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Global exponential stability of positive periodic solutions for an epidemic model with saturated treatment

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Abstract. This paper is concerned with an SIR model with periodic incidence rate and saturated treatment function. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential stability of positive periodic solutions for this model. The result obtained improves and supplements existing ones. We also use numerical simulations to illustrate our theoretical results.

1. Introduction. In [LBJ], Li et al. proposed the following system of differential equations:

$$(1.1)\qquad \begin{cases} S'(t) = A - \mathbf{d}S(t) - \dfrac{\lambda(t)S(t)I(t)}{1+kI(t)}, \\ I'(t) = \dfrac{\lambda(t)S(t)I(t)}{1+kI(t)} - (\mathbf{d}+\varepsilon+\mu)I(t) - \dfrac{\gamma I(t)}{1+\alpha I(t)}, \\ R'(t) = \mu I(t) + \dfrac{\gamma I(t)}{1+\alpha I(t)} - \mathbf{d}R(t), \end{cases}$$

to describe the dynamics of an SIR model with periodic incidence rate and saturated treatment function. Here S represents the number of individuals susceptible to the disease, I represents the number of infected individuals infectious and able to spread the disease by contacting with the susceptibles, and R is the number of the infectives removed or recovered. Moreover, the contact rate $\lambda(t)$ is $a+\eta\,\mathbf{term}(t)$ and $\mathbf{term}(t) = \sin(\pi t/6)$ is a periodic function, and a, η and the other parameters are positive. The interpretations and values of parameters are described in of [LBJ, Table 1]. The detailed biological explanations of the parameters of (1.1) can be found in [AS, GF, KRG].

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For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we let $|x|$ denote the absolute-value vector $|x| = (|x_1|, \dots, |x_n|)$ and define $\|x\| = \max_{i\in\{1,\dots,n\}} |x_i|$. Also, let $\mathbb{R}_+$ be the nonnegative real number space.

The initial conditions associated with (1.1) are as follows:

$$S(t_0) > 0, \quad I(t_0) > 0, \tag{1.2}$$

$$R(t_0) \geq 0. \tag{1.3}$$

Because the third equation in (1.1) is independent of the first two, the authors in [LBJ] established some sufficient conditions for the existence of positive periodic solutions of the reduced system

$$\begin{cases} S'(t) = A - \mathbf{d}S(t) - \dfrac{\lambda(t)S(t)I(t)}{1+kI(t)}, \\ I'(t) = \dfrac{\lambda(t)S(t)I(t)}{1+kI(t)} - (\mathbf{d}+\varepsilon+\mu)I(t) - \dfrac{\gamma I(t)}{1+\alpha I(t)}, \end{cases} \tag{1.4}$$

with initial value (1.2). However, it is difficult to obtain the stability of positive periodic solutions for (1.4), which is formulated as a challenging problem at the end of [LBJ]. Moreover, it is well known that the global exponential convergence behavior of solutions plays a key role in characterizing the behavior of a dynamical system since the exponential convergence rate can be estimated (see [BBI, L, S, X, YGW, ZYP]). This motivates us to study the global exponential stability of positive periodic solutions for SIR model (1.1).

As is well known, many infectious diseases exhibit periodic fluctuations and there is a saturation phenomenon during the treatment. Therefore, the coefficients in the differential equations of population and ecology problems are usually periodic. So we consider the following nonautonomous SIR model:

$$\begin{cases} S'(t) = A(t) - \mathbf{d}(t)S(t) - \dfrac{\lambda(t)S(t)I(t)}{1+k(t)I(t)}, \\ I'(t) = \dfrac{\lambda(t)S(t)I(t)}{1+k(t)I(t)} - (\mathbf{d}(t)+\varepsilon(t)+\mu(t))I(t) - \dfrac{\gamma(t)I(t)}{1+\alpha(t)I(t)}, \\ R'(t) = \mu(t)I(t) + \dfrac{\gamma(t)I(t)}{1+\alpha(t)I(t)} - \mathbf{d}(t)R(t), \end{cases} \tag{1.5}$$

where $A, \mathbf{d}, k, \alpha, \varepsilon, \mu, \gamma : \mathbb{R} \to (0, \infty)$ and $\lambda : \mathbb{R} \to \mathbb{R}_+$ are continuous T-periodic functions with $T > 0$. Obviously, (1.1) is a special case of (1.5).

For simplicity of notation, for a bounded continuous function g defined on $\mathbb{R}$, we denote

$$g^+ = \sup_{t\in\mathbb{R}} |g(t)| \quad \text{and} \quad g^- = \inf_{t\in\mathbb{R}} |g(t)|.$$

It will always be assumed that

$$k(t) \le \alpha(t) \quad \text{for all } t \in \mathbb{R}. \tag{1.6}$$

2. Preliminaries and lemmas

LEMMA 2.1. *Every solution of* (1.5) *with initial value conditions* (1.2) *and* (1.3) *is positive and bounded on* (t_0, ∞).

Proof. From [H, Theorem 1.3.1], we can deduce that there exists a unique solution $(S(t,t_0,x_0), I(t,t_0,x_0), R(t,t_0,x_0))$ of (1.5) passing through (t_0, x_0) with initial value $x_0 = (S(t_0), I(t_0), R(t_0))$ satisfying (1.2) and (1.3). Let $[t_0, T^*)$ be the maximal right-interval of existence of

$$(S(t), I(t), R(t)) = (S(t,t_0,x_0), I(t,t_0,x_0), R(t,t_0,x_0)).$$

We first prove that

$$S(t) > 0 \quad \text{for all } t \in [t_0, T^*). \tag{2.1}$$

Assume, by way of contradiction, that (2.1) does not hold. Then there must exist $T_1 \in (t_0, T^*)$ such that

$$S(T_1) = 0, \qquad S(s) > 0 \quad \text{for all } s \in [t_0, T_1), \qquad S'(T_1) \le 0.$$

But from the first equation of (1.5), we have

$$S'(T_1) = A(T_1) - \mathbf{d}(T_1)S(T_1) - \frac{\lambda(T_1)S(T_1)I(T_1)}{1 + k(T_1)I(T_1)} = A(T_1) > 0,$$

a contradiction. Hence (2.1) holds.

Next, we claim that $I(t) > 0$ for $t \in [t_0, T^*)$. Otherwise, there must exist $T_2 \in (t_0, T^*)$ such that

$$I(T_2) = 0, \qquad I(s) > 0 \quad \text{for all } s \in [t_0, T_2).$$

In view of the second equation of (1.5), we get

$$\begin{aligned} I'(v) &= \frac{\lambda(v)S(v)I(v)}{1 + k(v)I(v)} - (\mathbf{d}(v) + \varepsilon(v) + \mu(v))I(v) - \frac{\gamma(v)I(v)}{1 + \alpha(v)I(v)} \\ &\ge \frac{\lambda(v)S(v)I(v)}{1 + k(v)I(v)} - (\mathbf{d}(v) + \varepsilon(v) + \mu(v) + \gamma(v))I(v) \\ &\ge \frac{\lambda(v)S(v)I(v)}{1 + k(v)I(v)} - (\mathbf{d}^+ + \varepsilon^+ + \mu^+ + \gamma^+)I(v) \quad \text{for all } v \in [t_0, T_2], \end{aligned}$$

and hence

$$\begin{aligned} I(T_2) \ge{}& e^{-(T_2 - t_0)(\mathbf{d}^+ + \varepsilon^+ + \mu^+ + \gamma^+)} I(t_0) \\ &+ \int_{t_0}^{T_2} e^{-(T_2 - v)(\mathbf{d}^+ + \varepsilon^+ + \mu^+ + \gamma^+)} \frac{\lambda(v)S(v)I(v)}{1 + k(v)I(v)}\, dv > 0. \end{aligned}$$

This contradicts $I(T_2) = 0$ and the claim is proved.

Now, we prove that $R(t) > 0$ for all $t \in (t_0, T^*)$. If $R(t_0) > 0$, then by continuity we can choose a small positive constant η^* such that

$$R(t) > 0 \quad \text{for all } t \in (t_0, t_0 + \eta^*] \subset (t_0, T^*). \tag{2.2}$$

If $R(t_0) = 0$, then

$$\begin{aligned} R'(t_0) &= \mu(t_0)I(t_0) + \frac{\gamma(t_0)I(t_0)}{1+\alpha(t_0)I(t_0)} - \mathbf{d}(t_0)R(t_0) \\ &= \mu(t_0)I(t_0) + \frac{\gamma(t_0)I(t_0)}{1+\alpha(t_0)I(t_0)} > 0, \end{aligned}$$

which implies that (2.2) also holds. Now, we claim that

$$R(t) > 0 \quad \text{for all } t \in (t_0 + \eta^*, T^*). \tag{2.3}$$

Otherwise, there must exist $T_3 \in (t_0 + \eta^*, T^*)$ such that

$$R(T_3) = 0 \quad \text{and} \quad R(s) > 0 \quad \text{for all } s \in (t_0 + \eta^*, T_3). \tag{2.4}$$

From (1.5) and (2.4), we have

$$0 \geq R'(T_3) = \mu(T_3)I(T_3) + \frac{\gamma I(T_3)}{1+\alpha(T_3)I(T_3)} > 0,$$

which is a contradiction, and hence (2.3) holds.

From the above discussion, we find that

$$S(t), I(t), R(t) > 0 \quad \text{for all } t \in (t_0, T^*),$$

which together with (1.5) yields

$$\begin{aligned} S'(t) &= A(t) - \mathbf{d}(t)S(t) - \frac{\lambda(t)S(t)I(t)}{1+k(t)I(t)} \leq A^+ - \mathbf{d}^- S(t), \\ I'(t) &= \frac{\lambda(t)S(t)I(t)}{1+k(t)I(t)} - (\mathbf{d}(t) + \varepsilon(t) + \mu(t))I(t) - \frac{\gamma(t)I(t)}{1+\alpha(t)I(t)} \\ &\leq \frac{\lambda^+ S(t)}{k^-} - (\mathbf{d}^- + \varepsilon^- + \mu^-)I(t), \\ R'(t) &= \mu(t)I(t) + \frac{\gamma(t)I(t)}{1+\alpha(t)I(t)} - \mathbf{d}(t)R(t) \\ &\leq (\mu^+ + \gamma^+)I(t) - \mathbf{d}^- R(t). \end{aligned}$$

Therefore,

$$
\begin{aligned}
S(t) &\le S(t_0)\frac{e^{\mathbf{d}^- t_0}}{e^{\mathbf{d}^- t}} + \frac{A^+}{\mathbf{d}^-}\,\frac{e^{\mathbf{d}^- t} - e^{\mathbf{d}^- t_0}}{e^{\mathbf{d}^- t}} \le S(t_0) + \frac{A^+}{\mathbf{d}^-} =: M_1,\\
I(t) &\le \frac{I(t_0)e^{(\mathbf{d}^- + \varepsilon^- + \mu^-)t_0}}{e^{(\mathbf{d}^- + \varepsilon^- + \mu^-)t}} + \frac{M_1\lambda^+(e^{(\mathbf{d}^- + \varepsilon^- + \mu^-)t} - e^{(\mathbf{d}^- + \varepsilon^- + \mu^-)t_0})}{k^-(\mathbf{d}^- + \varepsilon^- + \mu^-)e^{(\mathbf{d}^- + \varepsilon^- + \mu^-)t}}\\
&\le I(t_0) + \frac{M_1\lambda^+}{k^-(\mathbf{d}^- + \varepsilon^- + \mu^-)} =: M_2,\\
R(t) &\le \frac{R(t_0)e^{\mathbf{d}^- t_0}}{e^{\mathbf{d}^- t}} + \frac{(\mu^+ + \gamma^+)M_2}{\mathbf{d}^-}\,\frac{e^{\mathbf{d}^- t} - e^{\mathbf{d}^- t_0}}{e^{\mathbf{d}^- t}}\\
&\le R(t_0) + \frac{(\mu^+ + \gamma^+)M_2}{\mathbf{d}^-} =: M_3,
\end{aligned}
$$

for all $t \in (t_0, T^*)$. It follows that $S(t)$, $I(t)$ and $R(t)$ are bounded on $[t_0, T^*)$. From [H, Theorem 1.2.1], we easily obtain $T^* = \infty$. ∎

LEMMA 2.2. *Let*

$$
L^S = \sup_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)} \ge l^S = \inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t) + \lambda(t)/k(t)} > 0,
$$

$$
l^I = \inf_{t\in\mathbb{R}} \frac{1}{k(t)}\left[\frac{\lambda(t)\inf_{t\in\mathbb{R}}\frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - \gamma(t)}{\mathbf{d}(t) + \varepsilon(t) + \mu(t)} - 1\right] > 0,
$$

and let $(S(t), I(t), R(t))$ *be a solution of system* (1.5) *with initial conditions* (1.2) *and* (1.3). *Then*

$$
l^S \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le L^S, \quad \liminf_{t\to\infty} I(t) \ge l^I, \quad \liminf_{t\to\infty} R(t) > 0.
$$

Proof. From Lemma 2.1, the solution $(S(t), I(t), R(t))$ is positive and bounded on (t_0, ∞). By the fluctuation lemma [S, Lemma A.1], there exist sequences $\{t_p^1\}_{p\ge1}$, $\{t_p^2\}_{p\ge1}$, $\{t_p^3\}_{p\ge1}$ and $\{t_p^4\}_{p\ge1}$ such that

$$
\left.
\begin{aligned}
&t_p^1 \to \infty,\ S(t_p^1) \to \limsup_{t\to\infty} S(t),\ S'(t_p^1) \to 0,\\
&t_p^2 \to \infty,\ S(t_p^2) \to \liminf_{t\to\infty} S(t),\ \ S'(t_p^2) \to 0,\\
&t_p^3 \to \infty,\ \ I(t_p^3) \to \liminf_{t\to\infty} I(t),\quad I'(t_p^3) \to 0,\\
&t_p^4 \to \infty,\ R(t_p^4) \to \liminf_{t\to\infty} R(t),\ R'(t_p^4) \to 0,
\end{aligned}
\right\}\ \text{as } p \to \infty. \tag{2.5}
$$

The first two lines in (2.5) yield

$$
\begin{aligned}
S'(t_p^1) &= A(t_p^1) - \mathbf{d}(t_p^1)S(t_p^1) - \frac{\lambda(t_p^1)S(t_p^1)I(t_p^1)}{1 + k(t_p^1)I(t_p^1)}\\
&\le \mathbf{d}(t_p^1)\left[\frac{A(t_p^1)}{\mathbf{d}(t_p^1)} - S(t_p^1)\right] \le \mathbf{d}(t_p^1)\left[\sup_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)} - S(t_p^1)\right],
\end{aligned}
$$

$$
\begin{aligned}
S'(t_p^2) &= A(t_p^2) - \mathbf{d}(t_p^2)S(t_p^2) - \frac{\lambda(t_p^2)S(t_p^2)I(t_p^2)}{1+k(t_p^2)I(t_p^2)} \\
&\geq A(t_p^2) - \mathbf{d}(t_p^2)S(t_p^2) - \frac{\lambda(t_p^2)S(t_p^2)}{k(t_p^2)} \\
&= A(t_p^2) - S(t_p^2)\left(\mathbf{d}(t_p^2) + \frac{\lambda(t_p^2)}{k(t_p^2)}\right) \\
&\geq \left(\mathbf{d}(t_p^2) + \frac{\lambda(t_p^2)}{k(t_p^2)}\right)\left[\inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - S(t_p^2)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{S'(t_p^1)}{\mathbf{d}(t_p^1)} \leq \sup_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)} - S(t_p^1), \\
&\frac{S'(t_p^2)}{\mathbf{d}(t_p^2)+\lambda(t_p^2)/k(t_p^2)} \geq \inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - S(t_p^2).
\end{aligned}
\tag{2.6}
$$

Letting $p \to \infty$ in (2.6) implies that

$$
l^S \leq \liminf_{t\to\infty} S(t) \leq \limsup_{t\to\infty} S(t) \leq L^S.
$$

Furthermore, we prove that there exists a positive constant l such that

$$
\liminf_{t\to\infty} I(t) \geq l. \tag{2.7}
$$

Otherwise, $\liminf_{t\to\infty} I(t) = 0$. For each $t \geq t_0$, we define

$$
m(t) = \max\left\{\xi : \xi \leq t,\ I(\xi) = \min_{t_0\leq s\leq t} I(s)\right\}.
$$

Observe that $m(t) \to \infty$ as $t \to \infty$ and that

$$
\lim_{t\to\infty} I(m(t)) = 0. \tag{2.8}
$$

However, $I(m(t)) = \min_{t_0\leq s\leq t} I(s)$, and so $I'(m(t)) \leq 0$ for all $m(t) > t_0$. Let $\epsilon > 0$ and $t_0^* > t_0$ be such that

$$
\inf_{t\in\mathbb{R}} \frac{1}{k(t)}\left[\frac{\lambda(t)\left(\inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - \epsilon\right) - \gamma(t)}{\mathbf{d}(t)+\varepsilon(t)+\mu(t)} - 1\right] > 0
$$

and

$$
S(t) > \inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - \epsilon > 0 \quad \text{for all } t \geq t_0^*.
$$

According to (1.5) and (1.6), we have

$$
\begin{aligned}
0 \geq I'(m(t)) = {} & \frac{\lambda(m(t))S(m(t))I(m(t))}{1+k(m(t))I(m(t))} \\
& - \big(\mathbf{d}(m(t)) + \varepsilon(m(t)) + \mu(m(t))\big)I(m(t)) - \frac{\gamma(m(t))I(m(t))}{1+\alpha(m(t))I(m(t))}
\end{aligned}
$$

$$\geq I(m(t))\Big\{\frac{\lambda(m(t))S(m(t))-\gamma(m(t))}{1+k(m(t))I(m(t))} \\ -\big(\mathbf{d}(m(t))+\varepsilon(m(t))+\mu(m(t))\big)\Big\} \quad \text{for all } m(t)\geq t_0^*.$$

Thus, for all $m(t)\geq t_0^*$,

$$\mathbf{d}(m(t))+\varepsilon(m(t))+\mu(m(t))\geq \frac{\lambda(m(t))S(m(t))-\gamma(m(t))}{1+k(m(t))I(m(t))}$$

and

$$\begin{aligned} I(m(t)) &\geq \frac{1}{k(m(t))}\Big\{\frac{\lambda(m(t))S(m(t))-\gamma(m(t))}{\mathbf{d}(m(t))+\varepsilon(m(t))+\mu(m(t))}-1\Big\} \\ &\geq \frac{1}{k(m(t))}\Big\{\frac{\lambda(m(t))\big(\inf_{t\in\mathbb{R}}\frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)}-\epsilon\big)-\gamma(m(t))}{\mathbf{d}(m(t))+\varepsilon(m(t))+\mu(m(t))}-1\Big\} \\ &\geq \inf_{t\in\mathbb{R}}\Big\{\frac{1}{k(t)}\Big[\frac{\lambda(t)\big(\inf_{t\in\mathbb{R}}\frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)}-\epsilon\big)-\gamma(t)}{\mathbf{d}(t)+\varepsilon(t)+\mu(t)}-1\Big]\Big\}>0, \end{aligned}$$

which contradicts (2.8). This proves (2.7).

By the continuity and boundedness of the coefficient functions in (1.5), we can select a subsequence, still denoted by $\{t_p^i\}_{p=1}^{\infty}$, such that

(2.9) the limits of $S(t_p^i)$, $\gamma(t_p^i)$, $\mathbf{d}(t_p^i)+\varepsilon(t_p^i)+\mu(t_p^i)$, $k(t_p^i)$, $\alpha(t_p^i)$ and $\lambda(t_p^i)$ as $p\to\infty$ exist for all $i=3,4$.

In view of (1.5), (1.6), (2.5) and (2.9), we obtain

$$\begin{aligned} \frac{I'(t_p^3)}{I(t_p^3)} &= \frac{\lambda(t_p^3)S(t_p^3)}{1+k(t_p^3)I(t_p^3)}-(\mathbf{d}(t_p^3)+\varepsilon(t_p^3)+\mu(t_p^3))-\frac{\gamma(t_p^3)}{1+\alpha(t_p^3)I(t_p^3)} \\ &\geq \frac{\lambda(t_p^3)S(t_p^3)-\gamma(t_p^3)}{1+k(t_p^3)I(t_p^3)}-(\mathbf{d}(t_p^3)+\varepsilon(t_p^3)+\mu(t_p^3)). \end{aligned} \tag{2.10}$$

Letting $p\to\infty$ in (2.9) and (2.10) implies that

$$\begin{aligned} \liminf_{t\to\infty} I(t) = \lim_{p\to\infty} I(t_p^3) &\geq \lim_{p\to\infty}\frac{1}{k(t_p^3)}\Big[\frac{\lambda(t_p^3)S(t_p^3)-\gamma(t_p^3)}{\mathbf{d}(t_p^3)+\varepsilon(t_p^3)+\mu(t_p^3)}-1\Big] \\ &\geq \inf_{t\in\mathbb{R}}\frac{1}{k(t)}\Big[\frac{\lambda(t)\inf_{t\in\mathbb{R}}\frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)}-\gamma(t)}{\mathbf{d}(t)+\varepsilon(t)+\mu(t)}-1\Big]=l^I>0. \end{aligned}$$

Similarly,

$$R'(t_p^4)=\mu(t_p^4)I(t_p^4)+\frac{\gamma(t_p^4)I(t_p^4)}{1+\alpha(t_p^4)I(t_p^4)}-d(t_p^4)R(t_p^4)\geq \mu(t_p^4)I(t_p^4)-d(t_p^4)R(t_p^4)$$

yields

$$\liminf_{t\to\infty} R(t) = \lim_{p\to\infty} R(t_p^4) \geq \inf_{t\in\mathbb{R}} \frac{\mu(t)}{\mathbf{d}(t)} \liminf_{t\to\infty} I(t) > 0.$$

The proof of Lemma 2.2 is now completed.

LEMMA 2.3. *Assume that*

$$\sup_{t\in\mathbb{R}} \left\{ -\mathbf{d}(t) + \frac{\lambda(t)L^S}{(1+k(t)l^I)^2} \right\} < 0, \tag{2.11}$$

$$\sup_{t\in\mathbb{R}} \left\{ -[\mathbf{d}(t)+\varepsilon(t)+\mu(t)] + \frac{\lambda(t)}{k(t)} + \frac{\lambda(t)L^S}{(1+k(t)l^I)^2} \right\} < 0, \tag{2.12}$$

and the assumptions of Lemma 2.2 hold. Let

$$(S(t), I(t), R(t)), \qquad (\hat{S}(t), \hat{I}(t), \hat{R}(t))$$

be the solutions of system (1.5) *with initial conditions* (1.2) *and* (1.3). *Then there exist* $\hat{t}_0 \geq t_0$ *and positive constants* ζ *and* K *such that*

$$|S(t)-\hat{S}(t)| \leq Ke^{-\zeta t}, \quad |I(t)-\hat{I}(t)| \leq Ke^{-\zeta t}, \quad \textit{for all } t \geq \hat{t}_0. \tag{2.13}$$

Moreover, there exist constants $t_R \geq \hat{t}_0$ *and* $K_R > 0$ *such that*

$$|R(t)-\hat{R}(t)| \leq K_R e^{-\zeta t} \quad \textit{for all } t \geq t_R. \tag{2.14}$$

Proof. Let

$$x(t) = (x_1(t), x_2(t)) = (S(t)-\hat{S}(t), I(t)-\hat{I}(t)) \quad \text{for all } t \in [t_0, \infty).$$

Then (1.5) gives

$$\begin{aligned}
x_1'(t) &= -\mathbf{d}(t)[S(t)-\hat{S}(t)] - \frac{\lambda(t)I(t)}{1+k(t)I(t)}[S(t)-\hat{S}(t)] \\
&\quad - \lambda(t)\hat{S}(t)\frac{I(t)-\hat{I}(t)}{(1+k(t)I(t))(1+k(t)\hat{I}(t))} \\
&= -\left[\mathbf{d}(t) + \frac{\lambda(t)I(t)}{1+k(t)I(t)}\right]x_1(t) - \lambda(t)\hat{S}(t)\frac{x_2(t)}{(1+k(t)I(t))(1+k(t)\hat{I}(t))}, \\
x_2'(t) &= \frac{\lambda(t)I(t)}{1+k(t)I(t)}x_1(t) + \lambda(t)\hat{S}(t)\frac{x_2(t)}{(1+k(t)I(t))(1+k(t)\hat{I}(t))} \\
&\quad - \left[\mathbf{d}(t)+\varepsilon(t)+\mu(t) + \frac{\gamma(t)}{(1+\alpha(t)I(t))(1+\alpha(t)\hat{I}(t))}\right]x_2(t),
\end{aligned}$$

which implies

$$\begin{aligned}
x_1(t) &= e^{-\int_{\hat{t}_0}^t [\mathbf{d}(\theta)+\frac{\lambda(\theta)I(\theta)}{1+k(\theta)I(\theta)}]\,d\theta} x_1(\hat{t}_0) + \int_{\hat{t}_0}^t e^{-\int_v^t [\mathbf{d}(\theta)+\frac{\lambda(\theta)I(\theta)}{1+k(\theta)I(\theta)}]\,d\theta} \\
&\qquad \times \left[-\lambda(v)\hat{S}(v)\frac{x_2(v)}{(1+k(v)I(v))(1+k(v)\hat{I}(v))}\right] dv,
\end{aligned} \tag{2.15}$$

and

$$x_2(t) = e^{-\int_{\hat{t}_0}^{t}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)+\frac{\gamma(t)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\hat{I}(\theta))}]\,d\theta}x_2(\hat{t}_0)$$
$$+\int_{\hat{t}_0}^{t} e^{-\int_{v}^{t}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)+\frac{\gamma(t)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\hat{I}(\theta))}]\,d\theta}\Bigg[\frac{\lambda(v)I(v)}{1+k(v)I(v)}x_1(v)$$
$$+\lambda(v)\hat{S}(v)\frac{x_2(v)}{(1+k(v)I(v))(1+k(v)\hat{I}(v))}\Bigg]\,dv, \tag{2.16}$$

for all $t \geq \hat{t}_0$. Let $\epsilon < \min\{l^I, l^S\}$ be a positive constant such that

$$\sup_{t\in\mathbb{R}}\left\{-\mathbf{d}(t)+\frac{\lambda(t)(L^S+\epsilon)}{[1+k(t)(l^I-\epsilon)]^2}\right\}<0,$$
$$\sup_{t\in\mathbb{R}}\left\{-[\mathbf{d}(t)+\varepsilon(t)+\mu(t)]+\frac{\lambda(t)}{k(t)}+\frac{\lambda(t)(L^S+\epsilon)}{[1+k(t)(l^I-\epsilon)]^2}\right\}<0.$$

This can be achieved because of (2.11) and (2.12). Consequently, we can choose positive constants ζ and τ such that

$$\sup_{t\in\mathbb{R}}\left\{\zeta-\mathbf{d}(t)+\frac{\lambda(t)(L^S+\epsilon)}{[1+k(t)(l^I-\epsilon)]^2}\right\}<-\tau, \tag{2.17}$$

$$\sup_{t\in\mathbb{R}}\left\{\zeta-[\mathbf{d}(t)+\varepsilon(t)+\mu(t)]+\frac{\lambda(t)}{k(t)}+\frac{\lambda(t)(L^S+\epsilon)}{[1+k(t)(l^I-\epsilon)]^2}\right\}<-\tau. \tag{2.18}$$

From Lemma 2.2, we can choose $\hat{t}_0 > t_0$ such that

$$S(t) \leq L^S+\epsilon,\ \hat{S}(t)\leq L^S+\epsilon,\ \hat{I}(t)\geq l^I-\epsilon,\qquad \text{for all } t\geq \hat{t}_0.$$

Let $\|x\|_0 = \max\{\sup_{t\in[t_0,\hat{t}_0]}|x_1(t)|, \sup_{t\in[t_0,\hat{t}_0]}|x_2(t)|\}$, and $K_0 > 1$ be a constant. It is obvious that

$$\|x(\hat{t}_0)\| < \|x\|_0+\epsilon < K_0(\|x\|_0+\epsilon) = K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\hat{t}_0}.$$

In the following, we will show

$$\|x(t)\| < K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta t}\qquad \text{for all } t>\hat{t}_0. \tag{2.19}$$

Otherwise, one of the following two cases must occur:

Case I: There exists $\theta_1 > 0$ such that

$$\begin{aligned}|x_1(\theta_1)| &= K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_1},\\ \|x(t)\| &< K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta t}\qquad \text{for all } t\in[\hat{t}_0,\,\theta_1).\end{aligned} \tag{2.20}$$

Case II: There exists $\theta_2 > 0$ such that

$$\begin{aligned}|x_2(\theta_2)| &= K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_2},\\ \|x(t)\| &< K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta t}\qquad \text{for all } t\in[\hat{t}_0,\theta_2).\end{aligned} \tag{2.21}$$

If Case I holds, then in view of (2.15), (2.17) and (2.20), we have

$$\begin{aligned}
|x_1(\theta_1)| &= \Bigg| e^{-\int_{\hat{t}_0}^{\theta_1}\left[\mathbf{d}(\theta)+\frac{\lambda(\theta)I(\theta)}{1+k(\theta)I(\theta)}\right]d\theta} x_1(t_0) \\
&\quad + \int_{\hat{t}_0}^{\theta_1} e^{-\int_v^{\theta_1}\left[\mathbf{d}(\theta)+\frac{\lambda(\theta)I(\theta)}{1+k(\theta)I(\theta)}\right]d\theta}\left[-\lambda(v)\hat{S}(v)\frac{x_2(v)}{(1+k(v)I(v))(1+k(v)\hat{I}(v))}\right]dv\Bigg| \\
&\le e^{-\int_{\hat{t}_0}^{\theta_1}\mathbf{d}(\theta)\,d\theta}|x_1(t_0)| \\
&\quad + \int_{\hat{t}_0}^{\theta_1} e^{-\int_v^{\theta_1}\mathbf{d}(\theta)\,d\theta}\lambda(v)(L^S+\epsilon)\frac{|x_2(v)|}{[1+k(v)(l^I-\epsilon)]^2}\,dv \\
&\le e^{-\int_{\hat{t}_0}^{\theta_1}\mathbf{d}(\theta)\,d\theta}(\|x\|_0+\epsilon) \\
&\quad + \int_{\hat{t}_0}^{\theta_1} e^{-\int_v^{\theta_1}\mathbf{d}(\theta)\,d\theta}\frac{\lambda(v)(L^S+\epsilon)}{[1+k(v)(l^I-\epsilon)]^2}K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta v}\,dv \\
&= K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_1}\Bigg\{\frac{1}{K_0}e^{-\int_{\hat{t}_0}^{\theta_1}(\mathbf{d}(\theta)-\zeta)\,d\theta} \\
&\qquad + \int_{\hat{t}_0}^{\theta_1} e^{-\int_v^{\theta_1}(\mathbf{d}(\theta)-\zeta)\,d\theta}\frac{\lambda(v)(L^S+\epsilon)}{[1+k(v)(l^I-\epsilon)]^2}\,dv\Bigg\} \\
&\le K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_1}\Bigg\{\frac{1}{K_0}e^{-\int_{\hat{t}_0}^{\theta_1}(\mathbf{d}(\theta)-\zeta)\,d\theta} \\
&\qquad + \int_{\hat{t}_0}^{\theta_1} e^{-\int_v^{\theta_1}(\mathbf{d}(\theta)-\zeta)\,d\theta}(\mathbf{d}(v)-\zeta)\,dv\Bigg\} \\
&= K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_1}\{1-(1-1/K_0)e^{-\int_{\hat{t}_0}^{\theta_1}(\mathbf{d}(\theta)-\zeta)\,d\theta}\} \\
&< K_0(\|x\|_0+\epsilon)e^{\zeta\hat{t}_0}e^{-\zeta\theta_1},
\end{aligned}$$

which contradicts the first equation in (2.20). Hence, (2.19) holds.

If Case II holds, then together with (2.16) and (2.18), (2.21) implies that

$$\begin{aligned}
|x_2(\theta_2)| &= \Bigg| e^{-\int_{\hat{t}_0}^{\theta_2}\left[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)+\frac{\gamma(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\hat{I}(\theta))}\right]d\theta} x_2(\hat{t}_0) \\
&\quad + \int_{\hat{t}_0}^{\theta_2} e^{-\int_v^{\theta_2}\left[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)+\frac{\gamma(\theta)}{(1+\alpha(\theta)I(\theta))(1+\alpha(\theta)\hat{I}(\theta))}\right]d\theta} \\
&\quad \times\left[\frac{\lambda(v)I(v)}{1+k(v)I(v)}x_1(v)+\lambda(v)\hat{S}(v)\frac{x_2(v)}{(1+k(v)I(v))(1+k(v)\hat{I}(v))}\right]dv\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le e^{-\int_{\hat t_0}^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta}|x_2(\hat t_0)| \\
&\quad+\int_{\hat t_0}^{\theta_2} e^{-\int_v^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta}\left[\frac{\lambda(v)}{k(v)}|x_1(v)|+\frac{\lambda(v)(L^S+\epsilon)}{[1+k(v)(l^I-\epsilon)]^2}|x_2(v)|\right]dv \\
&\le e^{-\int_{\hat t_0}^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta}(\|x\|_0+\epsilon) \\
&\quad+\int_{\hat t_0}^{\theta_2} e^{-\int_v^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta} \\
&\qquad\qquad\times\left[\frac{\lambda(v)}{k(v)}+\frac{\lambda(v)(L^S+\epsilon)}{(1+k(v)(l^I-\epsilon))^2}\right]K_0(\|x\|_0+\epsilon)e^{\zeta\hat t_0}e^{-\zeta v}\,dv \\
&\le e^{-\int_{\hat t_0}^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta}(\|x\|_0+\epsilon)+\int_{\hat t_0}^{\theta_2} e^{-\int_v^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)]\,d\theta} \\
&\qquad\qquad\times[\mathbf{d}(v)+\varepsilon(v)+\mu(v)-\zeta]K_0(\|x\|_0+\epsilon)e^{\zeta\hat t_0}e^{-\zeta v}\,dv \\
&=K_0(\|x\|_0+\epsilon)e^{\zeta\hat t_0}e^{-\zeta\theta_2}\bigg\{\frac{1}{K_0}e^{-\int_{\hat t_0}^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)-\zeta]\,d\theta} \\
&\qquad\qquad+\int_{\hat t_0}^{\theta_2} e^{-\int_v^{\theta_2}[\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta)-\zeta]\,d\theta}[\mathbf{d}(v)+\varepsilon(v)+\mu(v)-\zeta]\,dv\bigg\} \\
&=K_0(\|x\|_0+\epsilon)e^{\zeta\hat t_0}e^{-\zeta\theta_2}\{1-(1-1/K_0)e^{-\int_{\hat t_0}^{\theta_2}[(\mathbf{d}(\theta)+\varepsilon(\theta)+\mu(\theta))-\zeta]\,d\theta}\} \\
&<K_0(\|x\|_0+\epsilon)e^{\zeta\hat t_0}e^{-\zeta\theta_2},
\end{aligned}
$$

which contradicts the first equation in (2.21) and proves (2.19).

Letting $\epsilon\to 0^+$, it follows from (2.19) that

$$
\|x(t)\|\le K_0\|x\|_0e^{\zeta\hat t_0}e^{-\zeta t}\quad\text{for all } t>\hat t_0, \tag{2.22}
$$

which proves (2.13).

Now, we prove that (2.14) holds. Without loss of generality, we assume that

$$
\|R\|_\infty=\sup_{t\ge t_0}|R(t)-\hat R(t)|>0.
$$

Let

$$
x_3(t)=R(t)-\hat R(t)\quad\text{for all } t\in(t_0,\infty).
$$

Then

$$
x_3'(t)=\mu(t)x_2(t)+\frac{\gamma(t)x_2(t)}{(1+\alpha(t)I(t))(1+\alpha(t)\hat I(t))}-\mathbf{d}(t)x_3(t)
$$

and

$$x_3(t) = e^{-\int_{\widetilde{t}_0}^{t} \mathbf{d}(\theta)\,d\theta} x_3(\widetilde{t}_0) + \int_{\widetilde{t}_0}^{t} e^{-\int_v^t \mathbf{d}(\theta)\,d\theta} \times \left[\mu(v)x_2(v) + \frac{\gamma(v)x_2(v)}{(1+\alpha(t)I(v))(1+\alpha(v)\hat{I}(v))}\right] dv \tag{2.23}$$

for all $t \geq \widetilde{t}_0 \geq \hat{t}_0$. For any $\epsilon > 0$, since $\mathbf{d}^- - \zeta > 0$, we can choose $t_R \geq \hat{t}_0$ and $K_0^* > K_0$ such that

$$-(\mathbf{d}^- - \zeta) + \frac{(\mu^+ + \gamma^+)K_0\|x\|_0 e^{\zeta\hat{t}_0}}{K_0^*\|R\|_\infty e^{\zeta t_R}} < 0$$

and

$$\begin{aligned} &-(\mathbf{d}^- - \zeta) + \frac{(\mu^+ + \gamma^+)K_0\|x\|_0 e^{\zeta\hat{t}_0}}{K_0^*(\|R\|_\infty + \epsilon) e^{\zeta t_R}} \\ &\qquad \leq -(\mathbf{d}^- - \zeta) + \frac{(\mu^+ + \gamma^+)K_0\|x\|_0 e^{\zeta\hat{t}_0}}{K_0^*\|R\|_\infty e^{\zeta t_R}} < 0. \end{aligned} \tag{2.24}$$

Consequently,

$$|x_3(t_R)| < \|R\|_\infty + \epsilon < K_0^*(\|R\|_\infty + \epsilon)e^{\zeta t_R}e^{-\zeta t_R}.$$

Now, we will show

$$|x_3(t)| < K_0^*(\|R\|_\infty + \epsilon)e^{\zeta t_R}e^{-\zeta t} \quad \text{for all } t > t_R. \tag{2.25}$$

Otherwise, there must exist $\theta^* > t_R$ such that

$$\begin{aligned} &|x_3(\theta^*)| = K_0^*(\|R\|_\infty + \epsilon)e^{\zeta t_R}e^{-\zeta\theta^*}, \\ &|x_3(t)| < K_0^*(\|R\|_\infty + \epsilon)e^{\zeta t_R}e^{-\zeta t} \quad \text{for all } t \in [t_R, \theta^*). \end{aligned} \tag{2.26}$$

From (2.23), (2.24) and (2.26), we have

$$\begin{aligned} |x_3(\theta^*)| &= \bigg| e^{-\int_{t_R}^{\theta^*} \mathbf{d}(\theta)\,d\theta} x_3(t_R) \\ &\quad + \int_{t_R}^{\theta^*} e^{-\int_v^{\theta^*} \mathbf{d}(\theta)\,d\theta} \left[\mu(v)x_2(v) + \frac{\gamma(v)x_2(v)}{(1+\alpha(v)I(v))(1+\alpha(v)\hat{I}(v))}\right] dv \bigg| \\ &\leq e^{-\mathbf{d}^-(\theta^* - t_R)}|x_3(t_R)| + \int_{t_R}^{\theta^*} e^{-\mathbf{d}^-(\theta^* - v)}(\mu^+ + \gamma^+)|x_2(v)|\,dv \\ &\leq e^{-\mathbf{d}^-(\theta^* - t_R)}(\|R\|_\infty + \epsilon) + \int_{t_R}^{\theta^*} e^{-\mathbf{d}^-(\theta^* - v)}(\mu^+ + \gamma^+)K_0\|x\|_0 e^{\zeta\hat{t}_0}e^{-\zeta v}\,dv \\ &\leq K_0^*(\|R\|_\infty + \epsilon)e^{\zeta t_R}e^{-\zeta\theta^*} \\ &\quad \times \left\{ \frac{1}{K_0^*}e^{-(\theta^* - t_R)(\mathbf{d}^- - \zeta)} + \int_{t_R}^{\theta^*} e^{-(\theta^* - v)(\mathbf{d}^- - \zeta)} \frac{(\mu^+ + \gamma^+)K_0\|x\|_0 e^{\zeta\hat{t}_0}}{K_0^*(\|R\|_\infty + \varepsilon)e^{\zeta t_R}}\,dv \right\} \end{aligned}$$

$$\le K_0^*(\|R\|_\infty+\epsilon)e^{\zeta t_R}e^{-\zeta\theta^*} \times\left\{\frac{1}{K_0^*}e^{-(\theta^*-t_R)(\mathbf{d}^- -\zeta)}+\int_{t_R}^{\theta^*}e^{-(\theta^*-v)(\mathbf{d}^- -\zeta)}(\mathbf{d}^- -\zeta)\,dv\right\}$$
$$= K_0^*(\|R\|_\infty+\epsilon)e^{\zeta t_R}e^{-\zeta\theta^*}\{1-(1-1/K_0^*)e^{-(\theta^*-t_R)(\mathbf{d}^- -\zeta)}\}$$
$$< K_0^*(\|R\|_\infty+\epsilon)e^{\zeta t_R}e^{-\zeta\theta^*},$$

which contradicts the first equation in (2.26). Hence, (2.25) holds. Letting $\epsilon\to 0^+$, we deduce from (2.25) that (2.14) holds, which ends the proof. ∎

REMARK 2.1. Lemma 2.3 shows that a T-periodic solution $(\hat{S}(t),\hat{I}(t),\hat{R}(t))$ of (1.5) is globally exponentially stable.

3. Main results

THEOREM 3.1. *Under the assumptions of Lemma* 2.3*, system* (1.5) *has exactly one positive T-periodic solution which is globally exponentially stable.*

Proof. Let $(\tilde{S}(t),\tilde{I}(t),\tilde{R}(t))$ be a solution of (1.5) with initial conditions

$$\tilde{S}(t_0),\tilde{I}(t_0)>0,\qquad \tilde{R}(t_0)\ge 0. \tag{3.1}$$

By Lemmas 2.1 and 2.2, the solution $(\tilde{S}(t),\tilde{I}(t),\tilde{R}(t))$ is bounded and

$$\liminf_{t\to\infty}\tilde{S}(t)>0,\qquad \liminf_{t\to\infty}\tilde{I}(t)>0,\qquad \liminf_{t\to\infty}\tilde{R}(t)>0. \tag{3.2}$$

By the periodicity of the coefficients of (1.5), one can easily see that, for any nonnegative integer h, $(\tilde{S}(t+hT),\tilde{I}(t+hT),\tilde{R}(t+hT))$ is a solution of (1.5) with initial values

$$(\tilde{S}(t_0+hT),\tilde{I}(t_0+hT),\tilde{R}(t_0+hT)).$$

In particular, $(\breve{S}(t),\breve{I}(t),\breve{R}(t))=(\tilde{S}(t+T),\tilde{I}(t+T),\tilde{R}(t+T))$ is a solution of (1.5) with initial values

$$(\breve{S}(t_0),\breve{I}(t_0),\breve{R}(t_0))=(\tilde{S}(t_0+T),\tilde{I}(t_0+T),\tilde{R}(t_0+T)).$$

It follows from Lemma 2.3 that there exist $\breve{t}_0>t_0$ and $\breve{K}$ such that, for any nonnegative integer h and $t+hT\ge\breve{t}_0$,

(3.3)
$$|\tilde{S}(t+(h+1)T)-\tilde{S}(t+hT)|=|\breve{S}(t+hT)-\tilde{S}(t+hT)|\le\breve{K}e^{-\zeta(t+hT)},$$
$$|\tilde{I}(t+(h+1)T)-\tilde{I}(t+hT)|=|\breve{I}(t+hT)-\tilde{I}(t+hT)|\le\breve{K}e^{-\zeta(t+hT)},$$
$$|\tilde{R}(t+(h+1)T)-\tilde{R}(t+hT)|=|\breve{R}(t+hT)-\tilde{R}(t+hT)|\le\breve{K}e^{-\zeta(t+hT)}.$$

Now, we show that $\{(\tilde{S}(t+qT),\tilde{I}(t+qT),\tilde{R}(t+qT))\}_q$ is convergent on any compact interval as $q\to\infty$. Let $[a,b]\subset\mathbb{R}$ be an arbitrary interval. Choose a nonnegative integer q_0 such that $t+q_0T\ge\breve{t}_0$ for $t\in[a,b]$. Then for $t\in[a,b]$ and $q>q_0$ we have

$$
\begin{aligned}
\tilde{S}(t+qT) &= \tilde{S}(t+q_0T) + \sum_{h=q_0}^{q-1}[\tilde{S}(t+(h+1)T) - \tilde{S}(t+hT)],\\
\tilde{I}(t+qT) &= \tilde{I}(t+q_0T) + \sum_{h=q_0}^{q-1}[\tilde{I}(t+(h+1)T) - \tilde{I}(t+hT)],\\
\tilde{R}(t+qT) &= \tilde{R}(t+q_0T) + \sum_{h=q_0}^{q-1}[\tilde{R}(t+(h+1)T) - \tilde{R}(t+hT)],
\end{aligned}
$$

which, together with (3.3), implies that $\{(\tilde{S}(t+qT), \tilde{I}(t+qT), \tilde{R}(t+qT))\}_q$ converges uniformly to a continuous function, say $(S^*(t), I^*(t), R^*(t))$, on $[a, b]$. Because of arbitrariness of $[a, b]$, $(\tilde{S}(t+qT), \tilde{I}(t+qT), \tilde{R}(t+qT)) \to (S^*(t), I^*(t), R^*(t))$ as $q \to \infty$ for $t \in \mathbb{R}$. Moreover, $(S^*(t), I^*(t), R^*(t))$ is bounded and

$$
S^*(t) \geq \liminf_{t\to\infty} \tilde{S}(t) > 0, \quad I^*(t) \geq \liminf_{t\to\infty} \tilde{I}(t) > 0, \quad R^*(t) \geq \liminf_{t\to\infty} \tilde{R}(t) > 0,
$$

for all $t \in \mathbb{R}$.

It remains to show that $(S^*(t), I^*(t), R^*(t))$ is a T-periodic solution of (1.5). The periodicity is obvious since

$$
\begin{aligned}
S^*(t+T) &= \lim_{q\to\infty} \tilde{S}((t+T)+qT) = \lim_{q+1\to\infty} \tilde{S}(t+(q+1)T) = S^*(t),\\
I^*(t+T) &= \lim_{q\to\infty} \tilde{I}((t+T)+qT) = \lim_{q+1\to\infty} \tilde{I}(t+(q+1)T) = I^*(t),\\
R^*(t+T) &= \lim_{q\to\infty} \tilde{R}((t+T)+qT) = \lim_{q+1\to\infty} \tilde{R}(t+(q+1)T) = R^*(t),
\end{aligned}
$$

for all $t \in \mathbb{R}$. Now, note that $(\tilde{S}(t+qT), \tilde{I}(t+qT), \tilde{R}(t+qT))$ is a solution to (1.5), that is,

$$
\begin{aligned}
\tilde{S}(t+qT) - \tilde{S}(t_0+qT) = \int_{t_0}^{t} \Big[&A(s+qT) - \mathbf{d}(s+qT)\tilde{S}(s+qT)\\
&- \frac{\lambda(s+qT)\tilde{S}(s+qT)\tilde{I}(s+qT)}{1+k(s+qT)\tilde{I}(s+qT)}\Big]\, ds,\\
\tilde{I}(t+qT) - \tilde{I}(t_0+qT) = \int_{t_0}^{t} \Big[&\frac{\lambda(s+qT)\tilde{S}(s+qT)\tilde{I}(s+qT)}{1+k(s+qT)\tilde{I}(s+qT)}\\
&- (\mathbf{d}(s+qT) + \varepsilon(s+qT) + \mu(s+qT))\tilde{I}(s+qT)\\
&- \frac{\gamma(s+qT)\tilde{I}(s+qT)}{1+\alpha(s+qT)\tilde{I}(s+qT)}\Big]\, ds,\\
\tilde{R}(t+qT) - \tilde{R}(t_0+qT) = \int_{t_0}^{t} \Big[&\mu(s+qT)\tilde{I}(s+qT) + \frac{\gamma(s+qT)\tilde{I}(s+qT)}{1+\alpha(s+qT)\tilde{I}(s+qT)}\\
&\times \mathbf{d}(s+qT)\tilde{R}(s+qT)\Big]\, ds,
\end{aligned}
$$

for $t \ge t_0$. Letting $q \to \infty$ gives

$$S^*(t) - S^*(t_0) = \int_{t_0}^{t} \left[A(s) - \mathbf{d}(s)S^*(s) - \frac{\lambda(s)S^*(s)I^*(s)}{1+k(s)I^*(s)}\right] ds,$$

$$I^*(t) - I^*(t_0) = \int_{t_0}^{t} \left[\frac{\lambda(s)S^*(s)I^*(s)}{1+k(s)I^*(s)} - (\mathbf{d}(s)+\varepsilon(s)+\mu(s))I^*(s) - \frac{\gamma(s)I^*(s)}{1+\alpha(s)I^*(s)}\right] ds,$$

$$R^*(t) - R^*(t_0) = \int_{t_0}^{t} \left[\mu(s)I^*(s) + \frac{\gamma(s)I^*(s)}{1+\alpha(s)I^*(t)} - \mathbf{d}(s)R^*(s)\right] ds,$$

for $t \ge t_0$, so $(S^*(t), I^*(t), R^*(t))$ is a solution to (1.5) on $[t_0, \infty)$.

Finally, by Lemma 2.3, $(S^*(t), I^*(t), R^*(t))$ is globally exponentially stable. This completes the proof of Theorem 3.1. ∎

Remark 3.1. Assume that all parameters are constants. Then the autonomous SIR model (1.1) has exactly one endemic equilibrium which is globally exponentially stable.

4. An example. In this section, we will illustrate the existence and global exponential stability of positive periodic solutions for system (1.1) by simulations.

Let $A(t) = 20$, $\mathbf{d}(t) = 0.02$, $\lambda(t) = 2 \times 10^{-3} + 5 \times 10^{-4} \sin(\pi t/3)$, $k(t) = \alpha(t) = 0.5$, $\varepsilon(t) = 0.05$, $\mu(t) = 0.02$ and $\gamma = 0.05$. Then

$$L^S = \sup_{t\in\mathbb{R}} A(t)/\mathbf{d}(t) = 1000, \qquad l^S = \inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} = 800 > 0,$$

$$l^I = \inf_{t\in\mathbb{R}} \frac{1}{k(t)}\left[\frac{\lambda(t)\inf_{t\in\mathbb{R}} \frac{A(t)}{\mathbf{d}(t)+\lambda(t)/k(t)} - \gamma(t)}{\mathbf{d}(t)+\varepsilon(t)+\mu(t)} - 1\right] \ge 21 > 0,$$

$$\sup_{t\in\mathbb{R}}\left\{-\mathbf{d}(t) + \frac{\lambda(t)L^S}{(1+k(t)l^I)^2}\right\} \le \frac{10}{529} - \frac{1}{50} < 0,$$

$$\sup_{t\in\mathbb{R}}\left\{-[\mathbf{d}(t)+\varepsilon(t)+\mu(t)] + \frac{\lambda(t)}{k(t)} + \frac{\lambda(t)L^S}{(1+k(t)l^I)^2}\right\} \le \frac{10}{529} - \frac{17}{200} < 0.$$

This implies that

$$\begin{aligned} S'(t) &= 20 - 0.02S(t) - \frac{(2\times 10^{-3} + 5\times 10^{-4}\sin(\pi t/3))S(t)I(t)}{1+0.5I(t)}, \\ I'(t) &= \frac{(2\times 10^{-3} + 5\times 10^{-4}\sin(\pi t/3))S(t)I(t)}{1+0.5I(t)} - 0.09I(t) - \frac{0.05I(t)}{1+0.5I(t)}, \\ R'(t) &= 0.02I(t) + \frac{0.05I(t)}{1+0.5I(t)} - 0.01R(t), \end{aligned} \tag{4.1}$$

satisfies all the conditions in Theorem 3.1. Hence, system (4.1) has exactly one positive 6-periodic solution $(S^*(t), I^*(t), R^*(t))$. Moreover, it is globally exponentially stable with exponential convergence rate $\zeta \approx 0.002$. This fact is confirmed by the numerical simulations in Figures 1–3.

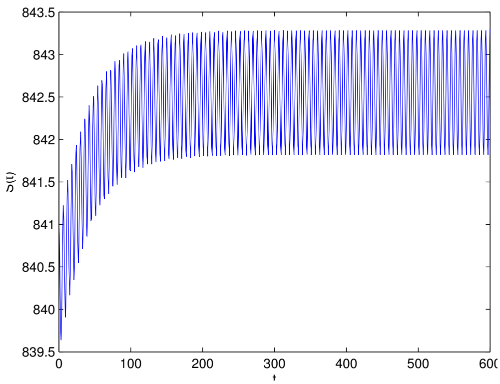

Fig. 1. Numerical solution $S(t)$ of (4.1) for $(S(0), I(0), R(0)) = (843, 28, 58)$

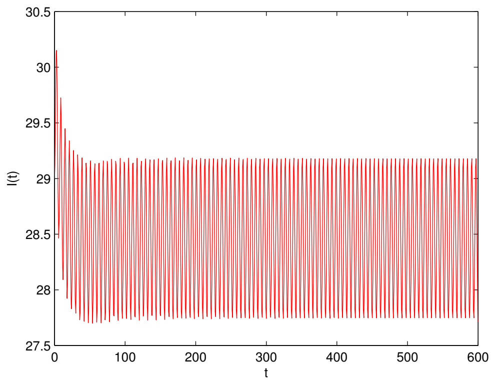

Fig. 2. Numerical solution $I(t)$ of (4.1) for $(S(0), I(0), R(0)) = (843, 28, 58)$

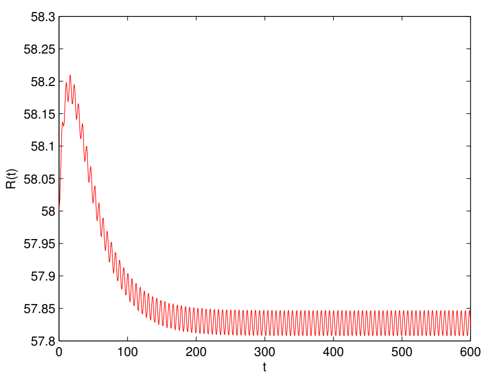

Fig. 3. Numerical solution $R(t)$ of (4.1) for $(S(0), I(0), R(0)) = (843, 28, 58)$

REMARK 4.1. To the best of our knowledge, there is no result on the global exponential stability of positive periodic solutions for the SIR model with periodic incidence rate and saturated treatment function. We also mention that the results in [AS, GF, KRG, LBJ] cannot be applied to the global exponential stability of positive periodic solutions for system (4.1). Here we employ a novel proof to establish some criteria which guarantee the existence and global exponential stability of positive periodic solutions for the SIR model.

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