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# Superelliptic equations arising from sums of consecutive powers

by

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**1. Introduction.** In 1964, LeVeque [21] applied a theorem of Siegel [28] to show that if $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree $k \geq 2$ with at least two simple roots, and $n \geq \max\{2, 5-k\}$ is an integer, then the *superelliptic* equation

$$f(x) = z^n \tag{1}$$

has at most finitely many solutions in integers $x$ and $z$. This result was extended by Schinzel and Tijdeman [27], through application of lower bounds for linear forms in logarithms, to show that equation (1) has in fact at most finitely many solutions in integers $x, z$ and *variable* $n \geq \max\{2, 5-k\}$ (where we count the solutions with $z^n = \pm 1, 0$ once).

While this latter result is effective (in the sense that the finite set of triples $(x, z, n)$ is effectively computable), in practice such a determination has infrequently been achieved, due to the extraordinary size of the bounds for $x$, $z$ and $n$ arising from the proof. The few cases that have been treated in the literature have been restricted to polynomials with very few monomials, or with multiple linear factors over $\mathbb{Q}$.

One class of polynomials that has proved, in certain cases at least, amenable to such an approach is that arising from sum of consecutive $k$th powers. Let us define

$$S_k(x) = 1^k + 2^k + \cdots + x^k,$$

where $x$ and $k$ are non-negative integers. Equations of the shape

$$S_k(x) - S_k(y) = z^n \tag{2}$$

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have been considered by a number of authors, under the hypotheses that $y = 0$ (see e.g. [10], [18], [19], [23], [24], [26], [34], [35]), that $y = [x/2]$ ([37]) and that $y = x - 3$ ([13], [37]). In the first two of these situations, the resulting polynomials on the left hand side of equation (2) have at least two distinct linear factors over $\mathbb{Q}$, which allows the problem to be reduced to one of binomial Thue equations.

Regarding the last of these cases, Cassels [13] solved the Diophantine equation $(x-1)^3 + x^3 + (x+1)^3 = z^2$ in integers $x$ and $z$, showing that the only solutions satisfy $x = 0, 1, 2$ and $24$; the same equation in a slightly disguised form is treated by Uchiyama [33]. Zhongfeng Zhang [37] subsequently considered the more general equation

$$(x-1)^k + x^k + (x+1)^k = z^n, \qquad x, z, k, n \in \mathbb{Z},\ k, n \geq 2. \tag{3}$$

Associating solutions to a Frey–Hellegouarch curve and applying standard level lowering arguments, he proved that the only solutions with $k \in \{2, 3, 4\}$ are $(x, z, k, n) = (1, \pm 3, 3, 2)$, $(2, \pm 6, 3, 2)$, $(24, \pm 204, 3, 2)$, $(\pm 4, \pm 6, 3, 3)$ and $(0, 0, 3, n)$.

In this paper, we extend Zhang's result, completely solving equation (3) in the cases $k = 5$ and $k = 6$. It should be emphasized that these results cannot apparently be obtained from the arguments of [37], using Frey–Hellegouarch curves over $\mathbb{Q}$. Indeed, the purpose of this paper is two-fold. On the one hand, we will use the case $k = 5$ to advertise the utility of the more powerful multi-Frey–Hellegouarch approach, pioneered in [12] (see also e.g. [2], [3], [4] and [11]). Our result here is as follows:

Theorem 1. *The only solutions to the equation*

$$(x-1)^5 + x^5 + (x+1)^5 = z^n, \qquad x, z, n \in \mathbb{Z},\ n \geq 2,$$

*satisfy* $x = z = 0$.

The other main purpose of this paper is to introduce a new computational approach to handle Diophantine problems where the problem of extracting information about associated forms arising from modularity is at the limit of current computational power. The method of Frey–Hellegouarch curves and Galois representations generally requires the explicit computation of weight 2 newforms of certain levels and also the computation of some of their Hecke eigenvalues. This computation can be completely impractical if the level is large. This turns out to be the case for equation (3) with $k = 6$, where required newforms have level $3^3 \cdot 3391$ and the newform space has dimension 4520. We develop a version of the standard 'method for bounding exponents' [29, Section 9] that does not require the computations of the newforms, but merely a few (computationally much less expensive) Hecke polynomials. This allows us to prove the following theorem.

THEOREM 2. *The equation*

$$(x-1)^6 + x^6 + (x+1)^6 = z^n, \qquad x, z, n \in \mathbb{Z},\ n \geq 2,$$

*has no solution.*

In a forthcoming paper we treat the equation $(x-1)^k + x^k = z^n$.

**2. The case $k = 5$ and two Fermat equations of signature $(p, p, 2)$.** The equation $(x-1)^5 + x^5 + (x+1)^5 = z^n$ can be rewritten as

$$x(3x^4 + 20x^2 + 10) = z^n. \tag{4}$$

It suffices to deal with the case $n = p$ where $p$ is a prime. We write $\alpha = \gcd(x, 10)$, whereby

$$x = \alpha^{p-1} z_1^p \quad \text{and} \quad 3x^4 + 20x^2 + 10 = \alpha z_2^p. \tag{5}$$

We shall use this factorization to construct two associated Fermat equations with signature $(p, p, 2)$. We make use of the identity

$$7x^4 + (3x^4 + 20x^2 + 10) = 10(x^2 + 1)^2.$$

Substituting from (5) and dividing by $\alpha$ we obtain

$$7\alpha^{4p-5} z_1^{4p} + z_2^p = (10/\alpha)(x^2 + 1)^2. \tag{6}$$

The reader will observe that this is a generalized Fermat equation with signature $(p, p, 2)$ where the three terms are coprime.

We also make use of the identity

$$3(3x^4 + 20x^2 + 10) + 70 = (3x^2 + 10)^2.$$

Again substituting from (5) and dividing by $\alpha$ we obtain

$$3z_2^p + \frac{70}{\alpha} = \alpha \left( \frac{3x^2 + 10}{\alpha} \right)^2. \tag{7}$$

Once again, the three terms in this equation are integral and coprime. We interpret this as a generalized Fermat equation with signature $(p, p, 2)$ by treating the term $70/\alpha$ as $(70/\alpha) \cdot 1^p$.

We will associate a Frey–Hellegouarch curve to each of the Fermat equations (6) and (7), and use the information derived simultaneously from both Frey–Hellegouarch curves to prove Theorem 1 for $n = p \geq 7$. We need to treat exponents $p = 2, 3$ and $5$ separately; we do this in the next sections.

**3. The case $k = 5$: small values of $p$**

LEMMA 3.1. *The only solution to* (4) *with $n = p = 2$ is $x = z = 0$.*

*Proof.* Write $X = 3\alpha x^2$ and $Y = 3\alpha x z_2$. From (5), it follows that $(X, Y)$ is an integral point on the elliptic curve

$$E_\alpha : Y^2 = X(X^2 + 20\alpha X + 30\alpha^2).$$

Using the computer algebra package Magma [8], we determine the integral points on $E_\alpha$. For this computation, Magma applies the standard linear forms in elliptic logarithms method [32, Chapter XIII]. The integral points on these curves are

$$(-6, \pm 18), \quad (-5, \pm 15), \quad (0,0) \quad \text{and} \quad (1080, \pm 35820)$$

for $\alpha = 1$; the points $(-54, \pm 306)$ and $(0,0)$ for $\alpha = 5$; and just the point $(0,0)$ for $\alpha = 2$ or $10$. The lemma follows immediately. ∎

LEMMA 3.2. *The only solution to* (4) *with* $n = p = 3$ *is* $x = z = 0$.

*Proof.* Let $X = 3\alpha z_2$ and $Y = 3\alpha(3x^2 + 10)$. From (7), we see that $(X, Y)$ is an integral point on the elliptic curve

$$E_\alpha : Y^2 = X^3 + 630\alpha^2.$$

Again using Magma, we determine the integral points on these four elliptic curves. The curve $E_1$ has no integral points. The integral points on $E_5$ are $(-5, \pm 125)$ and $(99, \pm 993)$, while those on $E_2$ are $(-6, \pm 48)$, $(9, \pm 57)$ and $(46, \pm 316)$. Finally, the integral points on $E_{10}$ are given by

$$(1, \pm 251), \quad (30, \pm 300), \quad (81, \pm 771) \quad \text{and} \quad (330, \pm 6000).$$

The lemma follows. ∎

LEMMA 3.3. *The only solution to* (4) *with* $n = p = 5$ *is* $x = z = 0$.

*Proof.* From (5) we have

$$x = \alpha^4 z_1^5, \quad (3x^2 + 10 + \sqrt{70})(3x^2 + 10 - \sqrt{70}) = 3\alpha z_2^5.$$

Let $K = \mathbb{Q}(\sqrt{70})$. This field has ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{70}]$ and fundamental unit $\varepsilon = 251 + 31\sqrt{70}$. We consider the following prime ideals:

$$\mathfrak{p}_2 = (2, \sqrt{70}), \quad \mathfrak{p}_3 = (3, 1 + \sqrt{70}), \quad \mathfrak{p}_3' = (3, 1 - \sqrt{70}),$$
$$\mathfrak{p}_5 = (25 + 3\sqrt{70})\mathcal{O}_K \quad \text{and} \quad \mathfrak{p}_7 = (7, \sqrt{70}).$$

These satisfy

$$\mathfrak{p}_2^2 = 2\mathcal{O}_K, \quad \mathfrak{p}_5^2 = 5\mathcal{O}_K, \quad \mathfrak{p}_7^2 = 7\mathcal{O}_K \quad \text{and} \quad \mathfrak{p}_3\mathfrak{p}_3' = 3\mathcal{O}_K.$$

The field $K$ has class number 2, with $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_3'$ and $\mathfrak{p}_7$ all representing the non-trivial ideal class. Observe that

$$\operatorname{ord}_{\mathfrak{p}_3}(10 + \sqrt{70}) = 1 \quad \text{and} \quad \operatorname{ord}_{\mathfrak{p}_3}(10 - \sqrt{70}) = 0.$$

Moreover,

$$\operatorname{ord}_{\mathfrak{p}_2}(10 + \sqrt{70}) = \operatorname{ord}_{\mathfrak{p}_5}(10 + \sqrt{70}) = 1.$$

Let $a = \operatorname{ord}_2(\alpha)$ and $b = \operatorname{ord}_5(\alpha)$, so that $a, b \in \{0, 1\}$. We deduce that

$$(3x^2 + 10 + \sqrt{70})\mathcal{O}_K = \mathfrak{a} \cdot \mathfrak{b}^5, \quad \text{where} \quad \mathfrak{a} = \mathfrak{p}_2^a \cdot \mathfrak{p}_5^b \cdot \mathfrak{p}_3$$

and $\mathfrak{b}$ is an ideal of $\mathcal{O}_K$. Observe that $\mathfrak{b}$ is principal if and only if $\mathfrak{a}$ is principal. Let

$$\mathfrak{q} = \begin{cases} 1 \cdot \mathcal{O}_K & \text{if } \mathfrak{a} \text{ is principal,} \\ \mathfrak{p}_7 & \text{if } \mathfrak{a} \text{ is non-principal.} \end{cases}$$

Then we can write

$$(3x^2 + 10 + \sqrt{70})\mathcal{O}_K = (\mathfrak{a}\mathfrak{q}^{-5}) \cdot (\mathfrak{b}\mathfrak{q})^5,$$

where both $\mathfrak{a}\mathfrak{q}^{-5}$ and $\mathfrak{b}\mathfrak{q}$ are principal; the former is a fractional ideal, while the latter is an integral ideal. Write

$$\mathfrak{a}\mathfrak{q}^{-5} = \frac{r + s\sqrt{70}}{d}\mathcal{O}_K$$

where $r, s, d \in \mathbb{Z}$, with $d \geq 1$ as small as possible. Now

$$3x^2 + 10 + \sqrt{70} = \frac{1}{d}(r + s\sqrt{70}) \cdot \varepsilon^c \cdot (u + v\sqrt{70})^5, \qquad -2 \leq c \leq 2,$$

with $u, v$ in $\mathbb{Z}$. Comparing coefficients of $1$, $\sqrt{70}$, and recalling that $x = \alpha^4 z_1^5$ we have

$$f(u,v) = d(3\alpha^8 z_1^{10} + 10) \quad \text{and} \quad g(u,v) = d, \tag{8}$$

where $f, g \in \mathbb{Z}[u,v]$ are homogeneous of degree 5. Observe that $d$ is determined by $\alpha$, while $f$ and $g$ are determined by $\alpha$ and $c$. For each possibility for $\alpha$ and $c$ we checked the system (8) for solubility modulo $2^6$, $3^3$, $5^3$, $7^3$ and all primes $11 \leq q < 100$. This allowed us to eliminate all possibilities except for $(\alpha, c) = (2, 2)$ and $(\alpha, c) = (10, 0)$. For both these possibilities $d = 1$. The second equation in (8) is in fact a Thue equation. We used `Magma` to solve both Thue equations; for the theory behind `Magma`'s Thue equation solver see [32, Chapter VII].

For $(\alpha, c) = (2, 2)$ this Thue equation is

$$\begin{aligned} 5521u^5 &+ 230960u^4v + 3864700u^3v^2 \\ &+ 32334400u^2v^3 + 135264500uv^4 + 226340800v^5 = 1, \end{aligned}$$

and we found that it has no solutions. For $(\alpha, c) = (10, 0)$ the corresponding Thue equation is

$$u^5 + 50u^4v + 700u^3v^2 + 7000u^2v^3 + 24500uv^4 + 49000v^5 = 1.$$

The only solution is $(u, v) = (1, 0)$. Since the first equation in (8) is, in this case,

$$\begin{aligned} 10u^5 + 350u^4v + 7000u^3v^2 + 49000u^2v^3 + 245000uv^4 + 343000v^5 \\ = 3 \cdot 10^8 \cdot z_1^{10} + 10, \end{aligned}$$

it follows that $z_1 = 0$, and hence $x = 0$ as required. ■

**4. The case $k = 5$: first Frey–Hellegouarch curve.** Henceforth we suppose that $p \geq 7$ and that $x \neq 0$. We apply the recipes of the first author and Skinner [7, Section 2] to equation (6) (see also [29]; this latter reference is a comprehensive tutorial on the modular approach). The recipes lead us to attach to (6) a Frey–Hellegouarch curve $E_{x,\alpha}$ which depends on $\alpha$ as well as on $x$. The possible values for $\alpha$ are 1, 5, 2 and 10. The corresponding Frey–Hellegouarch elliptic curves are

$$E_{x,1} : Y^2 = X^3 + 20(x^2+1)X^2 + 10(3x^4 + 20x^2 + 10)X, \tag{9}$$

$$E_{x,5} : Y^2 = X^3 + 4(x^2+1)X^2 + \frac{2(3x^4 + 20x^2 + 10)}{5}X, \tag{10}$$

$$E_{x,2} : Y^2 + XY = X^3 + \frac{5x^2+4}{4}X^2 + \frac{35x^4}{128}X, \tag{11}$$

$$E_{x,10} : Y^2 + XY = X^3 + \frac{x^2}{4}X^2 + \frac{7x^4}{640}X. \tag{12}$$

For a non-zero integer $u$ and a set $S$ of primes, we define $\mathrm{Rad}_S(u)$ to be the product of the distinct prime divisors of $u$ that do not belong to $S$. For an elliptic curve $E/\mathbb{Q}$, we denote its minimal discriminant and conductor by $\Delta(E)$ and $N(E)$.

Lemma 4.1. *The elliptic curves $E_{x,\alpha}$ have non-trivial 2-torsion over $\mathbb{Q}$. Their discriminants and conductors are*

$$\begin{aligned}
\Delta(E_{x,1}) &= 2^9 \cdot 5^3 \cdot 7 \cdot z_1^{4p} \cdot z_2^{2p},\\
\Delta(E_{x,5}) &= 2^9 \cdot 5^{4p-5} \cdot 7 \cdot z_1^{4p} \cdot z_2^{2p},\\
\Delta(E_{x,2}) &= 2^{8p-22} \cdot 5^3 \cdot 7^2 \cdot z_1^{8p} \cdot z_2^{p},\\
\Delta(E_{x,10}) &= 2^{8p-22} \cdot 5^{8p-10} \cdot 7^2 \cdot z_1^{8p} \cdot z_2^{p},\\
N(E_{x,1}) &= 2^8 \cdot 5^2 \cdot 7 \cdot \mathrm{Rad}_{\{2,5,7\}}(z_1 z_2),\\
N(E_{x,5}) &= 2^8 \cdot 5 \cdot 7 \cdot \mathrm{Rad}_{\{2,5,7\}}(z_1 z_2),\\
N(E_{x,2}) &= 2 \cdot 5^2 \cdot 7 \cdot \mathrm{Rad}_{\{2,5,7\}}(z_1 z_2),\\
N(E_{x,10}) &= 2 \cdot 5 \cdot 7 \cdot \mathrm{Rad}_{\{2,5,7\}}(z_1 z_2).
\end{aligned}$$

*Proof.* This follows from [7, Lemma 2.1]. ∎

We note in passing that we have already used the assumption $x \neq 0$. If $x = 0$, then $z_1 = 0$ and the curves $E_{x,\alpha}$ are not elliptic curves but merely singular Weierstrass equations (i.e. with discriminant $\Delta(E_{x,\alpha}) = 0$). We maintain the assumption $x \neq 0$ throughout.

For an elliptic curve $E/\mathbb{Q}$, we write $\overline{\rho}_{E,p}$ for the modulo $p$ representation giving the action of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $p$-torsion $E[p]$:

$$\overline{\rho}_{E,p} : G_{\mathbb{Q}} \to \mathrm{Aut}(E[p]) \cong \mathrm{GL}_2(\mathbb{F}_p).$$

If $\overline{\rho}_{E,p}$ arises from a newform $f$, then we write $E \sim_p f$.

LEMMA 4.2. *Let $E_{x,\alpha}$ be one of the Frey–Hellegouarch curves in* (9)–(12). *Then $E_{x,\alpha} \sim_p f$ where $f$ is a newform of weight* 2 *and level $L_\alpha$:*

$$L_1 = 2^8 \cdot 5^2 \cdot 7, \quad L_5 = 2^8 \cdot 5 \cdot 7, \quad L_2 = 2 \cdot 5^2 \cdot 7 \quad \text{and} \quad L_{10} = 2 \cdot 5 \cdot 7.$$

*Proof.* This is immediate from [7, Lemma 3.2], which in turn relies on modularity of elliptic curves over $\mathbb{Q}$ due to Wiles, Breuil, Conrad, Diamond and Taylor [36], [9], on Ribet's level lowering theorem [25] and also on irreducibility theorems for modulo $p$ representations of elliptic curves due to Mazur [22]. It is here that the assumption $p \geq 7$ is used to ensure the irreducibility of the representation $\overline{\rho}_{E_{x,\alpha},p}$. ■

Using `Magma`, we computed the weight 2 newforms of levels $L_\alpha$. The results of this computation are summarized in Table 1. For the computation we used `Magma`'s highly optimized Hilbert modular forms package (the classical newforms we are computing can be regarded as Hilbert newforms over $\mathbb{Q}$). The theory and algorithms behind this package are described in [15].

**Table 1.** Information for weight 2 newforms of level $L_\alpha$, where $L_\alpha$ is given by Lemma 4.2

| $\alpha$ | $\dim S_2^{\mathrm{new}}(L_\alpha)$ | Number of conjugacy classes of forms | $(d$, number of newforms of degree $d)$ |
|---|---|---|---|
| 1 | 912 | 196 | $(1,52)$, $(2,32)$, $(3,12)$, $(4,22)$, $(5,8)$, $(6,28)$, $(8,12)$, $(9,8)$, $(12,16)$, $(16,2)$, $(18,4)$ |
| 5 | 192 | 64 | $(1,20)$, $(2,12)$, $(3,12)$, $(4,4)$, $(6,16)$ |
| 2 | 10 | 8 | $(1,6)$, $(2,2)$ |
| 10 | 1 | 1 | $(1,1)$ |

**5. The case $k = 5$: second Frey–Hellegouarch curve.** Applying the recipes of Bennett and Skinner [7, Section 2] to equation (7) leads us to associate to this the Frey–Hellegouarch elliptic curve

$$F_{x,\alpha} : Y^2 = X^3 + 2(3x^2 + 10)X^2 + 70X. \tag{13}$$

Although this equation is independent of $\alpha$, the discriminant and conductor do depend on $\alpha$.

LEMMA 5.1. *The elliptic curve $F_{x,\alpha}$ has non-trivial* 2*-torsion over $\mathbb{Q}$. Its discriminant and conductor are given by*

$$\Delta(F_{x,1}) = 2^8 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot z_2^p, \qquad N(F_{x,1}) = 2^7 \cdot 3 \cdot 5 \cdot 7 \cdot \mathrm{Rad}_{\{2,3,5,7\}}(z_2),$$
$$\Delta(F_{x,5}) = 2^8 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^p, \qquad N(F_{x,5}) = 2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot \mathrm{Rad}_{\{2,3,5,7\}}(z_2),$$
$$\Delta(F_{x,2}) = 2^9 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot z_2^p, \qquad N(F_{x,2}) = 2^8 \cdot 3 \cdot 5 \cdot 7 \cdot \mathrm{Rad}_{\{2,3,5,7\}}(z_2),$$
$$\Delta(F_{x,10}) = 2^9 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot z_2^p, \qquad N(F_{x,10}) = 2^8 \cdot 3 \cdot 5^2 \cdot 7 \cdot \mathrm{Rad}_{\{2,3,5,7\}}(z_2).$$

*Proof.* Again this follows from [7, Lemma 2.1]. ∎

LEMMA 5.2. *Let $F_{x,\alpha}$ be the Frey–Hellegouarch curve in* (13). *Then $F_{x,\alpha} \sim_p g$ where $g$ is a newform of weight* 2 *and level $M_\alpha$, where*

$$M_1 = 2^7\cdot 3\cdot 5\cdot 7, \quad M_5 = 2^7\cdot 3\cdot 5^2\cdot 7, \quad M_2 = 2^8\cdot 3\cdot 5\cdot 7, \quad M_{10} = 2^8\cdot 3\cdot 5^2\cdot 7.$$

*Proof.* This is immediate from [7, Lemma 3.2]. ∎

Table 2 gives information about the spaces of newforms of weight 2 and level $M_\alpha$.

**Table 2.** Information for weight 2 newforms of level $M_\alpha$, where $M_\alpha$ is given by Lemma 5.2

| $\alpha$ | $\dim S_2^{\mathrm{new}}(M_\alpha)$ | Number of conjugacy classes of forms | ($d$, number of newforms of degree $d$) |
|---|---|---|---|
| 1 | 192 | 112 | $(1,64)$, $(2,28)$, $(3,12)$, $(4,4)$, $(5,4)$ |
| 5 | 912 | 356 | $(1,176)$, $(2,64)$, $(3,12)$, $(4,36)$, $(5,28)$, $(6,8)$, $(7,24)$, $(9,8)$ |
| 2 | 384 | 128 | $(1,48)$, $(2,16)$, $(3,16)$, $(4,28)$, $(6,8)$, $(8,12)$ |
| 10 | 1824 | 396 | $(1,124)$, $(2,60)$, $(3,20)$, $(4,52)$, $(5,8)$, $(6,40)$, $(8,28)$ |

**6. Proof of Theorem 1.** The following standard lemma [29, Proposition 5.1] will be helpful in exploiting Lemmata 4.2 and 5.2.

LEMMA 6.1. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$, and $f = q + \sum_{i\ge 2} c_i q^i$ be a newform of weight* 2 *and level $N' \mid N$. Write $K = \mathbb{Q}(c_1, c_2, \dots)$ for the totally real number field generated by the Fourier coefficients of $f$. Suppose $E \sim_p f$ for some prime $p$. Then there is some prime ideal $\mathfrak{p} \mid p$ of $K$ such that, for all primes $\ell$:*

- *if $\ell \nmid pNN'$ then $a_\ell(E) \equiv c_\ell \pmod{\mathfrak{p}}$,*
- *if $\ell \nmid pN'$ and $\ell \,\|\, N$ then $\pm(\ell+1) \equiv c_\ell \pmod{\mathfrak{p}}$.*

*Proof of Theorem 1.* Fix a possible value for $\alpha \in \{1, 2, 5, 10\}$. For convenience, we write $E_x$ and $F_x$ for the curves $E_{x,\alpha}$ and $F_{x,\alpha}$. Note that the levels $L_\alpha$ and $M_\alpha$ in Lemmata 4.2 and 5.2 depend only on $\alpha$. Now fix a weight 2 newform $f = q + \sum c_i q^i$ of level $L_\alpha$ and another $g = q + \sum d_i q^i$ of level $M_\alpha$. Suppose $E_x \sim_p f$ and $F_x \sim_p g$. Write $K_1 = \mathbb{Q}(c_1, c_2, \dots)$ and $K_2 = \mathbb{Q}(d_1, d_2, \dots)$, and let $\ell > 7$ be a prime. We would like to apply Lemma 6.1 to obtain information about $p$. Suppose for now that $\ell \ne p$. The Frey–Hellegouarch curves $E_x$ and $F_x$ depend on the unknown $x$. However, their traces modulo $\ell$ depend only on $x$ modulo $\ell$.

Let $0 \le a \le \ell - 1$ and suppose $x \equiv a \pmod{\ell}$. We shall write $\Delta_1(x)$ for the discriminant of the Weierstrass model $E_x$ and $\Delta_2(x)$ for the discriminant of the Weierstrass model $F_x$ (these are polynomials in $x$). Let

$$R_\ell(f,a) = \begin{cases} \mathrm{Norm}_{K_1/\mathbb{Q}}((\ell+1)^2 - c_\ell^2) & \text{if } \ell \mid \Delta_1(a), \\ \mathrm{Norm}_{K_1/\mathbb{Q}}(a_\ell(E_a) - c_\ell) & \text{if } \ell \nmid \Delta_1(a). \end{cases} \tag{14}$$

It follows from Lemmata 6.1 and 4.2 that $p \mid R_\ell(f,a)$. Let

$$S_\ell(g,a) = \begin{cases} \mathrm{Norm}_{K_2/\mathbb{Q}}((\ell+1)^2 - d_\ell^2) & \text{if } \ell \mid \Delta_2(a), \\ \mathrm{Norm}_{K_2/\mathbb{Q}}(a_\ell(F_a) - d_\ell) & \text{if } \ell \nmid \Delta_2(a). \end{cases}$$

It further follows from Lemmata 6.1 and 5.2 that $p \mid S_\ell(g,a)$. Now let

$$T_\ell(f,g,a) = \gcd(R_\ell(f,a), S_\ell(g,a)).$$

Then $p \mid T_\ell(f,g,a)$. Observe that while $a$ is unknown, as it is the residue of $x$ modulo $\ell$, we may suppose that $0 \le a \le \ell - 1$. Let

$$T_\ell(f,g) = \ell \prod_{0 \le a \le \ell-1} T_\ell(f,g,a).$$

Then $p \mid T_\ell(f,g)$. We had assumed above that $\ell \ne p$. However as $\ell$ is a factor in the product defining $T_\ell(f,g)$, the conclusion $p \mid T_\ell(f,g)$ is true even if $\ell = p$. Finally we let

$$U(f,g) = \gcd_{11 \le \ell < 100} T_\ell(f,g)$$

where the gcd is taken over all primes $\ell$ in the range $11 \le \ell < 100$. It follows that $p \mid U(f,g)$. To complete the proof of Theorem 1, we employ a simple `Magma` script that computes for each pair $(f,g)$ the quantity $U(f,g)$ and verifies that it is not divisible by primes $\ge 7$. The computation took roughly four days on a 2500 MHz AMD Opteron, dominated by the computation of the newforms. ∎

Remark. It is appropriate to comment at this stage as to whether the single Frey–Hellegouarch approach (using either of the Frey–Hellegouarch curves $E_{x,\alpha}$ or $F_{x,\alpha}$ on its own) would have allowed us to establish Theorem 1. The above argument is a multi-Frey–Hellegouarch version of the standard single Frey–Hellegouarch method for bounding exponents (see [29, Section 9]). With notation as above, let

$$B_\ell(f) = \ell \prod_{0 \le a \le \ell-1} R_\ell(f,a)$$

for $\ell \ne 2, 5, 7$ (note that 3 does not divide the possible levels of $f$). Under the assumption $E_x \sim f$, the single Frey–Hellegouarch method for bounding exponents asserts that $p \mid B_\ell(f)$ and succeeds in bounding $p$ if we can find a prime $\ell \ne 2, 5, 7$ such that $B_\ell(f) \ne 0$. Likewise, let

$$C_\ell(g) = \ell \prod_{0\le a\le \ell-1} S_\ell(g,a)$$

for $\ell \neq 2,3,5,7$. Under the assumption $F_x \sim g$, we have $p \mid C_\ell(g)$. We first note that the solution $(x,z,n) = (0,0,p)$ of equation (4) leads to an elliptic curve $F_{0,10}$ (i.e. a non-singular Weierstrass equation) with Cremona reference `134400BG1`. Let $g$ be the eigenform (of level $M_{10} = 134400$) corresponding to $F_{0,10}$. Then $a_\ell(F_{0,10}) = d_\ell$ where $g = q + \sum d_i q^i$. Hence $S_\ell(g,0) = 0$ and so $C_\ell(g) = 0$ for all possible $\ell$. Thus the single Frey–Hellegouarch method with the second Frey–Hellegouarch curve $F_{x,\alpha}$ fails to bound the exponent $p$.

The single Frey–Hellegouarch approach succeeds with the first Frey–Hellegouarch curve $E_{x,\alpha}$ in the sense that for all possible eigenforms $f$, we are able to find some prime $\ell \neq 2,5,7$ such that $B_\ell(f) \neq 0$. For any $\ell$, the bound $B_\ell(f)$ can be very large (especially if the field of coefficients of $f$ has large degree). However, we consider instead

$$B(f) = \gcd_{\ell\in\{3,11,13,\dots,97\}} B_\ell(f).$$

If $E_x \sim f$ then $p \mid B(f)$. We computed the $B(f)$ for the possible newforms $f$, and found many of them to be divisible by 7 and 13 though not by larger primes. It is possible to reduce the cases $p = 7$ and $p = 13$ to Thue equations as in the proof of Lemma 3.3. However the coefficients of these Thue equations will be so unpleasant that we do not expect to be able to solve them (uncondionally).

**7. Dealing with small exponents for $k = 6$.** We now consider the equation

$$(x-1)^6 + x^6 + (x+1)^6 = z^n, \qquad x,z,n \in \mathbb{Z},\ n \ge 2,$$

which corresponds to the case $k = 6$ of (3). This can be rewritten as

$$3x^6 + 30x^4 + 30x^2 + 2 = z^n,$$

whence necessarily $z^n \equiv 2 \pmod 3$ and so $n$ is odd. Moreover the polynomial $3t^6 + 30t^4 + 30t^2 + 2$ only takes values 2 and 3 as $t$ ranges over $\mathbb{F}_7$. As these values are not cubes in $\mathbb{F}_7$, we see that $3 \nmid n$. Thus to prove Theorem 2 for $k = 6$ it is sufficient to show that the equation

$$3x^6 + 30x^4 + 30x^2 + 2 = z^p \tag{15}$$

has no solutions with prime exponent $p \ge 5$.

Lemma 7.1. *Equation* (15) *has no solutions with* $p = 5,7,11,13$.

*Proof.* Write $f = 3t^6 + 30t^4 + 30t^2 + 2$. The polynomial $f$ is irreducible over $\mathbb{Q}$. Let $\theta$ be a root of the equation $f(t) = 0$, write $K = \mathbb{Q}(\theta)$ and let

$\mathcal{O}_K$ be the ring of integers of $K$. The field $K$ has unit rank 2 with $-1$ as a generator for the roots of unity, and class group $\cong (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/36\mathbb{Z})$.

Let $g(t) = f(t)/(x-\theta) \in K[t]$. There are prime ideals $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$, $\mathfrak{r}_4$ such that

$$2 \cdot \mathcal{O}_K = \mathfrak{p}^6, \quad 3 \cdot \mathcal{O}_K = \mathfrak{q}^6, \quad 3391 \cdot \mathcal{O}_K = \mathfrak{r}_1^2\mathfrak{r}_2^2\mathfrak{r}_3\mathfrak{r}_4,$$
$$\theta \cdot \mathcal{O}_K = \mathfrak{p}\mathfrak{q}^{-1} \quad \text{and} \quad g(\theta) \cdot \mathcal{O}_K = \mathfrak{p}^{11}\mathfrak{q}^3\mathfrak{r}_1\mathfrak{r}_2.$$

From (15), we know that

$$(x-\theta)g(x) = z^p.$$

Now $\mathrm{ord}_{\mathfrak{q}}(\theta) = -1$. As $x \in \mathbb{Z}$, we have $\mathrm{ord}_{\mathfrak{q}}(x-\theta) = -1$. Let $\mathcal{P} \neq \mathfrak{q}$ be a prime ideal and suppose that $\mathrm{ord}_{\mathcal{P}}(x-\theta) \not\equiv 0 \pmod p$, whence $\mathrm{ord}_{\mathcal{P}}(g(x)) \not\equiv 0 \pmod p$. From the factorization of $\theta\cdot\mathcal{O}_K$ we know that $\mathrm{ord}_{\mathcal{P}}(x-\theta) > 0$. It is easy to see that $\mathcal{P} \mid g(\theta)$. But $\mathrm{ord}_2(z) = 0$, so $\mathcal{P} = \mathfrak{r}_1$ or $\mathfrak{r}_2$. Let $S = \{\mathfrak{q}, \mathfrak{r}_1, \mathfrak{r}_2\}$. Hence $(x-\theta)K^{*p}$ belongs to the '$p$-Selmer group'

$$K(S,p) = \{\alpha \in K^*/K^{*p} : \mathrm{ord}_{\mathcal{P}}(\alpha) \equiv 0 \pmod p \text{ for all } \mathcal{P} \notin S\}.$$

This is an $\mathbb{F}_p$-vector space of finite dimension and, for a given $p$, easy to compute from the class group and unit group information (see [30, proof of Proposition VIII.1.6]). Let

$$\mathfrak{S}_p = \{\alpha \in K(S,p) : \mathrm{Norm}(\alpha) = (1/3)\mathbb{Q}^{*p}\}.$$

Observe that $\mathrm{Norm}(x-\theta) = z^p/3$ so that $x-\theta \in \mathfrak{S}_p$. Using `Magma`, we compute $K(S,p)$ and $\mathfrak{S}_p$ for $p = 5, 7, 11, 13$. In all cases, $K(S,p)$ has $\mathbb{F}_p$-dimension equal to 5, and the set $\mathfrak{S}_p$ has $p^3$ elements.

It follows that $x-\theta = \alpha\xi^p$ for some $\alpha \in \mathfrak{S}_p$ and $\xi \in K^*$. We are now in a position to finally obtain a contradiction. Fix an $\alpha \in \mathfrak{S}_p$ such that $x-\theta = \alpha\xi^p$. Let $\ell \neq 3$ be a rational prime and $\mathfrak{l}_1, \ldots, \mathfrak{l}_r$ be the prime ideals of $K$ dividing it. Suppose that none of the $\mathfrak{l}_i$ belong to the support of $\alpha$. Let $x \equiv a \pmod \ell$ where $a \in \{0, 1, \ldots, \ell-1\}$. Then $(a-\theta)/\alpha \equiv \xi^p \pmod{\mathfrak{l}_i}$ for $i = 1, \ldots, r$. Thus we may eliminate $\alpha$ if for each $a \in \{0, 1, \ldots, \ell-1\}$ there is some $i$ such $(a-\theta)/\alpha$ is not a $p$th power modulo $\mathfrak{l}_i$. For this to succeed, $\#\mathbb{F}_{\ell_i} = \mathrm{Norm}(\mathfrak{l}_i)$ needs to be $\equiv 1 \pmod \ell$. For $p = 5, 7, 11$ and $13$, we have, in each case, been able to find a set of primes $\ell$, which we denote by $\mathcal{T}_p$, allowing us to eliminate all $\alpha \in \mathfrak{S}_p$. The sets $\mathcal{T}_p$ are recorded in Table 3. ∎

**Table 3.** The sets $\mathcal{T}_p$ appearing in the proof of Lemma 7.1

| $p$ | $\mathcal{T}_p$ |
|---|---|
| 5 | $\{11, 191, 251, 691\}$ |
| 7 | $\{11, 337, 421, 491, 547\}$ |
| 11 | $\{397, 727, 859\}$ |
| 13 | $\{859, 1249\}$ |

**8. Frey–Hellegouarch curve for $k = 6$.** In this section, we construct a Frey–Hellegouarch curve attached to equation (15). In view of the previous section, we may suppose that the exponent $p$ in (15) is a prime $\geq 17$. The first author and Dahmen [5] attach a Frey–Hellegouarch curve to any equation of the form $F(u, v) = z^p$ where $F$ is a homogeneous cubic form. We now reproduce their recipe. Let

$$H(u, v) = -\frac{1}{4}\begin{vmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{vmatrix} \quad \text{and} \quad G(u, v) = \begin{vmatrix} F_u & F_v \\ H_u & H_v \end{vmatrix}.$$

Associate to the solution $(u, v, z)$ of the equation $F(u, v) = z^p$ the Frey–Hellegouarch elliptic curve

$$E'_{u,v} : Y^2 = X^3 - 3H(u, v)X + G(u, v). \tag{16}$$

This model has discriminant $2^4 \cdot 3^6 \cdot \Delta_F \cdot z^{2p}$, where $\Delta_F$ is the discriminant of the binary form $F$. Now consider the homogeneous cubic form

$$F(u, v) = 3u^3 + 30u^2v + 30uv^2 + 2v^3.$$

We note that $F(x^2, 1) = 3x^6 + 30x^4 + 30x^2 + 2$. Thus we may obtain a Frey–Hellegouarch curve for (15) by letting $(u, v) = (x^2, 1)$ in (16). In turns out that the model $E'_{x^2,1}$ has bad reduction at 2, but its quadratic twist by 2 has good reduction at 2, and we choose this to be the Frey–Hellegouarch curve associated to (15). A model which is minimal at 2 for this Frey–Hellegouarch curve is

$$\begin{aligned} E_x : Y^2 + Y = X^3 &+ \frac{-945x^4 - 1269x^2 - 1080}{2} X \\ &+ \frac{-15093x^6 - 18630x^4 + 26730x^2 + 19061}{4}. \end{aligned} \tag{17}$$

LEMMA 8.1. *The model $E_x$ is integral, minimal and has discriminant and conductor*

$$\Delta_x = 3^9 \cdot 3391 \cdot z^{2p} \quad \textit{and} \quad N = 3^3 \cdot 3391 \cdot \mathrm{Rad}_{\{3,3391\}}(z).$$

*Proof.* It is clear from (15) that $x$ is odd, whence one deduces that $E_x$ is integral. The discriminant for this model is

$$\Delta_x = 3^9 \cdot 3391 \cdot (3x^6 + 30x^4 + 30x^2 + 2)^2 = 3^9 \cdot 3391 \cdot z^{2p},$$

and the usual $c_4$-invariant is

$$c_4 = 2^3 \cdot 3^4 \cdot (35x^4 + 47x^2 + 40).$$

We find that

$$\mathrm{Res}(c_4, \Delta_x) = 2^{40} \cdot 3^{84} \cdot 3391^{12}\,.$$

Thus $E_x$ is minimal and semistable except possibly at $p \in \{2, 3, 3391\}$. Since $\Delta_x$ is odd, $E_x$ in fact has good reduction at 2.

We now show that $E_x$ has multiplicative reduction at 3391. The solutions to $c_4 \equiv 0 \pmod{3391}$ are

$$x \equiv 983, 2408 \pmod{3391}.$$

Both of these are roots to $3x^6 + 30x^4 + 30x^2 + 2$ modulo 3391. However, for a solution $(x, z)$ to (15) we know that

$$3x^6 + 30x^4 + 30x^2 + 2 \equiv 0 \pmod{3391^2}.$$

We checked that 983, 2408 do not lift to roots for this congruence. Hence $3391 \nmid c_4$. It follows that $E_x$ has multiplicative reduction at 3391.

Applying Tate's algorithm [31, Chapter IV], we found that the $E_x$ has reduction type $\mathrm{IV}^*$ at 3 with the valuation of the conductor equal to 3. The lemma follows. ∎

Lemma 8.2. *Let $(x, z, p)$ be a solution to (15) with $p \geq 17$ prime. Let $E = E_x$ as in (17). Then $\overline{\rho}_{E,p}$ is irreducible.*

*Proof.* Suppose $\overline{\rho}_{E,p}$ is reducible. As $p \geq 17$, it follows from the proof of Mazur's famous theorem on isogenies of elliptic curves that the $j$-invariant of $E$ belongs to $\mathbb{Z}[1/2]$ (see [22, Corollary 4.4]). However, $E$ has multiplicative reduction at 3391, and so 3391 appears in the denominator of its $j$-invariant. This contradiction shows that $\overline{\rho}_{E,p}$ is irreducible. ∎

## 9. Proof of Theorem 2

Lemma 9.1. *Let $(x, z, p)$ be a solution to (15) with prime exponent $p \geq 17$. Then $E_x \sim_p f$ for some newform $f$ of weight 2 and level $3^3 \cdot 3391$.*

*Proof.* This follows from Lemmata 8.1 and 8.2 together with Ribet's level lowering theorem [25] (the special case [29, Section 5] is enough for our purpose). ∎

From Cremona's database [14], there are precisely four elliptic curves having conductor $3^3 \cdot 3391$:

$$\begin{aligned}
F_1 &: y^2 + y = x^3 + 405x + 22673,\\
F_2 &: y^2 + y = x^3 + 45x - 840,\\
F_3 &: y^2 + y = x^3 - 42x - 104,\\
F_4 &: y^2 + y = x^3 - 378x + 2801.
\end{aligned}$$

Lemma 9.2. *$E_x \not\sim_p F_i$ for $i = 1, 2, 3, 4$.*

*Proof.* Suppose $E_x \sim_p F_i$. As 2 is a prime of good reduction for both elliptic curves, we have $a_2(E_x) \equiv a_2(F_i) \pmod{p}$. From (17) and the fact that $x$ is odd, we find that

$$E/\mathbb{F}_2 : Y^2 + Y = X^3 + X + 1.$$

It follows that $a_2(E_x) = 2$. Since

$$a_2(F_1) = 2, \quad a_2(F_2) = -2, \quad a_2(F_3) = a_2(F_4) = 0,$$

we thus have $i = 1$.

Next we apply the method of bounding the exponents. For a prime $\ell \neq 3, 3391$, let

$$R_\ell(a) = \begin{cases} (\ell+1)^2 - a_\ell(F_1)^2 & \text{if } \ell \mid \Delta_a, \\ a_\ell(E_a) - a_\ell(F_1) & \text{if } \ell \nmid \Delta_a, \end{cases} \tag{18}$$

and

$$B_\ell = \ell \prod_{0 \le a \le \ell-1} R_\ell(a).$$

It follows from Lemma 6.1 that $p \mid B_\ell$. We find that $B_{11} = 5^4 \cdot 7^3 \cdot 11$. As $p \geq 17$, we obtain a contradiction. ∎

The space $S_2^{\mathrm{new}}(3^3 \cdot 3391)$ has dimension 4520. Using `Magma`, we compute the conjugacy classes of eigenforms belonging to this space and find that these have degrees 1, 1, 1, 1, 554, 556, 564, 564, 565, 565, 574 and 574. The four rational eigenforms, of course, correspond to the four elliptic curves $F_i$. Unfortunately we have found it impossible to compute the coefficients of the irrational eigenforms due to the enormous size of their fields of coefficients. For a prime $\ell \neq 3, 3391$, write $T_\ell$ for the Hecke operator acting on $S_2^{\mathrm{new}}(3^3 \cdot 3391)$, and let $C_\ell \in \mathbb{Z}[t]$ be the characteristic polynomial of $T_\ell$ (i.e. the $\ell$th Hecke polynomial); this is a polynomial of degree 4520. Using `Magma`, we found it straightforward (though somewhat time-consuming) to compute the polynomials $C_\ell(t)$ for $\ell < 100$. The polynomial $C_\ell$ satisfies

$$C_\ell(t) = \prod (t - a_\ell(f))$$

where $f$ runs through the eigenforms of weight 2 and level $3^3 \cdot 3391$. Note that $t - a_\ell(F_i)$ divides $C_\ell(t)$ for $i = 1, 2, 3, 4$. We let

$$C'_\ell(t) = \frac{C_\ell(t)}{\prod_{1 \le i \le 4} (t - a_\ell(F_i))}.$$

We now let

$$R_\ell(a) = \begin{cases} C'_\ell(\ell+1) \cdot C'_\ell(\ell-1) & \text{if } \ell \mid \Delta_a, \\ C'_\ell(a_\ell(E_a)) & \text{if } \ell \nmid \Delta_a. \end{cases}$$

If $\ell \neq 2$, we let

$$B_\ell = \ell \cdot \prod_{0 \le a \le \ell} R_\ell(a)$$

and set $B_2 = C'_2(2)$.

Lemma 9.3. *Let $(x, z, p)$ be a solution to* (15) *with $p \geq 17$ prime. Let $\ell \neq 3, 3391$ be prime. Then $p \mid B_\ell$.*

*Proof.* By Lemmata 9.1 and 9.2, we know that $E_x \sim_p f$ where $f$ is an irrational eigenform of weight 2 and level $3^3 \cdot 3391$. It follows from the above that $t - a_\ell(f)$ is a factor of $C'_\ell$. The lemma now follows from Lemma 6.1 (for $\ell = 2$ we are making use of the fact that $E_x$ has good reduction at 2 and that $a_2(E_x) = 2$). ∎

*Proof of Theorem 2.* Let $(x, z, p)$ be a solution to (15) with $p \geq 11$. Let

$$P = \{2\} \cup \{5, 7, 11, \ldots, 97\}$$

be the set of primes less than 100 and excluding 3. Using `Magma`, we find that

$$\gcd\{B_\ell : \ell \in P\} = 2^{27} \cdot 3^{28} \cdot 5^3 \cdot 7.$$

This computation took roughly 21 hours on a 2500 MHz AMD Opteron. The computation time was dominated by the computation of the polynomials $C'_\ell$. The desired result then follows from Lemma 9.3. ∎

We remark in passing that the integers $B_\ell$ are extremely large, which is why we do not reproduce any of them here. By way of example, $|B_2| \approx 1.1 \cdot 10^{569}$.

**10. The equation $(x-1)^k + x^k + (x+1)^k = y^p$ with $k \geq 7$.** It is natural to wonder if it is possible to attach a Frey–Hellegouarch curve to a solution of the equation $(x-1)^k + x^k + (x+1)^k = z^p$ for exponents $k \geq 7$. It is easy to see that

$$(x-1)^k + x^k + (x+1)^k = \begin{cases} f_k(x^2) & \text{if } k \text{ is even,} \\ x f_k(x^2) & \text{if } k \text{ is odd,} \end{cases}$$

where $f_k \in \mathbb{Z}[x]$. For $7 \leq k \leq 50$, say, we find that the polynomials $f_k$ are irreducible and all their roots are real. We are unable to prove that this is true in general for higher values of $k$ (and, indeed, this property is not shared by the polynomials arising from the analogous equation $(x-1)^k + x^k = y^n$). Suppose now that $f_k$ is indeed a totally real irreducible polynomial, let $\theta$ be a root, and let $K = \mathbb{Q}(\theta)$. By a standard descent argument, $x^2 - \theta = \alpha\xi^p$ where $\alpha$ belongs to a finite set and $\xi$ is an integer in $K$. This can be viewed as a $(p, p, 2)$ Fermat equation to which one can apply modularity and level-lowering results over the totally real field $K$, in a similar manner to that of several recent papers, e.g. [6], [16], [17]. We hope to pursue this approach in a forthcoming paper.

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