THE v-RADIAL PALEY–WIENER THEOREM FOR THE HELGASON FOURIER TRANSFORM ON DAMEK–RICCI SPACES

BY

ROBERTO CAMPORESI (Torino)

Abstract. We prove the Paley–Wiener theorem for the Helgason Fourier transform of smooth compactly supported v-radial functions on a Damek–Ricci space $S = NA$.

1. Introduction. Let $S = NA$ be a Damek–Ricci space, i.e., the semidirect product of a (connected and simply connected) nilpotent Lie group $N$ of Heisenberg type [17] and the one-dimensional Lie group $A \cong \mathbb{R}^+$ acting on $N$ by anisotropic dilations. When $S$ is equipped with a suitable left-invariant Riemannian metric $\gamma_S$, $S$ becomes a (noncompact, simply connected) homogeneous harmonic Riemannian space [9, 10]. Conversely, every such space is a Damek–Ricci space if we exclude $\mathbb{R}^n$ and the “degenerate” case of real hyperbolic spaces (see [14, Corollary 1.2]). We refer to [23] for a nice introduction to the geometry and harmonic analysis on Damek–Ricci spaces.

We use the ball model $B$ of $S$, namely we identify $S$ with the unit ball $B$ in the Lie algebra $s$ via the Cayley transform $C$ (see [7]):

$$S = NA \cong B = \{(V, Z, t) \in s : |V|^2 + |Z|^2 + t^2 < 1\}.$$ 

Here $s = n \oplus a = v \oplus \mathfrak{z} \oplus a$, where $\mathfrak{z}$ is the center of $n$ and $v$ its orthogonal complement in $n$. We let $p = \dim v$, $q = \dim \mathfrak{z}$, $Q = p/2 + q$, and let $S^{p+q}$ be the unit sphere in $s$,

$$S^{p+q} = \partial B = \{\omega = (V, Z, t) \in s : |V|^2 + |Z|^2 + t^2 = 1\}.$$ 

Let $f \in C_0^\infty(B)$. The Helgason Fourier transform of $f$ is defined by

$$\tilde{f}(\lambda, \omega) = \int_B f(b) Q_\lambda(b, \omega) \, db \quad (\lambda \in \mathbb{C}, \omega \in S^{p+q}),$$

where $Q_\lambda(b, \omega)$ is the normalized Poisson kernel with parameter $\lambda$ on $B$ (see

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The inversion formula is \[ f(b) = \frac{c_{p,q}}{2\pi} \int_0^{\infty} \int_{S^{p+q}} Q_{-\lambda}(b,\omega) \tilde{f}(\lambda,\omega) \, d\omega \, d\mu(\lambda), \]

where \( d\omega \) is the normalized Euclidean surface measure on \( S^{p+q} \), and \( d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda \) with

\[ c(\lambda) = \frac{2^{Q-2i\lambda} \Gamma(2i\lambda) \Gamma(d/2)}{\Gamma(i\lambda + Q/2) \Gamma(i\lambda + p/4 + 1/2)}, \]

and

\[ c_{p,q} = 2^{q-1} \Gamma(d/2) \pi^{-d/2}, \quad d = p + q + 1. \]

A \( C^\infty \) function \( \psi(\lambda,\omega) \) on \( \mathbb{C} \times \partial B \), holomorphic in \( \lambda \), is called a \textit{holomorphic function of uniform exponential type} if there exists a constant \( R > 0 \) such that, for each integer \( j \geq 0 \),

\[ \sup_{(\lambda,\omega) \in \mathbb{C} \times \partial B} e^{-R|\text{Im}\lambda|} (1 + |\lambda|)^j |\psi(\lambda,\omega)| < \infty. \]

**Theorem 1.1.** The Fourier transform \( f(b) \mapsto \tilde{f}(\lambda,\omega) \) is a bijection of \( C_0^\infty(B) \) onto the set of holomorphic functions \( \psi(\lambda,\omega) \) of uniform exponential type satisfying the condition

\[ \int_{\partial B} Q_{-\lambda}(b,\omega) \psi(\lambda,\omega) \, d\omega = \int_{\partial B} Q_{\lambda}(b,\omega) \psi(-\lambda,\omega) \, d\omega \]

for any \( b \in B \) and \( \lambda \in \mathbb{C} \). Moreover, \( \tilde{f} \) satisfies (1.4) if and only if \( f \) has support in the closed ball \( B_R = \{ b \in B : d(b,C(e)) \leq R \} \).

The direct part of this theorem, asserting that \( \text{supp} f \subset B_R \) implies \( \tilde{f} \) holomorphic of uniform exponential type \( R \), was proved in \cite{2} (in the open model). Here we prove the \( \nu \)-radial case of Theorem 1.1. The converse part, in particular the surjectivity statement, is proved for \( \nu \)-radial functions \( f \) on \( B \), i.e., functions that are radial in the variable \( V \) and thus depend only on \( |V|, Z, \) and \( t \). In this case we show that \( f \mapsto \tilde{f} \) is a bijection onto the set of functions \( \psi(\lambda,\omega) \) that are holomorphic of uniform exponential type, \( \nu \)-radial in \( \omega \), and satisfy (1.5).

The case of biradial functions \( f = f(|V|,|Z|,t) \) on \( B \) was treated recently in \cite{5}. Here we extend the results of \cite{5}, by generalizing to \( \nu \)-radial functions on \( B \) the well known expansion into \( K \)-types of the symmetric case \cite{15}.

By working in geodesic polar coordinates \((r,\omega) \in (0,\infty) \times S^{p+q} \) around the origin in \( B \), we expand both functions \( \omega \mapsto f(r,\omega) \) and \( \omega \mapsto \tilde{f}(\lambda,\omega) \) in Fourier series with respect to an orthogonal system of \( \nu \)-radial eigenfunctions of the angular Laplacian \( L_{S(r)} \) in \( L^2(S^{p+q}) \). Here \( S(r) \simeq S^{p+q} \) is the geodesic sphere of radius \( r > 0 \) centered at the origin. The Fourier coefficients are then
functions of $r$ and $\lambda$, respectively, related by a suitable Jacobi transform. Using well known estimates for Jacobi functions, we prove the result.

Our method should generalize to arbitrary functions on $B$. The problem in the nonsymmetric case is that there is no analogue of the group $K$ acting transitively by isometries on the geodesic spheres. This makes more difficult the identification of non-$v$-radial eigenfunctions of $L_{S(r)}$, as it requires the explicit form of the full angular Laplacian, which is not yet available.

The outline of this paper is as follows. In Section 2 we first obtain a formula for the $v$-radial part of the angular Laplacian. It generalizes the formula obtained in [6, 5] in the biradial case. Then, using results of Koornwinder [19], we write down a decomposition of the space $H^{p+q+1,n}$ of spherical harmonics of degree $n$ on $S^{p+q}$ as an orthogonal direct sum of subspaces invariant and irreducible under the group $SO(p) \times SO(q)$. This enables us to identify the $v$-radial eigenfunctions of $L_{S(r)}$ in terms of spherical harmonics on $S^{p+q}$. We then compute the $v$-radial eigenfunctions of the full Laplacian $L_B$ on $B$ that separate in geodesic polar coordinates. The radial part of these eigenfunctions is given by associated Jacobi functions. We also obtain a Poisson integral representation for these $v$-radial eigenfunctions of $L_B$.

In Section 3 we prove that $v$-radiality is preserved by the Helgason Fourier transform, i.e., $f$ $v$-radial on $B$ implies $\omega \mapsto \tilde{f}(\lambda, \omega)$ $v$-radial on $S^{p+q}$. The proof involves the Radon transform and the method of “descent” to complex hyperbolic spaces [21 Proposition 5.1]. Then we write down the Fourier transform and prove the $v$-radial case of Theorem 1.1.

Let us mention some earlier results on the Paley–Wiener theorem for Damek–Ricci spaces. For radial functions on $NA$, the Helgason Fourier transform reduces to the spherical transform [2]. The Paley–Wiener theorem for the spherical transform follows from the general theory developed in [20] (Jacobi function analysis): see, for instance, [11] pp. 649–650. For nonradial functions, a partial result that uses the Radon transform and reduction to complex hyperbolic spaces appears in [24]. A Paley–Wiener theorem for nonradial functions on $NA$ supported in a set whose boundary is a horocycle was obtained in [3]. A Paley–Wiener theorem for the inverse Fourier transform on $NA$ was proved in [4].

2. $v$-radial eigenfunctions on $B$

2.1. The Cayley transform and the $v$-radial Laplacian. We denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the fixed inner product and associated norm on $s$, and by $(V, Z, t) \in s$ the element $\exp(V + Z) \exp(tH)$ of $S$, where $V \in v$, $Z \in z$, $t \in \mathbb{R}$, and $H \in a$ is a unit vector. For each $Z \in z$ we have the linear map $J_Z : v \to v$ defined by $\langle J_Z V, V' \rangle = \langle Z, [V, V'] \rangle$ for $V, V' \in v$. The Cayley
transform is defined by

\[ C : NA \to B, \quad (V, Z, t) \mapsto (V', Z', t'), \]

where

\[
\begin{align*}
V' &= \frac{(1 + e^t + \frac{1}{4}|V|^2)V - JZV}{(1 + e^t + \frac{1}{4}|V|^2)^2 + |Z|^2}, \\
Z' &= \frac{2Z}{(1 + e^t + \frac{1}{4}|V|^2)^2 + |Z|^2}, \\
t' &= \frac{-1 + (e^t + \frac{1}{2}|V|^2)^2 + |Z|^2}{(1 + e^t + \frac{1}{4}|V|^2)^2 + |Z|^2},
\end{align*}
\tag{2.1}
\]

with inverse

\[
\begin{align*}
V &= 2\frac{(1 - t')V' + JZ'V'}{(1 - t')^2 + |Z'|^2}, \\
Z &= \frac{2Z'}{(1 - t')^2 + |Z'|^2}, \\
e^t &= \frac{1 - R^2}{(1 - t')^2 + |Z'|^2},
\end{align*}
\tag{2.2}
\]

where \( R = \sqrt{|V'|^2 + |Z'|^2 + t'^2} \) (see \cite{23} (18), (19), Sect. 4.4]).

We also have a generalized stereographic projection

\[ C_0 : N \to S^{p+q} \setminus \{H\} \quad (H = (0, 0, 1)) \]

obtained by letting \( a_t = e^t = 0 \), i.e., \( t = -\infty \), in (2.1):

\[ C_0(n) = \lim_{t \to -\infty} C(na_t) \in \partial B \]

(see \cite{23} Section 4.6]). Its inverse \( C_0^{-1} \) is given by the first two lines in (2.2).

In the ball model \( B \) of \( S \), equipped with the transported metric \( \gamma_B = C^{-1*}(\gamma_S) \), we have \( C(\text{Exp}_e r\omega) = \text{th}(r/2)\omega \) for \( r \geq 0 \) and \( \omega \in S^{p+q} \). Thus the geodesics through the origin are the diameters, and the Riemannian sphere \( S(r) \) of radius \( r \) (centered at the origin) is just the Euclidean sphere \( S(R) \) of radius \( R = \text{th}(r/2) \) \cite{23} Thm. 10.

Let \( \text{vol}(S^n) = 2\pi^{(n+1)/2}/\Gamma((n + 1)/2) \) be the Euclidean surface measure of the \( n \)-sphere. In geodesic polar coordinates \((r, \omega)\) around the origin \( x_0 = C(e) = (0, 0, 0) \) in \( B \), the Riemannian measure is given by

\[
\text{db} = 2^{p+q}(\text{sh}(r/2))^{p+q}(\text{ch}(r/2))^{q}\text{vol}(S^{p+q})drd\omega =: J(r)drd\omega.
\tag{2.3}
\]

Let \( M \) be the group of orthogonal automorphisms of \( NA \), namely the automorphisms of \( S \) that preserve the inner product on the Lie algebra \( \mathfrak{s} \). Using the exponential map, we can identify \( M \) as the group of orthogonal automorphisms of the \( H \)-type Lie algebra \( \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z} \), i.e., the elements \((\psi, \phi)\)
in $O(\mathfrak{v}) \times O(\mathfrak{z})$ such that
\[
[\psi(V), \psi(V')] = \phi([V, V']),
\]
or equivalently,
\[
\psi(J_Z V) = J_{\phi(Z)} \psi(V), \quad \forall V, V' \in \mathfrak{v}, \forall Z \in \mathfrak{z}.
\]
By conjugating with the Cayley map $C$, we obtain a group of transformations of $B$ that we still denote by $M$. It is easy to check that this action of $M$ on $B$ is just the action of $M$ on $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, i.e., $m \cdot (V', Z', t') = (m \cdot V', m \cdot Z', t')$, where $M$ is trivial on $\mathfrak{a}$ and leaves $\mathfrak{v}$ and $\mathfrak{z}$ invariant. It is known that $M$ acts transitively on the unit sphere $S^{q-1}$ in $\mathfrak{z}$ (see \cite[Remark 6.3]{7}). However, $M$ may or may not be transitive on the unit sphere $S^p-1$ in $\mathfrak{v}$, depending on the Heisenberg-type group $N$ (see \cite{22}).

Let $f$ be a $\mathfrak{v}$-radial function on $B$, i.e., $f(V_1', Z', t') = f(V_2', Z', t')$ if $|V_1'| = |V_2'|$. Then $f$ depends only on the variables $|V|$, $Z'$ and $t'$, and we write $f = f(|V|, Z', t')$. We denote by $C^{\infty}_0(B)^{\mathfrak{v}}$ the subspace of $\mathfrak{v}$-radial functions in $C^{\infty}_0(B)$. In geodesic polar coordinates we write
\[
f(b) = f(\theta(r/2)\omega) = f(r, \omega).
\]
For each $r > 0$, the function $\omega = (V, Z, t) \mapsto f(r, \omega)$ is $\mathfrak{v}$-radial on $S^{p+q}$, i.e., it depends only on $Z$ and $t$. We use the following notations:
\[
\begin{align*}
S^{p-1} &= \{(V, 0, 0) : |V| = 1\} = S^{p+q} \cap \mathfrak{v} \quad \text{(unit sphere in $\mathfrak{v}$)}, \\
S^q &= \{(0, Z, t) : |Z|^2 + t^2 = 1\} = S^{p+q} \cap \mathfrak{z} \oplus \mathfrak{a} \quad \text{(unit sphere in $\mathfrak{z} \oplus \mathfrak{a}$)}, \\
S^{q-1} &= \{(0, Z, 0) : |Z| = 1\} = S^{p+q} \cap \mathfrak{z} \quad \text{(unit sphere in $\mathfrak{z}$)}.
\end{align*}
\]
Every $\omega \in S^{p+q}$ can be written as $\omega = \sqrt{1 - \rho^2} \omega_1 + \rho \omega_2$, where $0 \leq \rho \leq 1$, $\omega_1 \in S^{p-1}$, and $\omega_2 \in S^q$. By writing $\tilde{\omega}_2$ as $\tilde{\omega}_2 = \cos \phi H + \sin \phi \omega_2$, where $\omega_2 \in S^{q-1}$, $H = (0, 0, 1)$, and $0 \leq \phi \leq \pi$, we see that every $\omega = (V, Z, t) \in S^{p+q}$ can be represented in the form
\[
\begin{align*}
V &= \sqrt{1 - \rho^2} \omega_1, \\
Z &= \rho \sin \phi \omega_2, \\
t &= \rho \cos \phi,
\end{align*}
\]
where
\[
0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi, \quad \omega_1 \in S^{p-1}, \quad \omega_2 \in S^{q-1}.
\]
We write $\omega = (\rho, \phi, \omega_1, \omega_2)$ and refer to this as a system of bispherical coordinates on $S^{p+q}$. The choices of $\omega_1$, $\omega_2$ and $\phi$ are unique except when $V = 0$, or $Z = 0$, or $(Z, t) = (0, 0)$. The coordinates $(\rho, \phi)$ can be regarded as polar coordinates in the space $([Z], t)$:
\[
\begin{align*}
t &= \rho \cos \phi, \\
|Z| &= \rho \sin \phi.
\end{align*}
\]
We let $D_+$ be the upper-half unit disk, defined by $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$. 


A \( v \)-radial function \( \chi \) on \( S^{p+q} \) depends only on \( \rho, \phi, \omega_2 \), and we write \( \chi = \chi(\rho, \phi, \omega_2) \). A \( v \)-radial function \( f \) on \( B \) depends only on \( r, \rho, \phi, \omega_2 \), and we write \( f = f(r, \rho, \phi, \omega_2) \).

The Laplace–Beltrami operator on \( B \) in geodesic polar coordinates reads
\[
L_B = L_{\text{rad}} + L_{S(r)},
\]
where \( L_{\text{rad}} \) is the radial part, given by
\[
L_{\text{rad}} = \partial_r^2 + \left( \frac{p}{2} \coth \frac{r}{2} + q \coth r \right) \partial_r \quad (\partial_r = \partial/\partial r),
\]
and \( L_{S(r)} \) is the angular part, i.e., the Laplacian on the Riemannian sphere \( S(r) \) with respect to the induced metric. We identify \( S(r) \) with \( S^{p+q} \) by the map \( C(\text{Exp}_r \omega) \rightarrow \omega \), i.e., th\((r/2)\omega \rightarrow \omega \), for any fixed \( r > 0 \).

Let \( L_{S^n} \) denote the round Laplacian on the unit sphere \( S^n \). Then the round Laplacian on \( S^{p+q} \) can be written in bispherical coordinates as
\[
L_{S^{p+q}} = (1 - \rho^2) \partial_\rho^2 + \left( \frac{q}{\rho} - (p + q)\rho \right) \partial_\rho + \frac{1}{\rho^2} L_{S^q} + \frac{1}{1 - \rho^2} L_{S^{p-1}}
\]
(see [6]), where the round Laplacian on \( S^q \) is
\[
L_{S^q} = \partial_\phi^2 + (q - 1) \cot \phi \partial_\phi + \frac{1}{\sin^2 \phi} L_{S^{q-1}},
\]
with \( \phi \) playing the role of “radial” coordinate on \( S^q \).

**Theorem 2.1.** Let \( \chi = \chi(\rho, \phi, \omega_2) \) be a \( v \)-radial function on \( S^{p+q} \simeq S(r) \). Then the angular Laplacian \( L_{S(r)} \) acting on \( \chi \) is given by
\[
L_{S(r)} \chi = \frac{1}{4 \sinh^2 (r/2)} L_{S^{p+q}} \chi - \frac{1}{4 \cosh^2 (r/2)} L_{S^q} \chi
= \frac{1}{4 \sinh^2 (r/2)} \left\{ (1 - \rho^2) \partial_\rho^2 + \left( \frac{q}{\rho} - (p + q)\rho \right) \partial_\rho + \frac{1}{\rho^2} L_{S^q} \right\} \chi
- \frac{1}{4 \cosh^2 (r/2)} L_{S^q} \chi.
\]

**Proof.** The idea is to change variables \((V, Z, t) \rightarrow (V', Z', t')\) directly in the known expression of the Laplacian on \( S = NA \). This can be carried out in a rather explicit way, up to some point, which is enough to obtain the \( v \)-radial part.

The Laplace–Beltrami operator on \( S \) is given in the usual \( NA \)-chart by
\[
L_S = e^t \sum_{i=1}^p E_i^2 + e^{2t} \sum_{j=1}^q Y_j^2 + H^2 - QH,
\]
where \( \{E_1, \ldots, E_p, Y_1, \ldots, Y_q, H\} \) is an orthonormal basis of \( \mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a} \) with respect to the Euclidean inner product \( \langle , \rangle \) on \( \mathfrak{s} \). We identify \( H = \partial_t \).
and regard $E_i, Y_j$ as left-invariant vector fields on the group $N$, given at $(V, Z)$ by

$$E_i = \partial v_i + \frac{1}{2} \sum_{j=1}^{q} \langle J_{Y_j} V, E_i \rangle \partial z_j, \quad Y_j = \partial z_j,$$

if we write $V = \sum_i v_i E_i$ and $Z = \sum_j z_j Y_j$. Then

$$L_S = e^{t} \sum_i \left( \partial v_i + \frac{1}{2} \sum_j \langle J_{Y_j} V, E_i \rangle \partial z_j \right) \left( \partial v_i + \frac{1}{2} \sum_k \langle J_{Y_k} V, E_i \rangle \partial z_k \right)$$

$$+ e^{2t} \sum_j \partial^2 z_j + \partial^2_t - Q \partial t$$

$$= e^{t} \sum_i \partial^2 v_i + \frac{1}{2} e^{t} \sum_{i,k} (\partial v_i \langle J_{Y_k} V, E_i \rangle) \partial z_k$$

$$+ \frac{1}{2} e^{t} \sum_{i,k} \langle J_{Y_k} V, E_i \rangle \partial v_i \partial z_k + \frac{1}{2} e^{t} \sum_{i,j} \langle J_{Y_j} V, E_i \rangle \partial z_j \partial v_i$$

$$+ \frac{1}{4} e^{t} \sum_{i,j,k} \langle J_{Y_j} V, E_i \rangle \langle J_{Y_k} V, E_i \rangle \partial z_j \partial z_k + e^{2t} \sum_j \partial^2 z_j + \partial^2_t - Q \partial t.$$

The second term vanishes since

$$\sum_{i,k} (\partial v_i \langle J_{Y_k} V, E_i \rangle) \partial z_k = \sum_{i,k} \left( \partial v_i \left[ \langle Y_k, \left[ \sum_l v_l E_l, E_i \right] \right] \right) \partial z_k$$

$$= \sum_{i,k} \langle Y_k, [E_i, E_i] \rangle \partial z_k = 0.$$

The third and fourth terms are equal, and in the fifth term we have

$$\sum_{i,j,k} \langle J_{Y_j} V, E_i \rangle \langle J_{Y_k} V, E_i \rangle \partial z_j \partial z_k = \sum_{j,k} \langle J_{Y_j} V, J_{Y_k} V \rangle \partial z_j \partial z_k$$

$$= \sum_{j,k} |V|^2 \langle Y_j, Y_k \rangle \partial z_j \partial z_k = |V|^2 \sum_j \partial^2 z_j.$$

Here we have used the identity $\langle J_Z V, J_{Z'} V \rangle = \langle Z, Z' \rangle |V|^2$.

We can rewrite $L_S$ as

$$L_S = e^{t} \left( \sum_i \partial^2 v_i + \frac{1}{4} |V|^2 \sum_j \partial^2 z_j \right) + e^{2t} \sum_j \partial^2 z_j + \partial^2_t - Q \partial t$$

$$+ e^{t} \sum_{i,j} \langle J_{Y_j} V, E_i \rangle \partial z_j \partial v_i.$$

The operators

$$L_v = \sum_i \partial^2 v_i, \quad L_z = \sum_j \partial^2 z_j.$$
are of course the Euclidean Laplacians on \( v \simeq \mathbb{R}^p \) and \( z \simeq \mathbb{R}^q \), respectively. In Euclidean polar coordinates they read
\[
L_v = \partial^2_{|V|} + \frac{p-1}{|V|} \partial_{|V|} + \frac{1}{|V|^2} L_{S^{p-1}},
L_z = \partial^2_{|Z|} + \frac{q-1}{|Z|} \partial_{|Z|} + \frac{1}{|Z|^2} L_{S^{q-1}},
\]
where \( L_{S^{p-1}} \) and \( L_{S^{q-1}} \) are the round Laplacians on the unit spheres \( S^{p-1} \) and \( S^{q-1} \) in \( v \) and \( z \), respectively. We denote by \( L_2 \) the last term in \( L_S \),
\[
L_2 = e^t \sum_{i,j} \langle J_y V, E_i \rangle \partial_{z_j} \partial_{v_i},
\]
and observe that \( L_2 \) gives zero when acting on a \( v \)-radial function on \( S \), i.e., \( f = f(|V|, Z, t) \). Indeed in this case we have
\[
\partial_{v_i} f = \frac{\partial |V|}{\partial v_i} \partial_{|V|} f = \frac{v_i}{|V|} \partial_{|V|} f,
\]
so that
\[
L_2 f = e^t \sum_{i,j} \langle J_y V, E_i \rangle \frac{v_i}{|V|} \partial_{z_j} \partial_{|V|} f = e^t \sum_j \frac{1}{|V|} \langle J_y V, V \rangle \partial_{z_j} \partial_{|V|} f = 0.
\]
If we define the structure constants \( C^k_{ij} \) by
\[
[E_i, E_j] = \sum_k C^k_{ij} Y_k,
\]
we can rewrite \( L_2 \) as
\[
L_2 = e^t \sum_{i,j,k} C^k_{ij} v_i \partial_{z_k} \partial_{v_j} = \frac{1}{2} e^t \sum_{i,j,k} C^k_{ij} (v_i \partial_{v_j} - v_j \partial_{v_i}) \partial_{z_k}.
\]
Note that \( v_i \partial_{v_j} - v_j \partial_{v_i} \) is a well defined differential operator on the unit sphere \( S^{p-1} \) for any \( i, j = 1, \ldots, p \).

The biradial part \( \tilde{L}_S \) of \( L_S \) is given by
\[
(2.9) \quad \tilde{L}_S = e^t \left( \partial^2_{|V|} + \frac{p-1}{|V|} \partial_{|V|} + \frac{1}{4 |V|^2} \left( \partial^2_{|Z|} + \frac{q-1}{|Z|} \partial_{|Z|} \right) \right) + e^{2t} \left( \partial^2_{|Z|} + \frac{q-1}{|Z|} \partial_{|Z|} \right) + (\partial_t^2 - Q \partial_t),
\]
and the Laplacian on \( S \) becomes
\[
(2.10) \quad L_S = \tilde{L}_S + e^t \left( \frac{1}{|V|^2} L_{S^{p-1}} + \frac{|V|^2}{4 |Z|^2} L_{S^{q-1}} \right) + \frac{e^{2t}}{|Z|^2} L_{S^{q-1}} + L_2.
\]

Consider now the change of variables \((V, Z, t) \overset{C}{\rightarrow} (V', Z', t')\) given by the Cayley map in (2.1)–(2.2). It is convenient to separate out the norms
of $V, V', Z, Z'$ from their respective angular variables, and to transform to Euclidean polar coordinates $(R, \omega) \in (0, 1) \times S^{p+q}$ on $B$. We get the following transformations:

$$(|V|, |Z|, t, \omega_1, \omega_2) \mapsto (|V'|, |Z'|, t', \omega_1', \omega_2') \mapsto (R, \rho, \phi, \omega_1', \omega_2'),$$

where $V = |V|\omega_1$, $V' = |V'|\omega_1' \in S^{p-1}$, $Z = |Z|\omega_2$, $Z' = |Z'|\omega_2'$ ($\omega_2, \omega_2' \in S^{q-1}$),

$$
\begin{align*}
|V'| &= \frac{|V|}{\left(1 + e^t + \frac{1}{4}|V|^2 + |Z|^2\right)^{1/2}}, \\
|Z'| &= \frac{2|Z|}{\left(1 + e^t + \frac{1}{4}|V|^2 + |Z|^2\right)^{1/2}}, \\
t' &= \frac{-1 + (e^t + \frac{1}{4}|V|^2 + |Z|^2)}{\left(1 + e^t + \frac{1}{4}|V|^2 + |Z|^2\right)^{1/2}}, \\
\omega_1' &= \frac{(1 + e^t + \frac{1}{4}|V|^2)\omega_1 - |Z|J_{\omega_2}\omega_1}{\left[1 + e^t + \frac{1}{4}|V|^2 + |Z|^2\right]^{1/2}}, \\
\omega_2' &= \omega_2,
\end{align*}
$$

with inverse

$$
\begin{align*}
|V| &= \frac{2|V'|}{\left(1 - t'^2 + |Z'|^2\right)^{1/2}}, \\
|Z| &= \frac{2|Z'|}{\left(1 - t'^2 + |Z'|^2\right)^{1/2}}, \\
e^t &= \frac{1 - R^2}{\left(1 - t'^2 + |Z'|^2\right)^2}, \\
\omega_1 &= \frac{(1 - t')\omega_1' + |Z'|J_{\omega_2'}\omega_1'}{\left[1 - t'^2 + |Z'|^2\right]^{1/2}}, \\
\omega_2 &= \omega_2',
\end{align*}
$$

where $R^2 = |V'|^2 + |Z'|^2 + t'^2$, and $(R, \rho, \phi)$ can be regarded as spherical coordinates in the space $(|V'|, |Z'|, t')$:

$$
\begin{align*}
|V'| &= R\sqrt{1 - \rho^2}, \\
|Z'| &= R\rho \sin \phi, \\
t' &= R\rho \cos \phi
\end{align*}
$$

The Jacobian of the change of variables

$$(|V|, |Z|, t, \omega_1, \omega_2) \mapsto (|V'|, |Z'|, t', \omega_1', \omega_2')$$

gives the transformation between the gradients. We write it symbolically as
Now we observe from (2.11)–(2.12) that $f$ is biradial on $B$ if and only if $f \circ C$ is biradial on $S$, and more generally, $f$ is $\nu$-radial on $B$ if and only if $f \circ C$ is $\nu$-radial on $S$. 

(2.13)
Consider then the \( v \)-radial part of \( L_S \), given by

\[
L_{v-rad} = \tilde{L}_S + \frac{e^t}{|Z|^2} \left( \frac{1}{4} |V|^2 + e^t \right) L_{S^{q-1}}. 
\]

When we transform \( L_{v-rad} \) to \( B \) we must get the \( v \)-radial part of the Laplacian on \( B \), plus some operator \( L_1 \) such that \( L_1 f = 0 \) for \( f \) \( v \)-radial on \( B \). Let us examine the transformation of the two terms in (2.14) separately. When we transform \( \tilde{L}_S \) (given by (2.9)) using (2.13), we get the biradial part \( \tilde{L}_B \) of the Laplacian on \( B \) (which is known, see below) plus an operator \( L_1' \) such that \( L_1' f = 0 \) for \( f \) biradial on \( B \). Moreover, \( L_1' f = 0 \) if \( f \) is \( v \)-radial on \( B \), since every term in \( L_1' \) will carry derivatives with respect to the angular variable \( \omega_1' \). Next, by the transformation

\[
\partial_2 \rho = \partial_2 \rho' + \frac{\partial \omega_1'}{\partial \rho} \partial_1 \omega',
\]

we see that under the Cayley map, \( L_{S^{q-1}} \overset{C}{\mapsto} L_{S^{q-1}} + L_1'' \), where \( L_1'' f = 0 \) for \( v \)-radial \( f \), since \( L_1'' \) carries derivatives with respect to \( \omega_1' \). The biradial part \( \tilde{L}_B \) is known, namely ([6, Theorem 4.1], [5, Theorem 2.1])

\[
\tilde{L}_B = L_{rad} + \frac{1 - R^2}{4|Z'|^2} \left( |V'|^2 + 1 - R^2 \right) L_{S^{q-1}} + L_1,
\]

where the first two terms give the \( v \)-radial part of the Laplacian on \( B \), and the operator

\[
L_1 = L_1' + \frac{1 - R^2}{4|Z'|^2} (|V'|^2 + 1 - R^2) L_1''
\]

satisfies \( L_1 f = 0 \) for \( f \) \( v \)-radial on \( B \).

The biradial part \( \tilde{L}_B \) is known, namely ([6, Theorem 4.1], [5, Theorem 2.1])

\[
\tilde{L}_B = L_{rad} + \frac{1 - R^2}{4R^2} D_1 - \frac{1 - R^2}{4} D_2,
\]

where

\[
D_1 = (1 - \rho^2) \partial^2_\rho + \left( \frac{q}{\rho} - (p + q) \rho \right) \partial_\rho + \frac{1}{\rho^2} (\partial^2_\phi + (q - 1) \cot \phi \partial_\phi),
\]

\[
D_2 = \partial^2_\phi + (q - 1) \cot \phi \partial_\phi.
\]

The coefficient of \( L_{S^{q-1}} \) in (2.15) can be rewritten in terms of \( R, \rho \) and \( \phi \) as

\[
1 - \frac{R^2}{4R^2 \rho^2 \sin^2 \phi} \left( R^2 (1 - \rho^2) + 1 - R^2 \right) = \frac{1 - R^2}{4R^2 \rho^2 \sin^2 \phi} - \frac{1 - R^2}{4 \sin^2 \phi}.
\]

Consider now the term \( L_{S^{p-1}} \) in (2.10). It is easy to check that \( L_{S^{p-1}} \overset{C}{\mapsto} L_{S^{p-1}} \), i.e., the round Laplacian on \( S^{p-1} \), is invariant under the Cayley transformation. For example, for \( p = 2 \) and \( q = 1 \) a direct computation
shows that if \( \omega_1 = e^{i\phi_1} \) and \( \omega_1' = e^{i\phi_1'} \), then the angular coordinates on \( S^{p-1} = S^1 \) are related by

\[
\phi_1 = \phi_1' + \arctan \frac{R\rho \sin \phi}{1 - R\rho \cos \phi},
\]

so that \( \partial_{\phi_1} = \partial_{\phi_1'} \) and \( \partial^2_{\phi_1} = \partial^2_{\phi_1'} \). In the general case we observe that the map \( R : v \rightarrow v \) induced by \( \omega_1 \rightarrow \omega_1' \), namely

\[
\tilde{V} \mapsto R\tilde{V} = \left(1 + e^t + \frac{1}{4} |V|^2 - |Z|J_{\omega_2}\right)\tilde{V}
\]

\[
\left[\left(1 + e^t + \frac{1}{4} |V|^2\right)^2 + |Z|^2\right]^{1/2},
\]

is a linear map preserving the Euclidean norm for any \( |V|, |Z|, t \) and \( \omega_2 \) fixed. Thus \( R \in O(v) \) and the round Laplacian \( L_{S^{p-1}} \) is invariant under \( R \), as claimed.

By transforming \( L_S \) in (2.10), we then obtain the Laplacian on \( B \) in the form

\[
L_B = L_{\text{rad}} + \frac{1 - R^2}{4R^2} \left( D_1 + \frac{1}{\rho^2 \sin^2 \phi} L_{S^{q-1}} + \frac{1}{1 - \rho^2} L_{S^{p-1}} \right)
- \frac{1 - R^2}{4} \left( D_2 + \frac{1}{\sin^2 \phi} L_{S^{q-1}} \right) + L_1 + L_2,
\]

where we write \( L_2 \) for the image of \( L_2 \) under the Cayley transform. Note that the operators in the round brackets of (2.16) are precisely the round Laplacians \( L_{S^{p+q}} \) and \( L_{S^q} \) (cf. (2.6), (2.7)). Defining \( L_3 \) by

\[
L_1 + L_2 = -\frac{1 - R^2}{4} L_3 = -\frac{1}{4 \text{sh}^2(r/2)} L_3,
\]

and recalling the relationship \( R = \text{th}(r/2) \) between the Euclidean and Riemannian distance in \( B \), we can rewrite the Laplacian on \( B \) in geodesic polar coordinates as

\[
L_B = L_{\text{rad}} + \frac{1 - R^2}{4R^2} L_{S^{p+q}} - \frac{1 - R^2}{4} (L_{S^q} + L_3)
- \frac{1}{4 \text{sh}^2(r/2)} L_{S^{p+q}} - \frac{1}{4 \text{ch}^2(r/2)} (L_{S^q} + L_3).
\]

The angular Laplacian \( L_{S(r)} \) is identified as

\[
L_{S(r)} = \frac{1}{4 \text{sh}^2(r/2)} L_{S^{p+q}} - \frac{1}{4 \text{ch}^2(r/2)} (L_{S^q} + L_3).
\]

Now \( L_3 f = 0 \) for \( f \) \( v \)-radial on \( B \), since both \( L_1 \) and \( L_2 \) have this property, so the result follows.

**Remark 2.2.** The unknown part in \( L_{S(r)} \) is the operator \( L_3 \). It will be some expression in the derivatives \( \partial_\phi, \partial_{\omega_1'}, \partial_{\omega_2'} \). (The derivative \( \partial_R \) must obviously cancel out in \( L_3 \). The derivative \( \partial_\rho \) cancels out in \( L_3 \) since the
\(\rho\)-coordinate decouples from the remaining coordinates, as we know from the explicit form of the induced metric \(\gamma_{S(r)}\) \cite{6}. Thus the derivatives with respect to \(\rho\) only occur in the term \((4\sh^2(r/2))^{-1}L_{S^p+q}\), corresponding to the constant curvature part of the induced metric, namely \(4\sh^2(r/2)\gamma_{S^p+q}\); see \cite{6} Theorem 3.1\).

Since \(L_3f = 0\) for \(f\) \(v\)-radial, every term of \(L_3\) will contain derivatives with respect to the angular variable \(\omega'_1\). Symbolically, \(L_3\) carries the derivatives \(\partial^2_{\omega'_1}, \partial_{\omega'_1}\partial_{\omega'_2}, \partial_{\omega'_1}\partial_{\phi}\).

**Remark 2.3.** In the symmetric case, i.e., when \(S\) is a rank-1 symmetric space \(G/K\), \(L_3\) is \(r\)-independent and the operator \(L' = L_{S^q} + L_3\) in (2.17) is the “vertical” Laplacian acting along the fibers of the Hopf fibration of \(S^{p+q}\). For example for \(p = 2\) and \(q = 1\) we have

\[
v_1\partial v_2 - v_2\partial v_1 = \partial \phi_1 = \partial \phi'_1 = v'_1\partial v'_2 - v'_2\partial v'_1
\]

and

\[
L_2 = e^t(v_1\partial v_2 - v_2\partial v_1)\partial z = e^t\partial \phi_1\partial z = e^t\partial \phi'_1\partial z,
\]

where \(\partial z = a\partial R + b\partial \rho + c\partial \phi + d\partial \phi'_1\), with \(a, b, c, d\) suitable functions of \(R, \rho\) and \(\phi\). Adding on the contribution from \(L_1\), we see that all terms with the derivatives \(\partial R, \partial \rho\) cancel out, and we get

\[
L_3 = \partial^2_{\phi'_1} + 2\partial \phi \partial \phi'_1 \Rightarrow L' = L_{S^q} + L_3 = \partial^2_{\phi} + L_3 = (\partial \phi + \partial \phi'_1)^2 = \partial^2_{\phi}.
\]

Here \(\partial \theta = \partial \phi + \partial \phi'_1 = t'\partial z' - z'\partial t + v'_1\partial v'_2 - v'_2\partial v'_1\) is the Hopf vector field, generating the Hopf action along the fibers isomorphic to \(S^1\) at each point of \(S^{p+q} = S^3\).

In the nonsymmetric case, \(S^{p+q}\) is no longer a fibration with fiber \(S^q\), and there does not seem to be a natural interpretation of the operator \(L' = L_{S^q} + L_3\) in (2.17). Moreover, \(L_3\) will generally depend on \(r\), since the term \(L'\) is due to the “perturbed” part of the induced metric (denoted \(4\sh^4(r/2)h_{\text{th}}(r/2)\) in \cite{6} Theorem 3.1\), which is a complicated differential expression on \(S^{p+q}\) explicitly depending on \(r\).

**2.2. Spherical harmonics on \(S^{p+q}\).** We recall some results of Koornwinder \cite{19}. Let \(H^{p+q+1,n}\) be the space of spherical harmonics of degree \(n\) on \(S^{p+q}\). Recall that every \(\omega \in S^{p+q}\) can be written as \(\omega = \sqrt{1 - \rho^2} \omega_1 + \rho \tilde{\omega}_2\), with \(0 \leq \rho \leq 1\), \(\omega_1 \in S^{p-1}\), and \(\tilde{\omega}_2 \in S^q\). By \cite{19} Theorem 4.2\) (with \(q \mapsto q + 1\), \(\cos \theta = \rho\), \(m \mapsto n\), \(k \mapsto r\), \(l \mapsto s\)\) we have the decomposition

\[
H^{p+q+1,n} = \sum_{0 \leq r, s \leq n, n-r-s \text{ even } \geq 0} H^{p+q+1,n}_{r,s},
\]

where \(H^{p+q+1,n}_{r,s}\) is the vector space which is spanned by the functions

\[
S(\omega) = \rho^s(1 - \rho^2)^{r/2} R^{(p/2-1+r,(q-1)/2+s)}(2\rho^2 - 1) S_r^{(1)}(\omega_1) S_s^{(2)}(\tilde{\omega}_2)
\]
with
\[ S_r^{(1)} \in \mathcal{H}^{p,r} = \text{spherical harmonics of degree } r \text{ on } S^{p-1}, \]
\[ S_s^{(2)} \in \mathcal{H}^{q+1,s} = \text{spherical harmonics of degree } s \text{ on } S^q, \]
and \( R_{m}^{(a,b)}(x) \) is a Jacobi polynomial normalized so that \( R_{m}^{(a,b)}(1) = 1 \). The spaces \( \mathcal{H}_{r,s}^{p+q+1,n} \) are mutually orthogonal and they are invariant and irreducible under \( \text{SO}(p) \times \text{SO}(q+1) \).

We now refine this decomposition by adapting it to the bispherical coordinate chart \((\rho, \phi, \omega_1, \omega_2)\) of \( S^{p+q} \). As before, we write \( \tilde{\omega}_2 \in S^q \) as \( \tilde{\omega}_2 = \cos \phi H + \sin \phi \omega_2 \), with \( \omega_2 \in S^{q-1}, H = (0, 0, 1) \), and \( 0 \leq \phi \leq \pi \). Then by [19, Theorem 2.4] (with \( q \to q+1 \)) we have the decomposition
\[ (2.18) \quad \mathcal{H}^{q+1,s} = \sum_{j=0}^{s} \mathcal{H}_{j}^{q+1,s}, \]
where \( \mathcal{H}_{j}^{q+1,s} \) is the linear span of the functions
\[ S(\tilde{\omega}_2) = (\sin \phi)^j R_{s-j}^{(q/2-1+j,q/2-1+j)}(\cos \phi)S_j^{(2)}(\omega_2) \]
with \( S_j^{(2)} \in \mathcal{H}^{q,j} = \text{spherical harmonics of degree } j \text{ on } S^{q-1} \).

The spaces \( \mathcal{H}_{j}^{q+1,s} \) are mutually orthogonal and they are invariant and irreducible under \( \text{SO}(q) \). Using this decomposition for the spherical harmonics \( S_j^{(2)}(\tilde{\omega}_2) \) in \( \mathcal{H}_{r,s}^{p+q+1,n} \) above, we obtain the following decompositions of the spaces \( \mathcal{H}_{r,s}^{p+q+1,n} \) and \( \mathcal{H}^{p+q+1,n} \):
\[ (2.19) \quad \mathcal{H}_{r,s}^{p+q+1,n} = \sum_{j=0}^{s} \mathcal{H}_{r,s,j}^{p+q+1,n}, \]
\[ (2.20) \quad \mathcal{H}^{p+q+1,n} = \sum_{0 \leq r,s \leq n \atop n-r-s \text{ even} \geq 0} \sum_{j=0}^{s} \mathcal{H}_{r,s,j}^{p+q+1,n}, \]
where \( \mathcal{H}_{r,s,j}^{p+q+1,n} \) is the linear span of the functions
\[ S(\omega) = \rho^s(1-\rho^2)^{r/2} R_{(n-r-s)/2}^{(p/2-1+r,q-1)/2+s}(2\rho^2 - 1) \]
\[ \times (\sin \phi)^j R_{s-j}^{(q/2-1+j,q/2-1+j)}(\cos \phi)S_r^{(1)}(\omega_1)S_j^{(2)}(\omega_2) \]
with \( S_r^{(1)} \in \mathcal{H}^{p,r} \) and \( S_j^{(2)}(\omega_2) \equiv 1 \). The spaces \( \mathcal{H}_{r,s,j}^{p+q+1,n} \) are mutually orthogonal and they are invariant and irreducible under \( \text{SO}(p) \times \text{SO}(q) \).

**Remark 2.4.** For \( q = 1 \) the decompositions (2.18)–(2.20) must be modified as follows. The index \( j \) must be restricted to take the values \( 0 \leq j \leq \min(s, 1) \) and \( S_j^{(2)}(\omega_2) \equiv 1 \). The remaining formulas correctly reproduce the decomposition of spherical harmonics of degree \( s \) on \( S^q = S^1 \).
example, if $s \geq 1$, then \((2.18)\) reads $\mathcal{H}^{2,s} = \mathcal{H}^{2,s}_0 \oplus \mathcal{H}^{2,s}_1$, where $\mathcal{H}^{2,s}_0$ and $\mathcal{H}^{2,s}_1$ are the linear spans of the functions $R^{(1/2,1/2)}_{s-1} (\cos \phi) = \cos(s\phi)$ and $(\sin \phi) R^{(1/2,1/2)}_{s-1} (\cos \phi) = s^{-1} \sin(s\phi)$, respectively.

2.3. Separation of variables. The $v$-radial eigenfunctions of the angular Laplacian $L_{S(r)}$ are those that are independent of $\omega_1$ in the bispherical coordinate chart $(\rho, \phi, \omega_1, \omega_2)$ of $S^{p+q} \simeq S(r)$. It follows from \((2.8)\) that the $v$-radial eigenfunctions of $L_{S(r)}$ in $\mathcal{H}^{p+q+1,n}_0$ are the elements of

$$
\mathcal{H}_{0,s}^{p+q+1,n} = \sum_{j=0}^{s} \mathcal{H}_{0,s,j}^{p+q+1,n} \quad (0 \leq s \leq n, \ n-s \text{ even } \geq 0),
$$

namely

$$
Y \in \mathcal{H}_{0,s}^{p+q+1,n} \Rightarrow L_{S(r)} Y = \left( -\frac{n(n+p+q-1)}{4 \text{sh}^2(r/2)} + \frac{s(s+q-1)}{4 \text{ch}^2(r/2)} \right) Y,
$$

and conversely, if $Y \in \mathcal{H}_{0,s}^{p+q+1,n}$ is a $v$-radial eigenfunctions of $L_{S(r)}$, then $Y \in \mathcal{H}_{0,s}^{p+q+1,n}$ for some $s$ with $0 \leq s \leq n$ and $s$ of the same parity of $n$. Letting $n = k + l$ and $s = k - l$, we find that the $v$-radial eigenfunctions of $L_{S(r)}$ with the eigenvalue

$$(2.21) \quad \lambda_{k,l} = -\frac{(k+l)(k+l+p+q-1)}{4 \text{sh}^2(r/2)} + \frac{(k-l)(k-l+q-1)}{4 \text{ch}^2(r/2)}$$

are the elements of $\mathcal{H}_{0,k-l}^{p+q+1,k+l}$. They belong to subspaces invariant and irreducible under $\text{SO}(p) \times \text{SO}(q+1)$. The $v$-radial eigenfunctions with the eigenvalue $\lambda_{k,l}$ that belong to subspaces invariant and irreducible under $\text{SO}(p) \times \text{SO}(q)$ are the elements of $\mathcal{H}_{0,k-l,j}^{p+q+1,k+l}$, with $0 \leq j \leq k - l$. A basis of $\mathcal{H}_{0,k-l,j}^{p+q+1,k+l}$ is given by the functions

$$(2.22) \quad \chi_{k,l,j,i}(\rho, \phi, \omega_2) = \rho^{k-l} R_{i}^{(p/2-1, (q-1)/2+k-l)}(2\rho^2 - 1) \times (\sin \phi)^j R_{k-l-j}^{(q/2-1+j, q/2-1+j)}(\cos \phi) S_{j,i}^{(2)}(\omega_2),$$

where $S_{j,i}^{(2)} \ (i = 1, \ldots, \dim \mathcal{H}^{q,j})$ is a basis of $\mathcal{H}^{q,j}$. For $k$ and $l$ fixed, the only eigenfunctions that are invariant under $\text{SO}(p) \times \text{SO}(q)$ are those with $j = 0$, namely the biradial eigenfunctions (cf. [6, 5])

$$
\chi_{k,l}(\rho, \phi) = \rho^{k-l} R_{i}^{(p/2-1, (q-1)/2+k-l)}(2\rho^2 - 1) R_{k-l}^{(q/2-1, q/2-1)}(\cos \phi) \\
\in \mathcal{H}_{0,k-l,0}^{p+q+1,k+l} \subset \mathcal{H}_{0,k-l}^{p+q+1,k+l}.
$$

The degeneracy of $\lambda_{k,l}$ is then at least $\dim \mathcal{H}_{0,k-l}^{p+q+1,k+l} = \sum_{j=0}^{k-l} \dim \mathcal{H}^{q,j}$. It will actually be bigger than this, since this number only depends on $k - l$ but not on $k + l$. 

\[\text{THE } v\text{-RADIAL PALEY–WIENER THEOREM}\]
Remark 2.5. In [18, pp. 27–28] it is observed that, for any noncompact harmonic space $X$, the number $(f'/f)'(r)$, where $f$ is the density function, is an eigenvalue of $L_{S(r)}$ with degeneracy $\geq \dim X$. For a Damek–Ricci space we have $f(r) \propto (\text{sh}(r/2))^{p+q}(\text{ch}(r/2))^{q}$, so that $(f'/f)'(r)$ is precisely the first nonzero $\nu$-radial eigenvalue $\lambda_{1,0} = \lambda_{1,0}(r)$ in (2.21). Thus the degeneracy of $\lambda_{1,0}$ is at least $p+q+1$, whereas $\dim H_{0,1}^{p+q+1,1} = q+1$. Note that $p+q+1 = \dim H_{0,1}^{p+q+1,1} + \dim H_{1,0}^{p+q+1,1}$.

The normalized Euclidean measure $d\omega$ on $S^{p+q}$ can be written in bispHERical coordinates as

$$d\omega = \frac{\text{vol}(S^{p-1}) \text{vol}(S^{q-1})}{\text{vol}(S^{p+q})} \rho^q (1 - \rho^2)^{p/2-1} (\sin \phi)^{q-1} d\rho d\phi d\omega_1 d\omega_2$$

where $d\omega_1$ and $d\omega_2$ are the normalized Euclidean measures on $S^{p-1}$ and $S^{q-1}$, respectively.

For a $\nu$-radial function $\chi$ on $S^{p+q}$ we get (writing $\chi(\omega) = \chi(\rho, \phi, \omega_2)$)

$$\int_{S^{p+q}} \chi(\omega) \, d\omega = \int_0^1 \int_0^{2\pi} \int_0^\pi \chi(\rho, \phi, \omega_2) \, d\rho d\phi d\omega_1 d\omega_2.$$ 

Suppose the basis $\{S^{(2)}_{j,i}\}$ of $H^{q,j}$ is orthonormal in $L^2(S^{q-1}, d\omega_2)$. Then the system $\{\chi_{k,l,j,i}\}$ is orthogonal on $D_+ \times S^{q-1}$ with respect to the measure $d\mu = d\rho d\phi d\omega_2$,

$$\int_{S^{p+q}} \chi_{k,l,j,i}(\omega) \chi_{k',l',j',i'}(\omega) \, d\omega = \int_{D_+ \times S^{q-1}} \chi_{k,l,j,i} \chi_{k',l',j',i'} \, d\mu$$

$$= \|\chi_{k,l,j,i}\|^2 \delta_{kk'} \delta_{ll'} \delta_{jj'} \delta_{ii'}.$$ 

The squared $L^2$-norm $\|\chi_{k,l,j,i}\|^2$ is computed to be $(\pi_{k,l,j})^{-1}$ with

$$\pi_{k,l,j} = \frac{(2k - 2l + 2\beta)(k + l + \alpha)(\alpha - \beta)l(2\beta + 1)_{k-l}(\alpha + 1)_k(k - l + 2\beta)_j}{22^j(k - l + 2\beta)(k + \alpha) l!(k - l - j)!((\beta + 1)k(\beta + 1/2)_j},$$

where $\alpha = (p + q - 1)/2$, $\beta = (q - 1)/2$, and $(a)_k$ is defined by $(a)_0 = 1$ and $(a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1) \cdots (a + n - 1)$.

A smooth $\nu$-radial function $\chi$ on $S^{p+q}$ can then be expanded as

$$\chi = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{j=0}^{k-l} \sum_{i=1}^{\dim H^{q,j}} \pi_{k,l,j} a_{k,l,j,i} \chi_{k,l,j,i},$$

where

$$a_{k,l,j,i} = \int_{S^{p+q}} \chi(\omega) \chi_{k,l,j,i}(\omega) \, d\omega.$$
Consider now the eigenvalue equation for the Laplacian on $B$:

$$L_B f = -(\lambda^2 + Q^2/4)f \quad (\lambda \in \mathbb{C}, Q = p/2 + q).$$

Here we can separate variables in geodesic polar coordinates, by looking for the $v$-radial solutions of the form $f(r, \omega) = \phi(r) \chi(\omega)$.

Recall that for $\lambda \in \mathbb{C}, t \in \mathbb{R}, \alpha > \beta > -1/2$, and $k, l \in \mathbb{Z}, k \geq l \geq 0$, one defines the associated Jacobi functions $(\text{see } [20])$:

$$\phi^{(\alpha, \beta)}_{\lambda, k, l}(t) = c(2 \text{sh } t)^{k+l}(2 \text{ch } t)^{k-l} \phi_{\lambda}^{(\alpha+k+l, \beta+k-l)}(t),$$

where $c$ is a normalization constant and $\phi^{(\alpha, \beta)}_{\lambda}$ is a Jacobi function:

$$\phi_{\lambda}^{(\alpha, \beta)}(t) = F\left(\frac{\alpha + \beta + 1 - i\lambda}{2}, \frac{\alpha + \beta + 1 + i\lambda}{2}, \alpha + 1, -\text{sh}^2 t\right)$$

($F(a, b, c, z)$ is the hypergeometric function). The functions $\phi = \phi^{(\alpha, \beta)}_{\lambda, k, l}$ are the unique solutions (up to normalization) of the following equation that are regular at $t = 0$:

$$\left\{ \partial_t^2 + ((2\alpha + 1) \text{cth } t + (2\beta + 1) \text{th } t) \partial_t \right. \right.$$
$$\left. \left. - \frac{(k+l)(k+l+2\alpha)}{\text{sh}^2 t} + \frac{(k-l)(k-l+2\beta)}{\text{ch}^2 t} \right\} \phi = -(\lambda^2 + (\alpha + \beta + 1)^2)\phi.$$

Putting together $(2.4)$, $(2.5)$, $(2.8)$, $(2.21)$, $(2.22)$ and $(2.24)$, we obtain the following result.

**Theorem 2.6.** Let $S = NA \cong B$ be a Damek–Ricci space. The $v$-radial eigenfunctions of the Laplacian, solutions of $(2.23)$ that separate in geodesic polar coordinates and belong to subspaces invariant and irreducible under $\text{SO}(p) \times \text{SO}(q)$, are given by

$$f_{\lambda, k, l, j, i}(b) = f_{\lambda, k, l, j, i}(\text{th}(r/2)\omega) = \phi_{\lambda, k, l}(r)\chi_{k, l, j, i}(\omega),$$

where the $\phi_{\lambda, k, l}$ are the associated Jacobi functions

$$\phi_{\lambda, k, l}(r) = \phi^{(\alpha, \beta)}_{2\lambda, k, l}(r/2)$$
$$= q_{k, l}(\lambda)(2 \text{sh}(r/2))^{k+l}(2 \text{ch}(r/2))^{k-l} \phi_{2\lambda}^{(\alpha+k+l, \beta+k-l)}(r/2).$$

Here $q_{k, l}(\lambda)$ is a normalization constant, the functions $\chi_{k, l, j, i}$ are given by $(2.22)$, and the indices are as follows:

$$k, l \in \mathbb{Z}, \quad k \geq l \geq 0, \quad 0 \leq j \leq k - l, \quad 1 \leq i \leq \dim \mathcal{H}^{q, j},$$

$$\alpha = (p + q - 1)/2, \quad \beta = (q - 1)/2.$$

The functions $f_{\lambda, k, l, j, i}$ are biradial if and only if $j = 0$, in which case they reduce to the biradial eigenfunctions

$$f_{\lambda, k, l}(b) = f_{\lambda, k, l}(\text{th}(r/2)\omega) = \phi_{\lambda, k, l}(r)\chi_{k, l}(\omega)$$
They are radial if and only if \( k = l = j = 0 \), in which case they reduce to the spherical functions \( \phi_\lambda(r) = \phi_{2\lambda}(r/2) \).

2.4. Poisson integral representation. Let \( \mathcal{P}(x, n) \) be the Poisson kernel on \( NA \) given by (see [8])

\[
\mathcal{P}(a_t, n) = c_{p,q} \left( \frac{e^t}{(e^t + \frac{1}{4}|V|^2 + |Z|^2)^Q} \right)
\]

for \( x = a_t = \exp(tH) \in A \), and by

\[
\mathcal{P}(na_t, n') = \mathcal{P}(a_t, n^{-1}n') \quad (n, n' \in N)
\]

for general \( x = na_t \in S \). Define the normalized Poisson kernel with parameter \( \lambda \in \mathbb{C} \) on \( NA \) as the following function on \( NA \times N \) (cf. [2]):

\[
Q_\lambda(x, n) = \frac{\mathcal{P}_\lambda(x, n)}{\mathcal{P}_\lambda(e, n)} \quad (x \in NA, n \in N),
\]

where

\[
\mathcal{P}_\lambda(x, n) = (\mathcal{P}(x, n))^{1/2-i\lambda/Q}.
\]

We define a kernel \( \tilde{Q}_\lambda \) on \( B \) by

\[
\tilde{Q}_\lambda(C(x), C_0(n)) = Q_\lambda(x, n) \quad (x \in NA, n \in N),
\]

that is,

\[
\tilde{Q}_\lambda(b, \omega) = Q_\lambda(C^{-1}(b), C_0^{-1}(\omega)) \quad (b \in B, \omega \in \partial B \setminus \{H\}).
\]

From now on we write \( Q_\lambda(b, \omega) \) in place of \( \tilde{Q}_\lambda(b, \omega) \). The kernel \( Q_\lambda(b, \omega) \) extends to a smooth kernel on \( B \times \partial B \). For example for \( b = C(a_t) = \text{th}(t/2)H \) and \( \omega = (\rho, \phi, \omega_1, \omega_2) \), we have

\[
Q_\lambda(C(a_t), \omega) = |\text{ch}(t/2) - \rho e^{i\phi} \text{sh}(t/2)|^{2i\lambda-Q}
\]

(cf. [6] (5.20))). In particular, for \( \omega = H = (0, 0, 1) \),

\[
Q_\lambda(C(a_t), H) = e^{t(Q/2-i\lambda)}.
\]

For a suitable choice of the constant \( q_{k,l}(\lambda) \) in (2.26) (see below for details), one has the following Poisson integral representation of the associated Jacobi functions (cf. [6] Theorem 5.2) :

\[
\phi_{\lambda,k,l}(t) = f_{\lambda,k,l}(\text{th}(t/2)H) = \int_{S^{p+q}} Q_\lambda(\text{th}(t/2)H, \omega)\chi_{k,l}(\omega) \, d\omega.
\]

This formula extends to the biradial eigenfunctions \( f_{\lambda,k,l}(b) \) at arbitrary points, namely for any \( b \) in \( B \),

\[
f_{\lambda,k,l}(b) = \int_{S^{p+q}} Q_\lambda(b, \omega)\chi_{k,l}(\omega) \, d\omega.
\]
See [5, Theorem 2.2] for a proof of this result involving the Radon transform and the method of “descent” to complex hyperbolic spaces, which was used also in [11, 21] for the radial case ($k = l = 0$). A different proof in the radial case appears in [1, pp. 654–655].

The expression of $q_{k,l}(\lambda)$ is (see [5])

$$q_{k,l}(\lambda) = \frac{(-i\lambda + Q/2)_k (-i\lambda + p/4 + 1/2)_l}{(d/2)_{k+l}}. \tag{2.31}$$

This can also be written as a ratio of $c$-functions, namely

$$q_{k,l}(\lambda) = \frac{c_{\alpha,\beta}(-2\lambda)}{c_{\alpha+k+l,\beta+k-l}(-2\lambda)}, \tag{2.32}$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1-i\lambda} \Gamma(i\lambda) \Gamma(\alpha+1) \Gamma(\frac{i\lambda+\alpha+\beta+1}{2}) \Gamma(\frac{i\lambda+\alpha-\beta+1}{2})}{\Gamma(\frac{\lambda+\alpha+\beta+1}{2}) \Gamma(\frac{\lambda+\alpha-\beta+1}{2})}. \tag{2.33}$$

(See [20, (2.18), (4.15), (8.5), and (8.7)]. For the symmetric case, see [20, (8.13) and the last part of Section 8.1]. See also [13, Theorem 7 and Remark on p. 277].)

Observe that the $c$-function $c(\lambda)$ in (1.3) is precisely $c_{\alpha,\beta}(2\lambda)$ if $\alpha = (p+q-1)/2$ and $\beta = (q-1)/2$, as we continue to assume.

We now have a result similar to (2.30) for the $v$-radial eigenfunctions $f_{\lambda,k,l,j,i}$ in (2.25). The proof can be given along the same lines of [5, Theorem 2.2] for the biradial case ($j = 0$). Since the proof is quite involved already in the biradial case, we shall omit it altogether.

**Theorem 2.7.** Let $q_{k,l}(\lambda)$ in (2.26) be given by (2.31). For all $b \in B$ we have

$$f_{\lambda,k,l,j,i}(b) = \int_{S^{p+q}} \mathcal{Q}(b,\omega) \chi_{k,l,j,i}(\omega) \, d\omega. \tag{2.34}$$

Equivalently, if we define $f_{\lambda,k,l,j,i}(b)$ by (2.34), then (2.25) holds, i.e.,

$$\int_{S^{p+q}} \mathcal{Q}(\text{th}(r/2)\omega,\omega') \chi_{k,l,j,i}(\omega') \, d\omega' = \phi_{\lambda,k,l}(r) \chi_{k,l,j,i}(\omega), \tag{2.35}$$

\[ \forall r \geq 0, \forall \omega \in S^{p+q}. \]

**Remark 2.8.** In the symmetric case, the functions $\chi_{k,l,j,i}$ on $S^{p+q} \simeq K/M$ can be identified with suitable matrix coefficients of a $K$-type $\delta_{k,l}$ containing an $M$-fixed vector. The result (2.34)–(2.35) then follows for this general class of matrix coefficients (not only the $v$-radial ones) by an easy change-of-variable argument, rewriting the integral over $K/M$ as an integral over $K$ (see [16, Lemma 4.2]). This can be interpreted by saying that for a symmetric space $G/K$ of rank one, the subspace $\mathcal{E}_{\lambda,\delta_{k,l}}$ of $K$-finite functions of type $\delta_{k,l}$ in $\mathcal{E}_{\lambda}$ (the smooth eigenfunctions of $L_B$ satisfying (2.23)) is
essentially determined by the single function $\phi_{\lambda,k,l}$ (up to angular functions). Formulas (2.34)–(2.35) generalize this to any Damek–Ricci space, but only for the class of $v$-radial functions.

3. The Helgason Fourier transform in the $v$-radial case

3.1. Fourier series expansion of $f$ and $\tilde{f}$. Let $f$ be a $v$-radial function in $C^\infty_0(B)^{v\text{-rad}}$, with supp$f \subset B_R$. The function $\omega \mapsto f(r, \omega) = f(\text{th}(r/2)\omega)$ can be expanded in the Fourier series

\begin{equation}
  f(r, \omega) = \sum_{k \geq l} \sum_{j} \sum_{i} \pi_{k,l,j} a_{k,l,j,i}(r) \chi_{k,l,j,i}(\omega),
\end{equation}

where the Fourier coefficients

\begin{equation}
  a_{k,l,j,i}(r) = \int_{S^{p+q}} f(r, \omega) \chi_{k,l,j,i}(\omega) d\omega
\end{equation}

are smooth functions of the geodesic distance $r$ supported in $[0, R]$.

Let $\tilde{f}(\lambda, \omega)$ be the Fourier transform of $f$ given by (1.1). By the direct part of Theorem 1.1 (that was proved in [2, Theorem 4.5]), the function $\tilde{f}(\lambda, \omega)$ is holomorphic of uniform exponential type with constant $R$.

Lemma 3.1. Let $f \in C^\infty_0(B)^{v\text{-rad}}$. Then, for each $\lambda \in \mathbb{C}$, the function $\omega \mapsto \tilde{f}(\lambda, \omega)$ is $v$-radial on $S^{p+q}$.

Proof. Consider the normalized Helgason Fourier transform of $f \circ C$ in $S = NA$ given by

\begin{equation}
  \frac{1}{S}(f \circ C)(\lambda, n) = \int_{S} (f \circ C)(x) \mathcal{Q}_{\lambda}(x, n) dx \quad (\lambda \in \mathbb{C}, n \in N).
\end{equation}

Then

\begin{equation}
  \tilde{f}(\lambda, C_0(n)) = \tilde{f} \circ C(\lambda, n),
\end{equation}

and we need to prove that $n \mapsto \tilde{f} \circ C(\lambda, n)$ is $v$-radial. For simplicity, we write $f$ in place of $f \circ C$ in the following. Let

\begin{equation}
  \hat{f}(\lambda, n) = \int_{S} f(x) \mathcal{P}_{\lambda}(x, n) dx
\end{equation}

be the unnormalized Helgason Fourier transform of $f$, so that (cf. (2.28))

\begin{equation}
  \hat{f}(\lambda, n) = \mathcal{P}_{\lambda}(e, n) \tilde{f}(\lambda, n).
\end{equation}

Since $n \mapsto \mathcal{P}_{\lambda}(e, n)$ is biradial (cf. (2.27) and (2.29)), it is enough to prove that $n \mapsto \hat{f}(\lambda, n)$ is $v$-radial. We use the Radon transform to reduce the problem to the case $q = 1$ of complex hyperbolic spaces.

Fix $\omega \in z$ with $|\omega| = 1$, let $z_0 = \mathbb{R}\omega$, and consider the subspaces $\mathfrak{n}_o = v \oplus z_0$ and $\mathfrak{s}_o = \mathfrak{n}_o \oplus a$ of $\mathfrak{s}$, with the scalar product induced from that on $\mathfrak{s}$. Then $\mathfrak{n}_o$ is a Heisenberg-type Lie algebra if one defines the commutator
[V, V']_o = π_o([V, V']), where π_o is the orthogonal projection of 3 onto 3_o. The associated Lie group N_o is the classical Heisenberg group of dimension p + 1. We recall that p is even, so we let p = 2(n − 1) (n ≥ 2). Then the Damek–Ricci space S_o = N_oA can be identified with the complex hyperbolic space H^n(ℂ) ∼ G_o/K_o, where G_o = SU(n, 1) ∼ N_oAK_o and K_o = S(U(n) × U(1)).

The centralizer M_o of A in K_o is connected and acts trivially on the center 3_o. In matrix form we have

\[
M_o = \left\{ m = \begin{pmatrix} u & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & u \end{pmatrix} : U \in U(n-1), u \in U(1), u^2 \det U = 1 \right\}.
\]

The map \( m \mapsto U \) is a 2 : 1 homomorphism of \( M_o \) onto \( U(n-1) \). The action of \( M_o \) on \( v \simeq ℝ^{2(n-1)} \simeq ℂ^{n-1} \) is given by \( m \cdot V = u^{-1}UV \), and \( M_o \) acts transitively on the unit sphere \( S^{p-1} = S^{2n-3} \) in \( v \).

The group \( \tilde{M}_o \) of orthogonal automorphisms of \( N_oA \), i.e., of the \( H \)-type Lie algebra \( n_o = v \oplus 3_o \), is given by

\[
\tilde{M}_o = M_o \cup (σM_o),
\]

where \( σ \) is the automorphism of \( v \oplus 3_o \) defined by \( σ = (ψ, -\text{Id}) \), with \( ψ \) any orthogonal isomorphism of \( v \) that anticommutes with \( J_ω \). For example if \( v = \text{span}\{X_i, Y_i\}_{i=1}^{n-1} \) with \( [X_i, Y_j] = δ_{ij}ω \), we can take \( ψ(X_i, Y_i) = (-Y_i, -X_i) \).

We denote by \( \mathcal{P}^{(o)}(x_o, n_o) \) the Poisson kernel on \( S_o \), and define

\[
\mathcal{P}_λ^{(o)}(x_o, n_o) = (\mathcal{P}^{(o)}(x_o, n_o))^{1/2 - iλ/n} \quad (λ ∈ ℂ, x_o ∈ S_o, n_o ∈ N_o).
\]

Given \( g ∈ C_0^∞(S) \) and \( ω ∈ S^{q-1} (q > 1) \), we define the Radon transform of \( g \) by

\[
R_ω g(V, η, t) = e^{(1-q)t/2} \int_{ω^⊥} g(V, ηω + Ž, t) \ dǔ,
\]

where \( η, t ∈ ℝ \) and \( ω^⊥ \) is the orthogonal complement of \( ω \) in \( 3 \), with Lebesgue measure \( dǔ \). The function \( R_ω g \) is in \( C_0^∞(S_o) \). Note that \( R_ω g = 0 \) for all \( ω ∈ S^{q-1} \) implies \( g = 0 \). (This corrects a wrong statement in [5] p. 440 about the injectivity of the maps \( R_ω \) for \( ω \) fixed.) We keep the same notation \( R_ω g \) for the Radon transform applied to functions \( g \) on \( N \). In this case the variable \( t \) is absent and the factor \( e^{(1-q)t/2} \) is omitted. We observe that \( g \) is \( v \)-radial in \( S \) (resp. \( N \)) if and only if \( R_ω g \) is \( v \)-radial in \( S_o \) (resp. \( N_o \)) for every \( ω ∈ 3 \cap S^{p+q} \).

Consider now the Radon transform in \( N \) of the function \( n \mapsto \hat{f}(λ, n) \). It can be shown that this is well defined and that it is related to the Helgason Fourier transform in \( S_o = N_oA \) of the function \( R_ω f \).
Indeed by [24] Proposition 5.1] we have, for all \( \omega \in S^{q-1} \),
\[
(3.3) \quad (\mathcal{R}_\omega f(\lambda, \cdot))(n_o) = c_q B_{p,q}(\lambda)(\mathcal{P}_\lambda^{(o)}(e, n_o)(\mathcal{R}_\omega f)(\lambda, n_o)
= c_q B_{p,q}(\lambda)(\mathcal{R}_\omega f)(\lambda, n_o) \quad (n_o \in N_o),
\]
where \( c_q \) is a constant depending only on \( q \), and \( B_{p,q}(\lambda) \) is the
meromorphic function
\[
B_{p,q}(\lambda) = \frac{\Gamma((q-1)/2)\Gamma(p/4+1/2-i\lambda)}{\Gamma(Q/2-i\lambda)}.
\]
We can describe (3.3) as follows: the Radon transform in \( N \) of the (un-
normalized) Helgason Fourier transform of \( f \) in \( NA \) is proportional to the
(normalized) Helgason Fourier transform in \( N_o A \) of the Radon transform
of \( f \) in \( NA \).

Since \( f \) is \( \mathfrak{v} \)-radial in \( S \), the function \( g = \mathcal{R}_\omega f \) is \( \mathfrak{v} \)-radial in \( S_o \), and we
need to prove that \( n_o \mapsto \hat{g}(\lambda, n_o) \) is \( \mathfrak{v} \)-radial in \( N_o \). Since \( M_o \) is transitive on
the unit sphere \( S^{p-1} \) in \( \mathfrak{v}_o = \mathfrak{v} \), it is enough to show that
\[
\hat{g}(\lambda, (m \cdot V, \eta)) = \hat{g}(\lambda, (V, \eta)), \quad \forall m \in M_o, \ V \in \mathfrak{v}, \ \eta \in \mathbb{R}.
\]

Now
\[
\hat{g}(\lambda, (V, \eta)) = \int_{\mathfrak{v} \times \mathbb{R} \times \mathbb{R}} g(\tilde{V}, \tilde{\eta}, \tilde{t}) \mathcal{P}_\lambda^{(o)}(\tilde{V}, \tilde{\eta}, \tilde{t}, (V, \eta)) d\mu(\tilde{V}, \tilde{\eta}, \tilde{t}),
\]
where \( d\mu(\tilde{V}, \tilde{\eta}, \tilde{t}) = e^{-nt}d\tilde{V}d\tilde{\eta}d\tilde{t} \) and
\[
\mathcal{P}_\lambda^{(o)}(\tilde{V}, \tilde{\eta}, \tilde{t}, (V, \eta)) = \left( \frac{(n-1)!}{\pi^n} \right)^{1/2-i\lambda/n} \left( \frac{e^t}{(e^t + \frac{1}{4}|V - \tilde{V}|^2)^2 + |(\eta - \tilde{\eta})\omega - \frac{1}{2}|V, \tilde{V}|^2} \right)^{n/2-i\lambda}.
\]
Since \( g \) is \( \mathfrak{v} \)-radial, we have, for all \( m \in M_o \),
\[
\hat{g}(\lambda, (m \cdot V, \eta)) = \int_{\mathfrak{v} \times \mathbb{R} \times \mathbb{R}} g(\tilde{V}, \tilde{\eta}, \tilde{t}) \mathcal{P}_\lambda^{(o)}(\tilde{V}, \tilde{\eta}, \tilde{t}, (m \cdot V, \eta)) d\mu(\tilde{V}, \tilde{\eta}, \tilde{t})
= \int_{\mathfrak{v} \times \mathbb{R} \times \mathbb{R}} g(m^{-1} \cdot \tilde{V}, \tilde{\eta}, \tilde{t}) \mathcal{P}_\lambda^{(o)}(\tilde{V}, \tilde{\eta}, \tilde{t}, (m \cdot V, \eta)) d\mu(\tilde{V}, \tilde{\eta}, \tilde{t})
= \int_{\mathfrak{v} \times \mathbb{R} \times \mathbb{R}} g(\tilde{V}, \tilde{\eta}, \tilde{t}) \mathcal{P}_\lambda^{(o)}(m \cdot \tilde{V}, \tilde{\eta}, \tilde{t}, (m \cdot V, \eta)) d\mu(\tilde{V}, \tilde{\eta}, \tilde{t}).
\]
But
\[
\mathcal{P}_\lambda^{(o)}(m \cdot \tilde{V}, \tilde{\eta}, \tilde{t}, (m \cdot V, \eta)) = \left( \frac{(n-1)!}{\pi^n} \right)^{1/2-i\lambda/n} \left( \frac{e^t}{(e^t + \frac{1}{4}|V - \tilde{V}|^2)^2 + |(\eta - \tilde{\eta})\omega - \frac{1}{2}m \cdot [V, \tilde{V}]|^2} \right)^{n/2-i\lambda} = \mathcal{P}_\lambda^{(o)}(\tilde{V}, \tilde{\eta}, \tilde{t}, (V, \eta)),
\]
since $M_o$ is trivial on $\mathfrak{z}_o$. It follows that $\hat{g}(\lambda, \cdot) = (R_\omega f)(\lambda, \cdot)$ is $\nu$-radial in $N_o$, so $(R_\omega \hat{f}(\lambda, \cdot))$ is $\nu$-radial in $N_o$ for all $\omega \in S^{q-1}$, and finally $\hat{f}(\lambda, \cdot)$ is $\nu$-radial in $N$. ■

By this lemma, we have the Fourier expansion

\begin{equation}
\tilde{f}(\lambda, \omega) = \sum_{k \geq 1} \sum_{j} \sum_{i} \pi_{k,l,j} b_{k,l,j,i}(\lambda) \chi_{k,l,j,i}(\omega), \tag{3.4}
\end{equation}

where the coefficients

\[ b_{k,l,j,i}(\lambda) = \int_{S^{p+q}} \frac{\tilde{f}(\lambda, \omega)}{\omega} \chi_{k,l,j,i}(\omega) d\omega \]

are holomorphic functions of $\lambda$ of exponential type $R$.

The functions $a_{k,l,j,i}(r)$ and $b_{k,l,j,i}(\lambda)$ are related as follows. Let $\alpha' = \alpha + k + l$ and $\beta' = \beta + k - l$. Define $c_{\alpha', \beta'}(\lambda)$ by (2.33). Let $J^{(\alpha', \beta')}(g)$ be the Jacobi transform of $g \in C^\infty_R(\mathbb{R})_{\text{even}}$, defined by (see [20])

\[ J^{(\alpha', \beta')}(g)(\lambda) = \int_0^\infty g(t) \phi_\lambda^{(\alpha', \beta')}(t)(2 \text{sh } t)^{2\alpha'+1}(2 \text{ch } t)^{2\beta'+1} dt, \]

with inverse (see [20] Theorem 2.3])

\[ (J^{(\alpha', \beta')})^{-1}(h)(t) = \frac{1}{2\pi} \int_0^\infty h(\lambda) \phi_\lambda^{(\alpha', \beta')}(t)|c_{\alpha', \beta'}(\lambda)|^{-2} d\lambda. \]

**Proposition 3.2.** We have

\begin{equation}
\begin{aligned}
&b_{k,l,j,i}(\lambda) = \frac{\text{vol}(S^{p+q})}{2^q} q_{k,l}(\lambda) \\
&\quad \times \int_0^\infty a_{k,l,j,i}(r) \phi_\lambda^{(\alpha', \beta')}(r) \left(2 \text{sh } \frac{r}{2}\right)^{p+q+k+l} \left(2 \text{ch } \frac{r}{2}\right)^{q+k-l} dr \\
&\quad = \frac{\text{vol}(S^{p+q})}{2^{q-1}} q_{k,l}(\lambda) J^{(\alpha', \beta')}(\frac{a_{k,l,j,i}(2t)}{(2 \text{sh } t)^{k+l}(2 \text{ch } t)^{k-l}})(2\lambda),
\end{aligned} \tag{3.5}
\end{equation}

where $q_{k,l}(\lambda)$ is given by (2.31), and conversely

\begin{equation}
\begin{aligned}
&a_{k,l,j,i}(r) \\
&\quad = \frac{c_{p,q}}{2\pi} \left(2 \text{sh } \frac{r}{2}\right)^{k+l} \left(2 \text{ch } \frac{r}{2}\right)^{k-l} \int_0^\infty q_{k,l}(-\lambda) \phi_\lambda^{(\alpha', \beta')}(r) \left(\frac{r}{2}\right)^{k+l} b_{k,l,j,i}(\lambda) d\mu(\lambda) \\
&\quad = \frac{2^{q-1}}{\text{vol}(S^{p+q})} \left(2 \text{sh } \frac{r}{2}\right)^{k+l} \left(2 \text{ch } \frac{r}{2}\right)^{k-l} (J^{(\alpha', \beta')})^{-1}(\frac{b_{k,l,j,i}(\lambda'/2)}{q_{k,l}(\lambda'/2)}) \left(\frac{r}{2}\right),
\end{aligned} \tag{3.6}
\end{equation}
(3.9) \[
\frac{2^q}{\text{vol}(S^{p+q})} \left( \frac{2 \text{sh} r}{2} \right)^{k+l} \left( \frac{2 \text{ch} r}{2} \right)^{k-l} \\
\times \frac{1}{2\pi} \int_0^\infty \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)} \phi_{2\lambda}^{(\alpha',\beta')}(\frac{r}{2}) \left| c_{\alpha',\beta'}(2\lambda) \right|^{-2} d\lambda.
\]

**Proof.** The proof uses (2.3), (2.25), (2.26), (2.32) and (2.34), and it is entirely analogous to [5, Proposition 3.2].

**Remark 3.3.** Since the function \( r \mapsto a_{k,l,j,i}(r) \) is in \( C_\infty([0, \infty)) \) and \( \lambda \mapsto \phi_{2\lambda}^{(\alpha+k+l,\beta+k-l)}(r/2) \) is entire, the integral in (3.5) is an entire function of \( \lambda \). Since \( \lambda \mapsto q_{k,l}(\lambda) \) is a polynomial (cf. (2.31)), both functions \( \lambda \mapsto b_{k,l,j,i}(\lambda) \) and \( \lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda) \) are entire of exponential type \( R \) (see [15, Lemma 5.13, p. 288]), the second one being even.

We also observe from (3.6)–(3.8) and the Paley–Wiener theorem for the Jacobi transform [20, Theorem 2.1] that, for all \( k,l,j,i \), the map \( a_{k,l,j,i}(r) \mapsto b_{k,l,j,i}(\lambda) \) is a bijection from the space of smooth functions \( a_{k,l,j,i} \) on \([0, \infty)\) compactly supported in \([0, R]\) and such that the function

\[
r \mapsto a_{k,l,j,i}(r) \frac{(\text{sh}(r/2))^{k+l}(\text{ch}(r/2))^{k-l}}{C_\infty(R)_{\text{even}}},
\]

extends to \( C_\infty(R)_{\text{even}} \), i.e., \( a_{k,l,j,i} \in (\text{sh}(r/2))^{k+l}(\text{ch}(r/2))^{k-l}C_\infty(R)_{\text{even}} \), onto the space of holomorphic functions \( b_{k,l,j,i} \) on \( \mathbb{C} \) such that the function \( \lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda) \) is in \( \text{PW}_R(\mathbb{C})_{\text{even}} \), i.e., \( b_{k,l,j,i} \in q_{k,l}(\cdot) \text{PW}_R(\mathbb{C})_{\text{even}} \). Here \( \text{PW}_R(\mathbb{C})_{\text{even}} \) is the space of even entire functions on \( \mathbb{C} \) of exponential type \( R \). The proof of the converse part of Theorem 1.1 in the \( \nu \)-radial case (see below) implies that the Fourier coefficients \( b_{k,l,j,i}(\lambda) \) of \( \psi(\lambda, \omega) \) satisfying (1.4), (1.5) and \( \nu \)-radial in \( \omega \) are indeed in \( q_{k,l}(\cdot) \text{PW}_R(\mathbb{C})_{\text{even}} \).

**3.2. The \( \nu \)-radial Paley–Wiener theorem.** We now prove the \( \nu \)-radial case of Theorem 1.1.

**Theorem 3.4.** The Fourier transform \( f(b) \mapsto \tilde{f}(\lambda, \omega) \) is a bijection from \( C_0^\infty(B)^{\nu \text{-rad}} \) onto the set of holomorphic functions \( \psi(\lambda, \omega) \) of uniform exponential type, \( \nu \)-radial in \( \omega \), and satisfying the condition

\[
(3.10) \int_{\partial B} Q_\lambda(b, \omega) \psi(\lambda, \omega) d\omega = \int_{\partial B} Q_\lambda(b, \omega) \psi(-\lambda, \omega) d\omega
\]

for any \( b \in B \) and \( \lambda \in \mathbb{C} \). Moreover, \( \tilde{f} \) satisfies (1.4) if and only if \( f \) has support in the closed ball \( B_R = \{ b \in B : d(b, C(e)) \leq R \} \).

**Proof.** In view of Lemma 3.1 and [2, Theorem 4.5], we only need to prove the converse part, in particular the onto statement. We proceed as in [5, Theorem 3.3] for the biradial case.
Let \( \psi(\lambda, \omega) \) be a holomorphic function of uniform exponential type such that \((1.4)\) and \((3.10)\) hold, and such that the map \( \omega \mapsto \psi(\lambda, \omega) \) is \( \upsilon \)-radial on \( S^p+q \) for all \( \lambda \in \mathbb{C} \). Define \( b_{k,l,j,i}(\lambda) = \int_{S^p+q} \psi(\lambda, \omega) x_{k,l,j,i}(\omega) \, d\omega \), so that (cf. \((3.4)\))

\[
\psi(\lambda, \omega) = \sum_{k \geq l} \sum_{j} \sum_{i} \pi_{k,l,j} b_{k,l,j,i}(\lambda) \chi_{k,l,j,i}(\omega).
\]

Then \( \lambda \mapsto b_{k,l,j,i}(\lambda) \) is holomorphic of exponential type \( R \). Using \((3.11)\) in the integral \( \int_{\partial B} Q_{-\lambda}(b, \omega) \psi(\lambda, \omega) \, d\omega \) we get, by \((2.34)\),

\[
\int_{\partial B} Q_{-\lambda}(b, \omega) \psi(\lambda, \omega) \, d\omega = \sum_{k \geq l} \sum_{j} \sum_{i} \pi_{k,l,j} b_{k,l,j,i}(\lambda) f_{-\lambda,k,l,j,i}(b).
\]

From \((3.10)\), \((2.25)\)–\((2.26)\) and \((3.12)\), it follows that the function \( \lambda \mapsto b_{k,l,j,i}(\lambda) q_{k,l}(\lambda) \) is even. Thus so is \( \lambda \mapsto \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)} \).

Define \( f \) by the inversion formula \((1.2)\):

\[
f(b) = \frac{c_{p,q}}{2\pi} \int_{0}^{\infty} \int_{S^p+q} Q_{-\lambda}(b, \omega) \psi(\lambda, \omega) \, d\omega \, d\mu(\lambda).
\]

Then \( f \) is smooth and \( \upsilon \)-radial on \( B \) (by \((3.12)\)). Define \( a_{k,l,j,i}(r) \) by \((3.1)\)–\((3.2)\). Then we get again \((3.7)\) and \((3.9)\).

By \((2.31)\) we see that the function \( \lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda) \) in \((3.9)\) has no poles for \( \text{Im} \lambda \geq 0 \). Then, using the exponential type conditions for the functions \( b_{k,l,j,i}(\lambda) \) and the well known asymptotic estimates for the functions \( \phi_{2\lambda}(\alpha',\beta')(r/2) \) in \((3.9)\), we can prove that \( a_{k,l,j,i}(r) = 0 \) for \( r > R \).

In more detail, we use

\[
\phi_{2\lambda}(\alpha',\beta')(r/2) = c_{\alpha',\beta'}(2\lambda) \phi_{2\lambda}(\alpha',\beta')(r/2) + c_{\alpha',\beta'}(-2\lambda) \phi_{-2\lambda}(\alpha',\beta')(r/2),
\]

where the function \( \lambda \mapsto \Phi_{\lambda}(\alpha',\beta')(t) \) is holomorphic in \( \mathbb{C} \setminus \{-i\mathbb{N}\} \) for each \( t > 0 \) (cf. [12 Proposition 1]), to rewrite the integral in \((3.9)\) for \( r > 0 \) as

\[
F(r) = \int_{-\infty}^{\infty} \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)} \frac{\phi_{2\lambda}(\alpha',\beta')(r/2)}{c_{\alpha',\beta'}(-2\lambda)} \, d\lambda.
\]

Now \( (c_{\alpha',\beta'}(-2\lambda))^{-1} \) has no poles for \( \text{Im} \lambda \geq 0 \), and the integrand is holomorphic for \( \text{Im} \lambda \geq 0 \). Thus we obtain, by Cauchy’s theorem,

\[
F(r) = \int_{-\infty}^{\infty} \frac{b_{k,l,j,i}(\xi + i\eta)}{q_{k,l}(\xi + i\eta)} \frac{\phi_{2(\xi+i\eta)}(r/2)}{c_{\alpha',\beta'}(-2(\xi + i\eta))} \, d\xi
\]

for any \( \eta \geq 0 \).
We now use the estimates for \( \Phi_\lambda^{(\alpha',\beta')}(t) \) and \( c_{\alpha',\beta'}(\lambda) \) given in [12, Theorem 2] (see also [20 (6.4) and (6.5)]), namely for any \( c > 0 \) there exists \( K_1 > 0 \) such that for all \( t \geq c \) and all \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda \geq 0 \),
\[
|\Phi_\lambda^{(\alpha',\beta')}(t)| \leq K_1 e^{-(\text{Im} \lambda + \alpha' + \beta' + 1)t}.
\]
Moreover, there exists \( K_2 > 0 \) such that for all \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda \geq 0 \),
\[
|c_{\alpha',\beta'}(-\lambda)|^{-1} \leq K_2 (1 + |\lambda|)^{\alpha' + 1/2}.
\]
Using the exponential type conditions for \( b_{k,l,j,i}(\lambda) \) and the inequality
\[
\left| \frac{1}{q_{k,l}(\xi + i\eta)} \right| \leq \frac{(d/2)_{k+l}}{(Q/2)_k (p/4 + 1/2)_l} \quad (\forall \xi, \forall \eta \geq 0),
\]
which is easily proved from [2.31], we find (as in [12 p. 157])
\[
|F(r)| \leq K e^{-(2\eta + Q + 2k)r/2} \int_{-\infty}^{\infty} |b_{k,l,j,i}(\xi + i\eta)|(1 + 2|\xi + i\eta|)^{(p+q)/2+k+l} d\xi
\]
\[
\leq K' e^{-(2\eta + Q + 2k)r/2} e^{\eta R} \leq K' e^{\eta (R-r)}
\]
for suitable constants \( K, K' \). Since this holds for all \( \eta \geq 0 \), we get \( F(r) = 0 \) for \( r > R \), as claimed. It follows from (3.1) that \( f(b) = f(r,\omega) \) has support in \( B_R \). The proof is completed by showing that the Fourier transform of \( f \) is just \( \hat{f}(\lambda, \omega) = \psi(\lambda, \omega) \). In fact, the Fourier coefficients of \( \omega \mapsto \hat{f}(\lambda, \omega) \) are just the \( b_{k,l,j,i}(\lambda) \).

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Roberto Camporesi
Dipartimento di Scienze Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
E-mail: camporesi@polito.it