. Let

$$u = \frac{1}{k} \sum_{j=1}^{k} v_j.$$

Then $u \in \mathcal{E}_1$ and by the assumption of the corollary there is $f = (f_1, \ldots, f_k) \in A \cap B_{\mathbf{u},p}(1)$, where $\mathbf{u} = (u, \ldots, u)$. Note that $v_j \geq ku$. By Corollary 3.2 in [8] $\|f_j\|_{v_j,p} \leq k \|f_j\|_{u,p}$, $1 \leq j \leq k$. Hence $f \in B_{v,p}(k)$ and $A \cap B_{v,p}(k) \neq \emptyset$. By Theorem 4.1 $A \cap B_\infty(k) \neq \emptyset$. ∎

5. Interpolation theorem. A sequence $\{z_j\}_1^\infty \subset \mathbb{D}$ is δ-sparse for $\delta > 0$ if

$$\inf_k \prod_{j \neq k} \Big| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \Big| \geq \delta$$

for all k.

A sequence $\{z_j\} \subset \mathbb{D}$ is called *interpolating* if for any sequence $s = \{s_j\} \in l^\infty$ there is a function $f \in H^\infty$ such that $f(z_j) = s_j$ for all j and $\|f\|_{H^\infty} \le C\|s\|_\infty$ and the constant C does not depend on $\|s\|_\infty$.

The famous theorem of Carleson states

THEOREM 5.1. *A sequence $\{z_j\} \subset \mathbb{D}$ is interpolating if and only if it is δ-sparse for some $\delta > 0$.*

Now we can present a shorter proof of the sufficiency part of Theorem 5.1. The proof of necessity is quite elementary and can be found in [4]. Theorem 3.2 in [5] that is a quick consequence of the general characterization of Carleson measures, states that if a sequence $\{z_j\} \subset \mathbb{D}$ is δ-sparse then the measure

$$\nu = \sum_{j=1}^{\infty} (1 - |z_j|^2)\delta_{z_j}$$

is Carleson with a constant C depending only on δ.

We take an integer $N > 1$ and denote by X_N the set of all functions $f \in H^2$ such that $f(z_j) = s_j$, $1 \le j \le N$. Clearly X_N is closed and convex.

Let

$$B(z) = \prod_{j=1}^{N} \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{and} \quad B_k(z) = \prod_{j=1, j\ne k}^{N} \frac{z - z_j}{1 - \bar{z}_j z}, \qquad k = 1, \dots, N.$$

Then any function f in X_N has the form

$$\sum_{j=1}^{N} \frac{s_j}{B_j(z_j)} B_j(z) + B(z)h(z) = \left(\sum_{j=1}^{N} \frac{s_j}{B_j(z_j)} \frac{1 - \bar{z}_j z}{z - z_j} + h(z) \right) B(z),$$

where $h \in H^2$.

We set $C_j = s_j B_j^{-1}(z_j)$ and let

$$\phi(z) = \sum_{j=1}^{N} C_j \, \frac{1 - \bar{z}_j z}{z - z_j} \, .$$

Let $u \in \mathcal{E}_1$ and let $a = a_u$ be the function introduced in Section 2. Then for $g \in H^2$ with $\|g\|_{H^2} = 1$ we have

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} \phi(z) a^{1/2}(z) g(z)\, dz &= \sum_{j=1}^{N} C_j (1 - |z_j|^2) g(z_j) a^{1/2}(z_j) \\ &\le \frac{\|s\|_\infty}{\delta} \int_{\mathbb{D}} |g a^{1/2}|\, d\nu \le \frac{\|s\|_\infty}{\delta} \Big(\int_{\mathbb{D}} |g|^2\, d\nu \Big)^{1/2} \Big(\int_{\mathbb{D}} |a|\, d\nu \Big)^{1/2} \\ &\le \frac{C^2 \|s\|_\infty}{\delta} \, \|g\|_{H^2} \|a^{1/2}\|_{H^2} = \frac{C^2 \|s\|_\infty}{\delta} = C' \|s\|_\infty. \end{aligned}$$

Hence by (1) $\operatorname{dist}(\phi, H^2_u) \le C'\|s\|_\infty$ and this means that $X_N \cap B_{u,2}(C'\|s\|_\infty) \ne \emptyset$. Thus by Theorem 4.1 there is $f_N \in X_N \cap B_\infty(C'\|s\|_\infty)$. Since C' does not depend on N there is $f \in B_\infty(C'\|s\|_\infty)$ interpolating s.

6. Corona theorem. The same method can be applied to the corona theorem.

THEOREM 6.1. *If the functions $f_1, \dots, f_n$ are in the unit ball of H^∞ and*

$$\sum_{j=1}^{n} |f_j|^2 \ge \delta > 0,$$

then there are functions $g_1, \dots, g_n$ in H^∞ such that

$$\sum_{j=1}^{n} f_j g_j = 1 \tag{5}$$

and $\|g_j\| \le C$, where C depends only on δ.

We will discuss only the case when $n = 2$. It suffices to prove this theorem for functions f_j that can be continuously extended to $\overline{\mathbb{D}}$ and have finitely many zeros in $\overline{\mathbb{D}}$. In this case one can easily find functions ϕ_1 and ϕ_2 smooth up to the boundary such that

$$f_1\phi_1 + f_2\phi_2 = 1.$$

To make them holomorphic we look for a function v such that

$$\bar\partial(\phi_1 + f_2 v) = \bar\partial\phi + f_2\,\bar\partial v = 0$$

and

$$\bar\partial(\phi_2 - f_1 v) = \bar\partial\phi - f_1\,\bar\partial v = 0.$$

Since $f_1\,\bar\partial\phi_1 + f_2\,\bar\partial\phi_2 = 0$ we see that

$$\bar\partial v = f_1^{-1}\,\bar\partial\phi_2 = -f_2^{-1}\,\bar\partial\phi_1 =: \psi.$$

The following lemma can be found in [4].

LEMMA 6.2. *There are solutions ϕ_1 and ϕ_2 to* (5) *continuous up to the boundary such that the measure $\nu = |\psi|\,dz\,d\bar z$ is Carleson with constant C depending only on δ and $|\phi_1| + |\phi_2| \le K(\delta)$.*

Proof. Let

$$\Psi(z) = \int_{\mathbb{D}} \frac{\psi(\zeta)}{\zeta - z}\,d\zeta\,d\bar\zeta.$$

Then $\bar\partial\Psi = \psi$ and for any $u \in \mathcal{E}_1$

$$\Big|\int_{\mathbb{T}} \Psi(z)a_u(z)g(z)\,dz\Big|^2 = \Big|\int_{\mathbb{D}} \psi(\zeta)a_u(\zeta)g(\zeta)\,d\zeta\,d\bar\zeta\Big|^2$$
$$\le \int_{\mathbb{D}} |a_u(\zeta)|\psi(\zeta)\,d\zeta\,d\bar\zeta \int_{\mathbb{D}} |\psi(\zeta)|\,|a_u(\zeta)g^2(\zeta)|\,d\zeta\,d\bar\zeta \le C^2\|g\|_{u,2}^2.$$

Thus by Theorem 2.1 $\operatorname{dist}(\Psi, H_u^2) \le C$. Hence there is $v = \Psi + h$ such that $h \in H_u^2$ and $\|h\|_{H_u^2} \le C$. Therefore the function $h_1 = \phi_1 + f_2 v$ is holomorphic, lies in H_u^2 and $\|h_1\|_{u,2} \le K(\delta) + C = R$. The same estimate holds for the function $h_2 = \phi_2 - f_1 v$.

Thus if $A \in (H^2)^2$ is the set of all solutions (g_1, g_2) to (5), then $A \cap B_{u,2}(R) \ne \emptyset$ for all pairs (u, u), where $u \in \mathcal{E}_1$. Since the set A is convex and closed, by Corollary 4.2 $A \cap B_\infty(2R) \ne \emptyset$. This ends the proof. ∎

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