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Traces of Besov spaces on fractal h-sets and dichotomy results

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Abstract. We study the existence of traces of Besov spaces on fractal h-sets Γ with a special focus on assumptions necessary for this existence; in other words, we present criteria for the non-existence of traces. In that sense our paper can be regarded as an extension of Bricchi (2004) and a continuation of Caetano (2013). Closely connected with the problem of existence of traces is the notion of *dichotomy* in function spaces: We can prove that—depending on the function space and the set Γ—there occurs an alternative: *either* the trace on Γ exists, *or* smooth functions compactly supported outside Γ are dense in the space. This notion was introduced by Triebel (2008) for the special case of d-sets.

1. Introduction. The paper is devoted to a detailed study of traces of regular distributions taken on fractal sets. Such questions are of particular interest in view of boundary value problems for elliptic operators, where the solutions belong to some appropriate Besov (or Sobolev) space. One standard method is to start with assertions about traces on hyperplanes and then to transfer these results to bounded domains with sufficiently smooth boundary. Further studies may concern compactness or regularity results, leading to the investigation of spectral properties. However, when it comes to irregular (or fractal) boundaries, following that way one has to circumvent a lot of difficulties, so that another method turned out to be more appropriate. It was proposed by Edmunds and Triebel [ET1] in connection with smooth boundaries and then extended to fractal d-sets in [ET3, ET2, Tr4]. Later the setting of d-sets was extended to (d,Ψ)-sets by Moura [Mo2] and finally to the more general h-sets by Bricchi [Br2].

The idea is rather simple to describe, but the details are complicated: first one determines the trace spaces of certain Besov (or Sobolev) spaces

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as precisely as possible, studies (compact) embeddings of such spaces into appropriate target spaces together with their entropy and approximation numbers, and finally applies Carl's or Weyl's inequalities to link eigenvalues and entropy or approximation numbers. If one is in the lucky situation that, on the one hand, one has atomic or wavelet decomposition results for the corresponding spaces, and, on the other hand, the irregularity of the fractal can be characterised by its local behaviour (within 'small' cubes or balls), then there is some chance to carry over all the arguments to appropriate sequence spaces which are usually easier to handle. This is one reason for us to stick to fractal h-sets and Besov spaces at the moment. But still the problem is not so simple and little is known so far.

For spaces on h-sets we refer to [CaL2, CaL1, KZ, Lo], and, probably closest to our approach here, [Tr6, Chapter 8]. It turns out that one first needs a sound knowledge about the existence and quality of the corresponding trace spaces. Returning to the first results in that respect in [Br2] (see also [Br3, Br4, Br1]), we found that the approach can (and should) be extended for later applications.

More precisely, for a positive continuous and non-decreasing function $h : (0,1] \to \mathbb{R}$ (a gauge function) with $\lim_{r\to 0} h(r) = 0$, a non-empty compact set $\Gamma \subset \mathbb{R}^n$ is called an *h-set* if there exists a finite Radon measure μ in $\mathbb{R}^n$ with $\operatorname{supp}\mu = \Gamma$ and

$$\mu(B(\gamma, r)) \sim h(r), \qquad r \in (0,1],\ \gamma \in \Gamma$$

(see also [Ro, Chapter 2] and [Ma, p. 60]). In the special case $h(r) = r^d$, $0 < d < n$, Γ is called a *d-set* (in the sense of [Tr4, Def. 3.1], see also [JW, Ma]—be aware that this is different from [Fa]). Recall that some self-similar fractals are outstanding examples of d-sets; for instance, the usual (middle-third) Cantor set in $\mathbb{R}^1$ is a d-set for $d = \ln 2/\ln 3$, and the Koch curve in $\mathbb{R}^2$ is a d-set for $d = \ln 4/\ln 3$.

The trace is defined by completion of pointwise traces of $\varphi \in \mathcal{S}(\mathbb{R}^n)$, assuming that for $0 < p < \infty$ we have in addition $\|\varphi|_\Gamma \mid L_p(\Gamma)\| \lesssim \|\varphi \mid B^t_{p,q}(\mathbb{R}^n)\|$ for suitable parameters $t \in \mathbb{R}$ and $0 < q < \infty$. In the case of a compact d-set Γ, $0 < d < n$, this results in

$$\operatorname{tr}_\Gamma B^{(n-d)/p}_{p,q}(\mathbb{R}^n) = L_p(\Gamma) \quad \text{if } 0 < q \le \min\{p,1\}, \tag{1.1}$$

and, for $s > (n-d)/p$,

$$\operatorname{tr}_\Gamma B^s_{p,q}(\mathbb{R}^n) = \mathbb{B}^{s-(n-d)/p}_{p,q}(\Gamma)$$

(see [Tr4] with some later additions in [Tr5, Tr6]). Here $B^s_{p,q}(\mathbb{R}^n)$ are the usual Besov spaces defined on $\mathbb{R}^n$. In the classical case $d = n-1$, $0 < p < \infty$, $0 < q \le \min\{p,1\}$ this corresponds to the well-known trace result $\operatorname{tr}_{\mathbb{R}^{n-1}} B^{1/p}_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1})$.

In the case of h-sets Γ one needs to consider Besov spaces of generalised smoothness $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ which naturally extend $B^s_{p,q}(\mathbb{R}^n)$: instead of the smoothness parameter $s \in \mathbb{R}$ one now admits sequences $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ of positive numbers which satisfy $\sigma_j \sim \sigma_{j+1}$, $j \in \mathbb{N}_0$. Such spaces are special cases of $B^{\boldsymbol{\sigma},N}_{p,q}(\mathbb{R}^n)$ studied in [FaLe] recently, but they have been known for a long time: apart from the interpolation approach (with a function parameter, see [Me, CoF]), there is the rather abstract approach (approximation by series of entire analytic functions and coverings) developed independently by Gol'dman and Kalyabin in the late 70's and early 80's of the last century; we refer to the survey [KL] and the appendix [Li] which cover the extensive (Russian) literature of that time. We shall rely on the Fourier-analytical approach as presented in [FaLe].

It turns out that the classical smoothness $s \in \mathbb{R}$ has to be replaced by certain regularity indices $\overline{\mathfrak{s}}(\boldsymbol{\sigma})$, $\underline{\mathfrak{s}}(\boldsymbol{\sigma})$ of $\boldsymbol{\sigma}$.

For $\boldsymbol{\sigma} = (2^{js})_j$ the spaces $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ coincide and $\overline{\mathfrak{s}}(\boldsymbol{\sigma}) = \underline{\mathfrak{s}}(\boldsymbol{\sigma}) = s$. Dealing with traces on h-sets Γ in a similar way as for d-sets, one obtains

$$\operatorname{tr}_\Gamma B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) = \mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma),$$

where the sequence $\boldsymbol{\tau}$ (representing smoothness) depends on $\boldsymbol{\sigma}$, h (representing the geometry of Γ) and the underlying $\mathbb{R}^n$; in particular, with $\boldsymbol{h} := (h(2^{-j}))_j$, $\boldsymbol{h}_p = (h(2^{-j})^{1/p}\, 2^{jn/p})_j$, the counterpart of (1.1) reads

$$\operatorname{tr}_\Gamma B^{\boldsymbol{h}_p}_{p,q}(\mathbb{R}^n) = L_p(\Gamma), \qquad 0 < p < \infty,\ 0 < q \le \min\{p, 1\}.$$

These results were already obtained in [Br4] under some additional restrictions. In [Ca2] we studied sufficient conditions for the existence of such traces again (in the course of dealing with growth envelopes, characterising some singularity behaviour) and return to the subject now to obtain 'necessary' conditions, or, more precisely, conditions for the non-existence of traces. This problem is closely connected with the so-called *dichotomy*: Triebel [Tr7] coined this term for, roughly speaking, the following alternative: the existence of a trace on Γ (by completion of pointwise traces) on the one hand, and the density of the set of smooth functions compactly supported outside Γ, denoted by $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$, on the other. Though it is rather obvious that the density of $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ in some space prevents the existence of a properly defined trace, it is not clear (and, in fact, not true in general) that there is some close converse connection. However, in some cases there appears an alternative that *either* we have an affirmative answer to the density question *or* traces exist. The criterion for which case occurs naturally depends on the function spaces and the set Γ. Our main outcome in this respect, Theorem 3.18, establishes the following: if the h-set Γ satisfies, in addition, some porosity condition, $\boldsymbol{\sigma}$ is an admissible sequence and either

$1 \le p < \infty$, $0 < q < \infty$, or $0 < q \le p < 1$, then

$$\begin{aligned} \textbf{either} \quad & \mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) = \operatorname{tr}_\Gamma B^{\boldsymbol{\sigma}\boldsymbol{h}_p}_{p,q}(\mathbb{R}^n) \text{ exists} \\ \textbf{or} \quad & \mathcal{D}(\mathbb{R}^n \setminus \Gamma) \text{ is dense in } B^{\boldsymbol{\sigma}\boldsymbol{h}_p}_{p,q}(\mathbb{R}^n), \\ & \text{and therefore } \operatorname{tr}_\Gamma B^{\boldsymbol{\sigma}\boldsymbol{h}_p}_{p,q}(\mathbb{R}^n) \text{ cannot exist.} \end{aligned}$$

This result is later reformulated in terms of the dichotomy introduced in [Tr7]. Note that there are further related approaches to trace and dichotomy questions in [Sch1, Sch2] for Besov spaces defined by differences, and in [Pi, Ha] referring to weighted settings.

The paper is organised as follows. In Section 2 we collect some fundamentals about h-sets and Besov spaces of generalised smoothness, including their atomic decomposition. In Section 3 we turn to trace questions with our main result being Theorem 3.18, before we finally deal with the dichotomy and obtain Corollary 3.30. Throughout the paper we add remarks, discussions and examples to illustrate the (sometimes technically involved) arguments and results.

2. Preliminaries

2.1. General notation. As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are positive numbers c_1 and c_2 such that

$$c_1 a_k \le b_k \le c_2 a_k \quad \text{or} \quad c_1\varphi(x) \le \psi(x) \le c_2\varphi(x)$$

for all allowable values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and φ, ψ are non-negative functions. If only one of the inequalities above is meant to hold, we use the symbol $\lesssim$ instead. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. If not otherwise indicated, log is always taken with respect to base 2. For $\varkappa \in \mathbb{R}$ let

$$\varkappa_+ = \max\{\varkappa, 0\} \quad \text{and} \quad \lfloor \varkappa \rfloor = \max\{k \in \mathbb{Z} : k \le \varkappa\}. \tag{2.1}$$

Moreover, for $0 < r \le \infty$ the number r' is given by $1/r' := (1 - 1/r)_+$.

For convenience, let both dx and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure. The notation $|\cdot|$ is also used for the size of an n-tuple in $\mathbb{N}_0^n$ and the Euclidean norm in $\mathbb{R}^n$, while $|\cdot|_\infty$ is reserved for the corresponding infinity norm.

Given $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ denotes the closed ball

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| \le r\}. \tag{2.2}$$

2.2. ***h*-sets $\mathit{\Gamma}$.** A central concept for us is that of h-sets and corresponding measures; we refer to a comprehensive treatment of this concept in [Ro]. Certainly one of the most prominent subclasses of these sets are the famous d-sets (see also Example 2.7 below), but it is also well-known that in many cases more general approaches are necessary (cf. [Ma, p. 60]). Here we essentially follow the presentation in [Br1–Br4]; see also [Ma] for basic notions and concepts.

DEFINITION 2.1.

(i) Let $\mathbb{H}$ denote the class of all positive continuous and non-decreasing functions $h : (0,1] \to \mathbb{R}$ (*gauge functions*) with $\lim_{r\to 0} h(r) = 0$.

(ii) Let $h \in \mathbb{H}$. A non-empty compact set $\Gamma \subset \mathbb{R}^n$ is called *h-set* if there exists a finite Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp} \mu = \Gamma, \tag{2.3}$$

$$\mu(B(\gamma, r)) \sim h(r), \qquad r \in (0,1],\ \gamma \in \Gamma. \tag{2.4}$$

If for a given $h \in \mathbb{H}$ there exists an h-set $\Gamma \subset \mathbb{R}^n$, we call h a *measure function* (*in* $\mathbb{R}^n$) and any related measure μ with (2.3) and (2.4) will be called an *h-measure* (*related to* Γ).

We quote some results on h-sets and give examples afterwards; we refer to the above-mentioned books and papers for proofs and a more detailed account of geometric properties of h-sets.

In view of (ii) the question arises which $h \in \mathbb{H}$ are measure functions. We give a necessary condition first (see [Br2, Thm. 1.7.6]).

PROPOSITION 2.2. *Let $h \in \mathbb{H}$ be a measure function. Then there exists some $c > 0$ such that for all $j, k \in \mathbb{N}_0$,*

$$\frac{h(2^{-k-j})}{h(2^{-j})} \ge c 2^{-kn}. \tag{2.5}$$

REMARK 2.3. Note that every h-set Γ satisfies the *doubling condition*, i.e. there is some $c > 0$ such that

$$\mu(B(\gamma, 2r)) \le c\mu(B(\gamma, r)), \qquad r \in (0,1],\ \gamma \in \Gamma. \tag{2.6}$$

Obviously one can regard (2.5) as a refined version of (2.6) for the function h, in which the dimension n of the underlying space $\mathbb{R}^n$ is taken into account (as expected).

A complete characterisation of functions $h \in \mathbb{H}$ that are measure functions is given in [Br3]: There is a compact set Γ and a Radon measure μ

with (2.3) and (2.4) if and only if there are constants $0 < c_1 \le c_2 < \infty$ and a function $h^* \in \mathbb{H}$ such that

$$c_1 h^*(t) \le h(t) \le c_2 h^*(t), \qquad t \in (0,1],$$

and

$$h^*(2^{-j}) \le 2^{kn} h^*(2^{-k-j}) \quad \text{for all } j,k \in \mathbb{N}_0. \tag{2.7}$$

Proposition 2.4. *Let Γ be an h-set in $\mathbb{R}^n$. All h-measures μ related to Γ are equivalent to $\mathcal{H}^h|\Gamma$, the restriction to Γ of the generalised Hausdorff measure with respect to the gauge function h.*

Remark 2.5. A proof of this result is given in [Br2, Thm. 1.7.6]. Concerning the theory of generalised Hausdorff measures $\mathcal{H}^h$ we refer to [Ro, Chapter 2] and [Ma, p. 60]; in particular, if $h(r) = r^d$, then $\mathcal{H}^h$ coincides with the usual d-dimensional Hausdorff measure.

We recall a description of measure functions and give a few examples.

Proposition 2.6. *Let $n \in \mathbb{N}$.*

(i) *Let $\xi : (0,1] \to [0,n]$ be a measurable function. Then the function*

$$h(r) = \exp\Big\{-\int_r^1 \xi(s)\,\frac{ds}{s}\Big\}, \qquad r \in (0,1], \tag{2.8}$$

is a measure function.

(ii) *Conversely, let h be a measure function. Then for any $\varepsilon > 0$ there exists a measurable function $\xi : (0,1] \to [-\varepsilon, n+\varepsilon]$ such that*

$$h(r) \sim \exp\Big\{-\int_r^1 \xi(s)\,\frac{ds}{s}\Big\}, \qquad r \in (0,1]. \tag{2.9}$$

This version of the theorem is given in [Br4, Thm. 3.7]; it can also be identified as a special case of a result in [BGT, pp. 74].

Example 2.7. We restrict ourselves to a few examples only, but in view of Proposition 2.6 one can easily find further examples [Br4, Ex. 3.8]. All functions are defined for $r \in (0,\varepsilon)$, suitably extended on $(0,1]$.

Let Ψ be a continuous admissible function or a continuous slowly varying function, respectively. An *admissible* function Ψ in the sense of [ET2, Mo2] is a positive monotone function on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}$. A positive and measurable function Ψ defined on $(0,1]$ is said to be *slowly varying* (in Karamata's sense) if

$$\lim_{t\to 0} \frac{\Psi(st)}{\Psi(t)} = 1, \qquad s \in (0,1]. \tag{2.10}$$

For such functions it is known, for instance, that for any $\delta > 0$ there exists $c = c(\delta) > 1$ such that

$$\frac{1}{c} s^{\delta} \le \frac{\Psi(st)}{\Psi(t)} \le c s^{-\delta} \quad \text{for } t, s \in (0,1],$$

and for each $\varepsilon > 0$ there is a non-increasing function ϕ and a non-decreasing function φ with $t^{-\varepsilon}\,\Psi(t) \sim \phi(t)$ and $t^{\varepsilon}\,\Psi(t) \sim \varphi(t)$; we refer to the monograph [BGT] for details and further properties; see also [Zy, Ch. V], [EKP], and [Ne1, Ne2]. In particular,

$$\Psi_b(x) = (1 + |\log x|)^b, \quad x \in (0,1],\ b \in \mathbb{R}, \tag{2.11}$$

may be considered a prototype for both an admissible function and a slowly varying function.

Let $0 < d < n$. Then

$$h(r) = r^d \Psi(r), \quad r \in (0,1], \tag{2.12}$$

is a typical example for $h \in \mathbb{H}$. The limiting cases $d = 0$ and $d = n$ can be included, assuming additional properties of Ψ in view of (2.5) and $h(r) \to 0$ for $r \to 0$, e.g.

$$h(r) = (1 + |\log r|)^b, \quad b < 0,\ r \in (0,1], \tag{2.13}$$

referring to (2.11).

Later on we shall need to consider measure functions of this kind when $b \in [-1, 0)$, so we mention that with this restriction the proof that the above is a measure function is a simple application of the characterisation given just before Proposition 2.4 (take $h^* = h$ and observe that, for $b \in [-1,0)$, $(1+x)^b 2^{xn} \ge 1$ for $x = 0$ and for $x \ge 1$, from which (2.7) follows by taking $x = k$).

Such functions h given by (2.12) are related to so-called (d, Ψ)-sets studied in [ET2, Mo2], whereas the special setting $\Psi \equiv 1$ leads to

$$h(r) = r^d, \quad r \in (0,1],\ 0 < d < n, \tag{2.14}$$

connected with the well-known d-sets. Apart from (2.12) also functions of type $h(r) = \exp(b|\log r|^{\varkappa})$, $b < 0$, $0 < \varkappa < 1$, are allowed.

We shall need another feature of h-sets, called 'porosity' (see also [Ma, p. 156] and [Tr5, Sects. 9.16–9.19]).

Definition 2.8. A Borel set $\Gamma \ne \emptyset$ satisfies the *porosity condition* if there exists $0 < \eta < 1$ such that for any ball $B(\gamma, r)$ with $\gamma \in \Gamma$ and $0 < r \le 1$, there is a ball $B(x, \eta r)$ centred at some $x \in \mathbb{R}^n$ satisfying

$$B(\gamma, r) \supset B(x, \eta r), \quad B(x, \eta r) \cap \bar{\Gamma} = \emptyset. \tag{2.15}$$

Replacing η by $\eta/2$, we can complement (2.15) by

$$\operatorname{dist}(B(x, \eta r), \bar{\Gamma}) \ge \eta r, \quad 0 < r \le 1. \tag{2.16}$$

This definition coincides with [Tr4, Def. 18.10]. In [Tr5, Prop. 9.18] there is a complete characterisation for measure functions h such that the corresponding h-sets Γ satisfy the porosity condition. We recall it for convenience.

PROPOSITION 2.9. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Then Γ satisfies the porosity condition if and only if there exist constants $c, \varepsilon > 0$ such that*

$$\frac{h(2^{-j-k})}{h(2^{-j})} \geq c 2^{-(n-\varepsilon)k}, \qquad j, k \in \mathbb{N}_0. \tag{2.17}$$

Note that an h-set Γ satisfying the porosity condition has Lebesgue measure $|\Gamma| = 0$, but the converse is not true. This can be seen from (2.17) and the result of [Tr6, Prop. 1.153],

$$|\Gamma| = 0 \quad \text{if and only if} \quad \lim_{r \to 0} \frac{r^n}{h(r)} = 0 \tag{2.18}$$

for all h-sets Γ.

REMARK 2.10. In view of our above examples and (2.17) it is obvious that the h from (2.12) and (2.14) with $d = n$ do not satisfy the porosity condition, in contrast to the case of $d < n$.

Let $L_p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $0 < p \leq \infty$, stand for the usual quasi-Banach space of p-integrable (measurable, essentially bounded if $p = \infty$) functions with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\Omega)\| := \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p},$$

with the obvious modification if $p = \infty$. Moreover, when $\Gamma \subset \mathbb{R}^n$ is an h-set in the sense of Definition 2.1, we consider $L_p(\Gamma) = L_p(\Gamma, \mu)$ as the usual quasi-Banach space of p-integrable (measurable, essentially bounded if $p = \infty$) functions on Γ with respect to the measure μ, quasi-normed by

$$\|f \mid L_p(\Gamma)\| = \Big(\int_\Gamma |f(\gamma)|^p \mu(d\gamma) \Big)^{1/p} < \infty$$

for $0 < p < \infty$, and

$$\|f \mid L_\infty(\Gamma)\| = \inf\{s > 0 : \mu(\{\gamma \in \Gamma : |f(\gamma)| > s\}) = 0\} < \infty.$$

In view of Proposition 2.4 all (possibly different) measures μ corresponding to h yield the same $L_p(\Gamma)$ space.

2.3. Function spaces of generalised smoothness

DEFINITION 2.11. A sequence $\boldsymbol{\sigma} = (\sigma_j)_{j \in \mathbb{N}_0}$ of positive numbers is called *admissible* if there are positive constants d_0, d_1 such that

$$d_0 \sigma_j \leq \sigma_{j+1} \leq d_1 \sigma_j, \qquad j \in \mathbb{N}_0. \tag{2.19}$$

REMARK 2.12. If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are admissible sequences, then $\boldsymbol{\sigma\tau} := (\sigma_j \tau_j)_j$ and $\boldsymbol{\sigma}^r := (\sigma_j^r)_j$, $r \in \mathbb{R}$, are admissible, too. For later use we introduce the notation

$$(a) := (2^{ja})_{j\in\mathbb{N}_0} \quad \text{for } a \in \mathbb{R}, \tag{2.20}$$

that is, $(a) = \boldsymbol{\sigma}$ with $\sigma_j = 2^{ja}$, $j \in \mathbb{N}_0$. Obviously, for $a, b \in \mathbb{R}$, $r > 0$, and $\boldsymbol{\sigma}$ admissible, we have $(a)(b) = (a+b)$, $(a/r) = (a)^{1/r}$, and $(a)\boldsymbol{\sigma} = (2^{ja}\sigma_j)_{j\in\mathbb{N}_0}$.

EXAMPLE 2.13. Let $s \in \mathbb{R}$, and let Ψ be an admissible function in the sense of Example 2.7 above. Then $\boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j$ is admissible, including, in particular, $\boldsymbol{\sigma} = (s)$, $s \in \mathbb{R}$. We refer to [FaLe] for a more general approach and further examples.

We introduce some 'regularity' indices for $\boldsymbol{\sigma}$.

DEFINITION 2.14. Let $\boldsymbol{\sigma}$ be an admissible sequence, and set

$$\underline{\mathfrak{s}}(\boldsymbol{\sigma}) := \liminf_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right), \tag{2.21}$$

$$\overline{\mathfrak{s}}(\boldsymbol{\sigma}) := \limsup_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right). \tag{2.22}$$

REMARK 2.15. These indices were introduced in [Br2]. For admissible sequences $\boldsymbol{\sigma}$ as in (2.19) we have $\log d_0 \le \underline{\mathfrak{s}}(\boldsymbol{\sigma}) \le \overline{\mathfrak{s}}(\boldsymbol{\sigma}) \le \log d_1$. One easily verifies that

$$\overline{\mathfrak{s}}(\boldsymbol{\sigma}) = \underline{\mathfrak{s}}(\boldsymbol{\sigma}) = s \quad \text{if } \boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j, \tag{2.23}$$

for all admissible functions Ψ and $s \in \mathbb{R}$. On the other hand, one can find examples in [FaLe], due to Kalyabin, showing that an admissible sequence does not necessarily have a fixed main order. Moreover, it is known that for any $0 < a \le b < \infty$, there is an admissible sequence $\boldsymbol{\sigma}$ with $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = a$ and $\overline{\mathfrak{s}}(\boldsymbol{\sigma}) = b$, that is, with prescribed upper and lower indices.

For later use we record some observations that are more or less immediate consequences of the definitions (2.21), (2.22). Let $\boldsymbol{\sigma}, \boldsymbol{\tau}$ be admissible sequences. Then

$$\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = -\overline{\mathfrak{s}}(\boldsymbol{\sigma}^{-1}), \quad \underline{\mathfrak{s}}(\boldsymbol{\sigma}^r) = r\underline{\mathfrak{s}}(\boldsymbol{\sigma}), \quad r \ge 0, \tag{2.24}$$

and

$$\overline{\mathfrak{s}}(\boldsymbol{\sigma\tau}) \le \overline{\mathfrak{s}}(\boldsymbol{\sigma}) + \overline{\mathfrak{s}}(\boldsymbol{\tau}), \quad \underline{\mathfrak{s}}(\boldsymbol{\sigma\tau}) \ge \underline{\mathfrak{s}}(\boldsymbol{\sigma}) + \underline{\mathfrak{s}}(\boldsymbol{\tau}). \tag{2.25}$$

In particular, for $\boldsymbol{\sigma} = (a)$, $a \in \mathbb{R}$, (2.25) can be sharpened to

$$\overline{\mathfrak{s}}(\boldsymbol{\tau}(a)) = a + \overline{\mathfrak{s}}(\boldsymbol{\tau}), \quad \underline{\mathfrak{s}}(\boldsymbol{\tau}(a)) = a + \underline{\mathfrak{s}}(\boldsymbol{\tau}). \tag{2.26}$$

Observe that, given $\varepsilon > 0$, there are positive constants $c_1 = c_1(\varepsilon)$ and

$c_2 = c_2(\varepsilon)$ such that

$$c_1 2^{(\underline{\mathfrak{s}}(\boldsymbol{\sigma})-\varepsilon)j} \le \sigma_j \le c_2 2^{(\overline{\mathfrak{s}}(\boldsymbol{\sigma})+\varepsilon)j}, \qquad j \in \mathbb{N}_0. \tag{2.27}$$

Plainly this implies that whenever $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$, then $\boldsymbol{\sigma}^{-1}$ belongs to any space ℓ_u, $0 < u \le \infty$, whereas $\overline{\mathfrak{s}}(\boldsymbol{\sigma}) < 0$ leads to $\boldsymbol{\sigma}^{-1} \notin \ell_\infty$.

Remark 2.16. Note that in some later papers (cf. [Br1]), instead of (2.21) and (2.22) the so-called *upper* and *lower Boyd indices* of $\boldsymbol{\sigma}$ are considered, given by

$$\alpha_{\boldsymbol{\sigma}} = \lim_{j\to\infty} \frac{1}{j} \log\left(\sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) = \inf_{j\in\mathbb{N}} \frac{1}{j} \log\left(\sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) \tag{2.28}$$

and

$$\beta_{\boldsymbol{\sigma}} = \lim_{j\to\infty} \frac{1}{j} \log\left(\inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) = \sup_{j\in\mathbb{N}} \frac{1}{j} \log\left(\inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) \tag{2.29}$$

respectively. In general we have

$$\underline{\mathfrak{s}}(\boldsymbol{\sigma}) \le \beta_{\boldsymbol{\sigma}} \le \alpha_{\boldsymbol{\sigma}} \le \overline{\mathfrak{s}}(\boldsymbol{\sigma}),$$

but one can construct admissible sequences with $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) < \beta_{\boldsymbol{\sigma}}$ and $\alpha_{\boldsymbol{\sigma}} < \overline{\mathfrak{s}}(\boldsymbol{\sigma})$.

To introduce function spaces of generalised smoothness, we need to recall some notation. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\widehat{\varphi}(\xi) \equiv (\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{2.30}$$

denotes the Fourier transform of φ. As usual, $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$ stands for the inverse Fourier transform, given by the right-hand side of (2.30) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ extend to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if } |x| \le 1 \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}, \tag{2.31}$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \qquad x \in \mathbb{R}^n. \tag{2.32}$$

Then the sequence $(\varphi_j)_{j=0}^\infty$ forms a smooth dyadic resolution of unity.

Definition 2.17. Let $\boldsymbol{\sigma}$ be an admissible sequence, $0 < p, q \le \infty$, and $(\varphi_j)_{j=0}^\infty$ a smooth dyadic resolution of unity (in the sense described above). Then

$$B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left(\sum_{j=0}^\infty \sigma_j^q \| \mathcal{F}^{-1}\varphi_j \mathcal{F} f \mid L_p(\mathbb{R}^n)\|^q \right)^{1/q} < \infty \right\} \tag{2.33}$$

(with the usual modification if $q = \infty$).

REMARK 2.18. These spaces are quasi-Banach spaces, independent of the resolution of unity chosen, and $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ when $p, q < \infty$. Taking $\boldsymbol{\sigma} = (2^{js})_j$, $s \in \mathbb{R}$, we obtain the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, whereas $\boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j$, $s \in \mathbb{R}$, Ψ an admissible function, leads to the spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$, studied in [Mo1, Mo2] in detail. Moreover, the above spaces $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ are special cases of the more general approach investigated in [FaLe]. For the theory of $B^s_{p,q}(\mathbb{R}^n)$ spaces we refer to the series of monographs [Tr2–Tr6].

We recall the atomic characterisation of $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ spaces for later use. Let $\mathbb{Z}^n$ stand for the lattice of all points in $\mathbb{R}^n$ with integer-valued components, and let Q_{jm} denote a cube in $\mathbb{R}^n$ with sides parallel to the coordinates axes, centred at $2^{-j}m = (2^{-j}m_1, \ldots, 2^{-j}m_n)$, and with side length 2^{-j}, where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ with sides parallel to the axes and $r > 0$, then rQ is the cube in $\mathbb{R}^n$ concentric with Q, with sides parallel to the sides of Q and r times their length.

DEFINITION 2.19. Let $K \in \mathbb{N}_0$ and $b > 1$.

(i) A K times differentiable complex-valued function $a(\cdot)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called an 1_K-*atom* if

$$\operatorname{supp} a \subset b\, Q_{0m} \quad \text{for some } m \in \mathbb{Z}^n, \tag{2.34}$$

and

$$|\mathrm{D}^\alpha a(x)| \le 1 \quad \text{for } |\alpha| \le K.$$

(ii) Let $L + 1 \in \mathbb{N}_0$, and $\boldsymbol{\sigma}$ be admissible. A K times differentiable complex-valued function $a(\cdot)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(\boldsymbol{\sigma}, p)_{K,L}$-*atom* if for some $j \in \mathbb{N}_0$,

$$\operatorname{supp} a \subset b\, Q_{jm} \quad \text{for some } m \in \mathbb{Z}^n, \tag{2.35}$$

$$|\mathrm{D}^\alpha a(x)| \le \sigma_j^{-1} 2^{j(n/p+|\alpha|)} \quad \text{for } |\alpha| \le K,\ x \in \mathbb{R}^n, \tag{2.36}$$

$$\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 \quad \text{if } |\beta| \le L. \tag{2.37}$$

We adopt the usual convention to denote atoms located at Q_{jm} (which means (2.34) or (2.35) holds) by a_{jm}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. For sequences $\lambda = (\lambda_{jm})_{j\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ of complex numbers the Besov sequence spaces $b_{p,q}$, $0 < p, q \le \infty$, are given by

(2.38)

$$\lambda \in b_{p,q} \quad \text{if and only if} \quad \|\lambda \,|\, b_{p,q}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty$$

(with the usual modification if $p = \infty$ or $q = \infty$). The atomic decomposition theorem for $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ reads as follows (see [FaLe, Thm. 4.4.3, Rem. 4.4.8]).

PROPOSITION 2.20. *Let $\boldsymbol{\sigma}$ be admissible, $b > 1$, $0 < p, q \leq \infty$, $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K > \overline{\mathfrak{s}}(\boldsymbol{\sigma}) \quad \text{and} \quad L > -1 + n\left(\frac{1}{p} - 1\right)_+ - \underline{\mathfrak{s}}(\boldsymbol{\sigma}). \tag{2.39}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}(x) \qquad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)), \tag{2.40}$$

where $\lambda \in b_{p,q}$ and a_{jm} are 1_K-atoms $(j = 0)$ or $(\boldsymbol{\sigma}, p)_{K,L}$-atoms $(j \in \mathbb{N})$ in the sense of Definition 2.19. Furthermore,

$$\inf \|\lambda \,|\, b_{p,q}\|,$$

where the infimum is taken over all admissible representations (2.40), *is an equivalent quasi-norm in $B^{\boldsymbol{\sigma}}_{p,q}(\mathbb{R}^n)$.*

REMARK 2.21. For later use it is useful to remark that, for fixed $j \in \mathbb{N}_0$, the family

$$\{Q_{jm} : m \in \mathbb{Z}^n\}$$

constitutes a tessellation of $\mathbb{R}^n$. We shall call it a *tessellation of step* 2^{-j} and refer to each cube of it as a *grid cube.*

REMARK 2.22. The following will also be of use later on: given the families of tessellations as above (for any possible $j \in \mathbb{N}_0$), there clearly exists, for each $j \in \mathbb{N}_0$, a partition of unity $\{\varphi_{jm} : m \in \mathbb{Z}^n\}$ on $\mathbb{R}^n$ by functions φ_{jm} supported on $\frac{3}{2}Q_{jm}$ and such that, for each $\gamma \in \mathbb{N}_0^n$, there exists $c_\gamma > 0$ independent of j such that

$$|\mathrm{D}^\gamma \varphi_{jm}(x)| \leq c_\gamma 2^{j|\gamma|}, \qquad x \in \mathbb{R}^n,\ m \in \mathbb{Z}^n. \tag{2.41}$$

We shall call it a *partition of unity of step* 2^{-j}.

3. Besov spaces on $\mathit{\Gamma}$. Let $\mathit{\Gamma}$ be some h-set, $h \in \mathbb{H}$. Following [Br2], we use the abbreviation

$$\boldsymbol{h} := (h_j)_{j \in \mathbb{N}_0} \quad \text{with } h_j := h(2^{-j}),\ j \in \mathbb{N}_0, \tag{3.1}$$

for the sequence connected with $h \in \mathbb{H}$.

3.1. Trace spaces $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\mathit{\Gamma})$. Recall that $L_p(\mathit{\Gamma}) = L_p(\mathit{\Gamma}, \mu)$, where $\mu \sim \mathcal{H}^h|\mathit{\Gamma}$ is related to the h-set $\mathit{\Gamma}$. Suppose there exists some $c > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi|_{\mathit{\Gamma}} \,|\, L_p(\mathit{\Gamma})\| \leq c\|\varphi \,|\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)\|, \tag{3.2}$$

where the restriction on Γ is taken pointwise. By the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ for $p,q<\infty$ and the completeness of $L_p(\Gamma)$ one can define for $f\in B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ its trace $\mathrm{tr}_\Gamma f$ on Γ by completion of pointwise restrictions.

This was the approach followed in [Br2, Def. 3.3.5], combining general embedding results for Besov spaces on $\mathbb{R}^n$ (as seen for example in [CaF, Thm. 3.7] or [Br2, Prop. 2.2.16]) with the fact that (3.2) above holds for $\boldsymbol{\tau}=\boldsymbol{h}^{1/p}(n)^{1/p}$, $0<p<\infty$, $0<q\le\min\{p,1\}$ (cf. [Br2, Thm. 3.3.1(i)]). Then, following [Br2, Def. 3.3.5], for $0<p,q<\infty$ and $\boldsymbol{\sigma}$ admissible with $\underline{\mathfrak{s}}(\boldsymbol{\sigma})>0$, we define Besov spaces on Γ

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma):=\mathrm{tr}_\Gamma\, B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n), \tag{3.3}$$

more precisely,

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma):=\{f\in L_p(\Gamma):\exists g\in B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n),\ \mathrm{tr}_\Gamma\, g=f\}, \tag{3.4}$$

equipped with the quasi-norm

$$\begin{aligned}&\|f\,|\,\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)\|\\&\quad=\inf\{\|g\,|\,B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)\|:\mathrm{tr}_\Gamma\, g=f,\ g\in B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)\}.\end{aligned} \tag{3.5}$$

This was extended in [Ca2, Def. 2.7] in the following way:

Definition 3.1. Let $0<p,q<\infty$, let $\boldsymbol{\sigma}$ be an admissible sequence, and Γ be an h-set. Assume that

(i) for $p\ge1$ or $q\le p<1$,

$$\boldsymbol{\sigma}^{-1}\in\ell_{q'}, \tag{3.6}$$

(ii) for $0<p<1$ and $p<q$,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{1/r-1/p}\in\ell_{v_r}\quad\text{for some } r\in[p,\min\{q,1\}] \text{ with } \frac{1}{v_r}=\frac{1}{r}-\frac{1}{q}. \tag{3.7}$$

Then we again define $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ as in (3.3), (3.4) with the quasi-norm given by (3.5).

Remark 3.2. The reasonability of declaring that smoothness (1) (that is, 0, in classical notation) on Γ corresponds to smoothness $\boldsymbol{h}^{1/p}(n)^{1/p}$ on $\mathbb{R}^n$—as is implicit in (3.3)—comes from the fact that, at least when Γ also satisfies the porosity condition, we indeed have $\mathrm{tr}_\Gamma\, B^{\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)=L_p(\Gamma)$ when $0<p<\infty$, $0<q\le\min\{p,1\}$—cf. [Br2, Thm. 3.3.1(ii)].

Remark 3.3. Both in [Br2] and in [Ca2] the definition of Besov spaces on Γ also covers the cases when p or q can be ∞, with some modifications of the approach given above. However, in view of the main results to be presented in this paper, the restriction to $0<p,q<\infty$ is natural.

Remark 3.4. We briefly compare the different assumptions in [Br2, Def. 3.3.5] and in Definition 3.1 above. Due to the observation following (2.27), $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$ implies $\boldsymbol{\sigma}^{-1} \in \ell_v$ for all v, i.e. (3.6) and (3.7) with $r = p$. The converse, however, is not true: take e.g. $\sigma_j = (1+j)^{\varkappa}$, $\varkappa \in \mathbb{R}$; then $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = 0$, but $\boldsymbol{\sigma}^{-1} \in \ell_{q'}$ for $\varkappa > 1/q'$, corresponding to (3.6). As for (3.7), say with $p = r$, for the same $\boldsymbol{\sigma}$ we also have $\boldsymbol{\sigma}^{-1} \in \ell_{v_p}$ for $\varkappa > 1/p - 1/q$, but still $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = 0$. So the above definition is in fact a proper extension of the one considered in [Br2] and, as we shall see, will (at least in some cases) be indeed the largest possible extension.

Remark 3.5. The definition above applies in particular when Γ is a d-set and $\boldsymbol{\sigma} = (s)$ with $s > 0$. In simpler notation, we can write in this case that

$$\mathbb{B}^s_{p,q}(\Gamma) = \mathrm{tr}_\Gamma\, B^{s+(n-d)/p}_{p,q}(\mathbb{R}^n). \tag{3.8}$$

This coincides with [Tr4, Def. 20.2].

3.2. Criteria for non-existence of trace spaces. Here we shall get necessary conditions for the existence of the trace. To this end, we explore the point of view that the trace cannot exist in the sense of Definition 3.1 when $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$, the set of test functions with compact support outside Γ, is dense in $B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$.

In fact, assume that $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$. Let $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi \equiv 1$ on a neighbourhood of Γ. Clearly, $\varphi \in B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$. Then there exists a sequence $(\psi_k)_k \subset \mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ with

$$\|\varphi - \psi_k \,|\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)\| \xrightarrow[k\to\infty]{} 0. \tag{3.9}$$

If the trace were to exist in the sense explained before, this would imply

$$0 = \psi_k|_\Gamma = \mathrm{tr}_\Gamma\, \psi_k \xrightarrow[k\to\infty]{} \mathrm{tr}_\Gamma\, \varphi = \varphi|_\Gamma = 1 \quad \text{in } L_p(\Gamma), \tag{3.10}$$

which is a contradiction.

Discussion 3.6. So in order to disprove the existence of the trace in certain cases it is sufficient to show the density of $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$. Further, we may restrict ourselves to functions $\varphi \in C^\infty_0(\mathbb{R}^n)$ because of their density in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$, $0 < p, q < \infty$, and approximate them by functions $\psi_k \in \mathcal{D}(\mathbb{R}^n \setminus \Gamma)$. We shall construct such ψ_k based on finite sums of the type

$$\sum_{r\in I_k} \lambda_r \varphi_r,$$

where I_k is some finite index set, $\lambda_r \in \mathbb{C}$ and φ_r are compactly supported smooth functions with

$$\sum_{r\in I_k} \lambda_r \varphi_r \varphi = \varphi \tag{3.11}$$

on a neighbourhood (depending on k) of Γ, φ as above, and

$$\Big\| \sum_{r\in I_k} \lambda_r \varphi_r \varphi \,\Big|\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)\Big\| \xrightarrow[k\to\infty]{} 0. \tag{3.12}$$

Plainly, then $\psi_k := \varphi - \sum_{r\in I_k} \lambda_r\varphi_r\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \Gamma)$, $k \in \mathbb{N}$, and

$$\|\varphi - \psi_k \,|\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)\| = \Big\| \sum_{r\in I_k} \lambda_r \varphi_r \varphi \,\Big|\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)\Big\| \xrightarrow[k\to\infty]{} 0,$$

and therefore the required density is proved.

The limit in (3.12) above is going to be computed with the help of atomic representations for the functions considered. In order to explain how this will be done, we need first to consider a preliminary result which connects the Definition 2.1 of h-sets with the atomic representation given in Proposition 2.20.

DEFINITION 3.7. Given $\Gamma \subset \mathbb{R}^n$ and $r > 0$, we shall denote by Γ_r the neighbourhood of radius r of Γ, that is,

$$\Gamma_r := \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Gamma) < r\}.$$

LEMMA 3.8. *Let Γ be an h-set in $\mathbb{R}^n$ with μ a corresponding Radon measure. Let $j \in \mathbb{N}$. There is a cover $\{Q(j,i)\}_{i=1}^{N_j}$ of $\Gamma_{2^{-j}}$ such that*

- (i) $N_j \sim h_j^{-1}$,
- (ii) *$Q(j,i)$ are cubes in $\mathbb{R}^n$ of side length $\sim 2^{-j}$,*
- (iii) *each $Q(j,i)$ is, in fact, the union of ~ 1 grid cubes of a tessellation of $\mathbb{R}^n$ of step 2^{-j} in the sense of Remark 2.21,*
- (iv) *each $Q(j,i)$ contains a ball of radius $\sim 2^{-j}$ centred at a point of Γ such that any two of them, for different values of i, are disjoint.*

Denote by $\mathcal{Q}(j)$ the family of all grid cubes obtained in this way (that is, $\mathcal{Q}(j) = \{Q_{jm} : Q_{jm} \subset Q(j,i)$ for some $i\}$) and consider the corresponding functions of a related partition of unity $\{\varphi_{jm}\}$ of step 2^{-j} of $\mathbb{R}^n$ in the sense of Remark 2.22. That is, consider only the functions of that partition which are supported on $\frac{3}{2}Q_{jm}$ such that $Q_{jm} \in \mathcal{Q}(j)$.

The cover of $\Gamma_{2^{-j}}$ considered above can be chosen in such a way that the family

$$\{\varphi_{jm} : Q_{jm} \in \mathcal{Q}(j)\}$$

is a partition of unity of $\Gamma_{2^{-j}}$.

All equivalence constants in the estimates above are independent of j and i.

Proof. For each $j \in \mathbb{N}$, start by considering an optimal cover of Γ in the sense of [Br2, Lemma 1.8.3], that is, a cover by balls $B(\gamma_i, 2^{-j-1})$ centred at points $\gamma_i \in \Gamma$ and such that $B(\gamma_i, 2^{-j-1}/3) \cap B(\gamma_k, 2^{-j-1}/3) = \emptyset$ for $i \neq k$.

As was pointed out in [Br2, Lemma 1.8.3], the number $N_{2^{-j-1}}$ of balls in such a cover satisfies

$$N_{2^{-j-1}} \sim h_{j+1}^{-1}. \tag{3.13}$$

It is not difficult to see that each $B(\gamma_i, 2^{-j-1})$ is contained in the union of 2^n grid cubes of the type Q_{jm} which together form a cube of side length 2^{-j+1} and which we shall provisionally denote by $Q(j,i)$. Then assertions (ii)–(iv) of the lemma are clear.

As to (i), it is also clear, using (3.13) and Proposition 2.2, that $N_j \lesssim h_j^{-1}$. The reverse inequality follows from the fact that there can only be ~ 1 different $B(\gamma_k, 2^{-j-1})$ giving rise to the same $Q(j,i)$, because each of those also produces a ball $B(\gamma_k, 2^{-j-1}/3)$ which is disjoint from all the other balls obtained in a similar way. Eliminating the repetitions among the previously considered $Q(j,i)$ and redefining the i accordingly, we get a possible cover of Γ satisfying (i)–(iv) of the lemma.

To get the corresponding cover of $\Gamma_{2^{-j}}$, we just need to enlarge each $Q(j,i)$ by adding all the $4^n - 2^n$ surrounding grid cubes which touch it. It is clear that the new cubes $Q(j,i)$ obtained in this way also satisfy (i)–(iv) above.

Finally, if we further enlarge the preceding $Q(j,i)$ by adding to it all the $6^n - 4^n$ surrounding grid cubes which touch it, clearly we do not destroy properties (i)–(iv) for the new cubes $Q(j,i)$, this cover of $\Gamma_{2^{-j}}$ satisfying also the property that

$$\{\varphi_{jm} : Q_{jm} \in \mathcal{Q}(j)\}$$

is a partition of unity of $\Gamma_{2^{-j}}$. ■

Remark 3.9. It follows from the construction above that $\mathcal{Q}(j)$ is also a cover of $\Gamma_{2^{-j}}$ with $\sharp\mathcal{Q}(j) \sim h_j^{-1}$ (though a property like (iv) above cannot be guaranteed for each grid cube in $\mathcal{Q}(j)$).

Remark 3.10. We shall use the expression 'optimal cover of $\Gamma_{2^{-j}}$' when referring to a cover of $\Gamma_{2^{-j}}$ satisfying all the requirements of the lemma above.

We return now to the question of calculating the limit (3.12) in Discussion 3.6.

Discussion 3.11. The index set I_k, $k \in \mathbb{N}$, will have the structure

$$I_k = \{(j,m) \in \mathbb{N} \times \mathbb{Z}^n : j \in J_k,\ m \in M_j\}, \tag{3.14}$$

where J_k and M_j are appropriate finite subsets of $\mathbb{N}$ and $\mathbb{Z}^n$ respectively. In any case, for each $(j,m) \in I_k$ the relation $\operatorname{dist}(Q_{jm}, \Gamma) \lesssim 2^{-j}$ should be satisfied. With $r = (j,m) \in I_k$, we shall also require that

$$\operatorname{supp} \varphi_r \subset bQ_{jm} \tag{3.15}$$

for some constant $b > 1$, and, for any fixed $K \in \mathbb{N}$,

$$|\mathrm{D}^\gamma \varphi_r(x)| \le c_K 2^{j|\gamma|}, \qquad x \in \mathbb{R}^n,\ \gamma \in \mathbb{N}_0^n \ \text{with}\ |\gamma| \le K. \tag{3.16}$$

We claim that then (up to constants) the functions $a_{(j,m)} = \tau_j^{-1} 2^{jn/p} \varphi_{(j,m)} \varphi$ are $(\boldsymbol{\tau}, p)_{K,-1}$-atoms (no moment conditions) located at Q_{jm}: the support condition is obvious; as for the derivatives we calculate for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le K$, that

$$\begin{aligned} |\mathrm{D}^\alpha a_{(j,m)}(x)| \le\ & \tau_j^{-1} 2^{jn/p} \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} |\mathrm{D}^\gamma \varphi_{(j,m)}(x)|\, |\mathrm{D}^{\alpha-\gamma} \varphi(x)| \\ \le\ & \tau_j^{-1} 2^{jn/p} c'_K 2^{j|\alpha|} \|\varphi \mid C^K(\mathbb{R}^n)\| \\ \le\ & C_{K,\varphi} \tau_j^{-1} 2^{j(n/p+|\alpha|)}, \end{aligned}$$

fitting the needs of Definition 2.19.

Since

$$\sum_{r \in I_k} \lambda_r \varphi_r \varphi = \sum_{(j,m) \in I_k} (\lambda_{(j,m)} C_{K,\varphi} \tau_j 2^{-jn/p})(C_{K,\varphi}^{-1} a_{(j,m)}),$$

to obtain (3.12) we can then estimate from above the quasi-norm of this function by applying Proposition 2.20, provided the parameters considered do not require moment conditions.

If moment conditions are required, and the h-set Γ satisfies the porosity condition, a standard procedure can be applied to identify the above sum at least in a smaller neighbourhood of Γ (than the one considered for (3.11)) with an atomic representation

$$\sum_{(j,m) \in I_k} (c_2^{-1} \lambda_{(j,m)} C_{K,\varphi} \tau_j 2^{-jn/p})(c_2 C_{K,\varphi}^{-1} \tilde{a}_{(j,m)})$$

satisfying appropriate moment conditions; this produces essentially the same upper estimate by applying Proposition 2.20. So, in this case we replace, in Discussion 3.6, $\sum_{r \in I_k} \lambda_r \varphi_r \varphi$ by the above sum, keeping a property like (3.11). In any case, following Proposition 2.20, the quasi-norms of both sums in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ are estimated by

$$\lesssim \Big(\sum_{j \in J_k} \Big(\sum_{m \in M_j} |\lambda_{(j,m)} \tau_j 2^{-jn/p}|^p \Big)^{q/p} \Big)^{1/q}. \tag{3.17}$$

The standard procedure referred to above, leading to the creation of atoms with appropriate moment conditions, though still coinciding with the former atoms in a somewhat smaller neighbourhood of a set satisfying the porosity condition, comes from [TW] and was used for example in [Ca1] and, more recently, in [Tr7]. It is quite technical, but we give here a brief description for the convenience of the reader.

For each Q_{jm} as above, namely with $\operatorname{dist}(Q_{jm},\Gamma) \lesssim 2^{-j}$, fix an element of Γ, $x_{j,m}$ say, at a distance $\lesssim 2^{-j}$ from Q_{jm}. Clearly, there is a constant $c_1 > 0$ such that $Q_{jm} \subset B(x_{j,m}, c_1 2^{-j})$. Without loss of generality, we can assume that $0 < c_1 2^{-j} < 1$, so that, as Γ satisfies the porosity condition of Definition 2.8, there exists $y_{j,m} \in \mathbb{R}^n$ such that $B(y_{j,m}, \eta c_1 2^{-j}) \subset B(x_{j,m}, c_1 2^{-j})$ and $B(y_{j,m}, \eta c_1 2^{-j}) \cap \Gamma = \emptyset$, where $0 < \eta < 1$ is as in Definition 2.8. Obviously, as mentioned in (2.16), also $\operatorname{dist}(B(y_{j,m}, \eta c_1 2^{-j-1}), \Gamma) \geq \eta c_1 2^{-j-1}$.

Fix a natural L as in (2.39)—with τ instead of σ—and let ψ_γ, with $\gamma \in \mathbb{N}_0^n$ and $|\gamma| \leq L$, be C^∞-functions with support in the open ball $\mathring{B}(0,1)$ and satisfying

$$\forall \beta,\gamma \in \mathbb{N}_0^n \text{ with } |\beta|,|\gamma| \leq L, \qquad \int_{\mathbb{R}^n} x^\beta \psi_\gamma(x)\,dx = \delta_{\beta\gamma},$$

where $\delta_{\beta\gamma}$ stands for the Kronecker symbol (for the existence of such functions, see [TW, p. 665]).

Define, for each j,m as above,

$$d_\gamma^{j,m} \equiv \int_{\mathbb{R}^n} x^\gamma a_{(j,m)}(\eta c_1 2^{-j-1} x + y_{j,m})\,dx, \qquad \gamma \in \mathbb{N}_0^n \text{ with } |\gamma| \leq L,$$

and

$$\tilde{a}_{(j,m)}(z) = a_{(j,m)}(z) - \sum_{|\gamma|\leq L} d_\gamma^{j,m} \psi_\gamma((\eta c_1)^{-1} 2^{j+1}(z - y_{j,m})), \qquad z \in \mathbb{R}^n.$$

It is easy to see that

$$\int_{\mathbb{R}^n} ((\eta c_1)^{-1} 2^{j+1}(z - y_{j,m}))^\beta \tilde{a}_{(j,m)}(z)\,dz = 0, \qquad \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq L,$$

and consequently (by Newton's binomial formula),

$$\int_{\mathbb{R}^n} z^\beta \tilde{a}_{(j,m)}(z)\,dz = 0, \qquad \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq L.$$

This means that each $\tilde{a}_{(j,m)}$ has the required moment conditions for the atoms in the atomic representations of functions of $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$. Actually, it is not difficult to see that there exists a positive constant c_2 such that $c_2 C_{K,\varphi}^{-1} \tilde{a}_{(j,m)}$ is a $(\tau,p)_{K,L}$-atom located at Q_{jm}.

Since, from the hypotheses and choices made, $\tilde{a}_{(j,m)} = a_{(j,m)}$ on $\Gamma_{\eta c_1 2^{-j-1}}$, we are done.

Proposition 3.12. *Let Γ be an h-set satisfying the porosity condition, let $0 < p,q < \infty$ and let $\boldsymbol{\tau}$ be an admissible sequence. Assume that*

$$\boldsymbol{\tau}^{-1} \boldsymbol{h}^{1/p}(n)^{1/p} \notin \ell_\infty. \tag{3.18}$$

Then $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$, therefore

$$\operatorname{tr}_\Gamma B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \quad \textit{cannot exist.} \tag{3.19}$$

Proof. By a standard reasoning, (3.18) implies that $\boldsymbol{\tau}^{-1}\boldsymbol{h}^{1/p}(n)^{1/p}$ has a divergent subsequence. More precisely, there is a strictly increasing sequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \tau_{j_k} h_{j_k}^{-1/p} 2^{-j_k n/p} = 0. \tag{3.20}$$

For each $k \in \mathbb{N}$, consider an optimal cover of $\Gamma_{2^{-j_k}}$ in the sense of Remark 3.10, and follow our Discussions 3.6 and 3.11 with I_k from (3.14) given by

$$I_k = \{(j,m) \in \mathbb{N}\times\mathbb{Z}^n : j = j_k,\ Q_{jm} \in \mathcal{Q}(j)\}$$

(that is, $J_k = \{j_k\}$ and $M_j = M_{j_k} = \{m \in \mathbb{Z}^n : Q_{j_k m} \in \mathcal{Q}(j_k)\}$), $\varphi_{(j,m)} = \varphi_{jm}$ and $\lambda_{(j,m)} = 1$. By Remark 2.22, this fits nicely into Discussions 3.6 and 3.11, in particular (3.11), (3.15) and (3.16) hold. It then follows, especially from (3.17), that the quasi-norm of the sum in (3.12) (or of an alternative similar sum, as discussed in Discussion 3.11 apropos of the moment conditions) is

$$\begin{aligned}\lesssim \Big(\sum_{Q_{j_k m}\in\mathcal{Q}(j_k)} \tau_{j_k}^p 2^{-j_k n}\Big)^{1/p} &= \tau_{j_k} 2^{-j_k n/p}(\sharp\mathcal{Q}(j_k))^{1/p} \\ &\sim \tau_{j_k} 2^{-j_k n/p} h_{j_k}^{-1/p} \xrightarrow[k\to\infty]{} 0,\end{aligned}$$

where we have also used Remark 3.9 and (3.20). Thus (3.12) holds (possibly with an alternative similar sum if moment conditions are required, as mentioned above), and Discussion 3.6 then concludes the proof. ∎

Remark 3.13. The porosity condition was only used in the proof above to guarantee that atoms with appropriate moment conditions can be considered. Thus the proposition holds without assuming porosity of Γ as long as L can be taken equal to -1 in the atomic representation theorem for $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ (cf. Proposition 2.20 with $\boldsymbol{\tau}$ instead of $\boldsymbol{\sigma}$).

The next result shows that when $p < q$ we can conclude as in the preceding proposition with a hypothesis weaker than (3.18).

Proposition 3.14. *Let Γ be an h-set satisfying the porosity condition, and let $0 < p < q < \infty$ and $\boldsymbol{\tau}$ be admissible. Assume that*

$$\limsup \boldsymbol{\tau}^{-1}\boldsymbol{h}^{1/p}(n)^{1/p} > 0. \tag{3.21}$$

Then $\mathcal{D}(\mathbb{R}^n\setminus\Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$, therefore

$$\mathrm{tr}_\Gamma\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \quad \textit{cannot exist.} \tag{3.22}$$

Proof. Denote $\boldsymbol{\sigma} := \boldsymbol{\tau}\boldsymbol{h}^{-1/p}(n)^{-1/p}$. From (3.21) there exists a constant $c > 0$ and a strictly increasing sequence $(j_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\sigma_{j_\ell}^{-1} \ge c, \qquad \ell \in \mathbb{N}. \tag{3.23}$$

Given $k \in \mathbb{N}$, let $k_2 \in \mathbb{N}$ be such that

$$\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k_2} \geq 2, \tag{3.24}$$

which clearly exists. Then choose $j(k)$ so large that there is an optimal cover of $\Gamma_{2^{-j(k)}}$ in the sense of Remark 3.10 with the number $N_{j(k)}$ of cubes such that $k_2^{-1} N_{j(k)} \geq 2$. Again, this is clearly possible, because of $N_{j(k)} \sim h_{j(k)}^{-1}$ (cf. Lemma 3.8) and the properties of gauge functions (cf. Definition 2.1). Actually, we shall also require that $j(k)$ coincides with one of the j_ℓ's above, which is also possible.

Hence, for $i = k, k+1, \ldots, k_2$,

$$\lfloor i^{-1} N_{j(k)} \rfloor > i^{-1} N_{j(k)} - 1 \geq \frac{1}{2} i^{-1} N_{j(k)},$$

and therefore, using (3.24),

$$\lfloor k^{-1} N_{j(k)} \rfloor + \lfloor (k+1)^{-1} N_{j(k)} \rfloor + \cdots + \lfloor k_2^{-1} N_{j(k)} \rfloor \geq N_{j(k)}.$$

Let $k_1 \in \mathbb{N}$ be smallest such that

$$\lfloor k^{-1} N_{j(k)} \rfloor + \lfloor (k+1)^{-1} N_{j(k)} \rfloor + \cdots + \lfloor k_1^{-1} N_{j(k)} \rfloor \geq N_{j(k)}. \tag{3.25}$$

On the other hand, following Lemma 3.8 and Discussion 3.6,

$$\sum_{Q_{j(k)m} \in \mathcal{Q}(j(k))} \varphi_{j(k)m} \varphi = \varphi \quad \text{on } \Gamma_{2^{-j(k)}}. \tag{3.26}$$

Now we partition the terms of this sum in the following way: first consider the m's such that $Q_{j(k)m}$ are grid subcubes of the first $\lfloor k^{-1} N_{j(k)} \rfloor$ cubes of the optimal cover above; next consider the m's such that $Q_{j(k)m}$ are grid subcubes of the following $\lfloor (k+1)^{-1} N_{j(k)} \rfloor$ cubes of the same cover; and so on until the m's such that $Q_{j(k)m}$ are grid subcubes of at most $\lfloor k_1^{-1} N_{j(k)} \rfloor$ cubes of the optimal cover. This process certainly leads to repetition of grid cubes, so, in order not to affect the sum above we make the convention that $\varphi_{j(k)m}$ is replaced by zero any time it corresponds to a grid cube that has already been considered before. That is, $\varphi_{j(k)m}$ is replaced by $\tilde{\varphi}_{j(k)m}$ which can be either $\varphi_{j(k)m}$ or zero, as just explained.

Denote by ψ_i, $i = k, k+1, \ldots, k_1$, the sum of the $\tilde{\varphi}_{j(k)m}$'s corresponding to each part above, so that the sum in (3.26) can be written as

$$\sum_{i=k}^{k_1} \psi_i \varphi, \tag{3.27}$$

which, of course, still equals φ on $\Gamma_{2^{-j(k)}}$.

Next we choose, for each $i = k+1, \ldots, k_1$, $j(i)$ in such a way that

$$j(k) < j(k+1) < \cdots < j(k_1) \tag{3.28}$$

and $j(i)$ coincides with one of the j_ℓ's used to get (3.23). Then consider optimal covers of $\Gamma_{2^{-j(i)}}$ and the sum

$$\sum_{i=k}^{k_1} \sum_{Q_{j(i)m} \in \mathcal{Q}(j(i))} \varphi_{j(i)m} \psi_i \varphi, \tag{3.29}$$

which, by Lemma 3.8 and (3.28), equals φ on $\Gamma_{2^{-j(k_1)}}$. We know, following Remark 3.9, that each inner sum above has $\sim h_{j(i)}^{-1}$ terms. However, because of the product structure of the latter and the support of each ψ_i, actually only a smaller number, namely $\lesssim i^{-1} h_{j(i)}^{-1}$ of terms are non-zero. This can be seen in the following way: for fixed $i = k, k+1, \ldots, k_1$, only the $\varphi_{j(i)m}$ such that

$$\operatorname{supp} \varphi_{j(i)m} \cap \operatorname{supp} \psi_i \neq \emptyset$$

are of interest; this implies that the cubes of the optimal cover of $\Gamma_{2^{-j(i)}}$ which contain the grid cubes at which such functions $\varphi_{j(i)m}$ are located are contained in a neighbourhood of radius $\sim 2^{-j(i)}$ of $\operatorname{supp} \psi_i$, which in turn is contained in the union of the neighbourhoods of radius $\sim 2^{-j(k)}$ of the $\lesssim i^{-1} h_{j(k)}^{-1}$ cubes of the optimal cover of $\Gamma_{2^{-j(k)}}$ which were used to form ψ_i; since these union measures $\lesssim i^{-1} h_{j(k)}^{-1} h_{j(k)}$, that is, $\lesssim i^{-1}$, and the cubes of the optimal cover of $\Gamma_{2^{-j(i)}}$ contain disjoint balls of radius $\sim 2^{-j(i)}$, so measuring $\sim h_{j(i)}$ each, the number of cubes that are being considered here must be $\lesssim i^{-1} h_{j(i)}^{-1}$; as clearly the same estimate holds for the family of grid subcubes of such cubes, our claim is proved.

We then write (3.29) in the form

$$\sum_{i=k}^{k_1} {\sum_{Q_{j(i)m} \in \mathcal{Q}(j(i))}}' \varphi_{j(i)m} \psi_i \varphi, \tag{3.30}$$

where the prime means that we are in fact taking only $\lesssim i^{-1} h_{j(i)}^{-1}$ terms, according to the discussion above, without changing the value of (3.29).

Now we remark that, for $i = k, k+1, \ldots, k_1$,

$$\operatorname{supp} \varphi_{j(i)m} \psi_i \subset \operatorname{supp} \varphi_{j(i)m} \subset \frac{3}{2} Q_{j(i)m},$$

and, due to the Leibniz formula, (3.28) and (2.41), for a fixed $K \in \mathbb{N}$ there is a positive constant c_K such that

$$|\mathrm{D}^\gamma (\varphi_{j(i)m} \psi_i)(x)| \leq c_K 2^{j(i)|\gamma|}, \qquad x \in \mathbb{R}^n,\ \gamma \in \mathbb{N}_0^n \text{ with } |\gamma| \leq K.$$

That is, (3.11), (3.15) and (3.16) hold for $\lambda_r = \lambda_{(j(i),m)} = 1$ and $\varphi_r = \varphi_{(j(i),m)} = \varphi_{j(i)m} \psi_i$ with $r = (j(i), m)$ belonging to a set I_k as in (3.14), though more involved to describe: here we have $J_k = \{j(k), j(k+1), \ldots, j(k_1)\}$

and each $M_{j(i)}$ with $i = k, k+1, \ldots, k_1$ is the set of m's considered in the corresponding inner sum in (3.30).

It then follows, especially from (3.17), that the quasi-norm of the sum in (3.12) (or of an alternative similar sum, as considered in Discussion 3.11 apropos of the moment conditions) is

$$\begin{aligned}
&\lesssim \Big(\sum_{i=k}^{k_1}\Big(\sum\nolimits'_{Q_{j(i)m}\in\mathcal{Q}(j(i))} \tau_{j(i)}^{p} 2^{-j(i)n}\Big)^{q/p}\Big)^{1/q} \\
&\lesssim \Big(\sum_{i=k}^{k_1} \tau_{j(i)}^{q} 2^{-j(i)nq/p} i^{-q/p} h_{j(i)}^{-q/p}\Big)^{1/q} \\
&= \Big(\sum_{i=k}^{k_1} i^{-q/p}\sigma_{j(i)}^{q}\Big)^{1/q} \lesssim \Big(\sum_{i=k}^{\infty} i^{-q/p}\Big)^{1/q},
\end{aligned}$$

the last estimate being a consequence of (3.23) and the choice of the $j(i)$ in (3.28).

Using now the hypothesis $0 < p < q < \infty$, we conclude that the last expression above tends to zero as $k \to \infty$, so that the result follows from Discussion 3.6. ∎

Remark 3.15. An observation corresponding to the one in Remark 3.13 also holds here.

The next result shows that when $q > 1$ we can also get the conclusion of Proposition 3.12 with a hypothesis weaker than (3.18).

Proposition 3.16. *Let Γ be an h-set satisfying the porosity condition, and let $1 < q < \infty$ and $\boldsymbol{\tau}$ be admissible. Assume that*

$$\boldsymbol{\tau}^{-1}\boldsymbol{h}^{1/p}(n)^{1/p} \notin \ell_{q'}. \tag{3.31}$$

Then $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$, therefore

$$\mathrm{tr}_\Gamma\, B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \quad \textit{cannot exist.}$$

Proof. Denote $\boldsymbol{\sigma} := \boldsymbol{\tau}\boldsymbol{h}^{-1/p}(n)^{-1/p}$. From (3.31) there exists a strictly increasing sequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\sum_{l=j_k}^{j_{k+1}-1} \sigma_l^{-q'} \geq 1, \qquad k \in \mathbb{N}. \tag{3.32}$$

For each $k \in \mathbb{N}$, consider optimal covers of $\Gamma_{2^{-i}}$, in the sense of Remark 3.10, for all $i = j_\nu, j_\nu + 1, \ldots, j_{\nu+1} - 1$ and all $\nu = 1, \ldots, k$, and follow Discussions 3.6 and 3.11 with I_k from (3.14) defined so that $J_k = \{j \in \mathbb{N} : \exists \nu = 1, \ldots, k : j \in \{j_\nu, j_\nu{+}1, \ldots, j_{\nu+1}{-}1\}\}$ and $M_j = \{m \in \mathbb{Z}^n : Q_{jm} \in \mathcal{Q}(j)\}$,

$\varphi_{(j,m)} = \varphi_{jm}$ and

$$\lambda_{(j,m)} = \frac{\sigma_j^{-q'}}{k} \Big(\sum_{l=j_\nu}^{j_{\nu+1}-1} \sigma_l^{-q'} \Big)^{-1}, \quad j = j_\nu, \dots, j_{\nu+1}-1,\ \nu = 1, \dots, k. \tag{3.33}$$

Due to Remark 2.22, this fits nicely into Discussions 3.6 and 3.11. In particular, (3.15) and (3.16) immediately hold. As for (3.11), we have, for $x \in \Gamma_{2^{-j_{k+1}}}$,

$$\begin{aligned}
&\sum_{j=j_1}^{j_{k+1}-1} \sum_{Q_{jm}\in\mathcal{Q}(j)} \lambda_{(j,m)}\varphi_{(j,m)}(x)\varphi(x) \\
&\qquad = \frac{1}{k} \sum_{\nu=1}^{k} \sum_{j=j_\nu}^{j_{\nu+1}-1} \sigma_j^{-q'} \Big(\sum_{l=j_\nu}^{j_{\nu+1}-1} \sigma_l^{-q'} \Big)^{-1} \sum_{Q_{jm}\in\mathcal{Q}(j)} \varphi_{jm}(x)\varphi(x) \\
&\qquad = \frac{\varphi(x)}{k} \sum_{\nu=1}^{k} \sum_{j=j_\nu}^{j_{\nu+1}-1} \sigma_j^{-q'} \Big(\sum_{l=j_\nu}^{j_{\nu+1}-1} \sigma_l^{-q'} \Big)^{-1} = \varphi(x).
\end{aligned}$$

It then follows, especially from (3.17), that the quasi-norm of the sum in (3.12) (or of an alternative similar sum, as considered in Discussion 3.11 apropos of the moment conditions) is

$$\begin{aligned}
&\lesssim \Big(\sum_{j=j_1}^{j_{k+1}-1} \Big(\sum_{Q_{jm}\in\mathcal{Q}(j)} |\lambda_{(j,m)}\tau_j 2^{-jn/p}|^p \Big)^{q/p} \Big)^{1/q} \\
&= \Big(\sum_{j=j_1}^{j_{k+1}-1} \lambda_{(j,m)}^q \tau_j^q 2^{-jnq/p} (\sharp\mathcal{Q}(j))^{q/p} \Big)^{1/q} \\
&\sim \Big(\sum_{\nu=1}^{k} \sum_{j=j_\nu}^{j_{\nu+1}-1} \frac{\sigma_j^{-q'q}}{k} \Big(\sum_{l=j_\nu}^{j_{\nu+1}-1} \sigma_l^{-q'} \Big)^{-q} \sigma_j^q \Big)^{1/q} \\
&= \frac{1}{k} \Big(\sum_{\nu=1}^{k} \Big(\sum_{l=j_\nu}^{j_{\nu+1}-1} \sigma_l^{-q'} \Big)^{1-q} \Big)^{1/q} \\
&\le \frac{1}{k} \Big(\sum_{\nu=1}^{k} 1 \Big)^{1/q} = k^{-1/q'} \xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

since $q > 1$, where we have also used Remark 3.9, (3.33) and (3.32). Thus (3.12) holds (possibly with an alternative similar sum if moment conditions are required, as mentioned above), and Discussion 3.6 then concludes the proof. ∎

REMARK 3.17. An observation corresponding to the one in Remark 3.13 also holds here.

THEOREM 3.18. *Let Γ be an h-set satisfying the porosity condition, $\boldsymbol{\sigma}$ an admissible sequence, and let either* $1 \le p < \infty$, $0 < q < \infty$, *or* $0 < q \le p < 1$. *Then* ***either***

(i) $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) = \operatorname{tr}_\Gamma B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ *exists,* ***or***

(ii) $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ *is dense in* $B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$, *and so* $\operatorname{tr}_\Gamma B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ *cannot exist.*

Proof. Either (3.6) holds or not. If it holds, then from Definition 3.1 it follows that (i) above holds. If (3.6) fails, then, with $\boldsymbol{\tau} := \boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}$,

$$\boldsymbol{\tau}^{-1}\boldsymbol{h}^{1/p}(n)^{1/p} \notin \ell_{q'},$$

therefore, from Propositions 3.12 (for $0 < q \le 1$) and 3.16 (for $1 < q < \infty$), (ii) holds. ■

REMARK 3.19. The only use of the porosity condition in the proof above is to guarantee the existence of suitable atoms when moment conditions for these are required. Therefore the conclusion of the theorem above holds without assuming porosity of Γ whenever the atomic representation of $B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ does not require atoms with moment conditions.

REMARK 3.20. From the proof above it also follows that, under the conditions of the theorem, (i) is equivalent to (3.6), that is, $\boldsymbol{\sigma}^{-1} \in \ell_{q'}$.

CONJECTURE 3.21. *Let Γ be an h-set satisfying the porosity condition, $\boldsymbol{\sigma}$ an admissible sequence, and* $0 < p < 1$, $0 < p < q < \infty$. *Define v_p by the identity* $1/v_p = 1/p - 1/q$. *With the extra assumption*

$$\lim_{j\to\infty} h_j \sigma_j^{v_p} = 0, \tag{3.34}$$

the alternative in the conclusion of the theorem above also holds. More precisely, assertion (i) *holds iff assertion* (ii) *does not hold iff*

$$\boldsymbol{\sigma}^{-1} \in \ell_{v_p}. \tag{3.35}$$

We discuss this conjecture a little. Clearly, if (3.35) holds then Definition 3.1 guarantees that (i) holds, and therefore (ii) does not, even without the extra assumption (3.34).

On the other hand, Proposition 3.14 with $\boldsymbol{\tau} := \boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}$ guarantees that at least in the special case of $\boldsymbol{\sigma}^{-1} \notin \ell_{v_p}$ with $\limsup \boldsymbol{\sigma}^{-1} > 0$, (ii) holds, and therefore (i) does not.

In other words, we can get the alternative in the conclusion of the theorem above in the case $0 < p < 1$, $0 < p < q < \infty$ if instead of the extra assumption (3.34) we assume that $\limsup \boldsymbol{\sigma}^{-1} > 0$. The drawback of this restriction is that it immediately implies that (3.35) never holds, so (3.35) is not a real alternative under that extra assumption. From this point of view, (3.34) is more interesting. Notice also that, for this conjecture not to contradict Definition 3.1, under (3.34) it must be true that (3.35) holds whenever any of the conditions (3.7) does. In fact, by standard comparison criteria for series it is easy to see directly that this is indeed the case (take also into account that the case $v_r = \infty$ in (3.7) never holds under (3.34)).

REMARK 3.22. It is worth noticing that when Γ is a d-set with $0 < d < n$, (3.34) holds whenever (3.35) fails, so in that setting, and by the above discussion, there is no need to impose the extra condition (3.34), the conjecture then turning out to be indeed a known result (cf. [Tr7, (1.5)]). However, for general h-sets we cannot dispense with an extra assumption (such as (3.34)), as the conjecture above then fails, at least as regards equivalence with (3.35). We now give a class of relevant examples:

Consider $h(r) = (1 + |\log r|)^b$, $r \in (0, 1]$, for any given $b \in (-1, 0)$; recall (2.13) and the discussion afterwards. Let $0 < p < 1$ and $0 < p < q < \infty$. Given any $\kappa \in \big(-b\big(\frac{1}{p} - \frac{1}{q}\big) + (1 + b)\frac{1}{q'}, \frac{1}{p} - \frac{1}{q}\big]$, consider $\boldsymbol{\sigma} = ((1 + j)^{\kappa})_j$. It is easy to see that, though (3.35) fails, by Definition 3.1(ii) with $r = \min\{q, 1\}$, (i) of the theorem above holds, i.e., the trace exists (and therefore the corresponding assertion (ii) fails, as discussed at the beginning of Section 3.2).

Nevertheless, our conjecture as stated above resists this class of counterexamples: as is easily seen, for this class the assumption (3.34) is violated.

3.3. Dichotomy results. We combine our results from the previous subsections and deal with the so-called *dichotomy* of trace spaces. First we briefly describe the idea.

Recall that $\mathcal{D}(\mathbb{R}^n)$ is dense in all spaces $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ with $0 < p, q < \infty$. So removing from $\mathbb{R}^n$ only a 'small enough' Γ one can ask whether (still)

$$\mathcal{D}(\mathbb{R}^n \setminus \Gamma) \text{ is dense in } B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n). \tag{3.36}$$

Conversely, we have the affirmative trace results mentioned in Section **??**, but one can also ask for what ('thick enough') Γ

$$\text{there exists a trace of } B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \text{ in } L_p(\Gamma) \tag{3.37}$$

(for sufficiently high smoothness and q-regularity). Though these questions may arise independently, it is at least clear that whenever $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$, there cannot exist a trace according to (3.2); see our discussion at the beginning of Section 3.2 and [Tr7] for the corresponding argument in the classical case.

REMARK 3.23. It is not always true that one really has an alternative in the sense that *either* there is a trace *or* $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$.

Triebel studied such questions in [Tr7] for spaces of type $B^s_{p,q}$ and described an example of a set Γ where a gap remains: traces can only exist for spaces $B^s_{p,q}$ with smoothness $s \geq s_0$, whereas density requires $s \leq s_1$ and $s_1 < s_0$.

However, if one obtains an alternative between (3.36) and (3.37), then following Triebel [Tr7] we call this phenomenon *dichotomy*. First we recall this notion for spaces of type $B^s_{p,q}(\mathbb{R}^n)$ and then we point out necessary modifications for our setting. Let

$$\mathrm{tr}_\Gamma : B^s_{p,q}(\mathbb{R}^n) \to L_p(\Gamma) \tag{3.38}$$

be the trace operator defined by completion from the pointwise trace according to (3.2), and

$$B_p(\mathbb{R}^n) = \{B^s_{p,q}(\mathbb{R}^n) : 0 < q < \infty,\, s \in \mathbb{R}\}, \qquad 0 < p < \infty. \tag{3.39}$$

Definition 3.24. Let $n \in \mathbb{N}$, $\Gamma \subset \mathbb{R}^n$, $0 < p < \infty$. The *dichotomy* of the scale $B_p(\mathbb{R}^n)$ with respect to $L_p(\Gamma)$, denoted by $\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma))$, is defined by

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma)) = (s_\Gamma, q_\Gamma), \qquad s_\Gamma \in \mathbb{R},\ 0 < q_\Gamma < \infty, \tag{3.40}$$

if

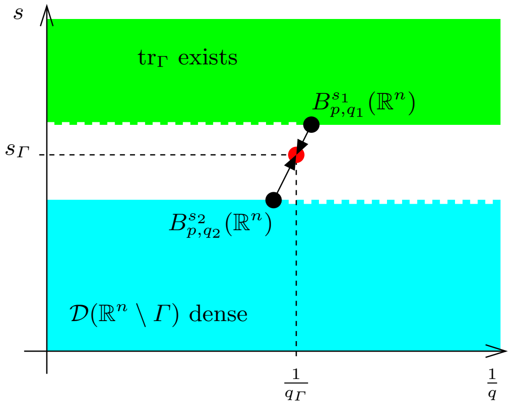
$$\text{(i) } \mathrm{tr}_\Gamma\, B^s_{p,q}(\mathbb{R}^n) \text{ exists for } \begin{cases} s > s_\Gamma, & 0 < q < \infty, \\ s = s_\Gamma, & 0 < q \leq q_\Gamma, \end{cases} \tag{3.41}$$

and

$$\text{(ii) } \mathcal{D}(\mathbb{R}^n \setminus \Gamma) \text{ is dense in } B^s_{p,q}(\mathbb{R}^n) \text{ for } \begin{cases} s = s_\Gamma, & q_\Gamma < q < \infty, \\ s < s_\Gamma, & 0 < q < \infty. \end{cases} \tag{3.42}$$

Remark 3.25. The notion applies to spaces of type $F^s_{p,q}$ in a similar way (cf. [Tr7]). Then one has to define the borderline cases $q_\Gamma = 0$ and $q_\Gamma = \infty$, too. But this will not be needed at the moment in our setting.

We briefly explain why it is reasonable to look for the 'breaking point' (s_Γ, q_Γ). In the diagram below we sketch this situation, where spaces of type $B^s_{p,q}(\mathbb{R}^n)$ are indicated by their parameters $(1/q, s)$ (while p is always assumed to be fixed). Assume that $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in some $B^{s_2}_{p,q_2}(\mathbb{R}^n)$. Then, since $\mathcal{D}(\mathbb{R}^n)$ is dense in all the spaces $B^t_{p,u}(\mathbb{R}^n)$, we immediately see that $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is also dense in all the spaces $B^t_{p,u}(\mathbb{R}^n)$ in which $B^{s_2}_{p,q_2}(\mathbb{R}^n)$ is continuously embedded (the shaded area left of and below $(1/q_2, s_2)$ referring to $B^{s_2}_{p,q_2}(\mathbb{R}^n)$ in the diagram). This explains why we look for the largest possible s_2 and smallest possible q_2 in (3.42). Conversely, if the trace exists for some $B^{s_1}_{p,q_1}(\mathbb{R}^n)$, then it exists likewise for all spaces which embed continuously into $B^{s_1}_{p,q_1}(\mathbb{R}^n)$ (the shaded area right of and above $(1/q_1, s_1)$ referring to $B^{s_1}_{p,q_1}(\mathbb{R}^n)$ in the diagram); hence we now search for the smallest possible s_1 and largest possible q_1 in (3.41). Dichotomy in the above-defined sense happens if the two 'extremal' points merge, that is, the common breaking point $(1/q_\Gamma, s_\Gamma)$ in the diagram exists. Then we denote the couple of parameters (s_Γ, q_Γ) by $\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma))$.



In [Tr8, Sect. 6.4.3] Triebel already mentioned that it might be more reasonable in general to exclude the limiting case $q = q_\Gamma$ in (3.41) or shift it to (3.42). But as will turn out below, (also) in our context the breaking point $q = q_\Gamma$ is always on the trace side.

Now we collect what is known in the situation of $B^s_{p,q}$ spaces for hyperplanes $\Gamma = \mathbb{R}^m$ or d-sets Γ, $0 < d < n$.

Proposition 3.26. *Let* $0 < p < \infty$.

(i) *Let* $m \in \mathbb{N}$, $m \leq n-1$. *Then*

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\mathbb{R}^m)) = \left(\frac{n-m}{p}, \min\{p,1\}\right).$$

(ii) *Let* Γ *be a compact* d*-set,* $0 < d < n$. *Then*

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma)) = \left(\frac{n-d}{p}, \min\{p,1\}\right).$$

Remark 3.27. The result (i) is proved in this explicit form in [Tr7] (see also [Tr8, Cor. 6.69] and [Sch1]). The second part (ii) can be found in [Tr7] and [Tr8, Thm. 6.68] (with forerunners in [Tr4, Thm. 17.6] and [Tr5, Prop. 19.5]). In the classical case of a bounded C^∞ domain Ω in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ (having $d = n-1$), $1 < p < \infty$, $1 \leq q < \infty$, $s \in \mathbb{R}$, the situation (whether $\mathcal{D}(\Omega)$ is dense in $B^s_{p,q}(\Omega)$) has been known for long (cf. [Tr1, Thm. 4.7.1]), even in a more general setting (we refer to [Sch2]). Moreover, dichotomy questions for weighted spaces corresponding to (ii) were dealt with in [Pi, Ha]. More precisely, when Γ is again a compact

d-set, $0 < d < n$, $0 < p < \infty$, and the weight $w_{\varkappa,\Gamma}$ is given by

$$w_{\varkappa,\Gamma}(x) = \begin{cases} \operatorname{dist}(x,\Gamma)^{\varkappa} & \text{if } \operatorname{dist}(x,\Gamma) \le 1, \\ 1 & \text{if } \operatorname{dist}(x,\Gamma) \ge 1, \end{cases}$$

with $\varkappa > -(n-d)$, then

$$\mathbb{D}(B_p(\mathbb{R}^n, w_{\varkappa,\Gamma}), L_p(\Gamma)) = \left(\frac{n-d+\varkappa}{p}, \min\{p,1\}\right). \tag{3.43}$$

In all the cases mentioned above there are parallel results for F-spaces, too.

We return to our setting of traces of $B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n)$ on fractal h-sets. Obviously, definitions (3.39)–(3.42) have to be modified. The following extension seems appropriate. (Recall that we set $v = \infty$ if $1/v = 0$.)

Definition 3.28. Let Γ be a porous h-set, and

$$B_p(\mathbb{R}^n) = \{B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) : 0 < q < \infty,\ \boldsymbol{\tau} \text{ admissible}\}, \qquad 0 < p < \infty. \tag{3.44}$$

Then the *dichotomy* of the scale $B_p(\mathbb{R}^n)$ with respect to $L_p(\Gamma)$ is defined by

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma)) = (\boldsymbol{\tau}_\Gamma, q_\Gamma) \tag{3.45}$$

if $\boldsymbol{\tau}_\Gamma$ is admissible, $0 < q_\Gamma < \infty$, and

$$\text{(i)}\ \ \operatorname{tr}_\Gamma B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \text{ exists for } \boldsymbol{\tau}^{-1}\boldsymbol{\tau}_\Gamma \in \ell_v, \text{ where } \frac{1}{v} = \left(\frac{1}{q_\Gamma} - \frac{1}{q}\right)_+, \tag{3.46}$$

and

$$\text{(ii)}\ \ \mathcal{D}(\mathbb{R}^n \setminus \Gamma) \text{ is dense in } B^{\boldsymbol{\tau}}_{p,q}(\mathbb{R}^n) \text{ for } \boldsymbol{\tau}^{-1}\boldsymbol{\tau}_\Gamma \notin \ell_v. \tag{3.47}$$

Remark 3.29. One immediately checks that (3.44)–(3.47) with $h(r) = r^d$, $\boldsymbol{\tau} = (s)$ for $s \in \mathbb{R}$, and $0 < p < \infty$, $0 < q < \infty$, coincides with (3.39)–(3.42), that is, the notion is extended (we may thus retain the same symbols with a slight abuse of notation). Using the continuous embedding

$$B^{\boldsymbol{\sigma}}_{p,q_1}(\mathbb{R}^n) \hookrightarrow B^{\boldsymbol{\tau}}_{p,q_2}(\mathbb{R}^n) \tag{3.48}$$

for $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau} \in \ell_{q^*}$ with $1/q^* = (1/q_2 - 1/q_1)_+$ (cf. [CaF, Thm. 3.7]), we can argue as in Remark 3.25 to motivate the definition.

Part of the results contained in Theorem 3.18 and Remark 3.20 can then be rephrased in terms of this notion of dichotomy in the following way.

Corollary 3.30. *Let Γ be an h-set satisfying the porosity condition and $1 \le p < \infty$. Then*

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\Gamma)) = (\boldsymbol{h}^{1/p}(n)^{1/p}, 1).$$

Remark 3.31. Again, for $h(r) = r^d$, $\boldsymbol{\tau} = (s)$, Corollary 3.30 coincides with Proposition 3.26(ii) for $p \ge 1$. So one might expect some parallel result with $q_\Gamma = p$ for $0 < p < 1$ (see also (3.43)). However, we have not been yet able to (dis)prove this claim. More precisely, if $0 < p < 1$ and $p < q < \infty$, (3.46) is satisfied with $\boldsymbol{\tau}_\Gamma = \boldsymbol{h}^{1/p}(n)^{1/p}$ and $q_\Gamma = p$ (see (3.7)). The gap that remains at the moment is to confirm (3.47) in that case (possibly with some additional assumptions); recall Conjecture 3.21 and the following discussion.

But we can obtain some weaker version as follows. Introducing a notion of dichotomy where also q can be fixed beforehand, hence adapting accordingly (3.44), (3.45) and denoting the new versions respectively by $B_{p,q}(\mathbb{R}^n)$ and $\mathbb{D}(B_{p,q}(\mathbb{R}^n), L_p(\Gamma)) = (\boldsymbol{\tau}_\Gamma, q_\Gamma)$, while keeping (3.46) and (3.47) unchanged, we can also recast the case $0 < q \le p < 1$ of Theorem 3.18 and Remark 3.20 in terms of dichotomy.

Corollary 3.32. *Let Γ be an h-set satisfying the porosity condition and $0 < q \le p < 1$. Then*

$$\mathbb{D}(B_{p,q}(\mathbb{R}^n), L_p(\Gamma)) = (\boldsymbol{h}^{1/p}(n)^{1/p},\ p).$$

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