

Monotonic rearrangements of functions with small mean oscillation

by

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Abstract. We obtain sharp bounds for the monotonic rearrangement operator from “dyadic-type” classes to “continuous” ones; in particular, for the BMO space and Muckenhoupt classes. The idea is to connect the problem with a simple geometric construction named α -extension.

1. Introduction. The BMO space has many nice properties. One of them is that the monotonic rearrangement operator is bounded on this space (see [5, 1]). In other words, the inequality

$$\|f^*\|_{\text{BMO}} \leq c\|f\|_{\text{BMO}}$$

holds true with some constant c . It is not hard to see that $c \geq 1$. Soon it was noticed that $c = 1$ when the dimension of the underlying space is one (see [7]).

The same boundedness is also present when the BMO space is replaced by its relatives: the Muckenhoupt classes (see [16, 17]) or the Gehring classes (see [4]). And again, if the underlying space is an interval, then the constant in the corresponding inequality equals one, i.e.

$$(1) \quad [f^*]_{A_p} \leq [f]_{A_p}$$

(see [2] for A_1 and [8] for the general case). On the other hand, see [2] for an example showing that inequality (1) does not hold in higher dimensions.

In [14] two of the present authors developed a setting that unifies the three cases (and, moreover, covers a more general situation described in [6] for a related extremal problem) and gave a proof of an inequality that

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generalizes (1) to the setting of that paper. The proof relied on passing to a certain class of martingales.

It seems a difficult problem to calculate the norm of the monotonic rearrangement operator in higher dimensions. Not being able to solve it, we deal with a problem that is a step to it: we calculate the aforementioned norm for the case when the BMO space (or any other class of similar nature) is dyadic. Dyadic classes seem to be a step towards the higher-dimensional case not only in our problem, but, for example, in the problem of finding the sharp constant in the John–Nirenberg inequality (see [13]). For numerous applications of monotonic rearrangements in various estimations see [3, 9, 10, 11, 12] and references therein.

We briefly formulate the corollaries of our abstract considerations that concern the classical cases of the BMO space and the Muckenhoupt class. Let $n \in \mathbb{N}$ and let \mathcal{D} be the set of all dyadic subcubes of $[0, 1]^n$. Consider the dyadic BMO space on $[0, 1]^n$ with the quadratic seminorm:

$$(2) \quad \text{BMO}^d([0, 1]^n) = \left\{ \varphi \in L^1([0, 1]^n) : \|\varphi\|_{\text{BMO}^d([0, 1]^n)}^2 = \sup_{I \in \mathcal{D}} (\langle \varphi^2 \rangle_I - \langle \varphi \rangle_I^2) < \infty \right\}.$$

If the above supremum is taken over all subcubes of $[0, 1]^n$, then we obtain the usual (“continuous”) BMO quadratic seminorm (in this paper, we consider only quadratic seminorms on BMO).

The *monotonic rearrangement* of a function φ from this space is a monotone (say, non-increasing) function φ^* on $[0, 1]$ with the same distribution as the function itself. The (non-linear) operator $\varphi \mapsto \varphi^*$ is called the *monotonic rearrangement operator*.

COROLLARY 1. *The monotonic rearrangement operator acts from the space $\text{BMO}^d([0, 1]^n)$ to $\text{BMO}([0, 1])$ with norm $(1 + 2^n)/(2^{1+n/2})$.*

Let us consider the dyadic Muckenhoupt class A_2^d on $[0, 1]^n$. A positive function φ on $[0, 1]^n$ is in A_2^d with constant Q if $\langle \varphi \rangle_I \langle \varphi^{-1} \rangle_I \leq Q$ for any $I \in \mathcal{D}$. Define

$$A_{2,Q}^d([0, 1]^n) = \left\{ \varphi \in L^1([0, 1]^n) : \sup_{I \in \mathcal{D}} (\langle \varphi \rangle_I \langle \varphi^{-1} \rangle_I) \leq Q \right\}.$$

Again, $A_{2,Q}([0, 1])$ stands for the set of functions on $[0, 1]$ for which a similar supremum taken over all subintervals of $[0, 1]$ does not exceed Q .

COROLLARY 2. *The monotonic rearrangement operator acts from $A_{2,Q}^d([0, 1]^n)$ to $A_{2,Q'}([0, 1])$ if and only if*

$$Q' \geq \frac{Q(2^n + 1)^2 - (2^n - 1)^2}{2^{n+2}}.$$

A similar statement can be obtained for the A_p class, $p \neq 2$, but the statement is not nice (it involves many solutions of implicit algebraic equations), so we do not dwell on this. An interested reader may calculate the sharp constant using Propositions 11 and 12 below.

Another motivation to write (and, we hope, to read) this paper is to demonstrate the strength of the martingale technique developed in [14]. More or less, the proof consists of an accurate manipulation with the definitions, a very simple geometric lemma from [13], and a martingale embedding theorem from [14].

In the next section, we state the main theorem in an abstract form (using the terminology from [6, 13]) and prove it. The last section consists of the examples of specific classes (in particular, it contains the proof of Corollaries 1 and 2).

2. Preliminaries and main theorem. Fix unbounded open strictly convex domains $\Omega_0, \Omega_1 \subset \mathbb{R}^2$ satisfying the following conditions:

- $\text{cl } \Omega_1 \subset \Omega_0$;
- (cone property) any ray lying inside Ω_0 can be shifted to lie inside Ω_1 .

Set $\Omega = \text{cl}(\Omega_0 \setminus \Omega_1)$. In what follows we will consider only domains of this type and call them lenses. The set $\partial\Omega_0$ is called the *fixed boundary* of the lens Ω and is denoted by $\partial_{\text{fixed}}\Omega$. The rest of the boundary, $\partial\Omega_1 = \partial\Omega \setminus \partial_{\text{fixed}}\Omega$, is called the *free boundary* of Ω and is denoted by $\partial_{\text{free}}\Omega$.

Recall a definition from [6].

DEFINITION 3. Let $J \subset \mathbb{R}$ be an interval and $\varphi: J \rightarrow \partial\Omega_0$ be a summable function. We say that φ belongs to the class \mathbf{A}_Ω if $\langle \varphi \rangle_I \in \Omega$ for every subinterval $I \subset J$.

Since the domain Ω_0 is unbounded and strictly convex, there exists at least one straight line $\ell \subset \mathbb{R}^2$ such that the orthogonal projection P_ℓ onto this line is injective on $\partial\Omega_0$. A function $\varphi \in \mathbf{A}_\Omega$ is called *monotone* if the composition $P_\ell \circ \varphi$ is monotone. The function $\varphi^*: J \rightarrow \partial\Omega_0$ is the *monotonic rearrangement* of φ if it is monotone (say, non-increasing) and has the same distribution as φ .

Let (X, \mathfrak{A}, μ) be a standard probability space. For any integrable vector-valued function φ on (X, \mathfrak{A}, μ) and for any subset $\omega \in \mathfrak{A}$ of positive measure we denote by $\langle \varphi \rangle_\omega$ the average of φ over ω , that is, $\langle \varphi \rangle_\omega = (1/\mu(\omega)) \int_\omega \varphi d\mu$.

We consider increasing discrete time filtrations $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ with finite algebras \mathcal{F}_n , starting with the trivial algebra $\mathcal{F}_0 = \{\emptyset, X\}$, such that \mathfrak{A} is generated by $\{\mathcal{F}_n\}_{n \geq 0} \bmod 0$. By Lévy's zero-one law, if $f \in L^1(X, \mathfrak{A}, \mu)$ and $\{F_n\}_{n \geq 0}$ is the martingale generated by f and \mathcal{F} (i.e. $F_n|_\omega = \langle f \rangle_\omega$ for any atom ω of the algebra \mathcal{F}_n), then F_n converges to f almost everywhere

and in $L^1(X, \mathfrak{A}, \mu)$. We write $A(\mathcal{F})$ for the set of all atoms of \mathcal{F}_n , $n \geq 0$. For a fixed filtration \mathcal{F} we introduce the class $\mathbf{A}_\Omega^\mathcal{F}$ of functions $\varphi: X \rightarrow \partial\Omega_0$ such that $\langle \varphi \rangle_\omega \in \Omega$ for every $\omega \in A(\mathcal{F})$.

DEFINITION 4. The lens $\tilde{\Omega}$ is called an *extension* of Ω if $\Omega \subset \tilde{\Omega}$ and $\partial_{\text{fixed}}\Omega = \partial_{\text{fixed}}\tilde{\Omega}$ ⁽¹⁾.

Let $\alpha \in (0, 1)$. The lens $\tilde{\Omega}$ is called an α -*extension* of Ω if for any x, y in Ω such that the straight line segment $[z, y]$ with $z = \alpha x + (1 - \alpha)y$ is contained in Ω , we have $[x, y] \subset \tilde{\Omega}$.

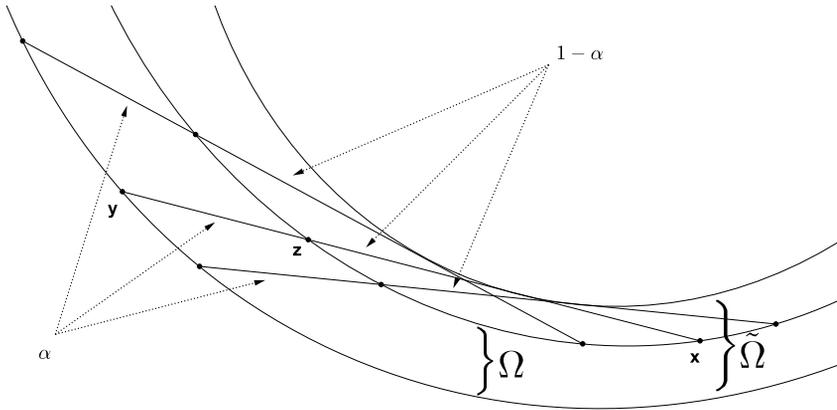


Fig. 1. Illustration to Definition 4: $\tilde{\Omega}$ is an α -extension of Ω

Note that if $z = \beta x + (1 - \beta)y$, $\beta > \alpha$, and $[z, y]$ is contained in Ω , then $[x, y]$ is contained in any α -extension of Ω , i.e. every α -extension is simultaneously a β -extension for any $\beta > \alpha$.

DEFINITION 5. We say that \mathcal{F} is an α -*filtration* if for any atoms $\omega_n \in \mathcal{F}_n$ and $\omega_{n+1} \in \mathcal{F}_{n+1}$ such that $\omega_{n+1} \subset \omega_n$ we have $\mu(\omega_{n+1}) \geq \alpha\mu(\omega_n)$. We refer to the pair (ω_n, ω_{n+1}) as above as a *parent* and a *child*. We say that \mathcal{F} is a *binary filtration* if every parent in \mathcal{F} has at most two children.

DEFINITION 6. Let \mathcal{F} be a binary filtration and let $\{F_n\}_{n \geq 0}$ be the martingale generated by a function $\varphi \in \mathbf{A}_\Omega^\mathcal{F}$ and \mathcal{F} . We say that $\{F_n\}_{n \geq 0}$ is an α -*martingale* if for all $\omega \in A(\mathcal{F})$ the following condition is fulfilled: if ω', ω'' are children of ω and the straight line segment $[\langle \varphi \rangle_{\omega'}, \langle \varphi \rangle_\omega]$ is not contained in Ω , then $|\langle \varphi \rangle_{\omega'} - \langle \varphi \rangle_\omega| \geq \alpha|\langle \varphi \rangle_{\omega'} - \langle \varphi \rangle_{\omega''}|$. For any filtration \mathcal{F} we say that $\{F_n\}_{n \geq 0}$ is an α -*martingale* if there exists a binary filtration $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_m\}_{m \geq 0}$ such that the martingale $\{\tilde{F}_m\}_{m \geq 0}$ generated by φ and $\tilde{\mathcal{F}}$ is an α -martingale and $F_n = \tilde{F}_{m_n}$ for some increasing sequence $\{m_n\}_n$.

⁽¹⁾ Note that this definition slightly differs from the one given in [14].

In other words, for an α -filtration, the condition $\mu(\omega_{n+1}) \geq \alpha\mu(\omega_n)$ should be fulfilled for all children of all subsets, whereas for α -martingales this condition is needed only in some cases. This relation is clarified in the simple lemma below.

LEMMA 7. *Let $\alpha \in (0, 1)$ and let \mathcal{F} be an α -filtration. Then the martingale generated by any $\varphi \in \mathbf{A}_\Omega^\mathcal{F}$ and \mathcal{F} is an α -martingale.*

Proof. If \mathcal{F} is binary, then the statement is evident. Indeed, for any $w \in A(\mathcal{F})$ we get

$$|\langle \varphi \rangle_w - \langle \varphi \rangle_{w'}| = \frac{\mu(\omega'')}{\mu(\omega)} |\langle \varphi \rangle_{w'} - \langle \varphi \rangle_{w''}|$$

and

$$|\langle \varphi \rangle_w - \langle \varphi \rangle_{w''}| = \frac{\mu(\omega')}{\mu(\omega)} |\langle \varphi \rangle_{w'} - \langle \varphi \rangle_{w''}|$$

from the identity $\mu(\omega)\langle \varphi \rangle_w = \mu(\omega')\langle \varphi \rangle_{w'} + \mu(\omega'')\langle \varphi \rangle_{w''}$ for the children ω', ω'' of ω . By Definition 5, both coefficients are not less than α .

So, for an arbitrary α -filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ we need to construct a binary one, $\tilde{\mathcal{F}}$, such that \mathcal{F} is a subfiltration of $\tilde{\mathcal{F}}$. We use induction. We start with the trivial algebra $\tilde{\mathcal{F}}_0 = \mathcal{F}_0$. Suppose that the sequence $\{\tilde{\mathcal{F}}_j\}_{j \leq m}$ has already been defined and has two properties: for some n we have $\tilde{\mathcal{F}}_m \subset \mathcal{F}_{n+1}$ and $\{\mathcal{F}_i\}_{i \leq n}$ is a subfiltration of $\{\tilde{\mathcal{F}}_j\}_{j \leq m}$; and $\langle \varphi \rangle_\omega \in \Omega$ for any atom ω of any algebra $\tilde{\mathcal{F}}_i, i \leq m$. We need to define the next algebra $\tilde{\mathcal{F}}_{m+1}$. We fix some atom ω of $\tilde{\mathcal{F}}_m$ which is not an atom of \mathcal{F}_{n+1} and take an arbitrary atom ω' of \mathcal{F}_{n+1} such that $\omega' \subset \omega$ and $\langle \varphi \rangle_{\omega''} \in \Omega$ where $\omega'' = \omega \setminus \omega'$.

The following simple argument (see [13, Lemma 2.3]) yields the existence of ω' . Let us enumerate all atoms of \mathcal{F}_{n+1} which are subsets of ω as $\omega'_1, \dots, \omega'_k$; and let $\omega''_j = \omega \setminus \omega'_j$ for $j = 1, \dots, k$. Since $\omega = \bigcup_{i=1}^k \omega'_i$ and $\omega'_i \cap \omega'_j = \emptyset$, we have

$$\mu(\omega)\langle \varphi \rangle_\omega = \sum_{i=1}^k \mu(\omega'_i)\langle \varphi \rangle_{\omega'_i} = \frac{1}{k-1} \sum_{i=1}^k \mu(\omega''_i)\langle \varphi \rangle_{\omega''_i}.$$

All the points $\langle \varphi \rangle_{\omega''_i}$ belong to $\text{cl}(\Omega_0)$ by convexity of this set. If we assume that none of the points $\langle \varphi \rangle_{\omega''_i}$ belongs to Ω , then they are all in Ω_1 . Since Ω_1 is convex, their convex combination $\langle \varphi \rangle_\omega$ should be in Ω_1 ; however, this is not the case. Thus, $\langle \varphi \rangle_{\omega''_i} \in \Omega$ for some i and we can take $\omega' = \omega'_i, \omega'' = \omega''_i$.

We now define $\tilde{\mathcal{F}}_{m+1}$ by replacing the atom ω of $\tilde{\mathcal{F}}_m$ by two new atoms ω', ω'' . We have thus made an induction step. Since the algebra \mathcal{F}_{n+1} is finite, after a finite number of steps we obtain $\tilde{\mathcal{F}}_{m_{n+1}} = \mathcal{F}_{n+1}$ for some m_{n+1} , and then we continue the induction with n increased by one.

Clearly, the resulting binary filtration $\tilde{\mathcal{F}}$ is an α -filtration. ■

DEFINITION 8. We say that a positive number α is *admissible* for the filtration \mathcal{F} if there exist $n \geq 0$, an atom $\omega \in \mathcal{F}_n$, and $\omega' \in \mathcal{F}_{n+1}$ such that $\omega' \subset \omega$ and $\mu(\omega') = \alpha\mu(\omega)$.

THEOREM 9. *Suppose that $\tilde{\Omega}$ is an extension of Ω . For a fixed filtration \mathcal{F} and a number α admissible for this filtration, the two assertions below are equivalent.*

- (1) *For every $\varphi \in \mathbf{A}_{\tilde{\Omega}}^{\mathcal{F}}$ such that the martingale $\{F_n\}_{n \geq 0}$ generated by φ and \mathcal{F} is an α -martingale, the monotonic rearrangement φ^* belongs to $\mathbf{A}_{\tilde{\Omega}}$.*
- (2) *The domain $\tilde{\Omega}$ is an α -extension of Ω .*

Proof. First we prove (2) \Rightarrow (1). Let $\varphi \in \mathbf{A}_{\tilde{\Omega}}^{\mathcal{F}}$ and let $\{F_n\}_{n \geq 0}$ be the α -martingale generated by φ and \mathcal{F} . Let $\{\tilde{F}_m\}$ be a binary α -martingale such that $\tilde{F}_{m_n} = F_n$. By Definition 4, for any $\omega \in A(\tilde{\mathcal{F}})$ and its children ω', ω'' , the whole segment $[\langle \varphi \rangle_{\omega'}, \langle \varphi \rangle_{\omega''}]$ is in $\tilde{\Omega}$ ⁽²⁾. By [14, Theorem 3.4] the monotonic rearrangement of $F_\infty = \lim \tilde{F}_m$ (that is, the function φ^*) belongs to $\mathbf{A}_{\tilde{\Omega}}$. The implication is proved.

Assume (2) is not fulfilled. Then we can find $x, y, z \in \Omega$ such that $z = \alpha x + (1 - \alpha)y$ with $[y, z] \subset \Omega$, but $[x, y] \not\subset \tilde{\Omega}$, i.e. $[x, z] \not\subset \tilde{\Omega}$. Without loss of generality we can suppose that $y \in \partial_{\text{fixed}}\Omega$. Indeed, if $y \notin \partial_{\text{fixed}}\Omega$, we can shift the points y and z to the new positions y' and z' along the line containing the segment $[x, y]$ so that $y' \in \partial_{\text{fixed}}\Omega$, $z' = \alpha x + (1 - \alpha)y'$, and $[y', z'] \subset \Omega$, but $[x, y'] \not\subset \tilde{\Omega}$. Take $a, b \in \partial_{\text{fixed}}\Omega$ such that $x \in [a, b] \subset \Omega$. Since the part of Ω between the chord $[a, b]$ and the corresponding arc of $\partial_{\text{fixed}}\Omega$ is a convex set, the point y cannot belong to this arc. Therefore, we can take for a the endpoint of the arc that is between y and b (see Figure 2).

Now we take a subset $\omega \in A(\mathcal{F})$ such that $\mu(\omega')/\mu(\omega) = 1 - \alpha$, where ω' is a union of several children of ω . Such ω and ω' do exist because α is admissible for \mathcal{F} . Define a function φ on X as follows. Set $\varphi = y$ on $\omega' \cup (X \setminus \omega)$, and on $\omega'' = \omega \setminus \omega'$ let φ take only values a and b in such a proportion that $\langle \varphi \rangle_{\omega''} = x$.

Let us check that $\varphi \in \mathbf{A}_{\tilde{\Omega}}^{\mathcal{F}}$. Let $\omega_1 \in A(\mathcal{F})$. If $\omega_1 \cap \omega = \emptyset$, then $\langle \varphi \rangle_{\omega_1} = y \in \Omega$. The average over ω is

$$\langle \varphi \rangle_\omega = \frac{\mu(\omega')}{\mu(\omega)} \langle \varphi \rangle_{\omega'} + \frac{\mu(\omega'')}{\mu(\omega)} \langle \varphi \rangle_{\omega''} = (1 - \alpha)y + \alpha x = z.$$

If $\omega_1 \supset \omega$, then

$$\langle \varphi \rangle_{\omega_1} = \frac{\mu(\omega)}{\mu(\omega_1)} z + \frac{\mu(\omega_1) - \mu(\omega)}{\mu(\omega_1)} y \in [y, z] \subset \Omega.$$

⁽²⁾ In the terminology of [14], the martingale $\{\tilde{F}_m\}$ is an $\tilde{\Omega}$ -martingale.

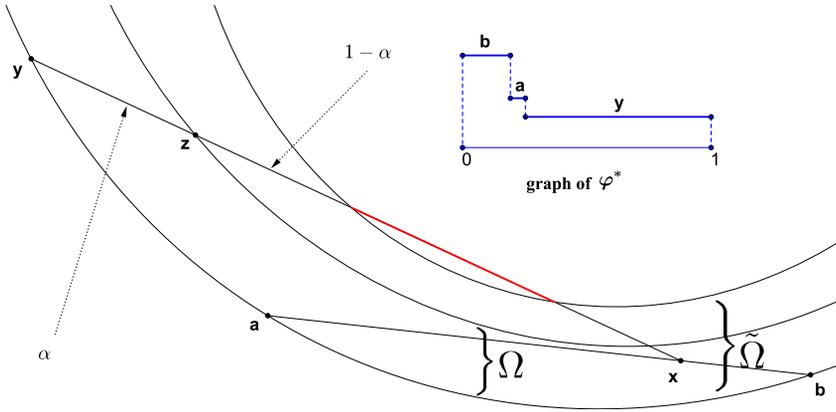


Fig. 2. Construction of φ

If $\omega_1 \subset \omega$, then either $\omega_1 \subset \omega'$ and $\langle \varphi \rangle_{\omega_1} = y \in \Omega$, or $\omega_1 \subset \omega''$ and $\langle \varphi \rangle_{\omega_1} \in [a, b] \subset \Omega$.

Let us check that $\varphi^* \notin \mathbf{A}_{\tilde{\Omega}}$. Without loss of generality we may assume that φ^* is defined on $[0, 1]$ and it is a step function taking three values b, a , and y . For definiteness we assume that $\varphi^*(0) = b$. Then $\langle \varphi^* \rangle_{[0, \mu(\omega'')]} = x$ and $\langle \varphi^* \rangle_{[0, t]}$ runs through the whole segment $[x, z]$ when t varies in $[\mu(\omega''), 1]$. Since by our assumption $[x, z] \not\subset \tilde{\Omega}$, we conclude that $\varphi^* \notin \mathbf{A}_{\tilde{\Omega}}$. Therefore, (1) \Rightarrow (2). ■

3. Examples. In this section we consider several examples of α -extensions of lenses which correspond to famous classes of functions.

In order to be an α -extension of Ω , $\tilde{\Omega}$ should contain all segments $[x, y]$ such that $x, y \in \Omega$ and $[z, y] \subset \Omega$, where $z = \alpha x + (1 - \alpha)y$. It is almost obvious that this property is satisfied if and only if it is fulfilled for such segments with $y \in \partial_{\text{fixed}}\Omega$ and $x, z \in \partial_{\text{free}}\Omega$ (in what follows we will call such segments *higher*). Therefore, the construction of the minimal α -extension is quite simple with α and Ω at hand.

3.1. BMO space. It is well known that the lenses

$$\Omega_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2\}, \quad \varepsilon > 0,$$

correspond to the BMO space. One can easily check that a function φ lies in the BMO space on some interval and has quadratic seminorm at most ε if and only if $(\varphi, \varphi^2) \in \mathbf{A}_{\Omega_\varepsilon}$.

PROPOSITION 10. *Let $\varepsilon, \alpha > 0$. The lens $\tilde{\Omega}$ is an α -extension of Ω_ε if and only if $\tilde{\Omega} \supset \Omega_{\varepsilon'}$ with $\varepsilon' = \frac{1+\alpha}{2\sqrt{\alpha}}\varepsilon$.*

Proof. Assume first that $y \in \partial_{\text{fixed}}\Omega_\varepsilon$, $x, z \in \partial_{\text{free}}\Omega_\varepsilon$, where $z = \alpha x + (1 - \alpha)y$, $[z, y] \subset \Omega_\varepsilon$ and the higher segment $[x, y]$ is horizontal. In that case $y = (y_1, y_1^2)$, $z = (z_1, y_1^2)$, $x = (x_1, y_1^2)$. Then $z_1 = -x_1$ and $z_1^2 + \varepsilon^2 = y_1^2$. But

$$y_1 = \frac{z_1 - \alpha x_1}{1 - \alpha} = \frac{1 + \alpha}{1 - \alpha} z_1,$$

therefore

$$z_1^2 \frac{(1 + \alpha)^2}{(1 - \alpha)^2} = z_1^2 + \varepsilon^2, \quad z_1^2 = \frac{(1 - \alpha)^2}{4\alpha} \varepsilon^2,$$

and

$$y_1^2 = \frac{(1 + \alpha)^2}{4\alpha} \varepsilon^2 = \varepsilon'^2.$$

Hence, the point $(0, \varepsilon'^2)$ lies on $[x, y]$ and in $\tilde{\Omega}$. Moreover, $[x, y] \subset \Omega_{\varepsilon'}$.

The lens Ω_ε is invariant under the affine transformations $\text{Aff}_t: (u_1, u_2) \mapsto (u_1 + t, u_2 + 2u_1 t + t^2)$, $t \in \mathbb{R}$, so $\tilde{\Omega}$ should contain all the images of the point $(0, \varepsilon'^2)$ under these maps, that is, $\{(t, t^2 + \varepsilon'^2)\}_{t \in \mathbb{R}}$. Thus $\Omega_{\varepsilon'} \subset \tilde{\Omega}$. Moreover, $\Omega_{\varepsilon'}$ is invariant under these maps as well and contains the horizontal higher segment, thus it contains all the higher segments, because they are nothing but the images of the appropriate horizontal higher segment under these affine maps. This proves that $\Omega_{\varepsilon'}$ is an α -extension of Ω_ε . ■

Proof of Corollary 1. Consider the filtration $\mathcal{F} = \{\mathcal{F}_k\}_{k \geq 0}$ on the probability space $[0, 1]^n$, where \mathcal{F}_k is the algebra generated by the family of cubes $\{I \in \mathcal{D}: |I| = 2^{-nk}\}$. First, note that it is a 2^{-n} -filtration. Second, note that $\|\varphi\|_{\text{BMO}^d([0,1]^n)} \leq \varepsilon$ if and only if $(\varphi, \varphi^2) \in \mathbf{A}_{\Omega_\varepsilon}^{\mathcal{F}}$. Theorem 9 states that the monotonic rearrangement operator acts from $\mathbf{A}_{\Omega_\varepsilon}^{\mathcal{F}}$ to $\mathbf{A}_{\tilde{\Omega}}$ if and only if $\tilde{\Omega}$ is a 2^{-n} -extension of Ω_ε , which, by Proposition 10, holds exactly when $\tilde{\Omega} \supset \Omega_{\varepsilon'}$ with $\varepsilon' = \frac{1+2^n}{2^{1+n/2}} \varepsilon$. ■

3.2. A_{p_1, p_2} classes. Consider the lenses

$$\Omega_C^q = \{(x_1, x_2): x_1, x_2 > 0, x_1^q \leq x_2 \leq Cx_1^q\}, \quad q \in \mathbb{R} \setminus \{0\}, C > 1.$$

They are closely related to the so-called A_{p_1, p_2} classes (see [15]), as we will see later.

We are going to find the minimal α -extension of the lens Ω_C^q . In what follows we assume $\partial_{\text{fixed}}\tilde{\Omega} = \partial_{\text{fixed}}\Omega_C^q$. We consider several cases.

PROPOSITION 11. *Let $q > 1$. Suppose that $\alpha > 1 - C^{-1/(q-1)}$. Then $\tilde{\Omega}$ is an α -extension of Ω_C^q if and only if $\tilde{\Omega} \supset \Omega_{C'}^q$, with*

$$(3) \quad C' = \frac{(1 - Ca^q)^q (q - 1)^{q-1}}{(1 - a)(a - Ca^q)^{q-1} q^q},$$

where a is the smallest of the two roots of the equation

$$(4) \quad C(\alpha a + (1 - \alpha))^q = \alpha Ca^q + (1 - \alpha).$$

If $\alpha \leq 1 - C^{-1/(q-1)}$, then the set $\{(x_1, x_2) : x_1 > 0, x_2 \geq x_1^q\}$ is the minimal α -extension of Ω_C^q .

Proof. First, we note that a segment $[x, y]$ is the higher one if and only if

$$(5) \quad C(\alpha x_1 + (1 - \alpha)y_1)^q = \alpha Cx_1^q + (1 - \alpha)y_1^q,$$

where $x = (x_1, Cx_1^q) \in \partial_{\text{free}}\Omega_C^q$, $y = (y_1, y_1^q) \in \partial_{\text{fixed}}\Omega_C^q$ and $z = \alpha x + (1 - \alpha)y \in \partial_{\text{free}}\Omega_C^q$. Since (5) is homogeneous, $a = x_1/y_1$ satisfies (4).

If $\alpha > 1 - C^{-1/(q-1)}$, then (4) has exactly two positive roots; one is greater than 1, and the other smaller. These two roots correspond to the higher segments with $x_1 > y_1$ and $x_1 < y_1$. If $\alpha \leq 1 - C^{-1/(q-1)}$, then (4) has only one root, which is greater than one. This means that there are no higher segments with $x_1 < y_1$. In that case any α -extension of Ω_C^q should contain the union of all segments $[x, y]$ such that $x_1 < y_1$ and $x \in \Omega_C^q$ and $[y, z] \subset \Omega_C^q$ for $z = \alpha x + (1 - \alpha)y$, which coincides with the set $\{(x_1, x_2) : x_1 > 0, x_2 \geq x_1^q\}$.

Let now $\alpha > 1 - C^{-1/(q-1)}$. The lens Ω_C^q is invariant under the affine transformations $\text{Aff}_t : (u_1, u_2) \mapsto (tu_1, t^q u_2)$, $t > 0$, each of which preserves the property of a segment to be higher. Thus, the minimal α -extension of Ω_C^q should be invariant under these transformations as well, and therefore should coincide with $\Omega_{C'}^q$, for some $C' > C$. For each higher segment $[x, y]$ we find a point (t_1, t_2) on it such that $t_2 t_1^{-q}$ is maximal. This maximal value is exactly the C' defined by (3). ■

Arguing in the same way one can obtain the following proposition.

PROPOSITION 12. *Let $q \leq -1$. The lens $\tilde{\Omega}$ is an α -extension of Ω_C^q if and only if $\tilde{\Omega} \supset \Omega_{C'}^q$, with*

$$(6) \quad C' = \frac{(a - Ca^q)^{1-q}(-q)^{-q}}{(a - 1)(1 - Ca^q)^{-q}(1 - q)^{1-q}},$$

where a is the larger of the two roots of equation (4).

Consider now $q \in (0, 1)$. In order to survey α -extensions of Ω_C^q , consider the affine transformation $T : (u, v) \mapsto (v/C, u)$. Then $T(\Omega_C^q) = \Omega_{C'}^{q'}$, where $q' = 1/q$. The lens $\tilde{\Omega}$ is an α -extension of Ω_C^q if and only if $T(\tilde{\Omega})$ is an α -extension of $\Omega_{C'}^{q'}$; but $q' > 1$, so one can use Proposition 11 to verify this property.

For $q \in (-1, 0)$ one can use the symmetry $T : (u, v) \mapsto (v, u)$ to reduce the question about an α -extension of Ω_C^q to the question about $\Omega_{C'}^{q'}$ with $q' = 1/q$ and use Proposition 12.

We have finished the description of α -extensions of the lenses Ω_C^q and we are ready to connect them with the A_{p_1, p_2} classes, where $p_1 > p_2$. Recall that a positive function φ is in A_{p_1, p_2} on an interval J with constant Q if

$\langle \varphi^{p_1} \rangle_I^{1/p_1} \langle \varphi^{p_2} \rangle_I^{-1/p_2} \leq Q$ for any subinterval $I \subset J$. Note that for $p_2 > 0$ this property is equivalent to the function $((Q^{-1}\varphi)^{p_1}, \varphi^{p_2})$ being in $\mathbf{A}_{\Omega_C^q}$, where $C = Q^{p_2}$, $q = p_2/p_1$, and for $p_2 < 0$ it is equivalent to the function $(\varphi^{p_1}, \varphi^{p_2})$ being in $\mathbf{A}_{\Omega_C^q}$, where $C = Q^{-p_2}$, $q = p_2/p_1$. Thus, the question about the monotonic rearrangement operator for dyadic-type A_{p_1, p_2} classes can be investigated via α -extensions of the corresponding lenses Ω_C^q . Unfortunately, for general p_1 and p_2 it does not seem possible to give a short answer for this question, but for the special case $p_1 = 1$ and $p_2 = -1$ (which corresponds to the A_2 class) we can follow the above procedure and obtain a short answer similar to Corollary 1 (note that it proves Corollary 2).

COROLLARY 13. *For any $\alpha \in (0, 1)$ and any $C \geq 1$ the lens $\tilde{\Omega}$ is an α -extension of Ω_C^{-1} if and only if $\tilde{\Omega} \supset \Omega_{C'}^{-1}$, where*

$$C' = \frac{C(\alpha + 1)^2 - (\alpha - 1)^2}{4\alpha}.$$

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