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INVARIANTS FOR QUASI-INJECTIVE MODULES OVER VALUATION DOMAINS

BY

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Abstract. Quasi-injective modules over valuation domains are classified by means of complete sets of cardinal invariants.

1. Introduction. A class of modules over valuation domains R that can be classified by means of cardinal invariants is the class of pure-injective modules without superdecomposable summands (see [4, XIII.5.13]), so, in particular, the class of injective R-modules, which are the injective hulls of direct sums of indecomposable modules (see [12]). The goal of this paper is to determine complete sets of invariants for quasi-injective modules over valuation domains. This also provides an answer to [4, Problem 32].

Recall that a module M over an arbitrary ring R is *quasi-injective* if every homomorphism from a submodule of M into M itself can be extended to an endomorphism of M. The quasi-injective modules form an important class generalizing that of injective modules; they are well studied, as well as their endomorphism rings (see [6, Sections 6G, 13A] and [4, IX.8]). A quasi-injective module which is not injective will be called *proper*.

Quasi-injective modules M are characterized by the property of being fully invariant in their injective hull $E(M)$, that is, $\phi(M) \leq M$ for every endomorphism ϕ of $E(M)$. Examples of proper quasi-injective modules over a commutative integral domain R are of the form

$$E[A] = \{x \in E : A \leq \operatorname{Ann}_R x\}$$

for E a torsion injective module and A a non-zero ideal of R. Actually, if R is a valuation domain, these are the only proper quasi-injective modules.

The invariants we will use to classify quasi-injective modules over valuation domains are a simplified version of the s-invariants used to classify pure-injective modules in [4].

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2. Basic submodules of quasi-injective modules. Problem 32 in [4] is to characterize quasi-injective modules over almost maximal valuation domains by invariants. In the following sections we provide such a characterization for quasi-injective modules over arbitrary valuation domains. So, from now on, R will denote a fixed but arbitrary valuation domain.

Our starting point is the characterization of quasi-injective R-modules obtained in [3, Chapter VI, Theorem 6.2].

PROPOSITION 2.1. *Let R be a valuation domain. An R-module M is quasi-injective if and only if $M = E[A]$, where $E = E(M)$ is the injective hull of M and $A = \operatorname{Ann}_R M$.*

Note that, if M is not a bounded module, that is, if $\operatorname{Ann}_R M = 0$, then Proposition 2.1 says that M is quasi-injective exactly if it is injective.

Recall that injective modules and finitely generated torsion modules over a valuation domain R share the following property:

(P) there exists an essential pure submodule $B = \bigoplus_{i\in I} U_i$ which is a direct sum of standard uniserial modules U_i.

The submodule B is called a *basic submodule*. It is well known that, in the case of injective modules, the standard uniserial modules U_i are of the form Q/J_i, for various ideals J_i, where Q denotes the field of quotients of R. In case of finitely generated modules, the uniserial modules U_i are cyclic, hence of the form R/J_i for various ideals J_i, and the index set I is finite (see [4, XI.5.6 and V.5.7]). Property (P) is not shared in general by pure-injective modules; actually, valuation domains which are not strongly discrete admit superdecomposable pure-injective modules, hence with no pure submodules isomorphic to Q/J for any $J \leq R$ (see [11] and [4]). For more information on indecomposable and superdecomposable pure-injective modules over different kinds of rings and algebras we refer to [8], [5], [9], [10] and [7].

In the next proposition we will see that quasi-injective modules have property (P).

PROPOSITION 2.2. *Let R be a valuation domain and M a quasi-injective module. Then M contains an essential pure submodule which is a direct sum of standard uniserial modules.*

Proof. We already mentioned that property (P) holds for M injective. If M is proper quasi-injective, then $M = E[A]$, where $E = E(M)$ and $0 \neq A = \operatorname{Ann} M$, by Proposition 2.1. Let $B \cong \bigoplus_{i\in I} Q/J_i$ be a basic submodule of E. Then $B \cap E[A] \cong \bigoplus_{i\in I}(Q/J_i)[A] = \bigoplus_{i\in I}(J_i : A)/J_i$ is a direct sum of non-zero standard uniserial modules (as usual, $J_i : A = \{q \in Q : qA \leq J_i\}$). $B \cap E[A]$ is clearly essential in $E[A]$; to conclude, we have to show that it is pure in $E[A]$, that is, $(B \cap E[A]) \cap r(E[A]) \leq r(B \cap E[A])$ for every $r \in R$.

If $r \in A$, then $r(E[A]) = 0$ and the inclusion is trivial. If $rR > A$, then $E[rR] \leq E[A]$. Pick any $b \in (B \cap E[A]) \cap r(E[A])$, so that $b = rx$ for an $x \in E[A]$; the purity of B in E implies that $b = rb'$ for some $b' \in B$. Thus $b' = (b' - x) + x \in (E[rR] + E[A]) \cap B = B \cap E[A]$, therefore $b \in r(B \cap E[A])$, as desired. ■

We now explain the problem we are going to investigate in the next sections. Let $M = E[A]$ be a proper quasi-injective R-module, where $E = E(M)$ is its injective hull and A is a proper non-zero ideal. The injective module E has a basic submodule $B \cong \bigoplus_{i \in I} Q/J_i$. Since $Q/J_i \cong Q/J_j$ if and only if $J_i \cong J_j$ (see [3, Theorem 1.4, p. 142]), we collect all the summands Q/J_i with $J_i \cong J$, thus obtaining $B \cong \bigoplus_{[J]} \bigoplus_{\sigma_{[J]}} Q/J$, where $[J]$ is an isomorphy class of ideals and the $\sigma_{[J]}$ are cardinal numbers, which coincide with the s-invariants of E (see [4, XI.4]). Since, up to isomorphism, E is determined by M and B is determined by E, the s-invariants are uniquely determined by M. In this setting, the following problem naturally arises.

Problem 2.3. *Can we detect the s-invariants $\sigma_{[J]}$ of $E(M)$ by just looking at the quasi-injective module M?*

Notice that, if we start from an injective module E with basic submodule $B \cong \bigoplus_{[J]} \bigoplus_{\sigma_{[J]}} Q/J$, and if we pick a proper non-zero ideal A of R, then the module $M = E[A]$ is quasi-injective, but E is no more its injective hull, in general. For instance, if the maximal ideal P of R is not principal, consider $B = (\bigoplus_{\alpha} Q/R) \oplus (\bigoplus_{\beta} Q/P)$, and let $E = E(B)$. Then $E[P] = \bigoplus_{\beta} R/P$ is not essential in E (see also the examples in Section 4).

As a byproduct of the results obtained in the next two sections we also obtain an answer to the above problem (see Lemma 4.1).

3. Invariants for quasi-injective modules. In order to classify quasi-injective modules over a valuation domain R we will use a simplified version of the s-invariants presented in [4, XI.4], which are cardinal invariants associated with arbitrary R-modules M, denoted by $\alpha_M[\sigma, I]$. These invariants are inspired by the Ulm–Kaplansky invariants for abelian p-groups (see [1]), and were originally presented in [2].

General s-invariants are defined by means of pairs (σ, I), where σ is a height and I is a proper ideal of R (for the notion of height and its properties we refer to [4, Chapter XI]). In the present context, where injective and quasi-injective modules are considered, we can disregard heights; thus we will consider invariants defined by means of proper ideals I only. In order to define them, we need to introduce some notions, following the notation of [4, Chapter XI].

Given a module M over a valuation domain R, and fixed a proper ideal I of R, set

$$M[I] = \{a \in M \mid \mathrm{Ann}_R\, a \geq I\}, \qquad M[I^+] = \{a \in M \mid \mathrm{Ann}_R\, a > I\}.$$

$M[I]$ and $M[I^+]$ are fully invariant submodules of M such that $M[I] \geq M[I^+]$. If $I > J$, then $M[I] \leq M[J^+]$; furthermore, $M[0^+] = tM$, the torsion submodule of M, and $M[0] = M$, therefore $M[0]/M[0^+] = M/tM$. It follows that, if M is a divisible module (in particular, an injective module) then $M[0]/M[0^+]$ is a divisible torsion-free module, hence a vector space over Q (the field of quotients of R), and $\dim(M[0]/M[0^+]) = rk(M)$, the torsion-free rank of M.

With a non-zero proper ideal I of R one can associate the following prime ideal containing it:

$$I^\sharp = \{r \in R \mid rI < I\}.$$

We also set $0^\sharp = 0$. The main properties of the ideal $I^\sharp$ are established in [4, Chapter II, Section 4]. We just recall here that the ideal I is in a canonical way a module over $R_{I^\sharp}$, the localization of R at the prime ideal $I^\sharp$, and that $I^\sharp$, which is the maximal ideal of $R_{I^\sharp}$, coincides with the union of all proper ideals of R isomorphic to I. Notice also that $R_{0^\sharp} = Q$. The isomorphy class of the non-zero proper ideal I is denoted by $[I]$ (this is a well established notation, even if it creates confusion with $M[I]$); note that we consider the isomorphy class $[I]$ only for I a proper ideal.

We can now define, for any R-module M and any non-zero proper ideal I of R, the factor module

$$\alpha_M(I) = M[I]/M[I^+].$$

It is straightforward to show that $\alpha_M(I)$ is a torsion-free module over the integral domain $R/I^\sharp$, since an element $a \in M$ represents a non-zero element of $\alpha_M(I)$ exactly if $\mathrm{Ann}_R\, a = I$ and $I = tI$ for all $t \in R \setminus I^\sharp$.

REMARK 3.1. If M is an h-divisible module, i.e., the quotient of an injective module, all its non-zero elements have height $\sigma = Q/R$, so the factor module $\alpha_M(I)$ defined above coincides with the factor module

$$\alpha_M(\sigma, I) = M^\sigma[I]/(M^\sigma[I^+] + M^{\sigma+}[I])$$

defined in [4, p. 390], which is a vector space over the field $R_{I^\sharp}/I^\sharp$ (see [4, Corollary 4.3, p. 391]). Recall that the s-invariant $\alpha_M[\sigma, I]$ is derived form the vector space $\alpha_M(\sigma, I)$ (see [4, p. 392]). This observation applies in particular to injective modules and will be used in the next lemma.

LEMMA 3.2. *Let M be a quasi-injective module over the valuation domain R, and let I be a proper non-zero ideal of R. Then the factor module $\alpha_M(I)$ is a vector space over the field $R_{I^\sharp}/I^\sharp$.*

Proof. By the preceding remark, we can assume that M is proper quasi-injective, hence of the form $M = E[A]$, where $E = E(M)$ is its injective hull and $A = \operatorname{Ann}_R M$ is a non-zero ideal. If $I \geq A$, then $M[I] = E[I]$ and $M[I^+] = E[I^+]$, therefore $\alpha_M(I) = \alpha_E(I)$ and we conclude by Remark 3.1. On the other hand, if $I < A$, then $M[I] = E[A] = M[I^+]$, hence $\alpha_M(I) = 0$ and the claim is trivial. ∎

If M is a quasi-injective module, we denote by

$$d_M(I) = \dim_{R_{I^\sharp}/I^\sharp} \alpha_M(I)$$

the dimension of the $R_{I^\sharp}/I^\sharp$-vector space $\alpha_M(I)$; $d_M(I)$ is a cardinal invariant associated with the module M.

For $I = 0$ we set

$$\alpha_M(0) = M[0]/M[0^+] = M/t(M)$$

and we have the cardinal invariant $d_M(0) = \operatorname{rk}(\alpha_M(0)) = \operatorname{rk}(M)$.

An element $a \in M$ represents a non-zero element of $\alpha_M(I)$ exactly if $\operatorname{Ann}_R a = I$; therefore, since $\operatorname{Ann}_R M = \bigcap_{a\in M} \operatorname{Ann}_R a$, we have

$$\operatorname{Ann}_R M = \bigcap\{I < R \mid \alpha_M(I) \neq 0\} = \bigcap\{I < R \mid d_M(I) > 0\}.$$

This shows that we can detect the ideal $\operatorname{Ann}_R M$ and the rank $rk(M)$ by the invariants $d_M(I)$ $(0 \leq I < R)$.

Injective modules over valuation domains can be characterized by the s-invariants. This fact is part of the main structure theorem for pure-injective modules over valuation domains, presented in [4, Chapter XIII, Theorem 5.13]. Since its proof is not explicitly given there, we include here a sketch of it, in a slightly modified form, taking care of the preceding Remark 3.1.

Proposition 3.3. *Let R be a valuation domain. Two injective R-modules E and E' are isomorphic if and only if $\alpha_E(I) \cong \alpha_{E'}(I)$ for all proper ideals I.*

Proof. The necessity is clear, since an isomorphism from E to E' induces an isomorphism from $\alpha_E(I)$ to $\alpha_{E'}(I)$ for all proper ideals I. Conversely, assume that $\alpha_E(I) \cong \alpha_{E'}(I)$ for all proper ideals I. Every injective module E contains a basic submodule B, which is an essential h-divisible submodule isomorphic to a direct sum of modules of the form Q/I, where I ranges over a family of proper ideals of R depending on E. By [4, XI.5.3], $\alpha_E(I) \cong \alpha_B(I)$ and, by [4, XI.4.6], two direct sums of divisible standard uniserial modules B and B' are isomorphic if and only if $\alpha_B(I) \cong \alpha_{B'}(I)$ for all proper ideals I. Hence, if B and B' are basic submodules of E and E', respectively, they are isomorphic. By the essentiality of the basic submodules and by injectivity, we infer that the isomorphism between B and B' extends to an isomorphism between E and E'. ∎

Notice that in the preceding proof, as mentioned above, $E[0]/E[0^+] = E/tE$ is a vector space over Q, and $\alpha_E(0) = \operatorname{rk}(E)$. We can now prove the main result of this section, extending Proposition 3.3 to proper quasi-injective modules.

THEOREM 3.4. *Two proper quasi-injective modules M and M' over a valuation domain R are isomorphic if and only if $\alpha_M(I) \cong \alpha_{M'}(I)$ for all proper non-zero ideals I of R.*

Proof. Only the proof of the sufficiency is needed, so assume that $\alpha_M(I) \cong \alpha_{M'}(I)$ for all proper non-zero ideals I of R. From the equality

$$\operatorname{Ann}_R M = \bigcap\{I < R \mid \alpha_M(I) \neq 0\}$$

we infer that $\operatorname{Ann}_R M = \operatorname{Ann}_R M'$. Therefore, by Proposition 2.1, $M = E[A]$ and $M' = E'[A]$, where E is the injective hull of M, E' is the injective hull of M', and $0 \neq A = \operatorname{Ann}_R M = \operatorname{Ann}_R M'$. It is enough to prove that $E \cong E'$, since from this isomorphism the isomorphism $E[A] \cong E'[A]$ obviously follows.

We claim that, for every non-zero proper ideal I, the isomorphism $\alpha_E(I) \cong \alpha_{E'}(I)$ holds, from which the desired isomorphism $E \cong E'$ follows by Proposition 3.3.

Assume first that $\alpha_E(I) \neq 0$, so that there exists $a \in E$ with $\operatorname{Ann}_R a = I$. There exists an $r \in R$ such that $0 \neq ra \in M = E[A]$, since M is essential in E, thus $\operatorname{Ann}_R ra = r^{-1}I$ and $R > r^{-1}I \geq A$. Now [4, XI.4.5] ensures that $\alpha_E(I) \cong \alpha_E(r^{-1}I)$. But $r^{-1}I \geq A$ implies that $M[r^{-1}I] = E[A][r^{-1}I] = E[r^{-1}I]$, and similarly $M[r^{-1}I^+] = E[r^{-1}I^+]$, hence $\alpha_E(r^{-1}I) = \alpha_M(r^{-1}I)$. Therefore we derive the desired isomorphism:

$$\alpha_E(I) \cong \alpha_E(r^{-1}I) = \alpha_M(r^{-1}I) \cong \alpha_{M'}(r^{-1}I) = \alpha_{E'}(r^{-1}I) \cong \alpha_{E'}(I).$$

Assume now $\alpha_E(I) = 0$. Then there are no elements in E with annihilator I. This implies that no element in M has annihilator isomorphic to I; in fact, if $\operatorname{Ann}_R x = tI$ for some $x \in M$ and $0 \neq t \in Q$, then, when $t \in R$, $\operatorname{Ann}_R tx = t^{-1}\operatorname{Ann}_R x = t^{-1}tI = I$, absurd; on the other hand, if $t^{-1} \in R$, there exists $y \in E$ such that $t^{-1}y = x$, so that $t\operatorname{Ann}_R y = \operatorname{Ann}_R t^{-1}y = \operatorname{Ann}_R x = tI$, which implies that $\operatorname{Ann}_R y = I$, again absurd. Hence $\alpha_M(r^{-1}I) = 0$ for $R > r^{-1}I \geq A$. But then also $\alpha_{M'}(r^{-1}I) = 0 = \alpha_{E'}(I)$. ■

4. Complete sets of cardinal invariants for quasi-injective modules. In order to classify quasi-injective modules by means of complete sets of invariants, and to make the statement of Theorem 3.4 more suitable to [4, Problem 32], we need to pass from the vector spaces $\alpha_M(I)$ (and their dimension $d_M(I)$) to their equivalence classes induced by the isomorphisms of ideals; this is the passage that leads from the vector spaces $\alpha_M(\sigma, I)$ to the s-invariants $\alpha_M[\sigma, I]$ (see [4, XI.4]). However, as we here disregard heights, we must define the equivalence classes in a more restrictive way.

Recall that two non-zero ideals I, J of R are isomorphic if either $I = rJ$, or $J = rI$ for a suitable element $0 \neq r \in R$. Given a module M such that $\mathrm{Ann}_R M = A \neq 0$, and an isomorphy class $[I]$ of non-zero ideals, set

$$[I]^{\geq A} = \{J \cong I \mid R > J \geq A\}.$$

Since $I^\sharp$ is the union of the proper ideals isomorphic to I, $J \in [I]$ implies $J \leq I^\sharp$; it follows that

$$[I]^{\geq A} \neq \emptyset \;\Leftrightarrow\; \text{either } A < I^\sharp \text{ or } A = I^\sharp \cong I.$$

If $J \cong rJ \in [I]^{\geq A}$ $(r \in R)$, then multiplication by r induces an isomorphism

$$\mu_r : M[rJ]/M[rJ^+] \to M[J]/M[J^+]$$

(see [4, XI.4.5]). Notice that, if $J \in [I] \setminus [I]^{\geq A}$ (that is, if $A > J \cong I$), then $M[J] = M[J^+] = M$, therefore $M[J]/M[J^+] = 0$.

If $[I]^{\geq A} \neq \emptyset$, we can consider the equivalence class $\alpha_M[I]^{\geq A}$ induced by isomorphisms of ideals in $[I]^{\geq A}$, consisting of all factor modules $\alpha_M(J)$ with J ranging over $[I]^{\geq A}$. All these factor modules are isomorphic vector spaces over the field $R_{I^\sharp}/I^\sharp$, so they have the same dimension and we set

$$d_M[I]^{\geq A} = \dim_{R_{I^\sharp}/I^\sharp} \alpha_M(J) \qquad (J \in [I]^{\geq A} \neq \emptyset).$$

We emphasize that the definition of the invariants $d_M[I]^{\geq A}$ depends only on M, and not on its injective hull.

Our next goal is to prove that the invariants $d_M[I]^{\geq A}$ for $[I]^{\geq A} \neq \emptyset$ form a complete and independent set of invariants for proper quasi-injective R-modules. First we need a result relating them to basic submodules of the injective hull.

LEMMA 4.1. *Let E be a torsion injective module over the valuation domain R, and let $B \cong \bigoplus_{[I]} \bigoplus_{\sigma_{[I]}} Q/I$ be a basic submodule of E. Let A be a proper non-zero ideal of R and let $M = E[A]$. Then, for every non-empty isomorphy class $[I]^{\geq A}$, $d_M[I]^{\geq A} = \sigma_{[I]}$.*

Proof. In view of Remark 3.1, our invariant $\alpha_E[I]$ coincides with the s-invariant $\alpha_E[Q/R, I]$ of [4, XI.4]. By [4, XI.5.3] we get $\alpha_E[Q/R, I] = \alpha_B[Q/R, I]$, that is, in our notation, $\alpha_E[I] = \alpha_B[I]$. Since we are assuming $[I]^{\geq A} \neq \emptyset$, $\dim(\alpha_B[I]) = d_M[I]^{\geq A}$, as proved in Theorem 3.4. But in [4, XI.4] it is proved that $\dim(\alpha_B[Q/R, I]) = \sigma_{[I]}$, therefore $d_M[I]^{\geq A} = \sigma_{[I]}$. ∎

Thus the invariant $d_M[I]^{\geq A}$ counts how many copies of Q/I are contained as summands in the injective hull E of $M = E[A]$, for those isomorphy classes $[I]$ such that $[I]^{\geq A} \neq \emptyset$. Note that, when passing from E to $M = E[A]$, the summands $\bigoplus_{\sigma_{[I]}} Q/I$ of B vanish for all isomorphy classes $[I]$ such that $[I]^{\geq A} = \emptyset$, because, under this assumption, $(Q/I)[A] = 0$, as is easily verified.

Using the above notation, we have the following result.

THEOREM 4.2.

(a) *Two proper quasi-injective modules M and M' over a valuation domain R are isomorphic if and only if* $\operatorname{Ann}_R M = A = \operatorname{Ann}_R M'$ *and, for all non-empty isomorphy classes $[I]^{\geq A}$, the cardinal numbers $d_M[I]^{\geq A}$ and $d_{M'}[I]^{\geq A}$ are equal.*

(b) *Fixed a non-zero ideal A, and given any family of cardinal numbers $\{\sigma_{[I]^{\geq A}}\}$ indexed by the non-empty isomorphy classes $[I]^{\geq A}$, there exists a quasi-injective R-module M such that $d_M[I]^{\geq A} = \sigma_{[I]^{\geq A}}$ for all isomorphy classes $[I]^{\geq A}$.*

Proof. (a) is just a restatement of Theorem 3.4.

(b) Choose, for each isomorphy class $[I]^{\geq A}$, a representative I; then we associate with the family $\{\sigma_{[I]^{\geq A}}\}$ of cardinal numbers the quasi-injective module $E[A]$, where E is the injective hull of the module $\bigoplus_{[I]^{\geq A}} \bigoplus_{\sigma_{[I]^{\geq A}}} Q/I$. Then Lemma 4.1 gives the conclusion. ∎

We provide examples of proper quasi-injective modules $M = E[A]$ and their invariants $d_M[I]^{\geq A}$ over three different kinds of valuation domains.

EXAMPLE 4.3. Let R be an archimedean valuation domain with value group isomorphic to the additive group of the real numbers, and let P be its maximal ideal. Then R has only two isomorphy classes of non-zero ideals: $[rR]$ and $[P]$, that is, the class of the principal ideals and that of the non-principal ones (see [4, III.4.1]). Given any proper non-zero ideal A of R, we have two possibilities:

(i) $A = P$, in which case $[rR]^{\geq A} = \emptyset$ and $[P]^{\geq A} = \{P\}$. So, if $M = E[P]$ is a proper quasi-injective module with annihilator ideal P, the only available cardinal invariant is $d_M[P]^{\geq A} = \dim_{R/P} M$.

(ii) $A < P$, in which case both $[rR]^{\geq A}$ and $[P]^{\geq A}$ are non-empty. If $M = E[A]$ is a proper quasi-injective module, there are two available invariants: $d_M[rR]^{\geq A}$ and $d_M[rP]^{\geq A}$.

Note that, if the injective module E has a basic submodule isomorphic to $(Q/R)^{(\alpha)} \oplus (Q/P)^{(\beta)}$, and $M = E[A]$, then $\beta = d_M[P]^{\geq A}$ in case (i), while in case (ii), $\alpha = d_M[rR]^{\geq A}$ and $\beta = d_M[P]^{\geq A}$.

EXAMPLE 4.4. Let R be an archimedean valuation domain with value group isomorphic to the additive group of the rational numbers. Then R has $2^{\aleph_0}$ isomorphy classes of proper ideals (see [4, III.4.2]). Given any proper non-zero ideal A of R, we have two possibilities:

(i) $A = P$, in which case everything is as in Example 4.3(i).

(ii) $A < P$, in which case $[I]^{\geq A} \neq \emptyset$ for every proper non-zero ideal I. So, if $M = E[A]$ is a proper quasi-injective module, there are $2^{\aleph_0}$ cardinal invariants $d_M[I]^{\geq A}$. If the injective module E has a basic submodule isomorphic to $\bigoplus_{[I]}(Q/I)^{(\alpha_{[I]})}$, then $\alpha_{[I]} = d_M[I]^{\geq A}$ for all isomorphy classes $[I]$.

Example 4.5. Let R be a valuation domain of Krull dimension 2, and let $pR > L$ be the two non-zero prime ideals of R, where L is a principal ideal of the localization R_L. There are only two isomorphy classes of proper non-zero ideals, namely, $[pR]$ and $[L]$. It follows that $[L]^{\geq rR} = \emptyset$ if $rR > L$ (i.e., if $r = p^n$, $n \geq 1$). Given any proper non-zero ideal A of R, we have two possibilities:

(i) $L < A = p^n R \leq pR$, in which case there is only one non-empty isomorphy class, $[pR]^{\geq A}$. So, if $M = E[A]$ is a proper quasi-injective module, the only available invariant is $d_M[pR]^{\geq A}$ and everything is as in case (i) of Example 4.3.

(ii) $A \leq L$, in which case there are two non-empty isomorphy classes, $[pR]^{\geq A}$ and $[L]^{\geq A}$. So, if $M = E[A]$ is a proper quasi-injective module, there are two available cardinal invariants: $d_M[pR]^{\geq A}$ and $d_M[L]^{\geq A}$.

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