FUNDAMENTA
MATHEMATICAE
235 (2016)

Shadowing for induced maps of hyperspaces

by

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Abstract. Given a nonempty compact metric space X and a continuous function $f : X \to X$, we study shadowing and h-shadowing for the induced maps on hyperspaces, particularly in symmetric products, $F_n(X)$, and the hyperspace 2^X of compact subsets of X. We prove that f has shadowing [h-shadowing] if and only if 2^f has shadowing [h-shadowing].

1. Introduction. A continuous function $f : X \to X$ on a compact metric space induces a number of maps on related spaces. There is a close relationship, for example, between the dynamical behaviour of f, the topological structure of the inverse limit space $\varprojlim(X, f)$ and the induced shift map on (X, f). This situation has been extensively studied (see for example [2, 14, 28] and the references contained therein). Over the past few years there has been increasing interest in the study of the induced map on the hyperspace of closed subsets and various of its subsets equipped with the Vietoris topology (or Hausdorff metric). This study was initiated by Bauer and Sigmund [5] and it has been argued [10] that, from a computational and domain-theoretic point of view, this is the natural approach to dynamical systems.

Given a compact metric space X, 2^X is the hyperspace of nonempty closed subsets of X with the Vietoris topology. A continuous map $f : X \to X$ induces a continuous map $2^f : 2^X \to 2^X$ defined by $2^f(A) = f(A)$. A number of well-studied subspaces (such as the collections $C_n(X)$ of closed sets with at most n components, $F(X)$ of finite subsets, or $F_n(X)$ of subsets with at most n points) are invariant under this map, and therefore form dynamical systems in their own right. It turns out that a number of dynamical properties lift between these systems. For example, 2^f is transitive if and only if it

2010 *Mathematics Subject Classification*: Primary 37B99, 37C50, 54B20, 54H20.
Key words and phrases: shadowing, pseudo orbit tracing, induced map, hyperspace.
Received 3 September 2015; revised 17 February 2016.
Published online 4 July 2016.

is weakly mixing if and only if f is weakly mixing [1, 26]. In [11] the authors study chain transitivity, chain recurrence and periodicity of induced maps on $F_n(X)$ and 2^X. Relationships between the entropy of the map f and the entropy of the induced maps on 2^X, $C_n(X)$, $F_n(X)$ and $F(X)$ are studied in [12] and [18]. In [13] the authors study periodicity, recurrence, quasi periodicity, wandering points, shadowing, exactness and nonwandering for the induced map in the hyperspace $F_n(X)$. Induced maps on the symmetric products $F_n(X)$ are also studied in [17] and [15].

Of particular relevance in the computation of a dynamical system is the notion of shadowing, which is the focus of this paper. Given a map f, a *δ-pseudo orbit* is a (finite or infinite) sequence of points such that the distance between $f(x_i)$ and x_{i+1} is less than δ. A typical example of a pseudo orbit would be the points produced computationally in calculating the orbit of a point where there is a round-off error. A pseudo orbit is said to be *ε-shadowed* if there is a real orbit whose points track the pseudo orbit within a distance of ε. The map f has the *shadowing property* if, for a given ε, there is a δ such that δ-pseudo orbits are ε-shadowed. Shadowing has been studied in the context of numerical analysis [8, 7, 24], at times being cited as a prerequisite to achieving accurate mathematical models, and extensively investigated as a property in its own right [9, 19, 21, 23, 25, 27, 29]. Bowen was one of the first to consider this property in [6], where he used it in the study of ω-limit sets of Axiom A diffeomorphisms.

Some work on the shadowing of induced hyperspace maps has been done. In [13] it is proved that, for any $n \geq 1$, if the restriction f_n of 2^f to $F_n(X)$ has shadowing, then f has shadowing. The authors also prove that if f has shadowing, then f_2 has shadowing, but give an example ($z \mapsto z^2$ on S^1) for which f has shadowing but f_n does not have shadowing for any $n \geq 3$. Interestingly, we prove below that the pseudo orbits in this example that cannot be shadowed in $F_n(X)$ can be shadowed in $F_m(X)$ for some $m > n$. Sakai [29] proves that a positively expansive map on a compact metric space has shadowing if and only if it is open. In [30] it is shown that the induced map 2^f of a positively expansive open map f is open but need not be positively expansive. However, the authors show that such induced maps do have shadowing. Se also [16]

In this paper we show that, in fact, 2^f has shadowing if and only if f has shadowing.

If a map $f : X \to X$ has shadowing, then the restriction $f^{<\omega}$ of 2^f to $F(X)$ has shadowing for finite pseudo orbits. Since $F(X)$ is not compact, this is not enough to show that $F(X)$ has shadowing. However $F(X)$ is dense in 2^X and invariant under 2^f, and this is enough, via a general result on shadowing in dense subspaces, to prove that f has shadowing if and only if 2^f has shadowing. Using slightly different arguments we prove a similar result which says that f has the much stronger property of h-shadowing if and only if 2^f does.

2. Preliminaries. We start with some definitions (see [22]).

DEFINITION 2.1. Let X be a compact metric space. Consider the following hyperspaces of X:

- $2^X = \{A \subseteq X : A$ is nonempty and closed$\}$ is the *hyperspace of closed nonempty subsets of* X.
- $C(X) = \{A \in 2^X : A$ is connected$\}$ is the *hyperspace of subcontinua of* X.
- $F_n(X) = \{A \in 2^X : A$ has at most n points$\}$ is the *n-fold symmetric product of* X.
- $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$ is the collection of all finite subsets of X.

DEFINITION 2.2. A map $f : X \to Y$ between compact metric spaces induces the following maps:

- $2^f : 2^X \to 2^Y$ given by $2^f(A) = f(A)$.
- $C(f) : C(X) \to C(Y)$ given by $C(f) = 2^f|_{C(X)}$.
- $f_n : F_n(X) \to F_n(Y)$ given by $f_n = 2^f|_{F_n(X)}$.
- $f^{<\omega} : F(X) \to F(X)$ given by $f^{<\omega} = 2^f|_{F(X)}$.

Given a metric space X with metric d, for any $r > 0$ and any $A \in 2^X$, we define the *open ball about A of radius r* by

$$N_X(A, r) = \{x \in X : d(x, A) < r\}.$$

For the special case when $A = \{x\}$ we write $N_X(x, r)$. If X is a compact metric space with metric d, then (see for example [20]) 2^X is a compact metric space when equipped with the Hausdorff metric

$$H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N_X(B, \varepsilon) \text{ and } B \subseteq N_X(A, \varepsilon)\}.$$

The topology generated by H coincides with the Vietoris topology.

3. Shadowing. It is shown in [30] that if f is a positively expansive open map, then 2^f has shadowing. Here we prove that if one of the induced maps f_n, $C(f)$, 2^f or $f^{<\omega}$ has shadowing, then f has shadowing. Also we prove that if f has shadowing, then $f^{<\omega}$ has finite shadowing, which, in turn, implies that 2^f has shadowing.

We start with basic definitions. Let X be a compact metric space and let $f : X \to X$ be a continuous function. For $\delta > 0$, a (finite or infinite) sequence $\Gamma = \langle x_0, x_1, \dots \rangle$ of points in X is a *δ-pseudo orbit* if $d(f(x_i), x_{i+1}) < \delta$ for every $i \geq 0$. If $\varepsilon > 0$, we say that the sequence $\langle y_0, y_1, \dots \rangle$ *ε-shadows Γ* provided $d(y_i, x_i) < \varepsilon$ for every i. If $y_i = f^i(y)$ for some $y \in X$, we say that *y shadows* the sequence Γ. We say that f *has shadowing* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit is ε-shadowed by some point in X. In the case that only finite pseudo orbits are shadowed, we say that f *has finite shadowing.* If X is compact, then f has shadowing if and only if f has finite shadowing (see, for example, [4, Remark 1]).

We first prove a general result about shadowing that we assume to be well known.

LEMMA 3.1. *Let X be a compact metric space, let $f : X \to X$ be a continuous function and let Y be a dense invariant subset of X. Then f has finite shadowing if and only if $f|_Y$ has finite shadowing.*

Proof. Assume first that f has shadowing. Let $\varepsilon > 0$ and choose δ such that every δ-pseudo orbit in X is $\varepsilon/2$-shadowed. Let $\Gamma = \langle y_0, y_1, \ldots, y_r\rangle$ be a δ-pseudo orbit in Y. Then Γ is a δ-pseudo orbit in X. Since f has shadowing, there is a point $x \in X$ which $\varepsilon/2$-shadows Γ, i.e., $d(f^i(x), y_i) < \varepsilon/2$ for every $i \in \{0, 1, \ldots, r\}$. Since f is continuous, there is $\eta_{r-1} > 0$ with $\eta_{r-1} < \varepsilon/2$ and $f(N_X(f^{r-1}(x), \eta_{r-1})) \subseteq N_X(f^r(x), \varepsilon/2)$. Also, there is $\eta_{r-2} > 0$ with $\eta_{r-2} < \eta_{r-1}$ and $f(N_X(f^{r-2}(x), \eta_{r-2})) \subseteq N_X(f^{r-1}(x), \eta_{r-1})$. Continuing this process, we arrive at $\eta_1 > 0$ with $\eta_1 < \eta_2$ and $f(N_X(f(x), \eta_1)) \subseteq N_X(f^2(x), \eta_2)$. Finally, there is $\eta_0 > 0$ with $\eta_0 < \eta_1$ and $f(N_X(x, \eta_0)) \subseteq N_X(f(x), \eta_1)$. By construction, every $y \in N_X(x, \eta_0) \cap Y$ ε-shadows Γ.

Now assume that $f|_Y$ has finite shadowing, let $\varepsilon > 0$ and let $\Gamma = \langle x_0, x_1, \ldots, x_r\rangle$ be a $\delta/3$-pseudo orbit in X, where δ is given by shadowing in $f|_Y$ for $\varepsilon/2$. Since f is continuous and X is compact, f is uniformly continuous and there exists $\eta > 0$ with $\eta < \delta/3$ and $\eta < \varepsilon/2$ such that if $d(x, y) < \eta$ then $d(f(x), f(y)) < \delta/3$. For each $i \in \{0, 1, \ldots, r\}$, let $y_i \in N_X(x_i, \eta) \cap Y$. Hence $d\,(f(x_i), f(y_i)) < \delta/3$. Thus, $\Gamma^* = \langle y_0, y_1, \ldots, y_r\rangle$ is a δ-pseudo orbit in Y because

$$\begin{aligned} d(f(y_i), y_{i+1}) &\le d(f(y_i), f(x_i)) + d(f(x_i), x_{i+1}) + d(x_{i+1}, y_{i+1}) \\ &< \delta/3 + \delta/3 + \delta/3 = \delta. \end{aligned}$$

Since $f|_Y$ has shadowing, there is a point $y \in Y$ which $\varepsilon/2$-shadows Γ^*. But then $d(f^i(y), x_i) < d(f^i(y), y_i) + d(y_i, x_i) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore, y ε-shadows Γ and f has finite shadowing. ■

Turning now to induced maps on hyperspaces, we start with a simple observation.

THEOREM 3.2. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. Let $n \ge 1$. If any of f_n, $C(f)$, 2^f or $f^{<\omega}$ has shadowing, then f has shadowing.*

Proof. The proof is identical in each case, so we present it for 2^f. Suppose that 2^f has shadowing. Let $\varepsilon > 0$ and let $\delta > 0$ be given by shadowing for 2^f. Let $\Gamma = \langle x_0, x_1, \dots, x_r\rangle$ be a δ-pseudo orbit in X. Then $\Gamma^* = \langle \{x_0\}, \{x_1\}, \dots, \{x_r\}\rangle$ is a δ-pseudo orbit in 2^X. Since 2^f has shadowing, there is a point $A \in 2^X$ which ε-shadows Γ^*. But then every point x of A ε-shadows Γ. ■

As mentioned above, in [13] it is shown that f has shadowing if and only if f_2 has shadowing but that there is a map f with shadowing for which certain pseudo orbits in $F_n(X)$ can only be shadowed in $F_m(X)$ for some $m > n$. The fact that finite sets can always be shadowed by larger finite sets turns out to be a general property of shadowing maps.

THEOREM 3.3. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. If f has shadowing, then $f^{<\omega}$ has finite shadowing.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by shadowing for f. Let $\Gamma = \langle A_0, A_1, \dots, A_r\rangle$ be a finite δ-pseudo orbit in $F(X)$ and assume that $|A_i| = n_i$ for each $i \in \{0, 1, \dots, r\}$. We will construct a family of δ-pseudo orbits in X, denoted $\{\Gamma_j : j \le n\}$, for some n, such that, writing

$$\Gamma_j = \langle a_0^j, a_1^j, \dots, a_r^j\rangle,$$

we have $A_i = \{a_i^j : j \le n\}$ for each $i \le r$.

To this end, suppose that $A_r = \{a_r^1, a_r^2, \dots, a_r^{n_r}\}$. For each j with $1 \le j \le n_r$, we first construct a δ-pseudo orbit in X with ith element in A_i, whose final element is a_r^j. Since Γ is a δ-pseudo orbit, we can choose $a_{r-1}^j \in A_{r-1}$ such that $d(f(a_{r-1}^j), a_r^j) < \delta$. Again, there is some $a_{r-2}^j \in A_{r-2}$ such that $d(f(a_{r-2}^j), a_{r-1}^j) < \delta$. Continuing in this way, we have δ-pseudo orbits

$$\Gamma_j = \langle a_0^j, a_1^j, \dots, a_r^j\rangle$$

for each $j \le n_r$, such that $A_r = \{a_r^j : j \le n_r\}$ and $\{a_i^j : j \le n_r\} \subseteq A_i$ for each $i \le r$.

Let $k = \max\{i < r : A_i \ne \{a_i^j : j \le n_r\}\}$ (if no such k exists, then we are done) and write $A_k - \{a_k^j : j \le n_r\} = \{a_k^j : n_r < j \le n_k'\}$. Exactly as for A_r, for each $n_r < j \le n_k'$, we can construct a δ-pseudo orbit $\Gamma_j' = \langle a_0^j, a_1^j, \dots, a_k^j\rangle$ such that $a_i^j \in A_i$ for $i \le k$. Clearly $A_k = \{a_k^j : j \le n_k'\}$. Now, since $f(a_k^j) \in f^{<\omega}(A_k)$ and $H(f^{<\omega}(A_k), A_{k+1}) < \delta$, there is $a_{k+1}^j \in A_{k+1}$ such that $d(f(a_k^j), a_{k+1}^j) < \delta$. Similarly, for each $n_r < j \le n_k'$ and $k < i < r$, there are $a_i^j \in A_i$ such that $d(f(a_i^j), a_{i+1}^j) < \delta$, so that we can extend Γ_j' to a δ-pseudo orbit Γ_j which starts in A_0 and ends in A_r.

Repeating this process, we clearly see that we can construct the collection $\{\Gamma_j : j \le n\}$ of δ-pseudo orbits in X. Since f has shadowing, for each Γ_j there is a point $b_j \in X$ which ε-shadows Γ_j. Let $B = \{b_0, b_1, \dots, b_m\}$. By construction, B ε-shadows Γ. ■

Our main theorem now follows easily.

THEOREM 3.4. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. Then f has shadowing if and only if 2^f has shadowing.*

Proof. By Theorem 3.2, if 2^f has shadowing, then f has shadowing. Conversely, if f has shadowing, then $f^{<\omega}$ has finite shadowing by Theorem 3.3, but $F(X)$ is an invariant dense subset of 2^X, so by Lemma 3.1, 2^f has shadowing. ∎

It also follows immediately from Theorems 3.2 and 3.3 that $f^{<\omega}$ has finite shadowing whenever f_n has shadowing for some positive integer n. In Example 3.5 below, f_n has shadowing for every positive integer n but $f^{<\omega}$ does not have infinite shadowing (recall that $F(X)$ is not a compact space). The proof of this fact isolates the fundamental idea in [13, Example 12]. The fact that this system has shadowing is well-known folklore, though we include a proof for completeness.

EXAMPLE 3.5. Let $X = \{1/2^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$, and let $f : X \to X$ be given by: $f(0) = 0$, $f(1) = 1$, and for every $n \in \mathbb{N}$, $n \geq 1$, $f(1/2^n) = 1/2^{n-1}$. To see that f has shadowing let $\varepsilon > 0$. Let k_0 be such that $1/2^{k_0+1} < \epsilon \leq 1/2^{k_0}$ and choose $\delta < 1/2^{k_0} - 1/2^{k_0+1}$. Let $\Gamma = \langle x_0, x_1, \ldots \rangle$ be a δ-pseudo orbit in X. Notice that if $x_m = 1/2^{k_0}$ for some $m \geq 0$, then $\langle x_m, x_{m+1}, \ldots \rangle$ must be a real orbit because of the choice of δ. There are two cases to consider: $\Gamma \subseteq [0, \varepsilon)$ or $\Gamma \cap [\varepsilon, 1] \neq \emptyset$. In the first case, $y = 0$ ε-follows Γ. In the second case, let m be the least nonnegative integer such that $x_m > \varepsilon$. Either $m = 0$ and Γ is a real orbit (which shadows itself), or $x_m = 1/2^{k_0}$ and so $y = 1/2^{k_0+m}$ ε-shadows Γ.

To see that $f^{<\omega}$ does not have infinite shadowing let $\varepsilon = 1/8$ and $\delta > 0$. There is $N \geq 3$ such that $1/2^N < \delta$. Let $A_0 = \{0, 1\}$, $A_1 = \{0, 1/2^N, 1\}$, $A_2 = f^{<\omega}(A_1) = \{0, 1/2^{N-1}, 1\}$, $A_3 = (f^{<\omega})^2(A_1) = \{0, 1/2^{N-2}, 1\}$, $\ldots$, $A_N = (f^{<\omega})^{N-1}(A_1) = \{0, 1/2^{N-(N-1)}, 1\} = \{0, 1/2, 1\}$, $A_{N+1} = (f^{<\omega})^N(A_1) = \{0, 1\} = A_0$. By construction

$$\Gamma = \langle A_0, A_1, \ldots, A_N, A_0, A_1, A_2, \ldots \rangle$$

is a δ-pseudo orbit in $F(X)$ (which actually is a δ-pseudo orbit in $F_3(X)$). It is not difficult to see that the sets that ε-shadow Γ are of the form

$$B = \left\{0, \frac{1}{2^{kN}}, \frac{1}{2^{(k-1)N}}, \frac{1}{2^{(k-2)N}}, \ldots, \frac{1}{2^N}, 1\right\}.$$

The number of iterations that B is going to ε-shadow Γ depends on k.

4. *h*-Shadowing.

The following definition was introduced in [4] and is motivated by the fact that shifts of finite type actually enjoy a stronger shadowing property, h-shadowing, or shadowing with exact hit, which happens to coincide with shadowing in shift spaces (but not necessarily in other systems). In fact (see [3]), it turns out that open maps that are expanding (in the sense that, for some $\mu > 1$ and small enough ε, $B_{\mu\varepsilon}(f(x)) \subseteq f(B_\varepsilon(x)))$ have h-shadowing.

DEFINITION 4.1. Let X be a compact metric space and let $f : X \to X$ be a continuous function. We say that f has *h-shadowing* if for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every finite δ-pseudo orbit $\Gamma = \langle x_0, x_1, \ldots, x_r\rangle$, there is a point $x \in X$ such that $d(f^i(x), x_i) < \varepsilon$ for every $i < r$ and $f^r(x) = x_r$.

The proofs of the following two theorems are similar to those of Theorems 3.2 and 3.3, respectively.

THEOREM 4.2. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. If f_n, 2^f or $f^{<\omega}$ has h-shadowing, then f has h-shadowing.*

THEOREM 4.3. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. If f has h-shadowing, then $f^{<\omega}$ has h-shadowing.*

Also, it follows immediately from Theorems 4.2 and 4.3 that if f_n has h-shadowing for every positive integer n, then $f^{<\omega}$ has h-shadowing.

LEMMA 4.4. *Let X be a compact metric space, let $f : X \to X$ be a continuous function and let Y be a dense invariant subset of X. If $f|_Y$ has h-shadowing, then f has h-shadowing.*

Proof. Suppose that $f|_Y$ has h-shadowing. Let $\varepsilon > 0$ and choose $\delta > 0$ so that every finite δ-pseudo orbit in Y is $\varepsilon/2$-h-shadowed by some $y \in Y$. Let $\Gamma = \langle x_0, x_1, \ldots, x_r\rangle$ be a $\delta/3$-pseudo orbit in X. By the proof of Lemma 3.1, for each $n > 0$ there is a δ-pseudo orbit in Y, $\Gamma_n^* = \langle y_{n,0}, y_{n,1}, \ldots, y_{n,r}\rangle$, such that $d(x_i, y_{n,i}) \le \varepsilon/2$ and $d(x_r, y_{n,r}) < 1/2^n$. By h-shadowing in Y, there is a point $y_n \in Y$ which $\varepsilon/2$-shadows Γ_n^*. Then, if y is the limit in X of a convergent subsequence from $\{y_n : n \ge 1\}$, y ε-h-shadows Γ. ∎

The converse of Lemma 4.4 is not true. To see this, let $f : [0,1] \to [0,1]$ be the full tent map with slope 2. Then, according to [3, Example 5.4], f has h-shadowing. Let $Y = ([0,1] - \mathbb{Q}) \cup \{0,1\}$. Then Y is a dense invariant (but not strongly invariant) subset of $[0,1]$, but $f|_Y$ does not have h-shadowing because for any δ there are δ-pseudo orbits ending in 1, which obviously cannot be shadowed by an orbit that ends in 1. However, it is true that f has h-shadowing if and only if 2^f has shadowing.

THEOREM 4.5. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. Then $2^f : 2^X \to 2^X$ has h-shadowing if and only if $f^{<\omega} : F(X) \to F(X)$ has h-shadowing.*

Proof. Assume first that 2^f has h-shadowing, let $\varepsilon > 0$, and let $\delta > 0$ be given by h-shadowing for 2^f. Let $\Gamma = \{A_0, A_1, \ldots, A_r\}$ be a δ-pseudo orbit in $F(X)$. Then Γ is a δ-pseudo orbit in 2^X. Since 2^f has h-shadowing, there is a point C in 2^X such that $H(f^i(C), A_i) < \varepsilon/2$ for $i \in \{0, 1, \ldots, r-1\}$ and $f^r(C) = A_r$. Let $B_r = A_r$ and assume that $B_r = \{b_r^1, \ldots, b_r^{n_r}\}$. Since $B_r = f^r(C)$, for each point b_r^j in B_r there is a point b_{r-1}^j in $f^{r-1}(C)$ such that $f(b_{r-1}^j) = b_r^j$. Let $B_{r-1}^* = \{b_{r-1}^1, \ldots, b_{r-1}^{n_r}\}$. If $H(B_{r-1}^*, f^{r-1}(C)) < \varepsilon/2$, let $B_{r-1} = B_{r-1}^*$. Otherwise, there are finitely many points $b_{r-1}^{n_r+1}, \ldots, b_{r-1}^{n_r+k}$ in $f^{r-1}(C) \backslash N_X(B_r, \varepsilon/2)$ such that if $B_{r-1} = \{b_{r-1}^1, \ldots, b_{r-1}^{n_r}, b_{r-1}^{n_r+1}, \ldots, b_{r-1}^{n_r+k}\}$, then $B_{r-1} \subseteq f^{r-1}(C)$ and $H(B_{r-1}, f^{r-1}(C)) < \varepsilon/2$, which implies $H(B_{r-1}, A_{r-1}) < \varepsilon$. Rename the points in B_{r-1} as follows: $B_{r-1} = \{b_{r-1}^1, \ldots, b_{r-1}^{n_{r-1}}\}$. Continuing this process we obtain $B_0 = \{b_0^1, \ldots, b_0^{n_0}\}$, a finite subset of C, which ε-shadows Γ and, by construction, $f^r(B_0) = A_r$. Thus, $f^{<\omega}$ has h-shadowing.

For the converse just recall that $2^f|_{F(X)} = f^{<\omega}$, therefore, if f has h-shadowing then so does 2^f by Lemma 4.4. ■

A consequence of Theorems 4.3 and 4.5 is the following result.

THEOREM 4.6. *Let X be a compact metric space and let $f : X \to X$ be a continuous function. Then f has h-shadowing if and only if 2^f has h-shadowing.*

Acknowledgements. The first author gratefully acknowledges support by CONACyT, scholarship for Postdoctoral Position no. 250254. The second author gratefully acknowledges support from the European Union through funding under FP7-ICT-2011-8 project HIERATIC (316705).

The authors would like thank the referee for a number of useful comments on the first draft of this paper. Also the authors would like to thank Héctor Méndez for useful comments on the preparation of the paper, particularly for discussions around Example 3.5.

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