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Blow-up of solutions for the non-Newtonian polytropic filtration equation with a generalized source

JUN ZHOU (Chongqing)

Abstract. This paper deals with the blow-up properties of the non-Newtonian polytropic filtration equation

$$u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = f(u)$$

with homogeneous Dirichlet boundary conditions. The blow-up conditions, upper and lower bounds of the blow-up time, and the blow-up rate are established by using the energy method and differential inequality techniques.

1. Introduction. Consider a compressible fluid flow in a homogeneous isotropic rigid porous medium. Then the volumetric moisture content $\theta(x)$, the macroscopic velocity $\vec{V}$ and the density of the fluid ρ are governed by the following equation [WZYL]:

$$\theta(x)\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{V}) - \eta(\rho) = 0, \tag{1.1}$$

where $\eta(\rho)$ is the source. For a non-Newtonian fluid, the linear Darcy law is no longer valid, because the influence of many factors such as molecular and ion effects has to be taken into account. Instead, one has the nonlinear relation

$$\rho\vec{V} = -\lambda|\nabla P|^{\sigma-2}\nabla P, \tag{1.2}$$

where $\rho\vec{V}$ and P denote the momentum velocity and pressure respectively, and $\lambda > 0$ and $\sigma \geq 2$ are some physical constants.

If the fluid considered is a polytropic gas, then the pressure and density satisfy the following equation of state:

$$P = c\rho^{\gamma}, \tag{1.3}$$

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where $c, \gamma > 0$ are some constants. Thus, it follows from (1.1)–(1.3) that

$$\theta(x)\frac{\partial\rho}{\partial t} = c^\alpha\lambda\,\mathrm{div}(|\nabla\rho^\gamma|^{\sigma-2}\nabla\rho^\gamma) + \eta(\rho). \tag{1.4}$$

The parabolic equation (1.4) also appears in population dynamics and chemical reactions, and it is usually called the *non-Newtonian polytropic filtration equation* (see [K, V1, WZYL] and references therein).

In this paper we consider (1.4) with $\theta(x) = 1$. Furthermore, we incorporate the zero boundary condition into this problem. Then we get the following initial-boundary problem after changing variables and notation:

$$\begin{cases} u_t - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) = f(u), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $m \geq 1$ and $p \geq 2$ are constants, u_0 is a non-negative function on Ω such that $u_0^m \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, and f is a continuous function on $\mathbb{R}$ that satisfies the following condition:

(H) $\limsup_{|s|\to\infty} |f(s)|/|s|^{m(r-1)} < \infty$ and $s^m f(s) \geq rF(s) \geq |s|^{mr}$, where r is a positive constant such that $p < r < \infty$ if $N \leq p$, while $p < r \leq Np/(N-p)$ if $N > p$, and

$$F(s) := m\int_0^s \tau^{m-1} f(\tau)\,d\tau.$$

For example, we can choose $f(s) = |s^m|^{r-2}s^m$ or $f(s) = s^q$ with $q > m(r-1) > 1$ to satisfy (H).

During the last decade, problem (1.5) has enjoyed a growing attention. Sattinger [Sa] constructed a stable set, which was used to construct global solutions (see [I, Ts, TY]). Furthermore, a lot of work has been devoted to singularity properties, such as blow-up, extinction and quenching (see [C, GWDW, Le, LM, V1, V2, WZYL, X, XCM, XWY, ZW, ZM1, ZM2, ZM3] and references therein).

In the above works, the authors discussed the blow-up properties by constructing upper and lower solutions. To the best of our knowledge, only a few papers deal with blow-up solutions when the initial energy is positive.

When $m = 1$, problem (1.5) degenerates to the p-Laplacian equation

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in S_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.6}$$

Zhao [Z] studied problem (1.6) and established global existence for f depending on u as well as on ∇u. He also proved a blow-up result for (1.6) under the condition

$$\frac{1}{p}\int_{\Omega}|\nabla u_0|^p\,dx-\int_{\Omega}F(u_0)\,dx\le-\frac{4(p-1)}{p\tilde{T}(p-2)^2}\int_{\Omega}u_0^2\,dx, \tag{1.7}$$

where

$$F(s)=\int_0^s f(\tau)\,d\tau.$$

More precisely, he showed that if there exists $\tilde{T}>0$ for which (1.7) holds, then the solution blows up before time $\tilde{T}$. This type of results have been extensively generalized and improved by Levine et al. [LPS], who proved some global, as well as blow-up, existence theorems. Their results, when applied to (1.6), require that

$$\frac{1}{p}\int_{\Omega}|\nabla u_0|^p\,dx-\int_{\Omega}F(u_0)\,dx<0. \tag{1.8}$$

Messaoudi [M] generalized the above result and proved that blow-up can be obtained even for vanishing initial energy. More precisely, he got blow-up under the condition

$$\frac{1}{p}\int_{\Omega}|\nabla u_0|^p\,dx-\int_{\Omega}F(u_0)\,dx\le 0. \tag{1.9}$$

Recently, the above results were improved by Liu and Wang [LW], who showed that certain solutions with positive initial energy can also blow up in finite time. Furthermore, the blow-up time T_* was estimated by

$$T_*\le\frac{2^r}{Cr}\|u_0\|_2^{2-r}. \tag{1.10}$$

When $p=2$, problem (1.5) degenerates to the porous medium equation

$$\begin{cases} u_t-\Delta u^m=f(u), & (x,t)\in Q_T,\\ u(x,t)=0, & (x,t)\in S_T,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.11}$$

The above problem was recently studied by Wu and Gao [WG]. As in [LW], they got blow-up criteria for positive initial energy. Furthermore, the blow-up time T_* was estimated by

$$T_*\le\frac{(m+1)^{mr}}{C(mr-m-1)}\|u_0\|_{m+1}^{m+1-mr}. \tag{1.12}$$

For the general case of $m\ge 1$ in problem (1.5), Yin, Li and Jin [YLJ] studied the problem

$$\begin{cases} u_t-\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)=\lambda u^q, & (x,t)\in Q_T,\\ u(x,t)=0, & (x,t)\in S_T,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.13}$$

where $q > m(p-1) \geq 1$ and $\lambda > 0$ are constants. They showed that if

$$\frac{1}{p}\int_{\Omega} |\nabla u_0^m|^p\,dx - \frac{\lambda m}{q+m}\int_{\Omega} u_0^{q+m}\,dx \leq 0, \tag{1.14}$$

then the solution of (1.13) blows up at a finite time. Furthermore, the blow-up time T_* was estimated by

$$T_* \leq \frac{(m+1)(q+m)|\Omega|^{\frac{q-1}{m+1}}}{\lambda(q-1)(q+m-mp)}\left(\frac{1}{m+1}\int_{\Omega} u_0^{m+1}\right)^{(1-q)(m+1)}. \tag{1.15}$$

Motivated by the above works, in this paper, we study the blow-up solutions of (1.5). We point out the following four problems:

1. In estimates (1.10) and (1.12), C is just some positive constant. Neither the exact value nor a lower bound of C were given in the above mentioned papers, so we cannot derive an exact upper bound of the blow-up time T_*.
2. In [YLJ], the authors give blow-up conditions when the initial energy is non-positive, but no blow-up conditions for positive initial energy are given.
3. When blow-up occurs, the blow-up time cannot usually be computed exactly. It is therefore of great importance in practice to bound it from above and below (see [BH, BS, Liu1, Liu2, LW, P, PP, PS, So] and references therein). However, no lower bound was given in the above papers.
4. As mentioned in [GV], when studying blow-up problems, it is important to calculate the blow-up rate. But none of the papers did it.

Based on the above questions, the main tasks of this paper are the following:

1. We will study the blow-up of solutions of (1.5) with positive initial energy.
2. We will give exact upper and lower bounds of the blow-up time T_*.
3. We will consider the blow-up rate.

It is well known that problem (1.5) is degenerate if $m > 1$ or $p > 2$, and therefore there is no classical solution in general. By a *solution* of (1.5), we mean a function $u(x,t)$ with $u^m \in L^\infty(Q_T) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $(u^m)_t \in L^2(Q_T)$ satisfying

$$\iint_{Q_T} (u\psi_t - |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\psi + f\psi)\,dx\,dt + \int_{\Omega} u_0\psi(x,0)\,dx = 0 \tag{1.16}$$

for all $\psi \in C^1(\overline{Q_T})$ such that $\psi(x,T) = 0$ and $\psi = 0$ on $\partial\Omega \times (0,T)$.

We first state a local existence theorem.

THEOREM 1.1. *Let $f \in C(\mathbb{R})$ satisfy* (H) *and*

$$|mu^{m-1}f(u)| \le g(u^m) \tag{1.17}$$

for some C^1 function g. Then for any non-negative function u_0 with $u_0^m \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$, there exists $T_1 \in (0,T]$ such that (1.5) *has a non-negative solution $u(x,t)$ satisfying*

$$u^m \in L^\infty(Q_{T_1}) \cap L^p(0,T_1;W_0^{1,p}(\Omega)), \qquad (u^m)_t \in L^2(Q_{T_1}).$$

The proof follows the methods of [WG], and will be given in the appendix for the readers' convenience.

Next, we give blow-up results for solutions of (1.5). By assumption (H) we know that there exist positive constants a and b such that

$$rF(s) \le |s|^m|f(s)| \le |s|^m(a+b|s|^{m(r-1)}). \tag{1.18}$$

Then it follows from the assumptions on r in (H) and (1.18) that

$$\begin{aligned}\frac{1}{B} &:= \inf_{u^m\in W_0^{1,p}(\Omega),\,u\not\equiv 0} \frac{\|\nabla u^m\|_p}{(\int_\Omega rF(u)\,dx)^{1/r}} \\ &\ge \frac{\|\nabla u^m\|_p}{(a\|u^m\|_{L^1(\Omega)}+b\|u^m\|^r_{L^r(\Omega)})^{1/r}} > 0,\end{aligned}$$

that is,

$$\Big(\int_\Omega rF(u)\,dx\Big)^{1/r} \le B\|\nabla u^m\|_p, \qquad \forall u^m \in W_0^{1,p}(\Omega). \tag{1.19}$$

Let $u(x,t)$ be the solution obtained in Theorem 1.1. The energy functional $E(t)$ related to (1.5) is

$$E(t) = \frac{1}{p}\int_\Omega |\nabla u^m(x,t)|^p\,dx - \int_\Omega F(u(x,t))\,dx. \tag{1.20}$$

It is easy to verify that

$$E'(t) = -m\int_\Omega u^{m-1}u_t^2\,dx = -\frac{4m}{(m+1)^2}\int_\Omega (u^{(m+1)/2})_t^2\,dx. \tag{1.21}$$

The next results are about the blow-up condition, the upper bound of the blow-up time and the blow-up rate.

THEOREM 1.2. *Let $f \in C(\mathbb{R})$ satisfy* (H) *and* (1.17)*. Assume the initial value u_0 with $0 \le u_0^m \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ satisfies*

$$E(0) < E_1, \tag{1.22}$$

$$\|\nabla u_0^m\|_p > \alpha_1, \tag{1.23}$$

where

$$\alpha_1 = B^{-r/(r-p)}, \qquad E_1 = \Big(\frac{1}{p}-\frac{1}{r}\Big)B^{-\frac{rp}{r-p}}. \tag{1.24}$$

Then the non-negative solution $u(x,t)$ of (1.5) blows up in finite time. Moreover, the blow-up time T_* and the blow-up rate can be estimated by

$$T_* \le \frac{(m+1)^{\frac{mr}{m+1}}}{\zeta(mr-m-1)} \|u_0\|_{m+1}^{m+1-mr} \tag{1.25}$$

and

$$\|u(\cdot,t)\|_{m+1} \le \frac{(m+1)^{\frac{mr}{(m+1)(mr-m-r)}}}{(\zeta(mr-m-1))^{\frac{1}{mr-m-1}}} (T_*-t)^{-\frac{1}{mr-m-r}}, \tag{1.26}$$

where

$$\zeta = \frac{(r-p)(m+1)^{\frac{mr}{m+1}}\big(1-(r/p-E(0)r\alpha_1^{-p})^{-\frac{r}{r-p}}\big)}{r|\Omega|^{\frac{mr}{m+1}-1}}.$$

REMARK 1.3. From $r>p$, (2.3) and (2.13), we know that $mr>m+1$ and $\zeta>0$, so (1.25) and (1.26) are valid.

Next we consider lower bounds of the blow-up time and the blow-up rate. Let $q=m(r-1)$. By assumption (H), there exist positive constants a and b such that

$$f(s) \le a+bs^{m(r-1)} = a+bs^q, \qquad s \ge 0. \tag{1.27}$$

To state the first result, we need the following constants:

$$\begin{aligned} C_1 &= k(k+1)m^{p-1}\bigg(\frac{p}{k+m(p-1)}\bigg)^p, \\ C_2(\epsilon_0) &= (k+1)\bigg(\frac{ak\epsilon_0}{k+q}+b\bigg), \\ C_3(\epsilon_0) &= \frac{(k+1)aq|\Omega|}{k+q}\epsilon_0^{-k/q}, \\ \epsilon_1 &= \frac{(N-p)(k+1)(q+1)}{pk+mN(p-1)+p-N}, \\ \epsilon_2 &= \frac{k+m(p-1)}{q-1+\epsilon_1}-1, \\ \varepsilon &= \frac{C_1(1+\epsilon_2)}{C_2(\epsilon_0)} C_S^{-\frac{p(q+1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}}, \end{aligned} \tag{1.28}$$

where ϵ_0 is any positive constant, k is a constant satisfying

$$k > \frac{\epsilon_1(k+m(p-1))}{q-1+\epsilon_1}, \tag{1.29}$$

and C_S is the best Sobolev constant of $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega)$, given by (see [Ta])

$$C_S = \pi^{-1/2} N^{-1/p} \left(\frac{p-1}{N-p}\right)^{1-1/p} \left[\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)}\right]^{1/N},$$

where $\Gamma(\cdot)$ is the Γ-function.

Let $Y_1 : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$Y_1(\rho) = \int_\rho^\infty \frac{d\eta}{C_3(\epsilon_0) + C_2(\epsilon_0)\frac{\epsilon_2}{1+\epsilon_2}\varepsilon^{-1/\epsilon_2}\eta^{\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{(k+1)\epsilon_2}}}. \tag{1.30}$$

Obviously, Y_1 is a decreasing function, which means its inverse function Y_1^{-1} exists and it is also decreasing.

THEOREM 1.4. *Assume $N > p$, and $f \in C(\mathbb{R})$ satisfies* (H) *and* (1.17). *Let k be so large that it satisfies* (1.29) *and*

$$k > \max\left\{0, \frac{(N-p)(q+1) - mN(p-1) - p + N}{p}\right\}, \tag{1.31}$$

$$k > \frac{(N-p)(q-1)}{N} - m(p-1) + \frac{N-p}{N}\epsilon_1, \tag{1.32}$$

$$k > q - 1 - m(p-1)\epsilon_1. \tag{1.33}$$

If the non-negative solution $u(x,t)$ of (1.5) *blows up at a finite time $t = T_*$, then $T_* \ge Y_1(\|u_0\|_{k+1}^{k+1})$ and $\|u(\cdot,t)\|_{k+1} \ge (Y_1^{-1}(T_* - t))^{1/(k+1)}$.*

REMARK 1.5. We give some remarks about the above theorem.

1. By (1.29), we know that $\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{(k+1)\epsilon_2} > 1$, so Y_1 is well-defined.

2. From

$$\lim_{k\to\infty} \epsilon_1 = \frac{(N-p)(q+1)}{p}, \tag{1.34}$$

we know that (1.32) and (1.33) hold if k is large enough. Since $q > 1$ and (1.34), we know that (1.29) holds if k is large enough.

3. If t is close enough to T_*, then

$$C_2(\epsilon_0)\frac{\epsilon_2}{1+\epsilon_2}\varepsilon^{-1/\epsilon_2}\|u(\cdot,t)\|_{k+1}^{\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{\epsilon_2}} \ge C_3(\epsilon_0),$$

and it follows from $\|u(\cdot,t)\|_{k+1} \ge (Y_1^{-1}(T_* - t))^{1/(k+1)}$ that

$$\begin{aligned} T_* - t &\ge Y_1(\|u(\cdot,t)\|_{k+1}^{k+1}) \\ &\ge \frac{(1+\epsilon_2)\epsilon_2^{1/\epsilon_2}}{2C_2(\epsilon_0)\epsilon_2} \int_{\|u(\cdot,t)\|_{k+1}^{k+1}}^\infty \eta^{-\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{(k+1)\epsilon_2}}\, d\eta \\ &= \frac{(k+1)(1+\epsilon_2)\epsilon_2^{1+1/\epsilon_2}}{2C_2(\epsilon_0)\epsilon_2(k+1-\epsilon_1(1+\epsilon_2))}\|u(\cdot,t)\|_{k+1}^{-\frac{k+1-\epsilon_1(1+\epsilon_2)}{\epsilon_2}}, \end{aligned}$$

which means that

$$\|u(\cdot,t)\|_{k+1} \geq \left(\frac{(k+1)(1+\epsilon_2)\epsilon_2^{1+1/\epsilon_2}}{2C_2(\epsilon_0)\epsilon_2(k+1-\epsilon_1(1+\epsilon_2))}\right)^{\frac{\epsilon_2}{k+1-\epsilon_1(1+\epsilon_2)}} \times (T^*-t)^{-\frac{\epsilon_2}{k+1-\epsilon_1(1+\epsilon_2)}}.$$

Theorem 1.4 does not give a lower bound of the blow-up time if $N \leq p$. But for $N = 3$, we can obtain the following result using the method of [LMX].

Let $Y_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$Y_2(\rho) = \int_\rho^\infty \frac{d\eta}{\ell_1\eta^{\delta_1} + \ell_2\eta^{\delta_2}}, \tag{1.35}$$

where ℓ_i and δ_i, $i = 1, 2$, are the positive constants given by

$$\begin{aligned} \sigma &= 3p - m(p-1) + 1,\\ \delta_1 &= \frac{\sigma - 1}{\sigma},\\ \delta_2 &= \frac{3p + 3m + 3q - 3pm}{3p + m + 1 - pm},\\ \ell_1 &= \sigma a|\Omega|^{1/\sigma},\\ \ell_2 &= \frac{16b^3\sigma\pi^2(3p)^{2p}\lambda_1^{3/2-p}|\Omega|^{2-\delta_2}}{27(\sigma-1)^2 4^p 3^{3/2} m^{2(p-1)}}; \end{aligned} \tag{1.36}$$

here λ_1 is the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta w(x) = \lambda w, & x \in \Omega,\\ w(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.37}$$

Obviously, Y_2 is a decreasing function, which means its inverse function Y_2^{-1} exists and is also decreasing.

THEOREM 1.6. *Let $3q > 2m(p-1) + 1$. Assume $N = 3$ and*

$$3 > \begin{cases} \dfrac{m(p-1)}{p} & \text{if } 2m(p-1)+1 < 3q \leq 2m(p-1)+2,\\ \dfrac{3q - m(p-1) - 2}{p} & \text{if } 3q > 2m(p-1)+2, \end{cases} \tag{1.38}$$

and $f \in C(\mathbb{R})$ satisfies (H) *and* (1.17). *If the non-negative solution $u(x,t)$ of* (1.5) *blows up at a finite time $t = T_*$, then $T_* \geq Y_2(\|u_0\|_\sigma^\sigma)$ and $\|u(\cdot,t)\|_\sigma \geq (Y_2^{-1}(T_* - t))^{1/\sigma}$.*

REMARK 1.7. We give some remarks about the above theorem.

1. Since $3q > 2m(p-1) + 1$, we know $\delta_2 > 1$. Thus $Y_2(\rho)$ is well-defined.
2. By (1.38), we know that $\sigma > 1$ and $\delta_2 < 2$.

3. If t is close enough to T_*, then $\ell_2\|u(\cdot,t)\|_\sigma^{\sigma\delta_2} \ge \ell_1\|u(\cdot,t)\|_\sigma^{\sigma\delta_1}$, and since $\delta_2 > 1 > \delta_1$, it follows from $\|u(\cdot,t)\|_\sigma \ge (Y_2^{-1}(T_*-t))^{1/\sigma}$ that

$$T_* - t \ge Y_2(\|u(\cdot,t)\|_\sigma^\sigma) \ge \frac{1}{2\ell_2}\int_{\|u(\cdot,t)\|_\sigma^\sigma}^{\infty} \eta^{-\delta_2}\,d\eta = \frac{1}{2\ell_2(\delta_2-2)}\|u(\cdot,t)\|_\sigma^{\sigma(1-\delta_2)},$$

which means that

$$\|u(\cdot,t)\|_\sigma \ge (2\ell_2(\delta_2-2))^{-\frac{1}{\sigma(\delta_2-1)}}(T_*-t)^{-\frac{1}{\sigma(\delta_2-1)}}.$$

The rest of this paper is organized as follows. In Section 2, we will give the proofs of the main results. The proof of the local existence will be given in Section 3.

2. Proofs of the main theorems. In this section, we prove the main theorems of Section 1. Firstly, we consider Theorem 1.2. To prove it, we need the following two lemmas by using the idea in [Vi].

LEMMA 2.1. *Let u be a solution of* (1.5). *Assume* (1.22) *and* (1.23) *hold. Then there exists a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u^m(\cdot,t)\|_p \ge \alpha_2, \qquad t \ge 0, \tag{2.1}$$

$$\Big(r\int_\Omega F(u(\cdot,t))\,dx\Big)^{1/r} \ge B\alpha_2, \qquad t\ge 0. \tag{2.2}$$

$$\frac{\alpha_2}{\alpha_1} \ge \left(\frac{r}{p} - E(0)r\alpha_1^{-p}\right)^{1/(r-p)} > 1. \tag{2.3}$$

Proof. Denote $\alpha = \|\nabla u^m\|_p$. We deduce from (1.19) and (1.20) that

$$\begin{aligned} E(t) &= \frac{1}{p}\|\nabla u^m\|_p^p - \int_\Omega F(u)\,dx \ge \frac{1}{p}\|\nabla u^m\|_p^p - \frac{1}{r}B^r\|\nabla u^m\|_p^r \\ &= \frac{1}{p}\alpha^p - \frac{1}{r}B^r\alpha^r =: h(\alpha). \end{aligned}\tag{2.4}$$

If $h'(\alpha) = 0$, then $\alpha = \alpha_1$. Set $E_1 = g(\alpha_1)$. From $r > p \ge 2$, we get $\lim_{\alpha\to\infty} h(\alpha) = -\infty$, $h(0) = 0$, h is increasing in $[0,\alpha_1]$ and decreasing in $[\alpha_1,\infty)$. Since $E(0) < E_1$, there exists a positive constant $\alpha_2 > \alpha_1$ such that $E(0) = h(\alpha_2)$. Let $\alpha_0 = \|\nabla u_0^m\|_p$. By (1.21), we have $g(\alpha_0) \le E(0) = g(\alpha_2)$. Since $\alpha_0, \alpha_2 \ge \alpha_1$, we get $\alpha_0 \ge \alpha_2$. So (2.1) holds for $t = 0$.

To prove (2.1), we suppose on the contrary that $\|\nabla u^m(\cdot,t_0)\|_p < \alpha_2$ for some $t_0 > 0$. Since $\alpha_1 < \alpha_2$, we may choose t_0 such that $\|\nabla u^m(\cdot,t_0)\|_p > \alpha_1$. Then it follows from (2.4) and monotonicity of h that

$$E(0) = h(\alpha_2) < h(\|\nabla u^m(\cdot,t_0)\|_p) \le E(t_0).$$

This is impossible since $E(t) \le E(0)$ for all $t > 0$. Hence (2.1) is established.

It follows from (1.20), (1.21), (2.1) and (2.4) that

$$\int_\Omega F(u)\,dx = \frac{1}{p}\|\nabla u^m\|_p^p - E(t) \ge \frac{1}{p}\|\nabla u^m\|_p^p - E(0)$$
$$\ge \frac{1}{p}\alpha_2^p - h(\alpha_2) = \frac{1}{r}(B\alpha_2)^r.$$

Thus (2.2) follows.

To prove (2.3), denote $\beta = \alpha_2/\alpha_1$. Then $\beta > 1$ since $\alpha_2 > \alpha_1$. So it follows from $E(0) = g(\alpha_2)$ and $B^r = \alpha_1^{p-r}$ that

$$E(0) = h(\beta\alpha_1) = (\beta\alpha_1)^p\left(\frac{1}{p} - \frac{1}{r}B^r(\beta\alpha_1)^{r-p}\right)$$
$$= (\beta\alpha_1)^p\left(\frac{1}{p} - \frac{1}{r}\beta^{r-p}\right) \ge \alpha_1^p\left(\frac{1}{p} - \frac{1}{r}\beta^{r-p}\right).$$

That is,

$$\frac{\alpha_2}{\alpha_1} = \beta \ge \left(\frac{r}{p} - E(0)r\alpha_1^{-p}\right)^{1/(r-p)}.$$

Since $r > p$, to complete the proof we only need to prove $r/p - E(0)r\alpha_1^{-p} > 1$. Since $E(0) < E_1$ and $B^r = \alpha_1^{p-r}$, it follows from (1.24) that

$$\frac{r}{p} - E(0)r\alpha_1^{-p} > \frac{r}{p} - E_1 r\alpha_1^{-p} = \frac{r}{p} - \left(\frac{1}{p} - \frac{1}{r}\right)B^{-\frac{rp}{r-p}} r\alpha_1^{-p} = 1. \blacksquare$$

Now we consider the case of $E(0) < E_1$ and $\|\nabla u_0^m\|_p > \alpha_1$. We set

$$H(t) = E_1 - E(t), \quad t \ge 0. \tag{2.5}$$

Then we have

LEMMA 2.2. *For all $t \ge 0$,*

$$0 < H(0) \le H(t) \le \int_\Omega F(u)\,dx. \tag{2.6}$$

Proof. By (1.21) we see that H is non-decreasing. Thus

$$H(t) \ge H(0) = E_1 - E(0) > 0. \tag{2.7}$$

Combining (1.20), (1.24), (2.1), (2.4) and $\alpha_2 > \alpha_1$, we obtain

$$H(t) = E_1 - \frac{1}{p}\|\nabla u^m\|_p^p + \int_\Omega F(u)\,dx \tag{2.8}$$
$$\le \left(\frac{1}{p} - \frac{1}{r}\right)\alpha_1^p - \frac{1}{p}\alpha_1^p + \int_\Omega F(u)\,dx \le \int_\Omega F(u)\,dx.$$

Then (2.6) follows from (2.7) and (2.8). $\blacksquare$

Proof of Theorem 1.2. We define

$$M(t)=\frac{1}{m+1}\int_{\Omega}u^{m+1}(x,t)\,dx. \tag{2.9}$$

Then by (H), (1.20) and (2.5), we obtain

$$\begin{aligned} M'(t)&=\int_{\Omega}u^{m}f(u)\,dx-\|\nabla u^{m}\|_{p}^{p} \\ &=\int_{\Omega}u^{m}f(u)\,dx-p\int_{\Omega}F(u)\,dx+pH(t)-pE_{1} \\ &\geq(r-p)\int_{\Omega}F(u)\,dx+pH(t)-pE_{1}. \end{aligned} \tag{2.10}$$

By using (1.24) and (2.2), we get

$$\begin{aligned} pE_{1}&=p\bigg(\frac{1}{p}-\frac{1}{r}\bigg)B^{r}B^{-\frac{rp}{r-p}-r}=\frac{r-p}{r}(B\alpha_{1})^{r} \\ &=\frac{r-p}{r}\bigg(\frac{\alpha_{1}}{\alpha_{2}}\bigg)^{r}(B\alpha_{2})^{r} \\ &\leq(r-p)\bigg(\frac{\alpha_{1}}{\alpha_{2}}\bigg)^{r}\int_{\Omega}F(u)\,dx. \end{aligned} \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$M'(t)\geq\tilde{C}\int_{\Omega}F(u)\,dx+pH(t), \tag{2.12}$$

where

$$\tilde{C}=(r-p)(1-(\alpha_{1}/\alpha_{2})^{r})>0.$$

Since $r>p\geq 2$ and $m>1$, we know that

$$\frac{mr}{m+1}>1. \tag{2.13}$$

By Hölder's inequality and (H), we have

$$M(t)^{\frac{mr}{m+1}}\leq\overline{C}\|u^{m}\|_{r}^{r}\leq r\overline{C}\int_{\Omega}F(u)\,dx, \tag{2.14}$$

where

$$\overline{C}=\bigg(\frac{1}{m+1}\bigg)^{\frac{mr}{m+1}}|\Omega|^{\frac{mr}{m+1}-1}.$$

So by (2.6), (2.12) and (2.14), we obtain

$$M'(t)\geq CM(t)^{\frac{mr}{m+1}}, \tag{2.15}$$

where $C=\tilde{C}/(r\overline{C})$. By (2.13) and (2.15),

$$M(t) \geq \left(M(0)^{1-\frac{mr}{m+1}} - \left(\frac{mr}{m+1} - 1\right)Ct\right)^{1-\frac{mr}{m+1}}$$
$$= \left(\left(\frac{1}{m+1}\|u_0\|_{m+1}^{m+1}\right)^{1-\frac{mr}{m+1}} - \frac{mr-m-1}{m+1}Ct\right)^{1-\frac{mr}{m+1}}.$$

Let

$$T^* := \frac{(m+1)^{\frac{mr}{m+1}}}{C(mr-m-1)}\|u_0\|_{m+1}^{m+1-mr}. \tag{2.16}$$

Then $M(t)$ blows up at time T^*. Therefore, $u(x,t)$ ceases to exist at some finite time $T_* \leq T^*$. That is, $u(x,t)$ blows up at a finite time T_*.

To derive an upper bound of T^*, we need to find a lower bound of C. It follows from (2.3) that

$$C = \frac{\tilde{C}}{r\overline{C}} = \frac{(r-p)(m+1)^{\frac{mr}{m+1}}(1-(\alpha_1/\alpha_2)^r)}{r|\Omega|^{\frac{mr}{m+1}-1}} \tag{2.17}$$
$$\geq \frac{(r-p)(m+1)^{\frac{mr}{m+1}}(1-(r/p-E(0)r\alpha_1^{-p})^{-\frac{r}{r-p}})}{r|\Omega|^{\frac{mr}{m+1}-1}}.$$

Then (1.25) follows from (2.16) and (2.17).

Since $\lim_{t\to T_*} M(t) = \infty$, we integrate (2.15) from t to T_* to get

$$\frac{1}{m+1}\|u(\cdot,t)\|_{m+1}^{m+1} = M(t) \leq \left(\frac{C(mr-m-1)}{m+1}(T_*-t)\right)^{-\frac{m+1}{mr-m-1}}. \tag{2.18}$$

Then (1.26) follows from (2.17) and (2.18). ∎

Proof of Theorem 1.4. Define

$$\phi(t) = \int_\Omega u^{k+1}\,dx, \tag{2.19}$$

where k is a constant satisfying (1.29) and (1.31)–(1.33). Multiplying the first equation of (1.5) by u^k, integrating by parts and using (1.27), we get

$$\frac{1}{k+1}\phi'(t) = -\int_\Omega |\nabla u^m|^{p-2}\nabla u^k \cdot \nabla u^m\,dx + \int_\Omega u^k f(u)\,dx$$
$$\leq -km^{p-1}\int_\Omega u^{k-1+(p-1)(m-1)}|\nabla u|^p\,dx + a\|u\|_k^k + b\|u\|_{k+q}^{k+q}$$
$$= -km^{p-1}\left(\frac{p}{k+m(p-1)}\right)^p \left\|\nabla u^{\frac{k+m(p-1)}{p}}\right\|_p^p + a\|u\|_k^k + b\|u\|_{k+q}^{k+q}.$$

The above inequality and Young's inequality imply

$$\phi'(t) \leq -C_1\left\|\nabla u^{\frac{k+m(p-1)}{p}}\right\|_p^p + C_2(\epsilon_0)\|u\|_{k+q}^{k+q} + C_3(\epsilon_0), \tag{2.20}$$

where ϵ_0 is any positive constant and C_1, $C_2(\epsilon_0)$, $C_3(\epsilon_0)$ are given in (1.28).

Let ϵ_1 be the constant defined in (1.28). By (1.31) and (1.32), we know that $0 < \epsilon_1 < k+1$ and

$$\frac{N(k+m(p-1))}{(N-p)(q-1+\epsilon_1)} > 1.$$

Thus Hölder's inequality gives

$$\|u\|_{k+q}^{k+q} \le \|u\|_{k+1}^{k+1-\epsilon_1} \left\|u^{\frac{k+m(p-1)}{p}}\right\|_{\frac{Np}{N-p}}^{\frac{p(q-1+\epsilon_1)}{k+m(p-1)}}. \tag{2.21}$$

By applying Young's inequality to (2.21), we obtain

$$\|u\|_{k+q}^{k+q} \le \frac{\epsilon_2}{1+\epsilon_2}\varepsilon^{-1/\epsilon_2}\|u\|_{k+1}^{\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{\epsilon_2}} + \frac{\varepsilon}{1+\epsilon_2}\left\|u^{\frac{k+m(p-1)}{p}}\right\|_{\frac{Np}{N-p}}^{\frac{p(q-1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}}, \tag{2.22}$$

where ε and ϵ_2 are the constants given in (1.28); ϵ_2 is positive by (1.33). Then a direct calculation shows that

$$\frac{(q-1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)} = 1.$$

Note that

$$\begin{aligned} \left\|u^{\frac{k+m(p-1)}{p}}\right\|_{\frac{Np}{N-p}}^{\frac{p(q-1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}} &\le C_S^{\frac{p(q+1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}} \left(\left\|\nabla u^{\frac{k+m(p-1)}{p}}\right\|_p^p\right)^{\frac{(q-1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}} \\ &= C_S^{\frac{p(q+1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}} \left\|\nabla u^{\frac{k+m(p-1)}{p}}\right\|_p^p. \end{aligned} \tag{2.23}$$

It follows from (2.20), (2.22) and (2.23) that

$$\begin{aligned} \phi'(t) \le& \left(C_2(\epsilon_0)C_S^{\frac{p(q+1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}}\frac{\varepsilon}{1+\epsilon_2} - C_1\right)\left\|\nabla u^{\frac{k+m(p-1)}{p}}\right\|_p^p \\ &+ \frac{C_2(\epsilon_0)\epsilon_2}{1+\epsilon_2}\varepsilon^{-1/\epsilon_2}\|u\|_{k+1}^{\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{\epsilon_2}} + C_3(\epsilon_0). \end{aligned} \tag{2.24}$$

By the definition of ε in (1.28),

$$C_2(\epsilon_0)C_S^{\frac{p(q+1+\epsilon_1)(1+\epsilon_2)}{k+m(p-1)}}\frac{\varepsilon}{1+\epsilon_2} - C_1 = 0.$$

Then we deduce from (2.24) that

$$\phi'(t) \le \frac{C_2(\epsilon_0)\epsilon_2}{1+\epsilon_2}\varepsilon^{-1/\epsilon_2}\phi(t)^{\frac{(k+1-\epsilon_1)(1+\epsilon_2)}{(k+1)\epsilon_2}} + C_3(\epsilon_0). \tag{2.25}$$

Since $\lim_{t\to T_*}\phi(t) = \infty$, integrating (2.25) from 0 to T_* we finally get $T_* \ge Y_1(\|u_0\|_{k+1}^{k+1})$. Similarly, we can integrate (2.25) from t to T_* to get $\|u(\cdot,t)\|_{k+1} \ge (Y_1^{-1}(T_*-t))^{1/(k+1)}$. ∎

Proof of Theorem 1.6. Define

$$\phi(t)=\int_\Omega u^\sigma\,dx, \tag{2.26}$$

where $\sigma=3p-m(p-1)+1$. Since $u^{(m-1)(p-1)+\sigma-2}|\nabla u|^p=3^{-p}|\nabla u^3|^p$, by a similar calculation to (2.20) and Hölder's inequality we obtain

$$\begin{aligned}\phi'(t)&=-\frac{\sigma(\sigma-1)m^{p-1}}{3^p}\|\nabla u^3\|_p^p+a\sigma\|u\|_{\sigma-1}^{\sigma-1}+b\sigma\|u\|_{\sigma+q-1}^{\sigma+q-1}\\&\le-\frac{\sigma(\sigma-1)m^{p-1}}{3^p}\|\nabla u^3\|_p^p+a\sigma|\Omega|^{1/\sigma}\phi(t)^{\frac{\sigma-1}{\sigma}}+b\sigma\|u\|_{\sigma+q-1}^{\sigma+q-1}.\end{aligned}\tag{2.27}$$

For simplicity, denote $v=u^3$ and $\gamma=q-m(p-1)>0$ (as $r>p$ and $q=m(r-1)$). Then (2.27) can be written as

$$\phi'(t)\le-\frac{\sigma(\sigma-1)m^{p-1}}{3^p}\|\nabla v\|_p^p+a\sigma|\Omega|^{1/\sigma}\phi(t)^{\frac{\sigma-1}{\sigma}}+b\sigma\|v\|_{p+\gamma/3}^{p+\gamma/3}. \tag{2.28}$$

Next we estimate the term $\|v\|_{p+\gamma/3}^{p+\gamma/3}$. By using the Sobolev inequality (see [Ta])

$$\|w\|_6\le4^{1/3}3^{-1/2}\pi^{2/3}\|\nabla w\|_2,\qquad\forall w\in H_0^1(\Omega),$$

Hölder's inequality and Schwarz's inequality, we get

$$\begin{aligned}\|v\|_{p+\gamma/3}^{p+\gamma/3}&\le\Big(\int_\Omega v^{2p}\,dx\Big)^{1/3}\Big(\int_\Omega v^{(p+\gamma)/2}\,dx\Big)^{2/3}\\&\le\Big(\int_\Omega v^{p}\,dx\Big)^{1/6}\Big(\int_\Omega v^{3p}\,dx\Big)^{1/6}\Big(\int_\Omega v^{(p+\gamma)/2}\,dx\Big)^{2/3}\\&\le4^{1/3}3^{-1/2}\pi^{2/3}\|\nabla v^{p/2}\|_2\Big(\int_\Omega v^{p}\,dx\Big)^{1/6}\Big(\int_\Omega v^{(p+\gamma)/2}\,dx\Big)^{2/3}.\end{aligned}\tag{2.29}$$

By virtue of the Rayleigh principle, we have

$$\lambda_1\|w\|_2^2\le\|\nabla w\|_2^2,\qquad\forall w\in H_0^1(\Omega).$$

Then it follows from (2.29) that

$$\|v\|_{p+\gamma/3}^{p+\gamma/3}\le4^{1/3}3^{-1/2}\pi^{2/3}\lambda_1^{-1/6}\|\nabla v^{p/2}\|_2^{4/3}\Big(\int_\Omega v^{(p+\gamma)/2}\,dx\Big)^{2/3}. \tag{2.30}$$

Since $|\nabla v^{p/2}|^2=(p^2/4)v^{p-2}|\nabla v|^2$, it follows by Hölder's inequality that

$$\|\nabla v^{p/2}\|_2\le\frac p2\|\nabla v\|_p\Big(\int_\Omega v^p\,dx\Big)^{1/2-1/p}\le\frac p2\lambda_1^{1/p-1/2}\|\nabla v\|_p\|\nabla v^{p/2}\|_2^{1-2/p},$$

which implies

$$\|\nabla v^{p/2}\|_2\le(p/2)^{p/2}\lambda_1^{1/2-p/4}\|\nabla v\|_p^{p/2}.$$

By the above inequality and (2.30), we have

$$\|v\|_{p+\gamma/3}^{p+\gamma/3} \le 4^{1/3}3^{-1/2}\pi^{2/3}(p/2)^{2p/3}\lambda_1^{1/2-p/3}\|\nabla v\|_p^{2p/3}\Big(\int_\Omega v^{(p+\gamma)/2}\,dx\Big)^{2/3}. \tag{2.31}$$

Now by Hölder's inequality, we obtain

$$\int_\Omega v^{(p+\gamma)/2}\,dx = \int_\Omega u^{3(p+\gamma)/2}\,dx \le |\Omega|^{1-\delta_2/2}\phi(t)^{\delta_2/2}, \tag{2.32}$$

where

$$\delta_2 = \frac{3p+3m+3q-3pm}{3p+m+1-pm} \in (1,2).$$

Then it follows from (2.31) and (2.32) that

$$\|v\|_{p+\gamma/3}^{p+\gamma/3} \le 4^{1/3}3^{-1/2}\pi^{2/3}(p/2)^{2p/3}\lambda_1^{1/2-p/3}|\Omega|^{(2-\delta_2)/3}\phi(t)^{\delta_2/3}\|\nabla v\|_p^{2p/3}. \tag{2.33}$$

We now use Young's inequality to get

$$\phi(t)^{\delta_2/3}\|\nabla v\|_p^{2p/3} \le \frac{1}{3\epsilon^2}\phi(t)^{\delta_2} + \frac{2\epsilon}{3}\|\nabla v\|_p^p, \tag{2.34}$$

where ϵ is any positive constant. Choosing

$$\epsilon = \frac{(\sigma-1)m^{p-1}\lambda_1^{p/3-1/2}}{3^{p-3/2}4^{5/6}\pi^{2/3}(p/2)^{2p/3}|\Omega|^{(2-\delta_2)/3}b}$$

yields $\sigma(\sigma-1)m^{p-1}/3^p = b\sigma 4^{1/3}3^{-1/2}\pi^{2/3}(p/2)^{2p/3}\lambda_1^{1/2-p/3}|\Omega|^{(2-\delta_2)/3}2\epsilon/3$. From (2.28), (2.33) and (2.34), we obtain

$$\phi'(t) \le \ell_1\phi(t)^{\delta_1} + \ell_2\phi(t)^{\delta_2}, \tag{2.35}$$

where

$$\delta_1 = \frac{\sigma-1}{\sigma},\ \ell_1 = a\sigma|\Omega|^{1/\sigma},\ \ell_2 = \frac{b\sigma 4^{1/3}3^{-1/2}\pi^{2/3}\big(\frac{p}{2}\big)^{2p/3}\lambda_1^{1/2-p/3}|\Omega|^{(2-\delta_2)/3}}{3\epsilon^2}.$$

Since $\lim_{t\to T_*}\phi(t) = \infty$, integrating (2.35) from 0 to T_*, we finally get $T_* \ge Y_2(\|u_0\|_\sigma^\sigma)$. Similarly, we can integrate (2.35) from t to T_* to get $\|u(\cdot,t)\|_\sigma \ge (Y_2^{-1}(T_*-t))^{1/\sigma}$. ■

3. Appendix. In this section we prove the local existence result by using the methods of [WG].

Proof of Theorem 1.1. In the proof, we denote by c a positive constant independent of n, which may change from line to line. Consider the problem

$$\begin{cases} u_t - \operatorname{div}((|\nabla u^m|^2 + 1/n)^{(p-2)/2}\nabla u^m) = f(u), & (x,t)\in Q_T,\\ u(x,t) = 1/n, & (x,t)\in S_T,\\ u(x,0) = u_{0n}(x) + 1/n, & x\in\Omega, \end{cases} \tag{3.1}$$

where $n = 1, 2, \ldots,\ 0 \le u_{0n} \in C^\infty(\overline{\Omega})$, such that

$$\begin{aligned}&\|(u_{0n} + 1/n)^m\|_\infty \le \|(u_0 + 1)^m\|_\infty,\\ &\|\nabla(u_{0n})^m\|_p \le c\|\nabla(u_0)^m\|_p,\\ &(u_{0n} + 1/n)^m \to (u_0)^m \quad \text{as } n \to \infty, \text{ strongly in } W^{1,p}(\Omega).\end{aligned}$$

By [LSU, Theorem 4.1], (3.1) has a classical solution u_n. By the maximum principle for parabolic equations, we obtain

$$u_n(x, t) \ge 1/n. \tag{3.2}$$

Now we claim that there exists $T_1 \in (0, T]$ such that

$$\|(u_n)^m\|_{L^\infty(Q_{T_1})} \le c \quad \text{for all } n = 1, 2, \ldots. \tag{3.3}$$

To see this, let $w(t)$ be the solution of the ODE

$$\begin{cases} \dfrac{dw}{dt} = g(w), \\ w(0) = \|(u_0 + 1)^m\|_\infty. \end{cases} \tag{3.4}$$

By [ZFC, Chapter 2, Theorem 5.1], there exists a $T_0 \in (0, T)$, which depends on the initial value $\|(u_0 + 1)^m\|_\infty$, such that w exists on $[0, T_0]$. Let $\varphi = (u_n)^m - w$. By (1.17), we have

$$\begin{aligned} m(u_n)^{m-1}f(u_n) - g(w) &\le ((u_n)^m - w)\int_0^1 (\theta(u_n)^m + (1-\theta)w)\,d\theta \\ &= C_n(x, t)\varphi. \end{aligned}$$

Then φ satisfies the inequalities

$$\begin{cases} \varphi_t - m(\varphi + w)^{(m-1)/m}\operatorname{div}((|\nabla\varphi|^2 + 1/n)^{(p-2)/p}\nabla\varphi) \\ \qquad\qquad\qquad - C_n(x, t)\varphi \le 0, \quad (x, t) \in Q_{T_0}, \\ \varphi(x, t) \le (1/n)^m - \|(u_0 + 1)^m\|_\infty \le 0, \qquad (x, t) \in S_{T_0}, \\ \varphi(x, 0) = (u_{0n}(x) + 1/n)^m - \|(u_0 + 1)^m\|_\infty \le 0, \qquad x \in \Omega. \end{cases}$$

By the comparison theorem, we have $\varphi(x, t) \le 0$ for $(x, t) \in Q_{T_0}$, which means that

$$\|(u_n)^m\|_{L^\infty(Q_{T_0})} \le \max_{[0, T_0]} w(t).$$

Setting $T_1 = T_0/2$ and $c = w(T_1)$ we obtain (3.3).

Multiplying the first equation of (3.1) by $(u_n)^m$ and integrating over Q_{T_1} yields

$$\frac{1}{m+1}\Big(\int_\Omega (u_n)^{m+1}(x,T_1)\,dx - \int_\Omega (u_{0n}+1/n)^{m+1}\,dx\Big)$$
$$= \iint_{S_{T_1}} (|\nabla(u_n)^m|^2+1/n)^{(p-2)/2}\nabla(u_n)^m\cdot\vec{\nu}(u_n)^m\,ds\,dt$$
$$- \iint_{Q_{T_1}} (|\nabla(u_n)^m|^2+1/n)^{(p-2)/2}|\nabla(u_n)^m|^2\,dx\,dt + \iint_{Q_{T_1}} f(u_n)^m\,dx\,dt,$$

where $\vec{\nu}$ is the unit outward normal vector on $\partial\Omega$. By the second equation of (3.1), we get

$$\iint_{Q_{T_1}} |\nabla(u_n)^m|^p\,dx\,dt \le \iint_{Q_{T_1}} (|\nabla(u_n)^m|^2+1/n)^{(p-2)/2}|\nabla(u_n)^m|^2\,dx\,dt \tag{3.5}$$
$$\le \frac{1}{m+1}\Big(\int_\Omega (u_{0n}+1/n)^{m+1}\,dx - \int_\Omega (u_n)^{m+1}(x,T_1)\,dx\Big) + c\iint_{Q_{T_1}} |f|\,dx\,dt \le c.$$

Multiplying the first equation of (3.1) by $((u_n)^m)_t$ and integrating, then using Young's inequality and (3.5), we obtain

$$\iint_{Q_{T_1}} m(u_n)^{m-1}|(u_n)_t|^2$$
$$= \iint_{Q_{T_1}} (|\nabla(u_n)^m|^2+1/n)^{(p-2)/2}\nabla(u_n)^m\cdot\nabla((u_n)^m)_t\,dx\,dt$$
$$+ \iint_{Q_{T_1}} m(u_n)^{m-1}(u_n)_t f\,dx\,dt$$
$$= -\frac{1}{2}\int_0^{T_1}\frac{\partial}{\partial t}\Big(\int_\Omega \int_0^{|\nabla(u_n)^m(x,t)|} (s+1/n)^{(p-2)/2}\Big)\,dt + \iint_{Q_{T_1}} m(u_n)^{m-1}(u_n)_t f\,dx\,dt$$
$$= \frac{1}{2}\int_\Omega \int_0^{|\nabla(u_{0n})^m|^2} (s+1/n)^{(p-2)/2}\,ds\,dx$$
$$- \frac{1}{2}\int_\Omega \int_0^{|\nabla(u_n)^m(x,T_1)|^2} (s+1/n)^{(p-2)/2}\,ds\,dx + \iint_{Q_{T_1}} m(u_n)^{m-1}(u_n)_t f\,dx\,dt$$
$$= \frac{1}{p}\int_\Omega (|\nabla(u_{0n})^m|^2+1/n)^{p/2}\,dx$$
$$- \frac{1}{p}\int_\Omega (|\nabla(u_n)^m(x,T_1)|^2+1/n)^{p/2}dx + \iint_{Q_{T_1}} m(u_n)^{m-1}(u_n)_t f\,dx\,dt$$
$$\le c + \frac{1}{2}\iint_{Q_{T_1}} m(u_n)^{m-1}f^2\,dx\,dt + \frac{1}{2}\iint_{Q_{T_1}} m(u_n)^{m-1}((u_n)_t)^2\,dx\,dt.$$

Then it follows from the above inequalities and (3.3) that

$$\iint_{Q_{T_1}} \left|\frac{\partial (u_n)^m}{\partial t}\right|^2 dx\,dt = m \iint_{Q_{T_1}} (u_n)^{m-1}\left(m(u_n)^{m-1}\left|\frac{\partial (u_n)}{\partial t}\right|^2\right) \le c. \tag{3.6}$$

Since $p \ge 2$, we deduce from (3.5) that

$$\iint_{Q_{T_1}} (1/n)^{(p-2)/2}|\nabla (u_n)^m|^{p/(p-1)} \le c \iint_{Q_{T_1}} (1/n)^{(p-2)/2}|\nabla (u_n)^m|^2 \le c.$$

The above inequality and (3.5) imply

$$\begin{aligned}\iint_{Q_{T_1}} &\left|(|\nabla (u_n)^m|^2 + 1/n)^{(p-2)/2}\frac{\partial (u_n)^m}{\partial x_i}\right|^{p/(p-1)} dx\,dt \\ &\le c\Big(\iint_{Q_{T_1}} |\nabla (u_n)^m|^p\,dx\,dt + (1/n)^{\frac{p(p-2)}{2(p-1)}} \iint_{Q_{T_1}} |\nabla (u_n)^m|^{p/(p-1)}\,dx\,dt\Big) \\ &\le c\Big(\iint_{Q_{T_1}} |\nabla (u_n)^m|^p\,dx\,dt + c(1/n)^{\frac{p-2}{2(p-1)}}\Big) \le c.\end{aligned} \tag{3.7}$$

Inequalities (3.2), (3.3) and (3.5)–(3.7) imply that there is a subsequence of $\{u_n\}$ (denoted by $\{u_n\}$ again) and a non-negative function $u \in L^\infty(Q_{T_1})$ such that:

- $u_n \to u$ and $f(u_n) \to f(u)$ a.e. on Q_{T_1} as $n \to \infty$,
- $\nabla (u_n)^m \rightharpoonup \nabla u^m$ weakly in $L^p(Q_{T_1})$ as $n \to \infty$,
- $\frac{\partial (u_n)^m}{\partial t} \rightharpoonup \frac{\partial u^m}{\partial t}$ weakly in $L^2(Q_{T_1})$ as $n \to \infty$,
- $(|\nabla (u_n)^m|^2 + 1/n)^{(p-2)/2}\frac{\partial (u_n)^m}{\partial x_i} \rightharpoonup w_i$ weakly in $L^{p/(p-1)}(Q_{T_1})$ as $n \to \infty$.

By [Li, p. 12, Lemma 1.3], we know that $w_i = |\nabla u^m|^{p-2}(u^m)_{x_i}$. Theorem 1.1 follows by a standard limiting process (see for example [Li, pp. 13–14]). ∎

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Jun Zhou
School of Mathematics and Statistics
Southwest University
Chongqing, 400715, P.R. China
E-mail: jzhouwm@163.com