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Dividing measures and narrow operators

by

Volodymyr Mykhaylyuk (Chernivtsi), Marat Pliev (Vladikavkaz),
Mikhail Popov (Słupsk and Chernivtsi) and
Oleksandr Sobchuk (Chernivtsi)

Abstract. We use a new technique of measures on Boolean algebras to investigate narrow operators on vector lattices. First we prove that, under mild assumptions, every finite rank operator is strictly narrow (before it was known that such operators are narrow). Then we show that every order continuous operator from an atomless vector lattice to a purely atomic one is order narrow. This explains in what sense the vector lattice structure of an atomless vector lattice given by an unconditional basis is far from its original vector lattice structure. Our third main result asserts that every operator such that the density of the range space is less than the density of the domain space, is strictly narrow. This gives a positive answer to Problem 2.17 from "Narrow Operators on Function Spaces and Vector Lattices" by B. Randrianantoanina and the third named author for the case of reals. All the results are obtained for a more general setting of (nonlinear) orthogonally additive operators.

1. Introduction. Narrow operators generalize compact operators defined on function spaces and vector lattices (see [PlichP] for the first systematic study in symmetric function spaces and [MMP] in vector lattices, and the recent monograph [PR]). Under mild assumptions on the domain space, every "small" operator is narrow (e.g., AM-compact and Dunford–Pettis operators, operators having "small" ranges, comparable to the domain). Some such assertions are easy to prove. For example, it is not hard to see that every AM-compact operator defined on a Köthe function space with an absolutely continuous norm on an atomless measure space is narrow [PR, Proposition 2.1]. However, to prove that every order-to-norm continuous AM-compact operator from an atomless Dedekind complete vector lattice to a Banach space is narrow requires much more effort [PR, Theorem 10.17].

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Recently the latter result was generalized to orthogonally additive laterally-to-norm continuous C-compact operators [PlievP]. Theorem 2.12 strengthens the latter result.

To make the introduction more self-contained, we recall some definitions. An F-*space* is a complete metric linear space X over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with an invariant metric, $\rho(x, y) = \rho(x+z, y+z)$ for all $x, y, z \in X$. The map $\| \cdot \| : X^2 \to \mathbb{K}$ given by $\|x\| \rho(x, 0)$ is called the F-*norm* of X. Since ρ is invariant, it is uniquely determined by the F-norm by the obvious formula $\rho(x, y) = \|x-y\|$. Let (Ω, Σ, μ) be a finite atomless measure space. We denote by $L_0(\mu)$ the linear space of all equivalence classes of measurable scalar-valued functions on Ω. An F-space E which is a linear subspace of $L_0(\mu)$ is called a *Köthe F-space* on (Ω, Σ, μ) if $\mathbf{1}_\Omega \in E$ and for every $x \in L_0(\mu)$ and every $y \in E$ the condition $|x| \le |y|$ (which is understood to hold μ-a.e. on Ω) implies that $x \in E$ and $\|x\| \le \|y\|$. Here and below, $\mathbf{1}_A$ denotes the characteristic function of a set $A \in \Sigma$. If, moreover, E is a Banach space and $E \subseteq L_1(\mu)$ then E is called a *Köthe Banach space* on (Ω, Σ, μ). For every $A \in \Sigma$ we set

$$E(A) = \{x \in E : \operatorname{supp} x \subseteq A\}.$$

A Köthe Banach space E on (Ω, Σ, μ) is said to have an *absolutely continuous norm* if $\lim_{\mu(A)\to 0} \|\mathbf{1}_A \cdot x\| = 0$ for all $x \in E$. If X, Y are F-spaces then $\mathcal{L}(X, Y)$ denotes the Banach space of all continuous linear operators from X to Y. We denote by $\operatorname{dens} X$ the *density* of an F-space X, that is, the minimal cardinality of a dense subset of X, and $\operatorname{H-dim} Z$ stands for the *algebraic dimension* of a linear space Z, that is, the cardinality of a Hamel basis of Z.

A function $f : E \to X$ from a Köthe F-space E on a finite atomless measure space (Ω, Σ, μ) to an F-space X is called *narrow* if for every $A \in \Sigma$ and every $\varepsilon > 0$ there is a decomposition $A = B \sqcup C$ (that is, $A = B \cup C$ and $B \cap C = \emptyset$) such that $\|f(\mathbf{1}_B) - f(\mathbf{1}_C)\| < \varepsilon$.

A function $f : E \to X$ is called *strictly narrow* if for every $A \in \Sigma$ there is a decomposition $A = B \sqcup C$ with $f(\mathbf{1}_B) = f(\mathbf{1}_C)$. Observe that a linear operator $T : E \to X$ is narrow if and only if for every $A \in \Sigma$ and every $\varepsilon > 0$ there is a decomposition $A = B \sqcup C$ with $\|Th\| < \varepsilon$ where $h = \mathbf{1}_B - \mathbf{1}_C$.

It is known that every continuous linear operator from L_p to ℓ_r is narrow, except for $p \ge 2 = r$ (see [MPRS] and [PR, Theorem 9.9]). Note that the proof for $p, r > 2$ is quite involved. We show (see Theorem 3.1) that for order bounded operators the same is true for arbitrary vector lattices provided the domain lattice is atomless and the range lattice is purely atomic.

It is well known that a Banach space X with a 1-unconditional basis (x_n) is a Banach lattice with respect to the order

$$\sum_{n=1}^{\infty} a_n x_n \ge 0 \quad \text{if and only if} \quad a_n \ge 0 \text{ for all } n \in \mathbb{N}.$$

As mentioned by Lindenstrauss and Tzafriri [LT2, p. 2], although the spaces $L_p = L_p[0,1]$ for $1 < p < \infty$ have an unconditional basis, the natural order in L_p (that is, $x \le y$ in L_p if and only if $x(t) \le y(t)$ for almost all $t \in [0,1]$) is completely different from the order induced by a basis. Theorem 3.1 makes this precise.

By [PR, Corollary 2.14], if a Köthe Banach space E on (Ω, Σ, μ) with an absolutely continuous norm satisfies $\operatorname{dens} X < \operatorname{dens} E(A)$ for every $A \in \Sigma$ with $\mu(A) > 0$ then every operator $T \in \mathcal{L}(E, X)$ is narrow. By [PR, Theorem 2.15], if, moreover, there is a reflexive Köthe Banach space E_1 on (Ω, Σ, μ) with continuous inclusion $E_1 \subseteq E$ and $\text{H-dim}\, X < \operatorname{dens} E(A)^{\aleph_0}$ for every $A \in \Sigma$ with $\mu(A) > 0$, then every operator $T \in \mathcal{L}(E, X)$ is strictly narrow. So, much restrictions give stronger result, and it was natural to ask the following question.

Problem 1.1 ([PR, Open Problem 2.17]). *Let E be a Köthe F-space with an absolutely continuous norm on (Ω, Σ, μ) and X be an F-space. Suppose that* $\operatorname{dens} E(A) > \operatorname{dens} X$ *for every $A \in \Sigma^+$. Does it follow that every operator $T \in \mathcal{L}(E, X)$ is strictly narrow?*

Our third main result (Theorem 4.1) partially answers this problem, for the case of reals, in a more general setting of orthogonally additive operators and vector lattices which are sublattices of $L_0(\mu)$. To prove this result, we generalize our main technical tool, the Lyapunov convexity theorem, to the nonseparable case. Finally, Theorem 4.11 gives a much more direct answer to Problem 1.1.

For standard information on Banach spaces we refer to [AlbKal], [LT1], [LT2], and for vector lattices to [AlBu], [Kus]. *All vector lattices below are assumed to be Archimedean.*

Investigation of narrow operators shows that one actually uses the additivity of an operator only for disjoint vectors. So, it is very natural to generalize results on narrow linear operators to orthogonally additive operators. The theory of orthogonally additive operators on vector lattices was developed by Mazón and Segura de León [MaLe1], [MaLe2], and of narrow orthogonally additive operators by the second and third named authors [PlievP]. Another kind of a generalization gives the notion of lateral continuity, weaker than order continuity.

The most general definitions of a narrow operator which are needed concern orthogonally additive operators from a vector lattice to a Banach space and to a vector lattice.

Let E be a vector lattice and X a vector space. A map $T : E \to X$ is called an *orthogonally additive operator*

$$\text{if } T(x+y) = T(x) + T(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

If, moreover, X is a vector lattice then an order bounded orthogonally additive operator is called an *abstract Urysohn operator*. An element y of a vector lattice E is called a *fragment* (in other terminology, a *component*) of an element $x \in E$ if $y \perp (x - y)$; we then write $y \sqsubseteq x$. A net $(x_\alpha)_{\alpha \in \Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{\mathrm{o}} x$) if there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in E such that $u_\alpha \downarrow 0$ and $|x_\beta - x| \le u_\beta$ for all $\beta \in \Lambda$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ if $i \neq j$. Note that in this case $x_i \sqsubseteq x$ for all i. If E is a vector lattice and $e \in E^+$ then we denote by $\mathfrak{F}_e$ the set of all fragments of e.

An element e of a vector lattice E is called a *projection element* if the band B_e generated by e is a projection band. A vector lattice E is said to have the *principal projection property* if every element of E is a projection element. For instance, every Dedekind σ-complete vector lattice has the principal projection property. Let E be a vector lattice with the principal projection property. For any $e \in E$, we denote by P_e the band projection of E onto the band B_e generated by e. So, P_e is a positive linear projection given by $P_e\, x = \bigvee_{n=1}^\infty (x \wedge ne)$ for all $x \in E^+$ [AlBu, p. 35]. One can easily verify that $P_e\, x \sqsubseteq x$ for all $e, x \in E$.

Definition 1.2. An element $u \neq 0$ of a vector lattice E is called

- an *atom* whenever $0 \le x \le |u|$, $0 \le y \le |u|$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$;
- a *weak atom* if $\mathfrak{F}_u = \{0, u\}$.

If $u \in E$ is an atom then either $u > 0$ or $u < 0$ [LuZa, Lemma 26.2(i)]. On the other hand, if u is an atom then $-u$ is obviously also an atom. So, for many purposes, it is enough to consider only positive atoms.

Proposition 1.3. *Let E be a vector lattice. Every atom $0 < u \in E$ is a weak atom. If E has the principal projection property then every weak atom in E is an atom.*

Proof. The first part is obvious. Let E have the principal projection property and let $0 < u \in E$ be a weak atom. Assume $0 \le x, y \le u$ and $x \wedge y = 0$. Then $P_x\, u, P_y\, u \in \mathfrak{F}_u = \{0, u\}$. Since $x \wedge y = 0$, we have $0 = P_{x \wedge y} = P_x P_y$ by [AlBu, Theorem 3.11]. Using the result of [AlBu, Ex. 9, p. 41], we obtain $P_x u \wedge P_y u = P_x P_y (u \wedge u) = 0$. Thus, $P_x\, u = P_y u = u$ is impossible. So, either $P_x\, u = 0$ or $P_y\, u = 0$. If $P_x\, u = 0$ then $0 \le x = u \wedge x \le P_x\, u = 0$, and hence $x = 0$. Analogously, if $P_y\, u = 0$ then $y = 0$. ∎

We need the following known property of atoms.

Proposition 1.4 ([LuZa, Theorem 26,4(ii)]). *For any atoms u, v in a vector lattice E, either $u \perp v$, or $v = \lambda u$ for some $0 \neq \lambda \in \mathbb{R}$.*

Definition 1.5. A vector lattice is said to be *atomless* if it has no atom. We say that a vector lattice E is *purely atomic* if there is a collection $(u_i)_{i \in I}$

of atoms in E^+, called a *generating collection of atoms*, such that $u_i \perp u_j$ for $i \neq j$ and for every $x \in E$, if $|x| \wedge u_i = 0$ for each $i \in I$ then $x = 0$.

By Proposition 1.4, a generating collection of atoms in a purely atomic vector lattice is unique up to a permutation and nonzero multiples.

Let E be a vector lattice. Consider any maximal collection of atoms $(u_i)_{i\in I}$ in E, the existence of which is guaranteed by Proposition 1.4 and Zorn's lemma. Let E_0 be the minimal band containing u_i for all $i \in I$. If E_0 is a projection band then $E = E_0 \oplus E_1$, where $E_1 = E_0^d$ is the disjoint complement to E_0 in E, which is an atomless sublattice of E. So, we obtain the following assertion.

Proposition 1.6. *Any vector lattice E with the principal projection property has a decomposition $E = E_0 \oplus E_1$ into mutually complemented bands where E_0 is a purely atomic vector lattice and E_1 is an atomless vector lattice.*

A net (x_α) in a vector lattice E *laterally converges* to $x \in E$ if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ for all $\alpha < \beta$ and $x_\alpha \xrightarrow{\text{o}} x$. In this case we write $x_\alpha \xrightarrow{\text{lat}} x$. For positive elements x_α, x the condition $x_\alpha \xrightarrow{\text{lat}} x$ means that $x_\alpha \sqsubseteq x$ and $x_\alpha \uparrow x$.

Following [PlievP] we introduce the next definition.

Definition 1.7. Let E be an atomless vector lattice and X a vector space. A map $T : E \to X$ is called:

- *strictly narrow* if for every $e \in E$ there exists a decomposition $e = f \sqcup g$ such that $T(f) = T(g)$;
- *narrow* if X is a normed space (or, more generally, an F-space), and for every $e \in E$ and every $\varepsilon > 0$ there exists a decomposition $e = f \sqcup g$ such that $\|T(e_1) - T(e_2)\| < \varepsilon$;
- *order narrow* if X is a vector lattice, and for every $e \in E$ there exists a net of decompositions $e = f_\alpha \sqcup g_\alpha$ such that $T(f_\alpha) - T(g_\alpha) \xrightarrow{\text{o}} 0$.

For linear maps Definition 1.7 is equivalent to the corresponding definitions of strictly narrow, narrow and order narrow operators given in [PR]. For general (not necessarily linear) maps Definition 1.7 is somewhat different, but it works in such a way that the main theorems on narrow linear operators remain true for orthogonally additive operators. The atomlessness condition on E in the above definition is not essential, but a narrow map sends atoms to zero, so the condition serves to avoid trivialities.

We remark that, for an orthogonally additive operator $T : E \to X$ in each part of Definition 1.7 it is sufficient to consider any $e \in E^+ \cup E^-$ instead of any $e \in E$ [PlievP, Proposition 2.3]. However, the map $T(e) = e^-$ of taking the negative part of an element $e \in E$ is an orthogonally additive operator satisfying each part of Definition 1.7 for any $e \in E^+$, but it is not narrow in any sense.

A map T from a vector lattice E to a Banach space X is called:

- *laterally-to-norm σ-continuous* if T sends laterally convergent sequences in E to norm convergent sequences in X;
- *laterally-to-norm continuous* provided T sends laterally convergent nets in E to norm convergent nets in X.

A map $T : E \to F$ between vector lattices E and F is called *laterally continuous* if T sends laterally convergent nets in E to order convergent nets in F.

The reader can find the necessary information on Boolean algebras for instance in [Jech], [Kus], [LuZa]. The most common example of a Boolean algebra is an algebra $\mathcal{A}$ of subsets of a set Ω, that is, a subset of the power-set $\mathcal{P}(\Omega)$ of all subsets of Ω, closed under union, intersection and complementation and containing $\emptyset$ and Ω. The Boolean operations on $\mathcal{A}$ are $A \vee B = A \cup B$, $A \wedge B = A \cap B$ and $\neg A = \Omega \setminus A$, and the constants are $\mathbf{0} = \emptyset$, $\mathbf{1} = \Omega$.

A map $h : \mathcal{A} \to \mathcal{B}$ between Boolean algebras is called a *Boolean homomorphism* if for all $x, y \in \mathcal{A}$:

(1) $h(\mathbf{0}) = \mathbf{0}$;
(2) $h(\mathbf{1}) = \mathbf{1}$;
(3) $h(x \vee y) = h(x) \vee h(y)$;
(4) $h(x \wedge y) = h(x) \wedge h(y)$;
(5) $h(\neg x) = \neg h(x)$.

A bijective Boolean homomorphism is called a *Boolean isomorphism.* Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Boolean isomorphic* if there is a Boolean isomorphism $h : \mathcal{A} \to \mathcal{B}$. The following remarkable result is known as the *Stone representation theorem.*

Theorem 1.8 ([Jech, Theorem 7.11]). *Every Boolean algebra is Boolean isomorphic to an algebra of subsets of some set.*

Any Boolean algebra $\mathcal{A}$ is a partially ordered set with respect to the partial order $x \le y$ if and only if $x \wedge y = x$, with respect to which $\mathbf{0}$ is the least element, $\mathbf{1}$ is the greatest element, $x \wedge y$ is the infimum and $x \vee y$ the supremum of the two-point set $\{x, y\}$ in $\mathcal{A}$. A Boolean algebra $\mathcal{A}$ is called *Dedekind complete* (resp., *σ-Dedekind complete*) if so is $\mathcal{A}$ as a partially ordered set, that is, every (resp., countable) order bounded nonempty subset of $\mathcal{A}$ has the least upper and the greatest lower bounds in $\mathcal{A}$. Obviously, a Boolean algebra is σ-Dedekind complete if and only if it is a σ-algebra.

There is a natural connection between Boolean algebras and vector lattices: if E is a vector lattice and $e \in E^+$ then the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with respect to the vector lattice operations $\vee$, $\wedge$ in E and $\mathbf{0} = 0$, $\mathbf{1} = e$, $\neg x = e - x$ [AlBu, Theorem 3.15]. It is then immediate to verify that for any $e \in E$ the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with respect to the operations $x \vee y = (x^+ \vee y^+) - (x^- \vee y^-)$, $x \wedge y = (x^+ \wedge y^+) - (x^- \wedge y^-)$ in E and $\mathbf{0} = 0$, $\mathbf{1} = e$, $\neg x = e - x$.

2. Dividing measures on Boolean algebras. By a *measure* on a Boolean algebra $\mathcal{A}$ we mean a finitely additive function $\mu : \mathcal{A} \to X$ of $\mathcal{A}$ to a vector space X, that is, a map satisfying

$$(\forall x, y \in \mathcal{A})\big(x \wedge y = \mathbf{0} \Rightarrow \mu(x + y) = \mu(x) + \mu(y)\big).$$

In other words, a measure is any orthogonally additive operator defined on a Boolean algebra. If moreover $\mathcal{A}$ is a Boolean σ-algebra and X is a topological vector space then a measure $\mu : \mathcal{A} \to X$ is σ-*additive* if whenever $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ with $x_n \uparrow x \in \mathcal{A}$ then $\lim_{n\to\infty} \mu(x_n) = \mu(x)$.

2.1. Definitions and simple properties. We give different definitions of dividing measures depending on the range space and the "degree" of divisibility.

Definition 2.1. Let $\mathcal{A}$ be a Boolean algebra and X a vector space. A measure $\mu : \mathcal{A} \to X$ is called *dividing* if for every $x \in \mathcal{A}$ there is a decomposition $x = y \sqcup z$ with $\mu(y) = \mu(z)$.

Observe that every atomless σ-additive scalar measure on a σ-algebra is dividing.

Definition 2.2. Let $\mathcal{A}$ be a Boolean algebra and X a normed space. A measure $\mu : \mathcal{A} \to X$ is called *almost dividing* if for every $x \in \mathcal{A}$ and every $\varepsilon > 0$ there is a decomposition $x = y \sqcup z$ with $\|\mu(y) - \mu(z)\| < \varepsilon$.

Definition 2.3. Let $\mathcal{A}$ be a Boolean algebra and X a vector lattice. A measure $\mu : \mathcal{A} \to X$ is called *order dividing* if for every $x \in \mathcal{A}$ there is a net of decompositions $x = y_\alpha \sqcup z_\alpha$ with $\mu(y_\alpha) - \mu(z_\alpha) \xrightarrow{\mathrm{o}} 0$.

Definition 2.4. Let E be a vector lattice and X a vector space. To every orthogonally additive operator $T : E \to X$ we associate a family $(\mu_e^T)_{e\in E}$ of measures as follows. Given any $e \in E$, we define a measure $\mu_e^T : \mathfrak{F}_e \to X$ on the Boolean algebra $\mathfrak{F}_e$ of fragments of e by setting $\mu_e^T x = T(x)$, called the *associated measure* of T at e.

The next proposition directly follows from the definitions.

Proposition 2.5. *Let E be an atomless vector lattice, X a vector space and $T : E \to X$ an orthogonally additive operator. Then:*

(1) *T is strictly narrow if and only if the measure μ_e^T is dividing for every $e \in E$.*
(2) *Let X be a normed space. Then T is narrow if and only if μ_e^T is almost dividing for every $e \in E$.*

(3) *Let X be a vector lattice. Then T is order narrow if and only if μ_e^T is order dividing for every $e \in E$.*

Obviously, a dividing measure is both almost dividing and order dividing, for an appropriate range space. The following three propositions are close to [PR, Propositions 10.7 and 10.9, and Example 10.8].

PROPOSITION 2.6. *Let $\mathcal{A}$ be a Boolean algebra and X a Banach lattice. Then every almost dividing measure $\mu : \mathcal{A} \to X$ is order dividing.*

Proof. Let $\mu : \mathcal{A} \to X$ be an almost dividing measure and $x \in \mathcal{A}$. Choose a sequence of decompositions $x = y_n \sqcup z_n$ with $\|\mu(y_n) - \mu(z_n)\| \le 2^{-n}$. Then for $z_n = \sum_{k=n}^{\infty} |\mu(y_n) - \mu(z_n)|$ one has $|\mu(y_n) - \mu(z_n)| \le z_n \downarrow 0$. Hence, $\mu(y_n) - \mu(z_n) \xrightarrow{\mathrm{o}} 0$. ∎

PROPOSITION 2.7. *Let Σ be the Boolean σ-algebra of Lebesgue measurable subsets of $[0, 1]$. Then there exists an order dividing measure $\mu : \Sigma \to L_\infty$ which is not dividing.*

Proof. Use Proposition 2.5 and [PR, Example 10.8]. ∎

PROPOSITION 2.8. *Let $\mathcal{A}$ be a Boolean algebra and X an order continuous Banach lattice. Then a measure $\mu : \mathcal{A} \to X$ is order dividing if and only if it is almost dividing.*

Proof. Let $\mu : \mathcal{A} \to X$ be order dividing. Given any $x \in \mathcal{A}$, let $x = y_\alpha \sqcup z_\alpha$ be a net of decompositions with $\mu(y_\alpha) - \mu(z_\alpha) \xrightarrow{\mathrm{o}} 0$. By the order continuity of X, $\|\mu(y_\alpha) - \mu(z_\alpha)\| \to 0$ and hence, μ is almost dividing by arbitrariness of $x \in \mathcal{A}$. By Proposition 2.6, the proof is complete. ∎

A nonzero element u of a Boolean algebra $\mathcal{A}$ is called an *atom* if for every $x \in \mathcal{A}$ the condition $0 < x \le u$ implies that $x = u$. Every dividing measure (of any type) sends atoms to zero.

PROPOSITION 2.9. *Let $\mathcal{A}$ be a Boolean algebra and X a vector space* (*a normed space, or a vector lattice*) *and $\mu : \mathcal{A} \to X$ a dividing measure* (*an almost dividing measure or an order dividing measure, respectively*). *If $a \in \mathcal{A}$ is an atom then $\mu(a) = 0$.*

The proof is an easy exercise.

2.2. The range convexity of measures. We need the following remarkable result known as the Lyapunov [1] convexity theorem.

THEOREM 2.10 ([LT2, Theorem 2.c.9]). *Let (Ω, Σ) be a measurable space, X a finite-dimensional normed space and $\mu : \Sigma \to X$ an atomless σ-additive measure. Then the range $\mu(\Sigma) = \{\mu(A) : A \in \Sigma\}$ of μ is a compact convex subset of X.*

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[1] Lyapounoff in the old spelling.

Using Stone's and Lyapunov's theorems we obtain the following result.

THEOREM 2.11. *Let $\mathcal{A}$ be a Boolean σ-algebra and X a finite-dimensional normed space. Then every atomless σ-additive measure $\mu : \mathcal{A} \to X$ is dividing.*

For a Boolean σ-algebra $\mathcal{A}$ and $x \in \mathcal{A} \setminus \{0\}$ we denote by $\mathcal{A}_x$ the Boolean σ-algebra $\{y \in \mathcal{A} : y \le x\}$ with unit $\mathbf{1}_{\mathcal{A}_x} = x$ and the operations induced from $\mathcal{A}$.

Proof. Let $\mu : \mathcal{A} \to X$ be an atomless σ-additive measure and $x \in \mathcal{A}$. If $x = 0$ then there is nothing to prove. Let $x \neq 0$. Then the restriction $\mu_x = \mu|_{\mathcal{A}_x} : \mathcal{A}_x \to X$ is an atomless σ-additive measure. By Theorem 1.8, $\mathcal{A}_x$ is Boolean isomorphic to some measurable space (Ω, Σ) by means of some Boolean isomorphism $J : \mathcal{A}_x \to \Sigma$. Since $\mathcal{A}_x$ is a Boolean σ-algebra, Σ is a σ-algebra. Then the map $\nu : \Sigma \to X$ given by $\nu(A) = \mu(J^{-1}(A))$ for all $A \in \Sigma$, is an atomless σ-additive measure. By Theorem 2.10, the range $\nu(\Sigma)$ of ν is a convex subset of X. In particular, since $0, \nu(J(x)) \in \nu(\Sigma)$, we see that $\nu(J(x))/2 \in \nu(\Sigma)$. Let $B \in \Sigma$ be such that $\nu(B) = \nu(J(x))/2 = \mu(x)/2$. Then for $y = J^{-1}(B)$ one has $y \le x$ and $\mu(y) = \nu(B) = \mu(x)/2$. Thus, for $z = x \wedge \neg y$ one has $x = y \sqcup z$ and $\mu(z) = \mu(x) - \mu(y) = \mu(x)/2 = \mu(y)$. ∎

2.3. Strict narrowness of laterally continuous finite rank operators. The main result of this subsection strengthens [MMP, Theorem 5.1] and [PlievP, Theorem 3.2]. Using a completely different method based on the Lyapunov theorem, we prove the strict narrowness of an operator.

THEOREM 2.12. *Let E be an atomless vector lattice with the principal projection property, and X a finite-dimensional normed space* (*resp., vector lattice*)*. Then every laterally-to-norm continuous* (*resp., laterally continuous*) *orthogonally additive operator $T : E \to X$ is strictly narrow.*

To use the technique of dividing measures, we preliminarily need the σ-additivity of a measure.

LEMMA 2.13. *Let E be an atomless Dedeking complete vector lattice, X a Banach space* (*resp., a vector lattice*) *and $T : E \to X$ a laterally-to-norm continuous* (*resp., laterally continuous*) *orthogonally additive operator. Then for every $0 \neq e \in E^+$ the associated measure μ_e^T is atomless and σ-additive.*

Proof. Fix any $0 \neq e \in E^+$. The σ-additivity of μ_e^T directly follows from the lateral continuity. We show that μ_e^T is atomless. Assume $x_0 \in \mathfrak{F}_e$ and $\mu_e^T(x_0) \neq 0$, that is, $T(x_0) \neq 0$. Set $Z = \{x \in \mathfrak{F}_{x_0} : T(x) = 0\}$. By the lateral continuity and Zorn's lemma, Z has a maximal element $z \in Z$. Since $T(z) = 0$, one has $T(x_0 - z) = T(z) + T(x_0 - z) = T(x_0) \neq 0$. Since E is atomless, we split $x_0 - z = u \sqcup v$ with $u, v \in \mathfrak{F}_{x_0} \setminus \{0\}$. By maximality of z,

$T(u) \neq 0$ and $T(v) \neq 0$. Thus, $x_0 = (z+u) \sqcup v$ is a decomposition with $\mu_e^T(z+u) = \mu_e^T(u) \neq 0$ and $\mu_e^T(y) \neq 0$. ∎

Proof of Theorem 2.12. Let $e \in E^+$. By Lemma 2.13, the associated measure $\mu_e^T : \mathfrak{F}_e \to X$ is atomless and σ-additive. By Theorem 2.11, μ_e^T is dividing. So, we split $e = f \sqcup g$ with $\mu_e^T(f) = \mu_e^T(g)$, that is, $T(f) = T(g)$. ∎

3. Operators from atomless to purely atomic vector lattices are order narrow. The following theorem is one of the main results of our paper.

THEOREM 3.1. *Let E, F be vector lattices possessing the principal projection property with E atomless and F purely atomic. Then every laterally continuous abstract Urysohn operator $T : E \to F$ is order narrow.*

For the proof we need an auxiliary result giving a representation of an element of a purely atomic vector lattice via atoms.

Let F be a purely atomic vector lattice with the principal projection property and a generating collection $(u_i)_{i\in I}$ of positive atoms. Let Λ denote the directed set of all finite subsets of I ordered by inclusion, that is, $\alpha \leq \beta$ for $\alpha, \beta \in \Lambda$ if and only if $\alpha \subseteq \beta$. For every $\alpha \in \Lambda$ we set

$$\mathbf{P}_\alpha = \sum_{i\in\alpha} P_{u_i}, \tag{3.1}$$

where P_{u_i} is the band projection of F onto the band generated by u_i. It is immediate that $\mathbf{P}_\alpha$ is the band projection of F onto the band generated by $\{u_i : i \in \alpha\}$, and $\mathbf{P}_\alpha = \bigvee_{i\in\alpha} P_{u_i}$.

THEOREM 3.2. *Let F be a purely atomic vector lattice with the principal projection property and a generating collection $(u_i)_{i\in I}$ of positive atoms. If $f \in F$ then*

(1) *for every $i \in I$ there is $a_i \in \mathbb{R}$ such that $P_{u_i} f = a_i u_i$;*
(2) *$\mathbf{P}_\alpha f \xrightarrow{\mathrm{o}} f$ for all $\alpha \in \Lambda$;*
(3) *$f = \bigvee_{i\in I} P_{u_i} f^+ - \bigvee_{i\in I} P_{u_i} f^-$.*

Proof. (1) is well known (see e.g. [AbAl, Lemma 2.10]).

(2) By the obvious argument, it is enough to consider the case where $f \geq 0$. Since $\mathbf{P}_\alpha f\uparrow$, it suffices to prove that $\bigvee_{\alpha\in\Lambda} \mathbf{P}_\alpha f = f$. Since $\mathbf{P}_\alpha$ is a band projection, $\mathbf{P}_\alpha f \leq f$ for all $\alpha \in \Lambda$. Let $\mathbf{P}_\alpha f \leq g$ for some $g \in F$ and all $\alpha \in \Lambda$. Our aim is to prove that $f \leq g$. In particular, for $\alpha = \{i\}$ we obtain $P_{u_i} f \leq g$ for all $i \in I$, hence $P_{u_i} f \leq P_{u_i} g$ for all $i \in I$. Therefore, by (1), there are reals $a_i \leq b_i$ such that $P_{u_i} f = a_i u_i$ and $P_{u_i} g = b_i u_i$ for all $i \in I$. Since $P_{u_i} h \leq h$ for all $h \in F^+$, P_{u_i} is a disjointness preserving operator, and hence $(P_{u_i} z)^+ = P_{u_i}(z^+)$ for every $z \in F$. Taking into account

that $P_{u_i}f \le P_{u_i}g$, we get

$$0 = (P_{u_i}f - P_{u_i}g)^+ = (P_{u_i}(f-g))^+ = P_{u_i}(f-g)^+ = \bigvee_{m=1}^{\infty} (f-g)^+ \wedge mu_i$$

for all $i \in I$. In particular, $(f-g)^+ \wedge u_i = 0$ for all $i \in I$. By the definition of a generating collection of atoms, $(f-g)^+ = 0$, that is, $f \le g$.

(3) Actually, we have proved right above that if $f \ge 0$ then $f = \bigvee_{i \in I} P_{u_i} f$, which is also enough to prove (3). ∎

Proof of Theorem 3.1. Let $T : E \to F$ be a laterally continuous abstract Urysohn operator. Fix any $e \in E^+$. Since the set $\mathfrak{F}_e$ of all fragments of e is order bounded in E, its image $T(\mathfrak{F}_e)$ is order bounded in F, say $|T(x)| \le f$ for some $f \in F^+$ and all $x \sqsubseteq e$.

Let $(u_i)_{i \in I}$ be a generating collection of positive atoms of F, Λ the directed set of all finite subsets of I ordered by inclusion, and $(\mathbf{P}_\alpha)_{\alpha \in \Lambda}$ the net of band projections of F defined by (3.1). By Theorem 3.2(1), $\mathbf{P}_\alpha$ is a finite rank operator for every $\alpha \in \Lambda$. Being a band projection, $\mathbf{P}_\alpha$ is order continuous. Then for each $\alpha \in \Lambda$ the composition operator $S_\alpha = \mathbf{P}_\alpha \circ T$ is a finite rank laterally continuous abstract Urysohn operator which is strictly narrow by Theorem 2.12. So, for each $\alpha \in \Lambda$ we choose a decomposition $e = e'_\alpha \sqcup e''_\alpha$ with $S_\alpha(e'_\alpha) = S_\alpha(e''_\alpha)$. Then

$$\begin{aligned} |T(e'_\alpha) - T(e''_\alpha)| &= |(I - \mathbf{P}_\alpha) \circ T(e'_\alpha) - (I - \mathbf{P}_\alpha) \circ T(e''_\alpha)| \\ &\le |(I - \mathbf{P}_\alpha) \circ T(e'_\alpha)| + |(I - \mathbf{P}_\alpha) \circ T(e''_\alpha)| \\ &\le (I - \mathbf{P}_\alpha)|T(e'_\alpha)| + (I - \mathbf{P}_\alpha)|T(e''_\alpha)| \\ &\le 2(I - \mathbf{P}_\alpha)(f) \xrightarrow{\text{o}} 0 \end{aligned}$$

by Theorem 3.2(2). ∎

Obviously, if $E \ne \{0\}$ is an atomless vector lattice then the identity operator on E is not order narrow, and neither is any lattice isomorphism from E to any vector lattice F. So, Theorem 3.1 makes it clear in what sense atomless vector lattices are far from purely atomic vector lattices. In particular, the new lattice structure on an atomless Banach lattice given by a 1-unconditional basis is far from its original lattice structure.

4. Operators with small ranges are strictly narrow. Let $|A|$ denote the cardinality of a set A. The main result of the section is the following.

THEOREM 4.1. *Let (Ω, Σ, μ) be a finite atomless measure space, E an atomless Dedekind complete vector sublattice of $L_0(\mu)$, and X a real F-space such that $|\mathfrak{F}_e| >$ H-dim X for all $e \in E$ with $e > 0$. Then every laterally continuous orthogonally additive operator $T : E \to X$ is strictly narrow.*

The proof needs some auxiliary work.

4.1. A generalization of the Lyapunov convexity theorem. Let (Ω, Σ, μ_0) be a finite measure space, X a Banach space and $\mu : \Sigma \to X$ a σ-additive measure. The *variation* of μ is the scalar measure $|\mu|$ defined on Σ by

$$|\mu|(A) = \sup\Big\{ \sum_{k=1}^{m} \|\mu(A_k)\| : m \in \mathbb{N},\, A_k \in \Sigma,\, A = \bigsqcup_{k=1}^{m} A_k \Big\},$$

which in general takes values from $[0, \infty]$. A σ-additive measure $\mu : \Sigma \to X$ is said to be

- of *finite variation* if $|\mu|(\Omega) < \infty$;
- *absolutely continuous* with respect to μ_0 if for every $A \in \Sigma$ the equality $\mu_0(A) = 0$ implies $\mu(A) = 0$.

Next we recall a generalization of the Lyapunov convexity theorem to measures valued in nonseparable Banach spaces, due to the third named author and V. Kadets.

THEOREM 4.2 ([KP]). *Let (Ω, Σ, μ_0) be a finite atomless measure space and*

$$\aleph_\alpha = \min\{\operatorname{dens} L_1(A) : A \in \Sigma,\, \mu_0(A) > 0\}.$$

Let X be a real Banach space with H-dim $X < \aleph_\alpha^{\aleph_0}$ *and let $\mu : \Sigma \to X$ be a σ-additive measure of finite variation absolutely continuous with respect to μ_0. Then the range $\mu(\Sigma)$ of μ is convex.*

We need a further generalization of the Lyapunov convexity theorem, which strengthens Theorem 4.2 by removing the condition of finiteness of $|\mu|$. Given a finite atomless measure space (Ω, Σ, μ_0) and $A \in \Sigma$, we denote by $\widetilde{\Sigma}(A)$ the set of all equivalence classes of μ_0-measurable subsets of A.

THEOREM 4.3. *Let (Ω, Σ, μ_0) be a finite atomless measure space and X a real F-space. Set $\aleph_\alpha = \min\{|\widetilde{\Sigma}(A)| : A \in \Sigma,\, \mu_0(A) > 0\}$. If* H-dim $X < \aleph_\alpha$ *then the range $\mu(\Sigma)$ of every σ-additive measure $\mu : \Sigma \to X$ absolutely continuous with respect to μ_0 is a convex subset of X.*

The following lemma is our main technical tool.

LEMMA 4.4. *Let (Ω, Σ, μ_0) be a finite atomless measure space, X a real F-space and $\mu : \Sigma \to X$ a σ-additive measure absolutely continuous with respect to μ_0. Then there exists a symmetric Banach space E on (Ω, Σ, μ_0) with an absolutely continuous norm such that*

1. *$L_\infty(\mu_0) \subseteq E$ and $L_\infty(\mu_0)$ is dense in E;*
2. *the linear operator $T : E \to X$ such that $T(\mathbf{1}_A) = \mu(A)$ for every $A \in \Sigma$ is bounded.*

Proof. It is sufficient to consider the case of a nonzero measure μ with $\mu_0(\Omega) = 1$. For every $t \in (0,1]$ set $\lambda(t) = \sup\{\|\mu(A)\| : \mu_0(A) \le t\}$. Note that $\lim_{t\to 0} \lambda(t) = 0$ and $\lambda(t) > 0$ for every $t \in (0,1]$. Moreover, if $s,t \in (0,1]$, $n \in \mathbb{N}$ and $s \ge (1/n)t$ then $\lambda(s) \ge (1/n)\lambda(t)$. Indeed, let $\mu_0(A) \le t$. Choose a family $(A_i : 1 \le i \le n)$ of measurable sets $A_i \subseteq A$ such that $A = \bigsqcup_{i=1}^n A_i$ and $\mu_0(A_i) = (1/n)\mu_0(A)$ for every $i \le n$. Then

$$\lambda(s) \ge \max\{\|\mu(\mathbf{1}_{A_i})\| : 1 \le i \le n\} \ge (1/n)\|\mu(A)\|.$$

Thus, $\lambda(s) \ge (1/n)\lambda(t)$.

Now let $0 < s < t \le 1$. We will show that $\lambda(s) \ge \frac{s}{2t}\lambda(t)$. Choose $n \in \mathbb{N}$ such that $s/t \in [1/(n+1), 1/n)$. Then we have

$$\lambda(s) \ge \frac{1}{n+1}\lambda(t) \ge \frac{1}{2n}\lambda(t) \ge \frac{s}{2t}\lambda(t).$$

Denote by K the convex hull of the set $\left\{\frac{1}{\lambda(\mu_0(A))}\mathbf{1}_A : A \in \Sigma^+\right\}$. Set

$$B_0 = \{y \in L_0(\mu_0) : \exists x \in K,\ |y| \le x\} \quad \text{and} \quad E_0 = \bigcup_{n=1}^{\infty} nB_0.$$

It is clear that E_0 is symmetric and $L_\infty(\mu_0) \subseteq E_0$. For every $x \in E_0$ set $\|x\| = \inf\{\alpha > 0 : x \in \alpha B_0\}$. Note that $\|\mathbf{1}_A\| \le \lambda(\mu_0(A))$. Since $\lambda(t) \ge (t/2)\lambda(1)$, we have

$$\int_\Omega \frac{1}{\lambda(\mu_0(A))}\mathbf{1}_A \, d\mu_0 \le \frac{2}{\lambda(1)}$$

and $\int_\Omega |y| \, d\mu_0 \le 2/\lambda(1)$ for every $y \in B_0$.

Therefore,

$$\|x\| \ge \frac{\lambda(1)}{2}\int_\Omega |x| \, d\mu_0 = \frac{\lambda(1)}{2}\|x\|_1.$$

Moreover, $\|x\|_\infty \cdot \mathbf{1}_\Omega \in \lambda(1)\|x\|_\infty B_0$ and $\|x\| \le \lambda(1)\|x\|_\infty$.

For every $x \in L_1(\mu_0)$ and $r > 0$ set

$${}^r x(t) = \begin{cases} x(t), & |x(t)| \le r, \\ r, & x(t) > r, \\ -r, & x(t) < -r. \end{cases}$$

Let E be the space of all $x \in L_1(\mu_0)$ for which $\sup\{\|{}^r x\| : r > 0\} < \infty$. Clearly E satisfies (1). We shall show that E is a Banach space with respect to the norm $\|x\| = \lim_{r\to\infty} \|{}^r x\|$. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in E. Note that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $L_1(\mu_0)$. Therefore there exists $x \in L_1(\mu_0)$ such that $x = \lim_{n\to\infty} x_n$ in $L_1(\mu_0)$. Since $\lim_{\mu_0(A)\to 0} \|\mathbf{1}_A\| = 0$, Egorov's theorem implies that $\lim_{n\to\infty} {}^r x_n = {}^r x$ for every $r > 0$. Thus $\sup\{\|{}^r x\| : r > 0\} \le \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$ and $x \in E$. It remains to prove that $\lim_{n\to\infty} \|y_n\| = 0$, where $y_n = x - x_n$. Fix any $\varepsilon > 0$ and choose $N \in \mathbb{N}$

so that $\|y_n - y_m\| < \varepsilon/2$ for all $n, m \geq N$. Suppose that $\|y_n\| \geq \varepsilon$ for some $n \geq N$ and choose $r > 0$ so that $\|{}^r y_n\| > 3\varepsilon/4$. Since $\lim_{m\to\infty} {}^r y_m = 0$, there exists $m \geq N$ such that $\|{}^r y_m\| < \varepsilon/4$. Then

$$\|y_n - y_m\| \geq \|{}^r y_n - {}^r y_m\| \geq \|{}^r y_n\| - \|{}^r y_m\| > \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

a contradiction.

Now we prove that $\lim_{\mu_0(A)\to 0} \|x\mathbf{1}_A\| = 0$ for every $x \in E$, that is, the norm of E is absolutely continuous. Let $x \in X$ and $\varepsilon, r > 0$ be such that $\|x - {}^r x\| < \varepsilon/2$. Since $\lim_{\mu_0(A)\to 0} \|\mathbf{1}_A\| = 0$, there exists $\delta > 0$ such that $\|\mathbf{1}_A\| < \varepsilon/(2r)$ if $\mu_0(A) < \delta$. Then

$$\|x\mathbf{1}_A\| \leq \|{}^r x\mathbf{1}_A\| + \|(x - {}^r x)\mathbf{1}_A\| \leq r\|\mathbf{1}_A\| + \|x - {}^r x\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

To verify condition (2), it is sufficient to prove that the operator $T : E \to X$ is bounded on the set

$$K_0 = \Big\{x = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k} \in E : \exists y \in K,\ |x| \leq y\}.$$

Let $x = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k} \in E$, $\alpha_k > 0$ for every $k \leq n$, $A_{k+1} \subseteq A_k$ for every $k \leq n-1$, $A \in \Sigma^+$ such that $x \leq \frac{1}{\lambda(\mu_0(A))}\mathbf{1}_A$, i.e.

$$\sum_{k=1}^{n} \alpha_k \leq \frac{1}{\lambda(\mu_0(A))}.$$

Since $\lambda(\mu_0(A_k)) \leq \lambda(\mu_0(A))$ for every $k \leq n$, we have

$$\|Tx\| \leq \sum_{k=1}^{n} \alpha_k \|T(\mathbf{1}_{A_k})\| = \sum_{k=1}^{n} \alpha_k \|\mu(A_k)\| \leq \sum_{k=1}^{n} \alpha_k \lambda(\mu_0(A_k)) \leq 1.$$

So, for any simple function $x = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k} \in E$ with $|x| \leq \frac{1}{\lambda(\mu_0(A))}\mathbf{1}_A$ we have $\|Tx\| \leq 2$.

Now let $x \in K_0$, $y \in K$ with $|x| \leq y$. Note that

$$y = \sum_{k=1}^{n} \frac{\alpha_k}{\mu_0(A_k)} \mathbf{1}_{A_k} \in E,$$

where $\alpha_k > 0$ and $\sum_{k=1}^{n} \alpha_k = 1$. Choose $x_1, \ldots, x_n \in K_0$ such that

$$x = \sum_{k=1}^{n} x_k \quad \text{and} \quad |x_k| \leq \frac{\alpha_k}{\mu_0(A_k)} \mathbf{1}_{A_k} \text{ for } k \leq n.$$

Finally we obtain

$$\|Tx\| \leq \sum_{k=1}^{n} \|Tx_k\| \leq 2 \sum_{k=1}^{n} \alpha_k = 2. \blacksquare$$

Proof of Theorem 4.3. Let $\mu : \Sigma \to X$ be a σ-additive measure absolutely continuous with respect to μ_0. By Lemma 4.4, there exists a symmetric Banach space E on (Ω, Σ, μ_0) with an absolutely continuous norm such that $L_\infty(\mu_0) \subseteq E$, $L_\infty(\mu_0)$ is dense in E and the linear operator $T : E \to X$ such that $T(\mathbf{1}_A) = \mu(A)$ for every $A \in \Sigma$ is bounded.

Now set $Y = \ker T$. We will show that $Y \cap L_\infty(A) \neq \{0\}$ for every $A \in \Sigma$ with $\mu_0(A) > 0$. Fix any $B \in \Sigma$ with $\mu_0(B) > 0$. Let $\mathcal{A}$ be a maximal system of sets $A \in \Sigma(B)$ such that the system $(\mu(A))_{A\in\mathcal{A}}$ is linearly independent. Observe that $|\mathcal{A}| \leq \text{H-dim}\, X < \aleph_\alpha \leq |\widetilde{\Sigma}(B)|$. Let $\mathcal{K}$ be the ring of sets generated by $\mathcal{A}$. If $\mathcal{K}$ is finite then $|\mathcal{K}| < |\widetilde{\Sigma}(B)|$. If $\mathcal{K}$ is infinite then $\mathcal{A}$ is infinite and hence $|\mathcal{K}| = |\mathcal{A}| < |\widetilde{\Sigma}(B)|$. Hence there are $m \in \mathbb{N}$, $a_1, \ldots, a_{m-1} \in \mathbb{R}$, $B_1, \ldots, B_{m-1} \in \mathcal{A}$ and $B_m \in \widetilde{\Sigma}(B) \setminus \mathcal{K}$ such that $\sum_{j=1}^{m-1} a_j \mu(B_j) = \mu(B_m)$. Then

$$y = \mathbf{1}_{B_m} - \sum_{j=1}^{m-1} a_j \cdot \mathbf{1}_{B_j} \neq 0 \quad \text{and} \quad Ty = \mu(B_m) - \sum_{j=1}^{m-1} a_j \mu(B_j) = 0.$$

Now we reason as in the proof of Theorem 4.2. According to [PlichP, Sec. 10, Theorem 1], for the real symmetric Banach space E and its closed subspace Y the following assertions are equivalent:

(i) $Y \cap L_\infty(A) \neq \{0\}$ for every $A \in \Sigma^+$:

(ii) for every $A \in \Sigma^+$ and for every $\alpha \in (0,1)$ there exists $A' \in \Sigma^+(A)$ such that $\mu_0(A') = \alpha\mu_0(A)$ and $x = (1-\alpha)(\mathbf{1}_{A'}) - \alpha(\mathbf{1}_{A\setminus A'}) \in Y$.

Using (ii) and the definition of T we find that for every $A \in \Sigma^+$ and for every $\alpha \in (0,1)$ there exists $A' \in \Sigma^+(A)$ such that $\mu_0(A') = \alpha\mu_0(A)$ and $\mu(A') = \alpha\mu(A)$.

Now let $A, B \in \Sigma^+$ and $\alpha \in (0,1)$. Choose $A' \in A \setminus B$ and $B' \in B \setminus A$ such that $\mu(A') = \alpha\mu(A \setminus B)$ and $\mu(B') = (1-\alpha)\mu(B \setminus A)$. Then

$$\mu(A' \cup B' \cup (A \cap B)) = \alpha\mu(A) + (1-\alpha)\mu(B). \ \blacksquare$$

In view of the following statement, Theorem 4.3 generalizes Theorem 4.2.

PROPOSITION 4.5. *Let (Ω, Σ, μ_0) be a finite atomless measure space. Set*

$$\aleph_\beta = \min\{|\widetilde{\Sigma}(A)| : A \in \Sigma,\ \mu_0(A) > 0\},$$
$$\aleph_\gamma = \min\{\operatorname{dens} L_1(A) : A \in \Sigma,\ \mu_0(A) > 0\}.$$

Then $\aleph_\beta = \aleph_\gamma^{\aleph_0}$.

Proof. The inequality $\aleph_\beta \leq \aleph_\gamma^{\aleph_0}$ follows from $|\widetilde{\Sigma}(A)| \leq (\operatorname{dens} L_1(A))^{\aleph_0}$, holding for all $A \in \Sigma$ with $\mu_0(A) > 0$.

Now we prove the reverse inequality. By the Maharam theorem [Lac, p. 122], there exists a unique at most countable set $\mathcal{M}$ of ordinals such that the measure space (Ω, Σ, μ_0) is isomorphic to the direct sum

$$\sum_{\alpha\in\mathcal{M}}\oplus\, \varepsilon_\alpha \cdot \big(\{-1,1\}^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha}\big),$$

where μ_{ω_α} is the Haar measure on $\{-1,1\}^{\omega_\alpha}$, Σ_{ω_α} is the set of all μ_{ω_α}-measurable subsets of $\{-1,1\}^{\omega_\alpha}$ and ε_α are some numbers with $\sum_{\alpha\in\mathcal{M}}\varepsilon_\alpha = \mu(\Omega)$. Note that $\aleph_\gamma = \min\{\aleph_\alpha : \alpha\in\mathcal{M}\}$, and $\aleph_\beta$ is the cardinality of the set of all equivalence classes of measurable subsets of $\{-1,1\}^{\omega_\gamma}$ with respect to the measure μ_{ω_γ}. So, it is enough to show that for $\Omega = \{-1,1\}^T$ with $|T| = \aleph_\gamma$ the collection $\widetilde{\Sigma}$ of all equivalence classes, with respect to the Haar measure, of subsets of Σ is of cardinality at least $\aleph_\gamma^{\aleph_0}$. Represent T in the form $T = \bigsqcup_{n=0}^\infty T_n$, where $|T_n| = 2n-1$ for all $n\in\mathbb{N}$. Denote by $\mathcal{F}$ the set of all countable subsets F of T_0. Observe that $|T_0| = \aleph_\gamma$ and $|\mathcal{F}| = \aleph_\gamma^{\aleph_0}$.

Now we construct an injective map $\varphi : \mathcal{F}\to\widetilde{\Sigma}$. Let $F = \{t_n : n\in\mathbb{N}\}\in\mathcal{F}$ with all t_n's distinct. For every $n\in\mathbb{N}$, let A_n denote the set of all functions $x : T\to\{-1,1\}$ such that $x(t) = 1$ for every $t\in T_n\cup\{t_n\}$. Set $\varphi(F) = \bigcup_{n=1}^\infty A_n$. Note that $\varphi(F)\in\widetilde{\Sigma}$ and

$$\mu(\varphi(F)) \le \sum_{n=1}^\infty \mu(A_n) = \sum_{n=1}^\infty \frac{1}{4^n} = \frac{1}{3}.$$

It remains to verify the injectivity of φ. Assume $F_1, F_2\in\mathcal{F}$ and $t\in F_1\setminus F_2$, and set

$$U = \{x\in\{-1,1\}^T : x(t) = 1\} \quad\text{and}\quad V = \{x\in\{-1,1\}^T : x(t) = -1\}.$$

Then $\mu(U\cap\varphi(F_2)) = \mu(V\cap\varphi(F_2))$. Since $\mu(\Omega\setminus\varphi(F_1)) > 0$ and $t\in F_1$, we obtain $\mu(U\cap\varphi(F_1)) \ne \mu(V\cap\varphi(F_1))$. Hence, $\varphi(F_1)\ne\varphi(F_2)$. ∎

4.2. Nowhere vanishing measures

Definition 4.6. Let $\mathcal{A}$ be a Boolean algebra and X a vector space. A measure $\mu : \mathcal{A}\to X$ is called *nowhere vanishing* if for every $x\in\mathcal{A}\setminus\{0\}$ there is $y\in\mathcal{A}$ such that $y\le x$ and $\mu(y)\ne 0$.

The following statement allows one to reduce, in a certain sense, any measure to a nowhere vanishing one.

Proposition 4.7. *Let $\mathcal{B}$ be a Boolean σ-algebra, X a topological vector space and $\mu : \mathcal{B}\to X$ an atomless σ-additive measure. Then there exist a Boolean σ-algebra $\mathcal{A}$, a Boolean homomorphism $h : \mathcal{B}\to\mathcal{A}$ and a nowhere vanishing atomless σ-additive measure $\nu : \mathcal{A}\to X$ such that $\mu(x) = \nu(h(x))$ for every $x\in\mathcal{B}$.*

Proof. Let $\mathcal{C}$ be the set of all $x\in\mathcal{B}$ such that $\mu(y) = 0$ for every $y\le x$. Given any $x\in\mathcal{B}$, we denote by $h(x)$ the set of all $y\in\mathcal{B}$ for which there are

$u, v \in \mathcal{C}$ such that $x \vee u = y \vee v$. It is easy to see that $\mathcal{A} = \{h(x) : x \in \mathcal{B}\}$ is a Boolean σ-algebra relative to the operations $h(x) \vee h(y) = h(x \vee y)$, $h(x) \wedge h(y) = h(x \wedge y)$ and $\neg h(x) = h(\neg x)$ with zero $h(\mathbf{0})$ and unit $h(\mathbf{1})$. Since $\mu(x) = \mu(y)$ for every $y \in h(x)$, the equation $\nu(h(x)) = \mu(x)$ well defines some measure $\nu : \mathcal{A} \to X$, which is nowhere vanishing, atomless and σ-additive. ∎

Now we are ready to generalize Theorem 2.11.

THEOREM 4.8. *Let $\mathcal{B}$ be a Boolean σ-algebra, $\mu_0 : \mathcal{B} \to [0, \infty)$ a positive σ-additive measure and X a real F-space with* H-dim $X < \aleph_\alpha$, *where $\aleph_\alpha = \min\{|\mathcal{B}_x| : x \in \mathcal{B} \setminus \{0\}\}$. Then every nowhere vanishing atomless σ-additive measure $\mu : \mathcal{B} \to X$ absolutely continuous with respect to μ_0 is dividing.*

Proof. Let $\mu : \mathcal{B} \to X$ be a nowhere vanishing atomless σ-additive measure and $x \in \mathcal{B} \setminus \{0\}$. Then the restriction $\mu_x = \mu|_{\mathcal{B}_x} : \mathcal{B}_x \to X$ is a nowhere vanishing atomless σ-additive measure. By Theorem 1.8, $\mathcal{B}_x$ is Boolean isomorphic to some measurable space (Ω, Σ) by means of some Boolean isomorphism $J : \mathcal{B}_x \to \Sigma$. Since $\mathcal{B}_x$ is a Boolean σ-algebra, Σ is a σ-algebra. Then the map $\nu : \Sigma \to X$ given by $\nu(A) = \mu\big(J^{-1}(A)\big)$ for all $A \in \Sigma$ is a nowhere vanishing atomless σ-additive measure. Note that ν is absolutely continuous with respect to the measure $\nu_0 : \Sigma \to [0, \infty)$, $\nu_0(A) = \mu_0(J^{-1}(A))$. Since μ is nowhere vanishing, $\nu_0(A) > 0$ for every nonempty $A \in \Sigma$. Thus,

$$\begin{aligned}\aleph_\beta &= \min\{|\widetilde{\Sigma}(A)| : A \in \Sigma,\, \nu_0(A) > 0\}\\ &= \min\{|\mathcal{B}_y \setminus \{0\}| : y \in \mathcal{B}_x \setminus \{0\}\} \ge \aleph_\alpha.\end{aligned}$$

By Theorem 4.3, $\nu(\Sigma)$ is a convex subset of X. In particular, since $0, \nu(J(x)) \in \nu(\Sigma)$, we have $\nu(J(x))/2 \in \nu(\Sigma)$. Let $B \in \Sigma$ be such that $\nu(B) = \nu(J(x))/2 = \mu(x)/2$. Then for $y = J^{-1}(B)$ one has $y \le x$ and $\mu(y) = \nu(B) = \mu(x)/2$. Thus, for $z = x \wedge \neg y$ one has $x = y \sqcup z$ and $\mu(z) = \mu(x) - \mu(y) = \mu(x)/2 = \mu(y)$. ∎

The following example shows that the nowhere vanishing assumption in Theorem 4.8 is essential.

EXAMPLE 4.9. Let (Ω, Σ) be a measurable space and $\nu : \Sigma \to [0, \infty)$ a finite atomless σ-additive measure with $\nu(A) = 0$ for every at most countable set $A \subseteq \Omega$. Let $\aleph_\alpha =$ H-dim $L_1(\Omega)$. For every $t \in \Omega$ set $\Omega_t = \{0, 1\}^{\omega_{\alpha+1}}$ and let $\mathcal{B}^t$ be the Boolean σ-algebra of all sets $x \subseteq \Omega_t$ depending on at most $\aleph_\alpha$ coordinates with $\mathbf{1}_t = \Omega_t$ and $\mathbf{0}_t = \emptyset$. Note that dens $\mathcal{B}^t_x > \aleph_\alpha$ for every nonempty $x \in \mathcal{B}^t$. Denote by $\mathcal{B}$ the Boolean σ-algebra of all sets

$$x = \bigsqcup_{t \in \Omega} x_t = \Big(\bigsqcup_{t \in A} \mathbf{1}_t\Big) \sqcup \Big(\bigsqcup_{t \in B} \mathbf{0}_t\Big) \sqcup \Big(\bigsqcup_{t \in C} x_t\Big),$$

where $\Omega = A \sqcup B \sqcup C$, $A, B \in \Sigma$, C is at most countable and $x_t \in \mathcal{B}_t$ for every

$t \in C$. The measure $\mu : \mathcal{B} \to L_1(\Omega)$ defined by $\mu(x) = \mathbf{1}_A$ is an atomless σ-additive measure of finite variation which is not almost dividing.

4.3. Proof of the main theorem. The last auxiliary step in the proof of Theorem 4.1 is the following lemma.

LEMMA 4.10. *Let E be an atomless Dedekind complete vector lattice and X a real F-space such that* $\text{H-dim}\, X < \aleph_\alpha$, *where*

$$\aleph_\alpha = \min\big\{|\mathfrak{F}_e| : 0 < e \in E\big\}.$$

Let $(\mu_e)_{0<e\in E}$ be a family of positive σ-additive measures $\mu_e : \mathfrak{F}_e \to [0,\infty)$. Then every laterally-to-norm σ-continuous orthogonally additive operator $T : E \to X$ such that for every $0 < e \in E$ the measure μ_e^T is absolutely continuous with respect to μ_e, is strictly narrow.

Proof. Let $e \in E^+$. Consider two cases.

(i) Assume $Te \neq 0$. By the lateral continuity and Zorn's lemma, we choose a maximal element $x \in \mathfrak{F}_e$ with $Tx = 0$. Set $z = e - x$ and observe that the measure μ_z^T is nowhere vanishing, and hence by Theorem 4.8 it is dividing.

(ii) Let $0 \neq e \in E^+$ and $Te = 0$. If $\mu_e^T \neq 0$ then there is a decomposition $e = u \sqcup v$ such that $Tu \neq 0$. Then $Tv \neq 0$, as well. Using (i), we split $u = x_1 \sqcup y_1$ and $v = x_2 \sqcup y_2$ so that $Tx_1 = Ty_1$ and $Tx_2 = Ty_2$. It remains to observe that $T(f) = T(g)$ for $f = x_1 \sqcup x_2$ and $g = y_1 \sqcup y_2$. ∎

Proof of Theorem 4.1. For every $e \in E^+$ and $u \in \mathfrak{F}_e$ we set

$$\mu_e(u) = \mu(\{t \in \Omega : u(t) > 0\}).$$

Then μ_e^T is absolutely continuous with respect to μ_e for every $e \in E^+$. ∎

The following corollary of Theorem 4.1 gives a positive answer to [PR, Problem 2.17] for the case of real scalars.

THEOREM 4.11. *Let E be a Köthe F-space on a finite atomless measure space (Ω, Σ, μ) with an absolutely continuous norm, and X a real F-space such that* $\operatorname{dens} E(A) > \operatorname{dens} X$ *for all $A \in \Sigma$ with $\mu(A) > 0$. Then every laterally continuous orthogonally additive operator $T : E \to X$ is strictly narrow.*

Proof. We show that E and X satisfy the assumptions of Theorem 4.1. Given $e \in E$ with $e > 0$, we show that $|\mathfrak{F}_e| > \text{H-dim}\, X$. Set $A = \operatorname{supp} e$. By the absolute continuity of the norm of E, the set $S(A) \subset E(A)$ of all simple functions is dense in $E(A)$ [PR, Proposition 2.10]. Hence,

$$\text{H-dim}\, E(A) \le |S|^{\aleph_0} \le |\mathfrak{F}_e|^{\aleph_0}. \tag{4.1}$$

By [PR, Lemma 2.16], if Y is an infinite-dimensional F-space then $\operatorname{dens} Y = (\text{H-dim}\, Y)^{\aleph_0}$. Assume, on the contrary, that $|\mathfrak{F}_e| \le \text{H-dim}\, X$. Then

$$\begin{aligned}\operatorname{dens} E(A) &= (\text{H-dim}\, E(A))^{\aleph_0}\\ &\le (|\mathfrak{F}_e|^{\aleph_0})^{\aleph_0} = |\mathfrak{F}_e|^{\aleph_0}\\ &\le (\text{H-dim}\, X)^{\aleph_0} = \operatorname{dens} X,\end{aligned}$$

which contradicts the theorem's assumption. ∎

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Volodymyr Mykhaylyuk
Department of Mathematics and Informatics
Chernivtsi National University
Kotsiubyns'koho 2
Chernivtsi 58012, Ukraine
E-mail: vmykhaylyuk@ukr.net

Marat Pliev
South Mathematical Institute
Vladikavkaz Scientific Center
Russian Academy of Sciences
Vladikavkaz 362027, Russian Federation
E-mail: maratpliev@gmail.com

Mikhail Popov
Institute of Mathematics
Pomeranian University in Słupsk
Arciszewskiego 22d
76-200 Słupsk, Poland
and
Department of Mathematics and Informatics
Chernivtsi National University
Kotsiubyns'koho 2
Chernivtsi 58012, Ukraine
E-mail: misham.popov@gmail.com

Oleksandr Sobchuk
Department of Mathematics and Informatics
Chernivtsi National University
Kotsiubyns'koho 2
Chernivtsi 58012, Ukraine
E-mail: ss220367@ukr.net