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# MONGE–AMPÈRE MEASURES AND POLETSKY–STESSIN HARDY SPACES ON BOUNDED HYPERCONVEX DOMAINS

SİBEL ŞAHİN

*Faculty of Engineering, Natural and Mathematical Sciences Department, Özyeğin University*
*E-mail: sahinsibel@sabanciuniv.edu*

**Abstract.** Poletsky–Stessin Hardy (PS–Hardy) spaces are the natural generalizations of classical Hardy spaces of the unit disc to general bounded, hyperconvex domains. On a bounded hyperconvex domain $\Omega$, the PS–Hardy space $H^p_u(\Omega)$ is generated by a continuous, negative, plurisubharmonic exhaustion function $u$ of the domain. Poletsky and Stessin considered the general properties of these spaces and mainly concentrated on the spaces $H^p_u(\Omega)$ where the Monge–Ampère measure $(dd^cu)^n$ has compact support for the associated exhaustion function $u$. In this study we consider PS–Hardy spaces in two different settings. In one variable case we examine PS–Hardy spaces that are generated by exhaustion functions with finite Monge–Ampère mass but $(dd^cu)^n$ does not necessarily have compact support. For $n > 1$, we focus on PS–Hardy spaces of complex ellipsoids which are generated by specific exhaustion functions. In both cases we will give results regarding the boundary value characterization and polynomial approximation.

**1. Introduction.** In their work of 2008, Poletsky and Stessin introduced Poletsky–Stessin Hardy spaces $H^p_u(\Omega)$ on hyperconvex domains and they unified the different definitions made for Hardy spaces for various domains. In this setting $\Omega$ is a hyperconvex domain and $u$ is a continuous, negative, plurisubharmonic exhaustion function for $\Omega$ which has finite Monge–Ampère mass and the growth condition is constructed using the Monge–Ampère measures defined in [D1]. The general framework of [PS] is based on the examination of PS–Hardy spaces when the exhaustion function $u$ belongs to a special class $\mathcal{E}_0$, i.e. the measure $(dd^cu)^n$ has compact support. In the first part of this paper we will examine the PS–Hardy spaces in a setting where $\Omega$ is a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve and $u$ is a continuous, negative,

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subharmonic exhaustion function for $\Omega$ which has finite Monge–Ampère mass. Different from Poletsky & Stessin's work, for the exhaustion function $u$, the measure $(dd^c u)^n$ does not necessarily have compact support. One of the main consequences of this choice of exhaustion function is that we can obtain PS–Hardy spaces which do not coincide with the classical ones so we have new Banach spaces to be explored inside the classical Hardy spaces. In this part we characterize PS–Hardy spaces through their boundary values and the corresponding Monge–Ampère boundary measure, and then using functional analysis techniques we obtain polynomial approximation in these spaces. In the second part we study PS–Hardy spaces on complex ellipsoids in $\mathbb{C}^n$, $n > 1$. Different from one variable case, classical Hardy spaces are strictly contained in Poletsky–Stessin Hardy spaces on complex ellipsoids so boundary values are not automatically obtained in this case. We have shown that functions belonging to Poletsky–Stessin Hardy spaces have boundary values also in the complex ellipsoid case. Moreover, we have obtained polynomial approximation in these spaces.

**2. Preliminaries and results.** In this section, we will give the preliminary definitions and some important results. Before proceeding with Poletsky–Stessin Hardy spaces let us first give the appropriate setting:

DEFINITION 1. A connected open subset $\Omega$ of $\mathbb{C}^n$ is called *hyperconvex* if there exists a plurisubharmonic function $g : \Omega \to [-\infty, 0)$ such that $\{z \in \Omega : g(z) < c\}$ is relatively compact for each $c < 0$. Here $g$ is called *an exhaustion* for $\Omega$.

Let us also give a special type of exhaustion functions, namely pluricomplex Green functions:

DEFINITION 2. A *pluricomplex Green function* of $\Omega \subset \mathbb{C}^n$ is defined as

$$g_\Omega(z, w) = \sup u(z)$$

where $u \in \mathrm{PSH}(\Omega)$ (including $u \equiv -\infty$), $u$ is non-positive and the function

$$t \mapsto u(t) - \log |t - w|$$

is bounded from above in a neighborhood of $w$. Pluricomplex Green function $g_\Omega(z, w)$ is a negative plurisubharmonic function with a logarithmic pole at $w$.

In [PS], Poletsky & Stessin introduced new Hardy type classes of holomorphic functions on hyperconvex domains in $\mathbb{C}^n$. Before defining these new classes let us first give some preliminary definitions. Let $\varphi : \Omega \to [-\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion function for $\Omega$. Following [D1] we define the pseudoball:

$$B(r) = \{z \in \Omega : \varphi(z) < r\},\ r \in [-\infty, 0),$$

and pseudosphere:

$$S(r) = \{z \in \Omega : \varphi(z) = r\},\ r \in [-\infty, 0),$$

and set

$$\varphi_r(z) = \max\{\varphi(z), r\}, \quad r \in (-\infty, 0).$$

In [D1], Demailly introduced the Monge–Ampère measures in the sense of currents as:

$$\mu_{\varphi,r} = (dd^c\varphi_r)^n - \chi_{\Omega\setminus B(r)}\,(dd^c\varphi)^n, \quad r \in (-\infty, 0).$$

It is clear from the definition that these measures are supported on $S(r)$. Demailly in [D2] proved the so-called Lelong–Jensen formula which we use throughout the sequel. Lelong–Jensen formula is stated as follows:

THEOREM 2.1. *Let $r < 0$ and $\phi$ be a plurisubharmonic function on $\Omega$ then for any negative, continuous, plurisubharmonic exhaustion function $u$*

$$\int_{S_u(r)} \phi\, d\mu_{u,r} - \int_{B_u(r)} \phi\,(dd^c u)^n = \int_{B_u(r)} (r-u)\, dd^c\,\phi\,(dd^c u)^{n-1}. \tag{1}$$

Now we can introduce the Poletsky–Stessin Hardy classes, which will be our main focus throughout this study. In [PS], Poletsky & Stessin gave the definition of new Hardy spaces using Monge–Ampére measures as:

DEFINITION 3. $H^p_\varphi(\Omega)$ for $p > 0$, is the space of functions $f \in \mathcal{O}(\Omega)$ such that

$$\limsup_{r\to 0^-} \int_{S_{\varphi},(r)} |f|^p\, d\mu_{\varphi,r} < \infty.$$

The norm on these spaces is given by

$$\|f\|_{H^p_\varphi} = \Big(\lim_{r\to 0^-} \int_{S_\varphi(r)} |f|^p\, d\mu_{\varphi,r}\Big)^{1/p}$$

and with respect to these norm the spaces $H^p_\varphi(\Omega)$ are Banach spaces [PS]. When the exhaustion function is chosen as the pluricomplex Green function, the PS–Hardy classes coincide with the classical Hardy spaces in the unit disc, polydisc and unitball cases.

Now let us continue with the one dimensional results concerning PS–Hardy spaces.

Recall that the classical Hardy space $H^p(\Omega)$ is defined as follows:

$$H^p(\Omega) = \{f \in \mathcal{O}(\Omega) : |f|^p \textit{ has a harmonic majorant in } \Omega\} \tag{2}$$

where $\mathcal{O}(\Omega)$ denotes the space of holomorphic function on $\Omega$. Next, we compare the PS–Hardy spaces with the classical Hardy spaces:

THEOREM 2.2. *Let $\Omega$ be a domain in $\mathbb{C}$ that contains $0$ and is bounded by an analytic Jordan curve. Suppose $\varphi$ is a continuous, negative, subharmonic exhaustion function for $\Omega$ such that $\varphi$ is harmonic outside of a compact set $K \subset \Omega$. Then for a holomorphic function $f \in \mathcal{O}(\Omega)$, $f \in H^p_\varphi(\Omega)$ if and only if $|f|^p$ has a harmonic majorant.*

When an exhaustion function $u$ has finite Monge–Ampère mass then by Theorem 3.1 in [PS] we have $H^p_u(\Omega) \subset H^p_{g_\Omega}(\Omega) = H^p(\Omega)$, however by explicitly constructing an exhaustion function $u$ on the unit disc $\mathbb{D}$, we showed that $H^p_u(\Omega)$ need not be equal to $H^p(\Omega)$:

THEOREM 2.3. *There exists an exhaustion function $u$ with finite Monge–Ampère mass such that the Hardy space $H^p_u(\mathbb{D}) \subsetneq H^p(\mathbb{D})$.*

As a consequence of this result we see that there are new Banach spaces to be explored inside the classical Hardy spaces of $\Omega \subset \mathbb{C}$.

One of the main concerns of this study is to understand the boundary behavior of Poletsky–Stessin Hardy spaces. For this we also need boundary measures which were introduced by Demailly in [D2]. Now let $\varphi : \Omega \to [-\infty, 0)$ be a continuous, plurisubharmonic exhaustion for $\Omega$ and suppose that the total Monge–Ampère mass is finite that is, we assume that

$$MA(\varphi) = \int_\Omega (dd^c\varphi)^n < \infty. \tag{3}$$

Then as $r$ approaches to 0, $\mu_{\varphi,r}$ converges to a positive measure $\mu_\varphi$ weak*-ly on $\Omega$ with total mass $\int_\Omega (dd^c\varphi)^n$ and supported on $\partial\Omega$. This measure $\mu_\varphi$ is called the *Monge–Ampère measure on the boundary associated with the exhaustion* $\varphi$.

Let $\Omega$ be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve and $u$ be a continuous, negative, subharmonic exhaustion function for $\Omega$ with finite Monge–Ampère mass. In the classical Hardy space theory on the unit disc $\mathbb{D}$ we can characterize the $H^p$ spaces through their boundary values inside the $L^p$ spaces of the unit circle. Since we have $H^p_u(\Omega) \subset H^p(\Omega)$, any holomorphic function $f \in H^p_u(\Omega)$ has the boundary value function $f^*$ from the classical theory ([ES], Theorem 10). In this section we will give an analogous characterization of the Poletsky–Stessin Hardy spaces through these boundary value functions and boundary Monge–Ampère measure. Now we will give the characterization of Poletsky–Stessin Hardy spaces $H^p_u(\Omega)$ through boundary value functions:

Theorem 2.4. *Let $f \in H^p(\Omega)$ be a holomorphic function and $u$ be a continuous, negative, subharmonic exhaustion function for $\Omega$. Then $f \in H^p_u(\Omega)$ if and only if $f^* \in L^p(d\mu_u)$ for $1 \le p < \infty$. Moreover $\|f^*\|_{L^p(d\mu_u)} = \|f\|_{H^p_u(\Omega)}$.*

Now we continue with the polynomial approximation in one dimension. Let $A(\Omega)$ denote the algebra of holomorphic functions on $\Omega$ which are continuous on $\partial\Omega$. We know that the algebra of holomorphic functions $A(\Omega)$ is dense in the classical Hardy spaces when $\Omega$ is a domain bounded by an analytic Jordan curve. Using functional analysis techniques we have shown the following result about the approximation of PS–Hardy classes $H^p_u(\Omega)$ where $u$ is a negative, continuous, subharmonic exhaustion function on $\Omega$ with finite Monge–Ampère mass:

Theorem 2.5. *The algebra $A(\Omega)$ is dense in $H^p_u(\Omega)$, $1 \le p < \infty$.*

Moreover, we know from Mergelyan's Approximation Theorem that the algebra $A(\Omega)$ can be uniformly approximated by polynomials therefore, we have the following corollary:

Corollary 2.1. *Polynomials are dense in $H^p_u(\Omega)$, $1 \le p < \infty$.*

Now we will continue with the $n > 1$ case, let us first recall the classical Hardy spaces given by [ES]. Let $\Omega$ be a smoothly bounded, hyperconvex domain in $\mathbb{C}^n$ and $\lambda$ be a characterizing function for $\Omega$ which is defined in a neighborhood of $\overline{\Omega}$, i.e. $\lambda$ is smooth, $\lambda(x) < 0$ if and only if $x \in \Omega$, $\partial\Omega = \{\lambda(x) = 0\}$ and $|\nabla\lambda(x)| > 0$ if $x \in \partial\Omega$. (The last condition is equivalent to $\frac{\partial\lambda}{\partial\nu_x} > 0$ where $\nu_x$ is the outward normal at $x$.) Let $\Omega_r = \{z : \lambda(z) < r, r < 0\}$ and $\partial\Omega_r = \{z : \lambda(z) = r\}$.

In [ES], E. M. Stein defines the class $H^p$ as

$$H^p(\Omega) \doteq \left\{ f : f \text{ holomorphic in } \Omega,\ \sup_{r<0} \int_{\partial\Omega_r} |f|^p \, d\sigma_r < \infty \right\}$$

where $d\sigma_r$ is the induced surface area measure on $\partial\Omega_r$. This space is equipped with the norm

$$\|f\|_p^p = \sup_{r<0} \int_{\partial\Omega_r} |f|^p \, d\sigma_r.$$

The space $H^p(\Omega)$ does not depend on the characterizing function used to define $\Omega$ and one gets equivalent norms for different characterizing functions. In this multidimensional case, we will focus on Poletsky–Stessin Hardy spaces on the complex ellipsoids in $\mathbb{C}^n$ which are considered as models for the domains of finite type. It should be noted that although complex ellipsoids are convex domains they are not strictly pseudoconvex since they have Levi flat points at the boundary. The complex ellipsoid $\mathbb{B}^{\mathbf{p}} \in \mathbb{C}^n$ is given as

$$\mathbb{B}^{\mathbf{p}} = \Big\{ z \in \mathbb{C}^n : \rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1 < 0 \Big\}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (\mathbb{Z}^n)^+$. One can easily see that $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \ldots + |z_n|^{2p_n})$ is a continuous, plurisubharmonic exhaustion function for $\mathbb{B}^{\mathbf{p}}$ so we can consider the Poletsky–Stessin Hardy spaces $H_u^p(\mathbb{B}^{\mathbf{p}})$ associated with this exhaustion function. We now have two Hardy type spaces on $\mathbb{B}^{\mathbf{p}}$, the first one is the Poletsky–Stessin Hardy space $H_u^p(\mathbb{B}^{\mathbf{p}})$ and the other one is $H^p(\mathbb{B}^{\mathbf{p}})$ which is defined with respect to surface area measure in accordance with Stein's definition. We will now show that these spaces are not equal. In fact, contrary to the one variable case Poletsky–Stessin Hardy class strictly contains the classical Hardy space.

PROPOSITION 2.1. *Let $\mathbb{B}^{\mathbf{p}}$ be the complex ellipsoid. Then there exists an exhaustion function $u$ such that $H^1(\mathbb{B}^{\mathbf{p}}) \subsetneq H_u^1(\mathbb{B}^{\mathbf{p}})$.*

As we have seen from the previous result, Poletsky–Stessin Hardy spaces $H_u^p(\mathbb{B}^{\mathbf{p}})$ are not included in the classical Hardy spaces $H^p(\mathbb{B}^{\mathbf{p}})$ on complex ellipsoids. Hence in this case we do not automatically inherit the existence of boundary values from the theory of classical Hardy spaces. Hence, we show the existence of the radial limits for holomorphic functions in $H_u^p(\mathbb{B}^{\mathbf{p}})$, $p \geq 1$:

THEOREM 2.6. *Let $f \in H_u^p(\mathbb{B}^{\mathbf{p}})$ be a holomorphic function. Then the radial limit function $f^*(\xi) = \lim_{\tilde{r}\to 1} f(\tilde{r}\xi)$, $\xi \in \partial\mathbb{B}^{\mathbf{p}}$ exists $\mu_u$-almost everywhere and $f^* \in L^p_{\mu_u}(\partial\mathbb{B}^{\mathbf{p}})$, $p \geq 1$.*

In the previous result we have shown that for the functions in the Poletsky–Stessin Hardy class $H_u^p(\mathbb{B}^{\mathbf{p}})$ we have the radial limit values and throughout the following arguments we will study the behavior of these boundary values in detail. In the classical Hardy space theory on strictly pseudoconvex domains, Stein showed the existence of boundary values along admissible approach regions that are more general than the radial approach. In [S1] and [S3], we showed that for the functions in the Poletsky–Stessin Hardy class $H_u^p(\mathbb{B}^{\mathbf{p}})$ boundary values along admissible approach regions exist. Although we use the general idea in Stein's classical method, our approach differs in two aspects, respectively the use of Cauchy–Fantappie kernel instead of Poisson kernel and the use of radial limits. In the study of the boundary behavior of holomorphic functions, having the boundary of the domain as a space of homogenous type seems to be a leitmotif because one of the most commonly used methods in order to understand boundary behavior is to use maximal functions ([ES], Theorem 3) and the natural setting for this type of analysis is homogenous spaces. Therefore we will start with recalling the properties of homogenous spaces and then as an application of this classical method we will show that polynomials are dense in the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^\mathbf{p})$ on complex ellipsoids. Before proceeding our arguments in $\mathbb{C}^n$ with maximal functions, let us first mention the spaces of homogenous type in $\mathbb{C}^n$:

Definition 4. Suppose that we are given a space $X$ which is equipped with a quasi-metric $\rho$ ([K], p. 145) and a regular Borel measure $\mu$ on $X$. Denote the balls in this quasi-metric by $B(x,r) = \{y \in X : \rho(x,y) < r\}$. We say that $(X,\rho,\mu)$ is a space of homogenous type if the following conditions are satisfied:

- For each $x \in X$ and $r > 0$, $0 < \mu(B(x,r)) < \infty$.
- (*Doubling Condition*) There is a constant $C_2 > 0$ such that for any $x \in X$ and $r > 0$ we have $\mu(B(x,2r)) \leq C_2\mu(B(x,r))$.

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain such that we have a quasi-metric $\rho$ on $\overline{\Omega}$ and a regular Borel measure $\mu$ on $\partial\Omega$. Let $K(z,\xi) : \Omega \times \partial\Omega \to \mathbb{C}$ be a kernel such that $K(z,\xi) \in L^1(d\mu)$ for $z \in \Omega$, $\xi \in \partial\Omega$. Let us consider the integral operator determined by $K(z,\xi)$ for an $L^p(d\mu)$ function $f^*$,

$$Kf^*(z) = \int_{\partial\Omega} f^*(\xi)K(z,\xi)\,d\mu(\xi)$$

and define the associated maximal function as

$$Mf^*(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu(B(\xi,\varepsilon))} \int_{B(\xi,\varepsilon)} |f^*|\,d\mu.$$

From the corresponding results in literature (see e.g. [ES], Theorem 2; [AZ], Chapter 14) the fundamental theorem of the theory of singular operators which is adopted to our setting can be stated as:

Theorem 2.7. *Suppose $f^* \in L^p(d\mu_u)$ and $1 \leq p \leq \infty$. Then*

(a) $\|Mf^*\|_p \leq A_p\|f^*\|_p$ *for* $1 < p \leq \infty$.
(b) *The mapping $f^* \to Mf^*$ is of weak type* (1-1), *i.e.* $\mu_u\{\xi : Mf^*(\xi) > \alpha\} \leq \frac{K}{\alpha}\|f^*\|_1$ *if* $f^* \in L^1(d\mu_u)$.

Now we further suppose that the following conditions are satisfied:

- $\rho$ is a quasi-metric on $\overline{\Omega}$,
- $(\partial\Omega,\rho,\mu)$ is a space of homogenous type,
- for all $z \in \Omega$, $\xi \in \partial\Omega$ with $\eta = \rho(z,\xi) > 0$ we have
$$|K(z,\xi)| \leq C\,\frac{1}{\mu(B(\xi,\eta))}$$
for some $C$ independent of $\xi$ and $\eta$ and dependence of $C$ to the point $z$ is given as in ([H], (3.3)). Such a kernel is called a standard kernel.

Following the method given in ([ES], Theorem 3), which was applied for the Poisson integrals of $L^p$ functions, we can estimate the integral operator given above in this general setting.

Theorem 2.8. *Suppose $Kf^*(z)$ is the $K(z,\xi)$-integral of an $L^p(d\mu)$ function $f^*$ where $K(z,\xi)$ satisfies the conditions given above. Let $Q_\alpha(y) = \{z \in \overline{\Omega} : \rho(y,z) < \alpha\delta_y(z)\}$ for $y \in \partial\Omega$, $z \in \Omega$ with $\delta_y(z) = \min\{\rho(z,\partial\Omega), \rho(z,T_y)\}$ ($T_y$ is the tangent plane at $y$), $\alpha > 0$, be the admissible approach region. Then*

- *When $\rho(y,z) = \varepsilon$ and $z \in Q_\alpha(y)$ the following inequality holds*
$$|Kf^*(z)| \le \tilde{A}\sum_{k=1}^{\infty}\big(\mu(B(y,2^k\varepsilon))\big)^{-1}\int_{B(y,2^k\varepsilon)}|f^*|\,d\mu,$$
- $\sup_{z\in Q_\alpha(y)}|Kf^*(z)| \le \tilde{A}Mf^*(y)$.

In [H], Hansson considered the boundedness of Cauchy–Fantappie integral operator, $H$, from $L^2_u(\partial\mathbb{B}^{\mathbf{p}})$ into $H^2_u(\mathbb{B}^{\mathbf{p}})$. In his work he applied an operator theory result known as $T1$-Theorem. In that result he showed the homogeneity of the boundary of the complex ellipsoid with respect to the quasi-metric $d$ and the boundary measure $\partial\rho\wedge(\overline{\partial}\partial\rho)^{n-1}$ where the function $\rho$ is defined as $\rho(z) = \sum_{j=1}^n |z_j|^{2p_j} - 1$. In fact an easy calculation shows that this measure is the boundary Monge–Ampère measure associated with the exhaustion function $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \ldots + |z_n|^{2p_n})$, $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (\mathbb{Z}^n)^+$ of the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$. Now let $d(\xi,z) \doteq |v(\xi,z)| + |v(z,\xi)|$ be the quasi-metric defined on $\overline{\mathbb{B}^{\mathbf{p}}}$ where $v(\xi,z) = \langle\partial\rho(\xi), \xi - z\rangle$. Then explicitly $v(\xi,z) = \sum_{j=1}^n p_j|\xi_j|^{2(p_j-1)}\bar{\xi_j}(\xi_j - z_j)$ and define the boundary balls as $B(z,\varepsilon) = \{\xi \in \partial\mathbb{B}^{\mathbf{p}} : d(\xi,z) < \varepsilon\}$. It is shown that $(\partial\mathbb{B}^{\mathbf{p}}, d, d\mu_u)$ is a space of homogenous type ([H], p. 1483) and $\frac{1}{(v(\xi,z))^n}$ is a standard kernel. In the following argument we will use this homogeneity result to apply the previous rather general procedure on the complex ellipsoid case with the so called Cauchy–Fantappie kernel:

The Cauchy–Fantappie integral (from now on we will refer as CF-integral) of an $L^p(d\mu_u)$ function $f^*$ is defined as
$$Hf(z) = \Big(\frac{1}{2\pi i}\Big)^n\int_{\partial\mathbb{B}^{\mathbf{p}}}\frac{f^*(\xi)\,d\mu_u(\xi)}{(v(\xi,z))^n}\,.$$
Before proceeding to further results let us briefly discuss the Cauchy–Fantappie kernel. In the theory of holomorphic functions in one variable a fundamental tool is Cauchy integral formula and in the case of several variables one wants a suitable generalization to Cauchy integral. One of the possible choices for the generalization is the so called Szegö kernel however except for a few domains Szegö kernel has no explicit formulation. One other choice is the well known Bochner–Martinelli kernel but the major shortcoming of this kernel is that it is not holomorphic in $z$ variable (for details see [R]). Contrary to Bochner–Martinelli kernel, Cauchy–Fantappie kernel is holomorphic in $z$ hence it is a natural generalization of Cauchy kernel to multivariable case and it has the reproducing property for the functions in the algebra $A(\mathbb{B}^{\mathbf{p}})$ ([R], Theorem 3.4). Hardy spaces which are examined in [H] are exactly the Poletsky–Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ that are generated by the exhaustion function $u$. At the beginning of this section it is shown that for the functions in $H^p_u(\mathbb{B}^{\mathbf{p}})$ the boundary value function $f^* \in L^p(d\mu_u)$ exists so the CF-integral of $f^*$ is well-defined. We show that CF-integral has reproducing property for the functions in $H^p_u(\mathbb{B}^{\mathbf{p}})$:

Proposition 2.2. *Let $f \in H^p_u(\mathbb{B}^p)$ be a holomorphic function, then*

$$f(z) = Hf(z) = \Big(\frac{1}{2\pi i}\Big)^n \int_{\partial \mathbb{B}^p} \frac{f^*(\xi)\, d\mu_u(\xi)}{(v(\xi,z))^n}.$$

Now define the maximal function for the functions in $L^p(d\mu_u)$ as follows:

$$Mf^*(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu_u(B(\xi,\varepsilon))} \int_{B(\xi,\varepsilon)} |f^*|\, d\mu_u.$$

The next result is a consequence of the general method for complex ellipsoid case and it gives the relation between the CF-integral and the maximal function of an $L^p(d\mu_u)$ function $f^*$:

Corollary 2.2. *Suppose $Hf(z)$ is the CF-integral of an $L^p(d\mu_u)$ function $f^*$. Let*

$$Q_\alpha(y) = \{z \in \overline{\mathbb{B}^p} : |v(y,z)| < \alpha \delta_y(z)\}$$

*for $y \in \partial \mathbb{B}^p$, $z \in \mathbb{B}^p$ with $\delta_y(z) = \min\{d(z,\partial X) : d(z,T_y)\}$ ($T_y$ is the tangent plane at $y$), $\alpha > 0$, be the admissible approach region. Then*

- *When $d(y,z) = \varepsilon$ and $z \in Q_\alpha(y)$ the following inequality holds*

$$|Hf(z)| \le \tilde{A} \sum_{k=1}^{\infty} \big(\mu_u(B(y,2^k\varepsilon))\big)^{-1} \int_{B(y,2^k\varepsilon)} |f^*|\, d\mu_u$$

- $\sup_{z\in Q_\alpha(y)} |Hf(z)| \le \tilde{A} Mf^*(y)$.

Next, as a very important application of these maximal function tools we see an approximation result on the Poletsky–Stessin Hardy spaces analogous to the one variable case:

Theorem 2.9. *Polynomials are dense in $H^p_u(\mathbb{B}^p)$.*

**3. Conclusion and comments.** In [S1], [S2] we have shown some results concerning the boundedness of composition operators in the unit disc case and in the most general hyperconvex domain case. In [S3] we have proved that in the complex ellipsoid case, the boundedness and compactness of composition operators are determined by the conditions which are known as Carleson conditions in the literature. Now in this last part we will briefly mention these results.

Let us first start with the unit disc case where the boundedness of composition operators is characterized through generalized Nevanlinna functions [PS]:

Notation. Let $\psi$ be a continuous, subharmonic exhaustion function for $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function then the generalized Nevanlinna function on unit disc $\mathbb{D}$ is given as

$$N^\varphi_\psi(w) = \int_{\mathbb{D}} (-\psi)\, dd^c \log|\varphi - w|.$$

Theorem 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\varphi(0) = 0$ and suppose that $\psi$ is a continuous, subharmonic exhaustion function for $\mathbb{D}$. If $\int_{\mathbb{D}} \frac{1}{(1-|\varphi|^2)^{p/2}}\, dd^c\psi < \infty$ and $\limsup_{|w|\to 1} \frac{N^\varphi_\psi(w)}{-\psi(w)} < \infty$ then $C_\varphi$ is bounded on $H^p_\psi(\mathbb{D})$, $p \ge 1$.*

Next, we will continue with the complex ellipsoid case where the boundedness and compactness criteria is given with respect to conditions known as Carleson conditions. For this we need some notation.

For a positive constant $\varepsilon > 0$ we set the balls as follows:

$$Q(\xi,\varepsilon) = \{z \in \overline{\mathbb{B}^{\mathbf{p}}} : d(z,\xi) < \varepsilon\},$$
$$B(\xi,\varepsilon) = Q(\xi,\varepsilon) \cap \partial\mathbb{B}^{\mathbf{p}}.$$

Theorem 3.2 (boundedness condition). *Let $\phi : \mathbb{B}^{\mathbf{p}} \to \mathbb{B}^{\mathbf{p}}$ be a holomorphic self map of $\mathbb{B}^{\mathbf{p}}$. For $1 \le p < \infty$, the composition operator $C_\phi(f) = f\circ\phi$ is bounded on $H^p_u(\mathbb{B}^{\mathbf{p}})$ if and only if $\mu(Q(\xi,\varepsilon)) \le C\mu_u(B(\xi,\varepsilon))$ for all $\xi \in \partial\mathbb{B}^{\mathbf{p}}$ and $\varepsilon > 0$ where $\mu(E) = \mu_u((\phi)^{-1}(E))$ for all measurable $E \subset \overline{\mathbb{B}^{\mathbf{p}}}$.*

For the compactness of composition operators:

Theorem 3.3 (compactness condition). *The composition operator $C_\phi(f) = f \circ \phi$ is compact on $H^p_u(\mathbb{B}^{\mathbf{p}})$ if and only if $\mu(Q(\xi,\varepsilon)) = o\big(\mu_u(B(\xi,\varepsilon))\big)$ as $\varepsilon \to 0$ uniformly on $\xi \in \partial\mathbb{B}^{\mathbf{p}}$.*

Lastly, let us mention the boundedness of compositions operators in the most general case of hyperconvex domains. For this, recall the following:

The Perron–Bremermann envelope for a given function $f : \partial\Omega \to \mathbb{R}$ is given by

$$PB_f(z) = \sup\big\{\omega(z) : \omega \in \mathrm{PSH}(\Omega),\ \limsup_{v\to\xi, v\in\Omega} \omega(v) \le f(\xi),\ \forall\xi \in \partial\Omega\big\}.$$

Definition 5. A continuous function $f : \partial\Omega \to \mathbb{R}$ which satisfies the following two conditions is called a *compliant function*:

- $\lim_{z\to\xi, z\in\Omega}(PB_f + PB_{-f})(z) = 0$ for every $\xi \in \Omega$,
- $\int_\Omega (dd^c(PB_f + PB_{-f}))^n < \infty$.

The set of all compliant functions is denoted by $\mathcal{CP}(\partial\Omega)$ and the set of functions for which $PB_{-f} = -PB_f$ is denoted by $\mathcal{CP}_0(\partial\Omega)$.

Now we have the following result:

Theorem 3.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded hyperconvex domain. Suppose $u$ is a continuous, negative, plurisubharmonic exhaustion function with finite Monge–Ampère mass and $\varphi : \overline{\Omega} \to \Omega$ is a one-to-one holomorphic self map of $\Omega$. If $u\circ\varphi \in \mathcal{CP}_0(\partial\Omega)$ then there exists a continuous exhaustion function $\psi$ with finite mass such that $C_\varphi : H^p_u(\Omega) \to H^p_\psi(\Omega)$ is continuous for $1 \le p < \infty$.*

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