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Proper subspaces and compatibility

by

Esteban Andruchow (Los Polvorines and Buenos Aires),
Eduardo Chiumiento (La Plata and Buenos Aires) and
María Eugenia Di Iorio y Lucero (Los Polvorines and Buenos Aires)

Abstract. Let $\mathcal{E}$ be a Banach space contained in a Hilbert space $\mathcal{L}$. Assume that the inclusion is continuous with dense range. Following the terminology of Gohberg and Zambickiĭ, we say that a bounded operator on $\mathcal{E}$ is a proper operator if it admits an adjoint with respect to the inner product of $\mathcal{L}$. A proper operator which is self-adjoint with respect to the inner product of $\mathcal{L}$ is called symmetrizable. By a proper subspace $\mathcal{S}$ we mean a closed subspace of $\mathcal{E}$ which is the range of a proper projection. Furthermore, if there exists a symmetrizable projection onto $\mathcal{S}$, then $\mathcal{S}$ belongs to a well-known class of subspaces called compatible subspaces. We find equivalent conditions to describe proper subspaces. Then we prove a necessary and sufficient condition for a proper subspace to be compatible. The existence of non-compatible proper subspaces is related to spectral properties of symmetrizable operators. Each proper subspace $\mathcal{S}$ has a supplement $\mathcal{T}$ which is also a proper subspace. We give a characterization of the compatibility of both subspaces $\mathcal{S}$ and $\mathcal{T}$. Several examples are provided that illustrate different situations between proper and compatible subspaces.

1. Introduction. Let $\mathcal{E}$ be a Banach space space which is continuously and densely included in a Hilbert space $\mathcal{L}$. A bounded operator on $\mathcal{E}$ is a *proper operator* if it admits an adjoint with respect to the inner product of $\mathcal{L}$. This definition goes back to Gohberg and Zambickiĭ [25], and it gives a simple condition under which they obtained several results on operators in spaces with two norms. In this context, we introduce the following class of subspaces: a subspace $\mathcal{S}$ of $\mathcal{E}$ is called a *proper subspace* if it is the range of a proper projection. If, in addition, the proper projection is self-adjoint with respect to the inner product of $\mathcal{L}$, then $\mathcal{S}$ is called a *compatible subspace*. The aim of the present work is to study proper subspaces and their relation to compatible subspaces.

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The notion of compatible subspace has been studied in recent years. It was in the paper [16] by Corach, Maestripieri and Stojanoff that the theory of compatibility was introduced; it was then studied systematically in [18, 19]. The usual setting to study problems concerning compatibility differs from our context. One has a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ and a positive semidefinite operator $A \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$. Then a bounded sesquilinear form can be defined by $\langle f, g\rangle_A = \langle Af, g\rangle_{\mathcal{H}}$, where $f, g \in \mathcal{H}$. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, the set of A-self-adjoint projections with range $\mathcal{S}$ is given by

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{B}(\mathcal{H}) : Q^2 = Q,\ AQ = Q^*A,\ R(Q) = \mathcal{S}\}.$$

The subspace $\mathcal{S}$ is compatible if $\mathcal{P}(A, \mathcal{S})$ is not empty. When A is an injective operator, this is a special case of the setting described in the first paragraph, where $\mathcal{E} = \mathcal{H}$ and $\mathcal{L}$ the completion of $\mathcal{H}$ with respect to the norm induced by the inner product defined by A. In this case, if the set $\mathcal{P}(A, \mathcal{S})$ is not empty, then it is a singleton. We remark that a definition of compatible subspace without assuming that $\mathcal{E}$ is a Hilbert space was already considered in [20, Remark 5.8], but it was not studied in further works.

It is interesting to note that compatible subspaces can be found in much earlier literature. Sard used an equivalent definition to give an operator-theoretic approach to problems in approximation theory (see [32, 15]). On the other hand, Hassi and Nordström [26] found conditions that guarantee the existence and uniqueness of self-adjoint projections with respect to an Hermitian form. More recently, the notion of compatibility has been related to different topics such as signal processing [23, 24], frame theory [6], de Branges complementation theory [1, 13, 20], sampling theory [5, 33] and abstract splines [17, 8, 12, 21].

Let us describe the contents of this paper. In Section 2 we establish notation and give the necessary background on proper operators. In Section 3 we prove elementary properties of proper subspaces. The set of all proper operators is an involutive Banach algebra, and thus proper subspaces are ranges of projections in a Banach algebra. We find equivalent conditions to describe proper subspaces in Theorem 3.7. One of these conditions asserts that a closed subspace $\mathcal{S}$ of $\mathcal{E}$ is a proper subspace if and only if there is another closed subspace $\mathcal{T}$ of $\mathcal{E}$ satisfying

$$\mathcal{S} \dotplus \mathcal{T} = (\mathcal{S}^{\perp} \cap \mathcal{E}) \dotplus (\mathcal{T}^{\perp} \cap \mathcal{E}) = \mathcal{E},$$

where the orthogonal complement is considered with respect to $\mathcal{L}$. This kind of supplements $\mathcal{T}$, which are also proper subspaces, will be called *proper companions* of $\mathcal{S}$.

We address the question of when a proper subspace is a compatible subspace in Section 4. Both notions coincide if the subspace has finite codimension, but they are different in general, as we shall see in concrete examples. In Theorem 4.8 we obtain a criterion for a proper subspace to be compatible. Let $\mathcal{S}$ be a proper subspace and $\mathcal{T}$ a proper companion of $\mathcal{S}$. Then the projection $P_{\mathcal{S}//\mathcal{T}}$ with range $\mathcal{S}$ and nullspace $\mathcal{T}$ is well-defined and continuous on $\mathcal{E}$. Our criterion basically asserts that $\mathcal{S}$ is compatible if and only if the operator

$$C_{\mathcal{S},\mathcal{T}} = P_{\mathcal{S}//\mathcal{T}} + P^{+}_{\mathcal{S}//\mathcal{T}} - I$$

has range equal to $\mathcal{T} \dotplus (\mathcal{T} \cap \mathcal{E})$. Here the superscript $+$ stands for the restriction to $\mathcal{E}$ of the adjoint in $\mathcal{L}$.

We prove in Theorem 4.9 different conditions equivalent to the compatibility of both a proper subspace and a fixed proper companion. Among other conditions, we find that a proper subspace $\mathcal{S}$ and a proper companion $\mathcal{T}$ are compatible subspaces exactly when the operator $C_{\mathcal{S},\mathcal{T}}$ is invertible on $\mathcal{E}$. Next we examine when the compatibility of a proper companion $\mathcal{T}$ implies the compatibility of another proper companion $\mathcal{T}_1$. As we shall show by examples in the next section, this does not hold in general. However, it holds in some special cases, for instance if the proper projections associated to a pair of companions are closed enough in a metric induced by the algebra of proper operators (Corollary 4.10). As a curious fact, we point out that the existence of non-compatible proper subspaces is closely related to spectral properties of symmetrizable operators (Theorem 4.11 and Corollary 4.14).

In Section 5 we give several examples. In particular, we provide examples of non-compatible proper subspaces and we show that the compatibility of one proper companion does not imply compatibility of all proper companions (see Subsection 5.1). We give two examples of compatible subspaces if $\mathcal{E}$ is the space of trace class operators and $\mathcal{L}$ is the space of Hilbert–Schmidt operators. Finally, we exhibit examples of proper invertible operators.

2. Preliminaries and notation. Let $(\mathcal{E}, \| \ \|_{\mathcal{E}})$ be a Banach space contained in a Hilbert space $(\mathcal{L}, \| \ \|_{\mathcal{L}})$. Denote by $\langle \, , \, \rangle$ the inner product of $\mathcal{L}$. We assume that the inclusion $\mathcal{E} \hookrightarrow \mathcal{L}$ is continuous with dense range. In order to simplify some computations, we further suppose that $\|f\|_{\mathcal{L}} \leq \|f\|_{\mathcal{E}}$ for all $f \in \mathcal{E}$.

REMARK 2.1. The Banach space $\mathcal{E}$ is continuously and densely contained in some Hilbert space $\mathcal{L}$ if and only if there exists a bounded conjugate-linear operator $J : \mathcal{E} \to \mathcal{E}^*$ such that $(Jf)(f) > 0$ for all $f \in \mathcal{E}$, $f \neq 0$. If this condition is fulfilled, $\mathcal{L}$ is the Hilbert space obtained as the completion of $\mathcal{E}$ with respect to the norm $\|f\|_{\mathcal{L}} = ((Jf)(f))^{1/2}$ and the inner product is given by the continuous extension of $\langle f,g\rangle = (Jg)(f)$, where $f,g \in \mathcal{E}$.

2.1. Subspaces and projections. Let $\mathcal{B}(\mathcal{E})$ denote the algebra of bounded linear operators on $\mathcal{E}$. The range of an operator $T \in \mathcal{B}(\mathcal{E})$ is denoted by $R(T)$, and its nullspace by $N(T)$. An operator $T \in \mathcal{B}(\mathcal{E})$ is a projection if $T^2 = T$. We denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{E}$. If these subspaces are closed and $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{E}$, the oblique projection $P_{\mathcal{S}//\mathcal{T}}$ onto $\mathcal{S}$ along $\mathcal{T}$ is the bounded projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. Given a subset $\mathcal{S}$ of $\mathcal{E}$, $\mathcal{S}^\perp$ is the usual orthogonal complement as a subspace of $\mathcal{L}$, that is,

$$\mathcal{S}^\perp = \{f \in \mathcal{L} : \langle f,g\rangle = 0, \forall\, g \in \mathcal{S}\}.$$

It is easily seen that $\mathcal{S}^\perp \cap \mathcal{E}$ is a closed subspace of $\mathcal{E}$. Moreover, we have $\mathcal{S}^\perp \cap \mathcal{E} = J^{-1}(\mathcal{S}^\circ)$, where J is the map defined in Remark 2.1 and $\mathcal{S}^\circ$ is the annihilator of $\mathcal{S}$.

Throughout, the closure $\bar{\mathcal{S}}$ of a closed subspace $\mathcal{S}$ of $\mathcal{E}$ is understood with respect to the topology of $\mathcal{L}$. The operator $P_{\bar{\mathcal{S}}}$ is the orthogonal projection onto $\bar{\mathcal{S}}$. The notation for the oblique projection $P_{\mathcal{S}//\mathcal{T}}$, introduced before for subspaces of $\mathcal{E}$, will also be used with the same meaning when $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{L}$. It will be useful to state here the following result on projections.

Theorem 2.2 (Ando [2]). *Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of a Hilbert space $\mathcal{L}$. If $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{L}$, then the operator $P_{\mathcal{S}} - P_{\mathcal{T}}$ is invertible and*

$$(P_{\mathcal{S}} - P_{\mathcal{T}})^{-1} = P_{\mathcal{S}//\mathcal{T}} + P^*_{\mathcal{S}//\mathcal{T}} - I, \qquad P_{\mathcal{S}//\mathcal{T}} = P_{\mathcal{S}}(P_{\mathcal{S}} - P_{\mathcal{T}})^{-1}.$$

We remark that the first formula above was first proved by Buckholtz [14].

2.2. Proper operators. In this subsection, we describe the basic properties of proper operators proved in [25]. An operator $T \in \mathcal{B}(\mathcal{E})$ is *proper* if for every $f \in \mathcal{E}$, there is a vector $g \in \mathcal{E}$ such that $\langle Th,f\rangle = \langle h,g\rangle$ for all $h \in \mathcal{E}$. This allows us to define $T^+ f = g$, and by the closed graph theorem, it follows that $T^+ \in \mathcal{B}(\mathcal{E})$.

Theorem 2.3 (Gohberg–Zambickiĭ [25]). *Let T be a proper operator. Then:*

(i) *T has a bounded extension $\bar{T}$ on $\mathcal{L}$. The usual operator norms of $\bar{T} \in \mathcal{B}(\mathcal{L})$ and $T \in \mathcal{B}(\mathcal{E})$ are related by*

$$\|\bar{T}\|_{\mathcal{B}(\mathcal{L})} \le \min\{\|T^+T\|^{1/2}_{\mathcal{B}(\mathcal{E})}, \|TT^+\|^{1/2}_{\mathcal{B}(\mathcal{E})}\}.$$

(ii) *If $\sigma_{\mathcal{E}}(T)$ and $\sigma_{\mathcal{L}}(\bar{T})$ denote the spectrum of T on $\mathcal{E}$ and the spectrum of $\bar{T}$ on $\mathcal{L}$, respectively, then*

$$\sigma_{\mathcal{L}}(\bar{T}) \subseteq \sigma_{\mathcal{E}}(T) \cup \overline{\sigma_{\mathcal{E}}(T^+)},$$

where the last bar indicates complex conjugation.

(iii) *If T is a compact operator on $\mathcal{E}$, then T is a compact operator on $\mathcal{L}$. Moreover, $\sigma_{\mathcal{L}}(\bar{T}) = \sigma_{\mathcal{E}}(T)$ and the eigenspaces of T in $\mathcal{E}$ and $\mathcal{L}$ corresponding to the non-zero eigenvalues coincide.*

When T is a proper operator, it turns out that $T^+ = \bar{T}^*|_{\mathcal{E}}$, where the last adjoint is the adjoint of $\bar{T}$ with respect to the inner product of $\mathcal{L}$. A *symmetrizable operator* is a proper operator T such that $T^+ = T$. This class of operators was studied independently by Dieudonné [22], Krein [29] and Lax [30].

We denote by $\mathfrak{P}$ the set of all proper operators. It is not difficult to see that $\mathfrak{P}$ is not closed in $\mathcal{B}(\mathcal{E})$. However, $\mathfrak{P}$ becomes an involutive unital Banach algebra with the norm defined by

$$\|T\|_{\mathfrak{P}} := \|T\|_{\mathcal{B}(\mathcal{E})} + \|T^+\|_{\mathcal{B}(\mathcal{E})}.$$

Remark 2.4. We shall need the notion of proper operators between different spaces. Consider a bounded operator $T : \mathcal{E}_1 \to \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are Banach spaces which are continuously and densely included in Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Then T is *proper* if for every $f \in \mathcal{E}_2$, there is a vector $g \in \mathcal{E}_1$ such that $\langle Th, f\rangle_2 = \langle h, g\rangle_1$ for all $h \in \mathcal{E}_1$. Here $\langle\ ,\ \rangle_i$ denotes the inner product of $\mathcal{L}_i$, $i = 1, 2$. The set of all proper operators from $\mathcal{E}_1$ to $\mathcal{E}_2$ is denoted by $\mathfrak{P}(\mathcal{E}_1, \mathcal{E}_2)$. As before, the operator $T^+ : \mathcal{E}_2 \to \mathcal{E}_1$ defined by $T^+ f = g$ is bounded and $T^+ \in \mathfrak{P}(\mathcal{E}_2, \mathcal{E}_1)$.

Remark 2.5. There are three different notions of groups of invertible operators: the group of invertible operators on $\mathcal{E}$, the group of invertible operators on $\mathcal{L}$ and the group of invertible operators of the algebra $\mathfrak{P}$. These groups are denoted by $Gl(\mathcal{E})$, $Gl(\mathcal{L})$ and $\mathfrak{P}^\times$, respectively. If T is a proper operator, we write $\sigma_{\mathcal{E}}(T)$ for the spectrum in the Banach space $\mathcal{E}$, $\sigma_{\mathcal{L}}(\bar{T})$ for the spectrum of the continuous extension $\bar{T}$ in the Hilbert space $\mathcal{L}$ and $\sigma_{\mathfrak{P}}(T)$ for the spectrum in the Banach algebra $\mathfrak{P}$. The relation between the first two notions of spectrum is stated in Theorem 2.3. Since T is invertible in the algebra $\mathfrak{P}$ if and only if T and T^+ are invertible on $\mathcal{E}$, one can see that

$$\sigma_{\mathfrak{P}}(T) = \sigma_{\mathcal{E}}(T) \cup \overline{\sigma_{\mathcal{E}}(T^+)}.$$

There are examples which show that the inclusion $\sigma_{\mathcal{E}}(T) \subseteq \sigma_{\mathfrak{P}}(T)$ may be strict.

3. Proper subspaces. A closed subspace $\mathcal{S} \subseteq \mathcal{E}$ is called *proper* if there exists a proper projection Q such that $R(Q) = \mathcal{S}$. In this section we prove basic facts on proper subspaces. We start with two examples.

EXAMPLE 3.1. Let $\mathcal{S}$ be a finite-dimensional subspace of $\mathcal{E}$. We can construct a basis $\{f_1, \dots, f_n\}$ of $\mathcal{S}$ satisfying $\langle f_i, f_j\rangle = \delta_{ij}$. In fact, note that we only have to apply the Gram–Schmidt process to any basis of $\mathcal{S}$ to get a new basis with this property. On the other hand, it is well known that as an operator on $\mathcal{L}$, any projection Q onto $\mathcal{S}$ can be written as

$$Q = \sum_{i=1}^{n} \langle \,\cdot\, , h_i\rangle f_i, \tag{3.1}$$

where $\{h_1, \dots, h_n\}$ is a basis of $N(Q)^{\perp}$ satisfying $\langle f_i, h_j\rangle = \delta_{ij}$. If we restrict this projection to $\mathcal{E}$, we find a characterization of an arbitrary proper projection Q with finite-dimensional range: Q is a proper projection if and only if $h_1, \dots, h_n \in \mathcal{E}$. Furthermore, by choosing $h_i = f_i$, $i = 1, \dots, n$, we have proved that any finite-dimensional subspace is proper.

EXAMPLE 3.2. Let T be a proper operator. Let λ be an isolated point in $\sigma_{\mathfrak{P}}(T)$. For the different notions of spectrum of a proper operator see Remark 2.5. Let $B_\epsilon(\lambda)$ be the open ball of radius ϵ centered at λ. Assume that $B_{2\epsilon}(\lambda) \cap \sigma_{\mathfrak{P}}(T) = \{\lambda\}$. In particular, this implies that $B_{2\epsilon}(\lambda) \cap \sigma_{\mathcal{E}}(T) = \{\lambda\}$ and $B_{2\epsilon}(\bar{\lambda}) \cap \sigma_{\mathcal{E}}(T^+) = \{\bar{\lambda}\}$. We claim that

$$Q = \frac{1}{2\pi i} \int\limits_{\partial B_\epsilon(\lambda)} (z - T)^{-1}\, dz$$

is a proper projection, and thus $R(Q)$ is a proper subspace.

To prove our claim, let $\gamma : [-\pi, \pi] \to \partial B_\epsilon(\lambda)$ be a smooth curve with the positive orientation. Pick a partition $0 = t_0 < t_1 < \dots < t_n = \pi$ of the interval $[0, \pi]$, and then consider the partition $t_{-k} = -t_k$ of $[-\pi, 0]$. For n large enough, the above defined integral can be approximated by the Riemann sum

$$\frac{1}{2\pi i} \sum_{i=-n}^{n} (\gamma(t_i) - T)^{-1} \dot{\gamma}(t_i) \Delta t_i.$$

On the other hand, if $z \in \partial B_\epsilon(\lambda)$, then $z - T$ and $\bar{z} - T^+$ are invertible in $\mathcal{E}$. We can define the following projection:

$$P = \frac{1}{2\pi i} \int\limits_{\partial B_\epsilon(\bar{\lambda})} (\bar{z} - T^+)^{-1}\, dz.$$

Then the curve $\beta(t) = \overline{\gamma(-t)}$ is positively oriented, and $\beta([-\pi, \pi]) = \partial B_\epsilon(\bar{\lambda})$. The projection P can be approximated by

$$\frac{1}{2\pi i} \sum_{i=-n}^{n} (\beta(t_i) - T^+)^{-1} \dot{\beta}(t_i) \Delta t_i.$$

Next note that

$$\Big\langle \frac{1}{2\pi i}\sum_{i=-n}^{n}(\gamma(t_i)-T)^{-1}\dot\gamma(t_i)\Delta t_i\, h, f\Big\rangle$$
$$= -\Big\langle h, \frac{1}{2\pi i}\sum_{i=-n}^{n}(\overline{\gamma(t_i)}-T^+)^{-1}\overline{\dot\gamma(t_i)}\Delta t_i\, f\Big\rangle$$
$$= \Big\langle h, \frac{1}{2\pi i}\sum_{i=-n}^{n}(\beta(-t_i)-T^+)^{-1}\dot\beta(-t_i)\Delta t_i\, f\Big\rangle$$
$$= \Big\langle h, \frac{1}{2\pi i}\sum_{i=-n}^{n}(\beta(t_i)-T^+)^{-1}\dot\beta(t_i)\Delta t_i\, f\Big\rangle,$$

where in the last step we have used the fact that the partition is symmetric with respect to the origin. Letting $n\to\infty$, we get $\langle Qh, f\rangle = \langle h, Pf\rangle$. Thus, Q is a proper projection and $Q^+ = P$.

Now we prove some elementary properties of proper operators.

Lemma 3.3. *Let T be a proper operator. Then:*

(i) $N(T^+) = R(T)^\perp \cap \mathcal{E}$.
(ii) $\overline{R(T^+)} \cap \mathcal{E} = N(T)^\perp \cap \mathcal{E}$.

Proof. (i) Let $f \in N(T^+)$. Then $\langle f, Tg\rangle = \langle T^+f, g\rangle = 0$ for all $g \in \mathcal{E}$, which means that $f \in R(T)^\perp \cap \mathcal{E}$. Conversely, suppose that $f \in R(T)^\perp \cap \mathcal{E}$. This is equivalent to $0 = \langle f, Tg\rangle = \langle T^+f, g\rangle$ for all $g\in\mathcal{E}$, that is, $f \in N(T^+)$.

(ii) Since $(T^+)^+ = T$, we know from (i) that $N(T) = R(T^+)^\perp \cap \mathcal{E}$. Then take the orthogonal complement in $\mathcal{L}$ and the intersection with $\mathcal{E}$. ∎

Remark 3.4. Let Q be a proper projection. By Lemma 3.3(i), $N(Q^+) = R(Q)^\perp \cap \mathcal{E}$, and since $R(Q^+) = N((Q-I)^+)$, we also have $R(Q^+) = N(Q)^\perp \cap \mathcal{E}$.

Remark 3.5. It will be useful to rephrase the definition of proper operator in terms of range inclusions: $T \in \mathcal{B}(\mathcal{E})$ is proper if and only if $R(T'J) \subseteq R(J)$, where J is defined in Remark 2.1 and T' is the transpose of T.

In the case where $\mathcal{E} = \mathcal{H}$ is a Hilbert space, there is an injective positive operator $A \in \mathcal{B}(\mathcal{H})$ such that $(Jg)(f) = \langle Af, g\rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. Then $T \in \mathcal{B}(\mathcal{H})$ is proper if and only if there exists $W \in \mathcal{B}(\mathcal{H})$ such that $AT = W^*A$, where the last adjoint is with respect to $\mathcal{H}$. In the case of projections, there is another useful characterization in [19, Lemma 2.1]: a projection Q is proper if and only if

$$R(A) = (R(A)\cap R(Q^*)) \dotplus (R(A)\cap N(Q^*)).$$

This can also be proved in our setting with the obvious modifications:

Lemma 3.6. *Let $Q \in \mathcal{B}(\mathcal{E})$ be a projection. Then Q is proper if and only if*

$$R(J) = (R(J) \cap R(Q')) \dot{+} (R(J) \cap N(Q')).$$

Proof. By Remark 3.5, Q is proper if and only if $R(Q'J) \subseteq R(J)$. Clearly, this is equivalent to the equality $R(J) = R(Q'J) + R((I - Q')J)$.

On the other hand, we claim that Q is proper if and only if $R(Q'J) = R(J) \cap R(Q')$. In fact, note that if $R(Q'J) = R(J) \cap R(Q')$, then $R(Q'J) \subseteq R(J)$, which implies that Q is proper. Conversely, suppose that Q is proper; then $R(Q'J) \subseteq R(J) \cap R(Q')$. But since Q' is a projection, any functional $\phi = Jf = Q'\phi$, where $f \in \mathcal{E}$ and $\phi \in R(Q')$, can be written as $\phi = Q'\phi = Q'Jf$. Therefore $R(Q'J) = R(J) \cap R(Q')$.

Applying the same argument to $I - Q$, we also find that Q is a proper projection if and only if $R((I - Q')J) = R(J) \cap N(Q')$. It follows that a projection Q is proper if and only if $R(J) = R(Q'J) + R((I - Q')J) = (R(J) \cap R(Q')) \dot{+} (R(J) \cap N(Q'))$. ■

We now give a characterization of proper subspaces:

Theorem 3.7. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{E}$. The following conditions are equivalent:*

(i) *$\mathcal{S}$ is a proper subspace.*
(ii) *There exists a projection $Q \in \mathcal{B}(\mathcal{E})$ such that $R(Q) = \mathcal{S}$ and*

$$R(J) = (R(J) \cap R(Q')) \dot{+} (R(J) \cap N(Q')).$$

(iii) *There exists a closed subspace $\mathcal{T}$ of $\mathcal{E}$ such that*

$$\mathcal{S} \dot{+} \mathcal{T} = (\mathcal{S}^\perp \cap \mathcal{E}) \dot{+} (\mathcal{T}^\perp \cap \mathcal{E}) = \mathcal{E}.$$

Proof. (i)⇔(ii). This follows immediately from Lemma 3.6.

(i)⇔(iii). Suppose that $\mathcal{S}$ is a proper subspace. Then there is a proper projection Q such that $R(Q) = \mathcal{S}$. According to Remark 3.4, we can take $\mathcal{T} = N(Q)$. In fact, we have $N(Q^+) = \mathcal{S}^\perp \cap \mathcal{E}$ and $R(Q^+) = \mathcal{T}^\perp \cap \mathcal{E}$. Since Q^+ is a projection in $\mathcal{B}(\mathcal{E})$, we get $(\mathcal{S}^\perp \cap \mathcal{E}) \dot{+} (\mathcal{T}^\perp \cap \mathcal{E}) = \mathcal{E}$.

Conversely, assume that there is a closed subspace $\mathcal{T}$ satisfying $\mathcal{S} \dot{+} \mathcal{T} = (\mathcal{S}^\perp \cap \mathcal{E}) \dot{+} (\mathcal{T}^\perp \cap \mathcal{E}) = \mathcal{E}$. Then we can define the continuous projections $Q = P_{\mathcal{S}//\mathcal{T}}$ and $P = P_{\mathcal{T}^\perp \cap \mathcal{E}//\mathcal{S}^\perp \cap \mathcal{E}}$. Note that for any $h, f \in \mathcal{E}$, we have

$$\begin{aligned}\langle Qh, f\rangle &= \langle Qh, Pf + (I - P)f\rangle \\ &= \langle Qh, Pf\rangle = \langle (I - Q)h + Qh, Pf\rangle = \langle h, Pf\rangle.\end{aligned}$$

This shows that Q and P are proper operators and $Q^+ = P$. Hence $\mathcal{S}$ is a proper subspace. ■

If $\mathcal{S}$ is a proper subspace, we have seen that there exists a closed subspace $\mathcal{T}$ of $\mathcal{E}$ such that $\mathcal{S} \dot{+} \mathcal{T} = (\mathcal{S}^\perp \cap \mathcal{E}) \dot{+} (\mathcal{T}^\perp \cap \mathcal{E}) = \mathcal{E}$. We refer to any such subspace $\mathcal{T}$ as a *proper companion of* $\mathcal{S}$.

COROLLARY 3.8. *If $\mathcal{S}$ is a proper subspace of $\mathcal{E}$ with a proper companion $\mathcal{T}$. Then $\mathcal{S}^\perp \cap \mathcal{E}$, $\mathcal{T}$ and $\mathcal{T}^\perp \cap \mathcal{E}$ are proper subspaces.*

Proof. Let Q and P be the proper projections defined in the proof of Proposition 3.7. The ranges of $I-Q$, P and $I-P$ are the subspaces $\mathcal{T}$, $\mathcal{T}^\perp \cap \mathcal{E}$ and $\mathcal{S}^\perp \cap \mathcal{E}$, respectively. Hence these three subspaces are proper. ∎

COROLLARY 3.9. *Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$. Then:*

(i) *If $\mathcal{T}$ is a proper companion of $\mathcal{S}$, then $\bar{P}_{\mathcal{S}//\mathcal{T}} = P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}}$. In particular, $\bar{\mathcal{S}} \dot{+} \overline{\mathcal{T}} = \mathcal{L}$.*
(ii) *$\bar{\mathcal{S}} \cap \mathcal{E} = \mathcal{S}$.*
(iii) *Let Q be a proper projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. Set $V = 2Q - I$. Then $V^2 = I$ on $\mathcal{E}$ and $\bar{V}^2 = I$ on $\mathcal{L}$.*

Proof. (i) First note that the bounded projection $Q := P_{\mathcal{S}//\mathcal{T}}$ is well defined because $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{E}$. In the proof of Theorem 3.7 we have seen that Q is a proper operator. Thus, Q has a bounded extension $\bar{Q}$ to the Hilbert space $\mathcal{L}$. Note that $\mathcal{S} \subseteq R(\bar{Q}) \subseteq \bar{\mathcal{S}}$, and $\bar{Q}$ has closed range, which implies $R(\bar{Q}) = \bar{\mathcal{S}}$. Similarly, one can check that $N(\bar{Q}) = \overline{\mathcal{T}}$.

(ii) The non-trivial inclusion is $\bar{\mathcal{S}} \cap \mathcal{E} \subseteq \mathcal{S}$. Pick $f \in \bar{\mathcal{S}} \cap \mathcal{E}$. Since $\mathcal{S}$ is a proper subspace, there is a proper projection Q with range $\mathcal{S}$ and nullspace $\mathcal{T}$. By the first item, we know that $\bar{Q} = P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}}$, so $f = \bar{Q}f = Qf \in \mathcal{S}$.

(iii) From $Q^2 = Q$, it follows that $V^2 = (2Q - I)^2 = I$ on $\mathcal{E}$. Since Q is a proper projection, by (i) we know that $\bar{Q}$ is a bounded projection on $\mathcal{L}$. Then $\bar{V}^2 = (2\bar{Q} - I)^2 = I$ on $\mathcal{L}$. ∎

COROLLARY 3.10. *Let $\mathcal{T}$ be a proper companion of a proper subspace $\mathcal{S}$, and let $G \in \mathfrak{P}^\times$. Then $G(\mathcal{S})$ and $G(\mathcal{T})$ are proper companions. Moreover, if $\mathcal{T}_1$ is another proper companion of $\mathcal{S}$, then there exists an operator $G \in \mathfrak{P}^\times$ such that $G(\mathcal{T}) = \mathcal{T}_1$ and $G(\mathcal{S}) = \mathcal{S}$.*

Proof. For the first assertion, we only have to note that the projection given by $P = GP_{\mathcal{S}//\mathcal{T}}G^{-1}$ is a proper operator with range $G(\mathcal{S})$ and nullspace $G(\mathcal{T})$. To show the second assertion, consider the bounded operator

$$G_0 = (P_{\mathcal{T}_1//\mathcal{S}})|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_1.$$

It is easy to check that G_0 is an isomorphism. Then the operator defined by

$$G(f_1 + f_2) = f_1 + G_0 f_2, \qquad f_1 \in \mathcal{S},\ f_2 \in \mathcal{T},$$

is invertible on $\mathcal{E}$, $G(\mathcal{T}) = \mathcal{T}_1$ and $G(\mathcal{S}) = \mathcal{S}$. To show that G is a proper operator, we note that it can be written as

$$G = P_{\mathcal{S}//\mathcal{T}} + P_{\mathcal{T}_1//\mathcal{S}} P_{\mathcal{T}//\mathcal{S}}.$$

Since each projection in this expression is a proper operator, so is G and

$$G^+ = P_{\mathcal{T}^\perp\cap\mathcal{E}//\mathcal{S}^\perp\cap\mathcal{E}} + P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}^\perp\cap\mathcal{E}} P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}_1^\perp\cap\mathcal{E}}.$$

It remains to prove that G is invertible in the Banach algebra $\mathfrak{P}$. Since G is invertible on $\mathcal{E}$, we have to show that G^+ is invertible on $\mathcal{E}$. Clearly, G^+ is injective. To see that it is surjective, given $g \in \mathcal{E}$, write $g = g_1 + g_2$, where $g_1 \in \mathcal{S}^\perp \cap \mathcal{E}$ and $g_2 \in \mathcal{T}^\perp \cap \mathcal{E}$. Then one can also write $g_2 = g_{2,1} + g_{2,2}$, where $g_{2,1} \in \mathcal{S}^\perp \cap \mathcal{E}$ and $g_{2,2} \in \mathcal{T}_1^\perp \cap \mathcal{E}$. Therefore, the vector $f = g - g_{2,1}$ satisfies $G^+ f = g$. ∎

4. Proper and compatible subspaces. A closed subspace $\mathcal{S} \subseteq \mathcal{E}$ is called *compatible* if there exists a proper projection Q such that $Q = Q^+$ and $R(Q) = \mathcal{S}$. The following elementary characterizations of compatible subspaces will be useful later.

LEMMA 4.1. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{E}$. The following conditions are equivalent:*

(i) *$\mathcal{S}$ is compatible.*
(ii) $\mathcal{S} \dotplus (\mathcal{S}^\perp \cap \mathcal{E}) = \mathcal{E}$.
(iii) *There exists a proper projection Q such that $R(Q) = \mathcal{S}$ and $N(Q) \subseteq \mathcal{S}^\perp \cap \mathcal{E}$.*
(iv) $R(J) = J(\mathcal{S}) \dotplus (R(J) \cap \mathcal{S}^\circ)$.
(v) *$\bar{\mathcal{S}} \cap \mathcal{E} = \mathcal{S}$ and $P_{\bar{\mathcal{S}}}(\mathcal{E}) \subseteq \mathcal{E}$.*

Proof. (i)⇔(ii). Suppose that $\mathcal{S}$ is a compatible subspace. Then there is a proper projection Q such that $Q = Q^+$ and $R(Q) = \mathcal{S}$. Using Remark 3.4, we get $N(Q) = N(Q^+) = \mathcal{S}^\perp \cap \mathcal{E}$, which yields $\mathcal{E} = R(Q) \dotplus N(Q) = \mathcal{S} \dotplus (\mathcal{S}^\perp \cap \mathcal{E})$.

To prove the converse, assume that $\mathcal{S} \dotplus (\mathcal{S}^\perp \cap \mathcal{E}) = \mathcal{E}$. Then the projection $Q = P_{\mathcal{S}//\mathcal{S}^\perp\cap\mathcal{E}}$ is continuous on $\mathcal{E}$, and $\langle Qh, f\rangle = \langle Qh, Qf\rangle = \langle h, Qf\rangle$ for all $f, h \in \mathcal{E}$. Thus, Q is a proper projection, $R(Q) = \mathcal{S}$ and $Q^+ = Q$.

(i)⇔(iii). This is a direct consequence of a result of Krein [27]. We refer to [19, Lemma 2.5] for a proof when $\mathcal{E}$ is a Hilbert space. It is not difficult to see that this proof can also be carried out in the Banach space setting.

(i)⇔(iv). Follow the proof in [19, Prop. 2.14(2)], taking into account the map J that shows up in the Banach setting.

(i)⇔(v). This was proved in [3, Prop. 3.5] in the setting of Hilbert spaces. The proof in our setting follows exactly the same lines. ∎

REMARK 4.2. If $\mathcal{S}$ is a compatible subspace, there exists a unique proper projection Q such that $Q^+ = Q$ and $R(Q) = \mathcal{S}$. This follows immediately from the fact that Q is uniquely determined as $(P_{\bar{\mathcal{S}}})|_{\mathcal{E}}$. We denote this projection by $Q_{\mathcal{S}}$.

Note that in Example 3.1 we actually show that every finite-dimensional subspace is compatible. Subspaces of finite codimension in $\mathcal{E}$ may not be compatible or proper, but both notions coincide for this type of subspaces.

PROPOSITION 4.3. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{E}$ with finite codimension. Then $\mathcal{S}$ is a proper subspace if and only if $\mathcal{S}$ is a compatible subspace.*

Proof. The "if" part is trivial. To prove the "only if" part, suppose that $\mathcal{S}$ is a proper subspace. Any supplement of $\mathcal{S}$ in $\mathcal{E}$ has to be finite-dimensional. In particular, a proper companion $\mathcal{T}$ is finite-dimensional, and then $\overline{\mathcal{T}} = \mathcal{T}$. Let Q be a proper projection such that $R(Q) = \mathcal{S}$. Then $P = I - Q$ is a proper projection with range $\mathcal{T}$. In Example 3.1 we saw that P can be described by formula (3.1). Note that $R(P) = \mathcal{T} = \operatorname{span}\{f_1, \dots, f_n\}$ and $R(P^+) = N(P)^\perp \cap \mathcal{E} = \mathcal{S}^\perp \cap \mathcal{E} = \operatorname{span}\{h_1, \dots, h_n\}$. From these facts, we get $\dim \mathcal{T} = \dim \mathcal{S}^\perp \cap \mathcal{E}$.

On the other hand, recall that $\bar{\mathcal{S}} \dot{+} \mathcal{T} = \mathcal{L}$ by Corollary 3.9. Since $\mathcal{S}^\perp$ is a supplement for $\bar{\mathcal{S}}$ in $\mathcal{L}$, it follows that $\dim \mathcal{T} = \dim \mathcal{S}^\perp$. Therefore, $\dim \mathcal{S}^\perp = \dim \mathcal{S}^\perp \cap \mathcal{E}$. Hence $\mathcal{S}^\perp = \mathcal{S}^\perp \cap \mathcal{E}$.

To prove that $\mathcal{S}$ is compatible, it suffices to show that $\bar{\mathcal{S}} \cap \mathcal{E} = \mathcal{S}$ and $P_{\bar{\mathcal{S}}}(\mathcal{E}) \subseteq \mathcal{E}$ by Lemma 4.1. According to Corollary 3.9, we have $\bar{\mathcal{S}} \cap \mathcal{E} = \mathcal{S}$. To prove the second condition, we note that $\mathcal{S}^\perp = \mathcal{S}^\perp \cap \mathcal{E} \subseteq \mathcal{E}$, and thus $P_{\bar{\mathcal{S}}}(\mathcal{E}) = (I - P_{\mathcal{S}^\perp})(\mathcal{E}) = (I - P_{\mathcal{S}^\perp \cap \mathcal{E}})(\mathcal{E}) \subseteq \mathcal{E}$. ∎

EXAMPLE 4.4. Given a vector $g \in \mathcal{L} \setminus \mathcal{E}$ with $\|g\|_{\mathcal{L}} = 1$, the subspace

$$\mathcal{S} = \{f \in \mathcal{E} : \langle f, g \rangle = 0\} = \{g\}^\perp \cap \mathcal{E}$$

is neither compatible nor proper. Note that $\mathcal{S}$ has finite codimension, so by Proposition 4.3 it is enough to prove that $\mathcal{S}$ is non-compatible. The first condition in Lemma 4.1(v), that is, $\bar{\mathcal{S}} \cap \mathcal{E} = \mathcal{S}$, clearly holds. On the other hand, the orthogonal projection onto $\bar{\mathcal{S}}$ is given by

$$P_{\bar{\mathcal{S}}}(f) = f - \langle f, g \rangle g.$$

Apparently, $P_{\bar{\mathcal{S}}}(\mathcal{E}) \not\subseteq \mathcal{E}$ by our choice of g. Hence $\mathcal{S}$ is non-compatible.

EXAMPLE 4.5. In [18, Example 4.3] the authors gave the following example of a non-compatible subspace. Let A be a positive injective non-invertible operator acting on $\mathcal{E} = \mathcal{H}$. As usual, $\mathcal{L}$ is the Hilbert space obtained by completing $\mathcal{H}$ with respect to the inner product $\langle f, h \rangle_A = \langle Af, h \rangle_{\mathcal{H}}$. Pick any $g \in \mathcal{H} \setminus R(A)$. Then they proved that the subspace $\mathcal{S} = \{g\}^{\perp_{\mathcal{H}}}$ is non-compatible. Furthermore, $\mathcal{S}$ has finite codimension in $\mathcal{H}$. Thus, by Proposition 4.3, $\mathcal{S}$ is not proper either.

We shall need the following algebraic result by Maestripieri. Its proof can be found in [8, Prop. 2.8].

Lemma 4.6. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{E})$ be such that $R(T_1) \cap R(T_2) = \{0\}$. Then $\mathcal{E} = N(T_1) + N(T_2)$ if and only if $R(T_1) + R(T_2) = R(T_1 + T_2)$.*

Remark 4.7. It will be useful to state the last condition of the above lemma in a slightly different way. We notice that $R(T_1)+R(T_2) = R(T_1+T_2)$ if and only if $R(T_1) \subseteq R(T_1 + T_2)$ (see [9, Prop. 2.4]).

Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$, and $\mathcal{T}$ a proper companion. As we shall see, the compatibility of these subspaces is related to properties of the following symmetrizable operator:

$$C_{\mathcal{S},\mathcal{T}} = P_{\mathcal{S}//\mathcal{T}} + P^{+}_{\mathcal{S}//\mathcal{T}} - I.$$

We observe that its extension $\bar{C}_{\mathcal{S},\mathcal{T}}$ is invertible on $\mathcal{L}$ if and only if $\bar{V} + \bar{V}^*$ is invertible on $\mathcal{L}$, where $V = 2P_{\mathcal{S}//\mathcal{T}} - I$. Since $V^2 = I$ on the Banach space $\mathcal{E}$, we see that $\bar{V}^2 = I$ on the Hilbert space $\mathcal{L}$. Therefore $\bar{V} + \bar{V}^*$ is invertible on $\mathcal{L}$ if and only if $-1 \notin \sigma_{\mathcal{L}}(\bar{V}^*\bar{V})$, which clearly holds since $\bar{V}^*\bar{V}$ is positive on $\mathcal{L}$. Thus, $\bar{C}_{\mathcal{S},\mathcal{T}}$ is invertible on $\mathcal{L}$. In particular, $C_{\mathcal{S},\mathcal{T}}$ is injective as an operator on $\mathcal{E}$.

Our main result enabling one to decide when a proper subspace is compatible now follows. Its proof is based on Lemma 4.6. This idea has been used in [8, Prop. 2.9] to relate compatible subspaces in Hilbert spaces and Bott–Duffin inverses.

Theorem 4.8. *Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$, and $\mathcal{T}$ a proper companion of $\mathcal{S}$. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is a compatible subspace.*
(ii) *$\mathcal{T} \dotplus (\mathcal{T}^{\perp} \cap \mathcal{E}) = R(C_{\mathcal{S},\mathcal{T}})$.*
(iii) *$\mathcal{T}^{\perp} \cap \mathcal{E} \subseteq R(C_{\mathcal{S},\mathcal{T}})$.*
(iv) *$\mathcal{T} \subseteq R(C_{\mathcal{S},\mathcal{T}})$.*

If any of these statements is satisfied, the unique proper projection $Q_{\mathcal{S}}$ such that $Q_{\mathcal{S}} = Q^{+}_{\mathcal{S}}$ and $R(Q_{\mathcal{S}}) = \mathcal{S}$ is given by

$$Q_{\mathcal{S}} = C^{-1}_{\mathcal{S},\mathcal{T}} P^{+}_{\mathcal{S}//\mathcal{T}}.$$

Proof. (i)⇔(ii). Set $Q = P_{\mathcal{S}//\mathcal{T}}$. We shall use Lemma 4.6 with $T_1 = Q^+$ and $T_2 = Q - I$. Note that $R(Q^+) = \mathcal{T}^{\perp} \cap \mathcal{E}$ and $R(Q - I) = \mathcal{T}$ have trivial intersection, and thus the lemma applies. Then, as shown before,

$$N(Q^+) = R(Q)^{\perp} \cap \mathcal{E} = \mathcal{S}^{\perp} \cap \mathcal{E}, \qquad N(Q - I) = R(Q) = \mathcal{S}.$$

According to Lemma 4.1, the fact that $\mathcal{S}$ is compatible is equivalent to $\mathcal{E} = \mathcal{S} + (\mathcal{S}^{\perp} \cap \mathcal{E})$.

Clearly, the equivalence of (ii), (iii) and (iv) follows from Remark 4.7.

Now assume that $\mathcal{S}$ is compatible. Before the statement of this theorem, we have shown that the operator $C_{\mathcal{S},\mathcal{T}} = Q + Q^+ - I$ is injective. Since we know that $R(Q^+) = \mathcal{T}^\perp \cap \mathcal{E}$, by (iii) the operator $(Q+Q^+-I)^{-1}Q^+$ is everywhere defined in $\mathcal{E}$. Apparently, it has closed graph: let $f_n \to f$ in $\mathcal{E}$ with $(Q+Q^+-I)^{-1}Q^+f_n \to g$. Then $Q^+f_n \to (Q+Q^+-I)g$. Also $Q^+f_n \to Q^+f$. It follows that

$$(Q+Q^+-I)g = Q^+f, \quad \text{i.e.} \quad g = (Q+Q^+-I)^{-1}Q^+f.$$

Thus, $(Q+Q^+-I)^{-1}Q^+$ is bounded. We claim that $(Q+Q^+-I)^{-1}Q^+ = Q_{\mathcal{S}}$. This is equivalent to proving that $Q^+ = (Q+Q^+-I)Q_{\mathcal{S}}$. Since $R(Q) = R(Q_{\mathcal{S}}) = \mathcal{S}$, one has $QQ_{\mathcal{S}} = Q_{\mathcal{S}}$, and thus

$$(Q+Q^+-I)Q_{\mathcal{S}} = Q^+Q_{\mathcal{S}}.$$

Note also that $Q^+(I-Q_{\mathcal{S}}) = 0$, because $R(I-Q_{\mathcal{S}}) = N(Q_{\mathcal{S}}) = \mathcal{S}^\perp \cap \mathcal{E} = N(Q^+)$. Then

$$Q^+Q_{\mathcal{S}} = Q^+(Q_{\mathcal{S}} + (I-Q_{\mathcal{S}})) = Q^+. \ \blacksquare$$

Of course, the operator $C_{\mathcal{S},\mathcal{T}}$ may not be invertible on $\mathcal{E}$, and $\mathcal{S}$ can be a compatible subspace (see Theorem 4.11). In fact, $C_{\mathcal{S},\mathcal{T}}$ is invertible on $\mathcal{E}$ exactly when $\mathcal{S}$ and $\mathcal{T}$ are both compatible subspaces.

THEOREM 4.9. *Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$, and $\mathcal{T}$ a proper companion of $\mathcal{S}$. The following conditions are equivalent:*

(i) *$\mathcal{S}$ and $\mathcal{T}$ are compatible subspaces.*
(ii) *$(P_{\bar{\mathcal{S}}} + P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$.*
(iii) *$(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$.*
(iv) *$C_{\mathcal{S},\mathcal{T}}$ is invertible on $\mathcal{E}$.*

Proof. (i)⇒(ii). This implication follows from Lemma 4.1(v).

(ii)⇒(iii). Since $\mathcal{S}$ is a proper subspace, we know that $\bar{\mathcal{S}} \dotplus \overline{\mathcal{T}} = \mathcal{L}$ by Corollary 3.9(i). Then the following formula (see Theorem 2.2) for the projection on a Hilbert space with range $\bar{\mathcal{S}}$ and nullspace $\overline{\mathcal{T}}$ can be used:

$$P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}} = P_{\bar{\mathcal{S}}}\,(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})^{-1}.$$

Equivalently, $P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}}\,(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}}) = P_{\bar{\mathcal{S}}}$. Interchanging the roles of the subspaces, we also get $P_{\overline{\mathcal{T}}//\bar{\mathcal{S}}}\,(P_{\overline{\mathcal{T}}} - P_{\bar{\mathcal{S}}}) = P_{\overline{\mathcal{T}}}$. Then we obtain

$$P_{\bar{\mathcal{S}}} + P_{\overline{\mathcal{T}}} = (2P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}} - I)\,(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}}). \tag{4.1}$$

Since $\mathcal{S}$ is a proper subspace, we have $\bar{P}_{\mathcal{S}//\mathcal{T}} = P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}}$ by Corollary 3.9(i). Then the symmetry $2P_{\bar{\mathcal{S}}//\overline{\mathcal{T}}} - I$ acting on $\mathcal{L}$ is an extension of the symmetry $2P_{\mathcal{S}//\mathcal{T}} - I$, which is an invertible operator on $\mathcal{E}$ (see also Corollary 3.9(iii)). From the equality (4.1), it is now clear that $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$ if and only if $(P_{\bar{\mathcal{S}}} + P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$.

(iii)⇒(i). We have shown that $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$ is equivalent to $(P_{\bar{\mathcal{S}}} + P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$. If we add or subtract $P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}}$ and $P_{\bar{\mathcal{S}}} + P_{\overline{\mathcal{T}}}$, we get this implication by Lemma 4.1 and Corollary 3.9(ii).

(iii)⇒(iv). Set $Q := P_{\mathcal{S}//\mathcal{T}}$. By Corollary 3.9(i), we have $\bar{\mathcal{S}} \dot{+} \overline{\mathcal{T}} = \mathcal{L}$. Therefore $P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}}$ is invertible on $\mathcal{L}$, and its inverse is given by

$$(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})^{-1} = \bar{Q} + \bar{Q}^* - I.$$

These facts can be found again in Theorem 2.2. If the operator $Q + Q^+ - I$ is invertible on $\mathcal{E}$, then its extension $\bar{Q} + \bar{Q}^* - I$ to $\mathcal{L}$ maps $\mathcal{E}$ onto $\mathcal{E}$. Therefore $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) = (P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(Q + Q^+ - I)(\mathcal{E}) = \mathcal{E}$.

To prove the converse, assume that $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) \subseteq \mathcal{E}$. Note that $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})^{-1}(\mathcal{E}) = (\bar{Q} + \bar{Q}^* - I)(\mathcal{E}) = (Q + Q^+ - I)(\mathcal{E}) \subseteq \mathcal{E}$, which implies that $\mathcal{E} \subseteq (P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E})$. Therefore $(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) = \mathcal{E}$. Now we have

$$(Q + Q^+ - I)(\mathcal{E}) = (\bar{Q} + \bar{Q}^* - I)(\mathcal{E}) = (\bar{Q} + \bar{Q}^* - I)(P_{\bar{\mathcal{S}}} - P_{\overline{\mathcal{T}}})(\mathcal{E}) = \mathcal{E}. \blacksquare$$

Let $\mathcal{S}$ be a compatible subspace of $\mathcal{E}$. By Corollary 3.10 any proper companion of $\mathcal{S}$ arises as the image of any other proper companion by an invertible operator in the Banach algebra $\mathfrak{P}$ given by the proper operators. In general, this invertible operator does not extend to a unitary operator on $\mathcal{L}$. Thus, if $\mathcal{T}$ and $\mathcal{T}_1$ are two proper companions of $\mathcal{S}$, and $\mathcal{T}$ is a compatible subspace, the subspace $\mathcal{T}_1$ may be non-compatible. For a concrete example of this situation see Examples 5.1 and 5.2. However, we shall give below two sufficient conditions to ensure the compatibility of $\mathcal{T}_1$. We first have to introduce the following metric in the set of all proper companions of $\mathcal{S}$:

$$d(\mathcal{T}_1, \mathcal{T}_2) = \|P_{\mathcal{T}_1//\mathcal{S}} - P_{\mathcal{T}_2//\mathcal{S}}\|_{\mathfrak{P}},$$

where $\mathcal{T}_i$, $i = 1, 2$, are proper companions of $\mathcal{S}$ and $\| \|_{\mathfrak{P}}$ is the norm of the algebra $\mathfrak{P}$.

Corollary 4.10. *Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$, and $\mathcal{T}$ a proper companion of $\mathcal{S}$. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are compatible subspaces. Then:*

(i) *There exists a constant $r > 0$, depending only on $\mathcal{S}$ and $\mathcal{T}$, such that $\mathcal{T}_1$ is a compatible subspace whenever $d(\mathcal{T}, \mathcal{T}_1) < r$.*
(ii) *Let $G \in \mathfrak{P}^\times$ be such that $G - I$ and $G^+ - I$ are compact operators on $\mathcal{E}$. Then $G(\mathcal{S})$ and $G(\mathcal{T})$ are compatible subspaces.*

Proof. (i) It is enough to show that the map

$$\{\mathcal{T}_1 \subseteq \mathcal{E} : \mathcal{T}_1 \text{ is a proper companion of } \mathcal{S}\} \to \mathcal{B}(\mathcal{E}), \qquad \mathcal{T}_1 \mapsto C_{\mathcal{S},\mathcal{T}_1}, \tag{4.2}$$

is continuous at $\mathcal{T}$, when the first space is endowed with the metric d defined above and $\mathcal{B}(\mathcal{E})$ is considered with its usual operator norm $\| \|$. In fact, if this map is continuous, then there is a constant $r > 0$ depending on $P_{\mathcal{T}//\mathcal{S}}$ such that

$$\|C_{\mathcal{S},\mathcal{T}_1} - C_{\mathcal{S},\mathcal{T}}\| \le 1/\|C_{\mathcal{S},\mathcal{T}}^{-1}\|$$

whenever $d(\mathcal{T}, \mathcal{T}_1) < r$. The displayed inequality implies that $C_{\mathcal{S},\mathcal{T}_1}$ is invertible on $\mathcal{E}$, and by Theorem 4.9, this is equivalent to the compatibility of $\mathcal{S}$ and $\mathcal{T}_1$.

Let $(\mathcal{T}_n)$ be proper companions of the compatible subspace $\mathcal{S}$ such that $d(\mathcal{T}_n,\mathcal{T})\to 0$. This means that

$$(4.3)\qquad \|P_{\mathcal{T}_n//\mathcal{S}}-P_{\mathcal{T}//\mathcal{S}}\|\to 0 \quad\text{and}\quad \|P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}_n^\perp\cap\mathcal{E}}-P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}^\perp\cap\mathcal{E}}\|\to 0.$$

On the other hand, we recall from Corollary 3.10 that there exist operators $G_n\in\mathfrak{P}^\times$ such that $G_n(\mathcal{T})=\mathcal{T}_n$ and $G(\mathcal{S})=\mathcal{S}$. In particular, it follows that $G_nP_{\mathcal{S}//\mathcal{T}}G_n^{-1}=P_{\mathcal{S}//\mathcal{T}_n}$. As we have shown in the proof of that corollary, G_n and G_n^+ are given by

$$G_n=P_{\mathcal{S}//\mathcal{T}}+P_{\mathcal{T}_n//\mathcal{S}}\,P_{\mathcal{T}//\mathcal{S}},$$
$$G_n^+=P_{\mathcal{T}^\perp\cap\mathcal{E}//\mathcal{S}^\perp\cap\mathcal{E}}+P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}^\perp\cap\mathcal{E}}\,P_{\mathcal{S}^\perp\cap\mathcal{E}//\mathcal{T}_n^\perp\cap\mathcal{E}}.$$

Using (4.3), we see that $\|G_n-I\|\to 0$ and $\|G_n^+-I\|\to 0$. Therefore,

$$\|C_{\mathcal{S},\mathcal{T}_n}-C_{\mathcal{S},\mathcal{T}}\|=\|G_nP_{\mathcal{S}//\mathcal{T}}G_n^{-1}+(G_n^+)^{-1}P^+_{\mathcal{S}//\mathcal{T}}G_n^+-P_{\mathcal{S}//\mathcal{T}}-P^+_{\mathcal{S}//\mathcal{T}}\|\to 0.$$

This completes the proof of the continuity of the map (4.2).

(ii) We set $\mathcal{T}_1=G(\mathcal{T})$, $\mathcal{S}_1=G(\mathcal{S})$ and $Q=P_{\mathcal{S}//\mathcal{T}}$. Then

$$\begin{aligned}C_{\mathcal{S}_1,\mathcal{T}_1}&=GQG^{-1}+(G^+)^{-1}Q^+G^+-I\\&=(G-I)QG^{-1}+Q(G^{-1}-I)+((G^+)^{-1}-I)Q^+G^+\\&\quad+(G^+)^{-1}Q^+(G^+-I)+C_{\mathcal{S},\mathcal{T}}=K+C_{\mathcal{S},\mathcal{T}},\end{aligned}$$

for some compact operator K on $\mathcal{E}$. This implies that the essential spectrum $\sigma_{\mathrm{ess},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})$ of $C_{\mathcal{S}_1,\mathcal{T}_1}$ coincides with that of $C_{\mathcal{S},\mathcal{T}}$. Thus, $0\notin\sigma_{\mathrm{ess},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})$.

Now we recall that the essential spectrum of a symmetrizable operator consists of those numbers in the spectrum over $\mathcal{E}$ which are not isolated eigenvalues of finite multiplicity (see [31, Thm. 1]). Applying this result to the operator $C_{\mathcal{S}_1,\mathcal{T}_1}$, we find that its spectrum can be written as the following disjoint union:

$$\sigma_{\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})=\sigma_{\mathrm{ess},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})\cup\sigma_{\mathrm{p},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1}),$$

where $\sigma_{\mathrm{p},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})$ is the point spectrum of $C_{\mathcal{S}_1,\mathcal{T}_1}$ on $\mathcal{E}$ consisting of the isolated eigenvalues of finite multiplicity. Since $\bar{C}_{\mathcal{S}_1,\mathcal{T}_1}$ is always invertible on $\mathcal{L}$, we know that zero does not belong to its point spectrum on $\mathcal{L}$. But the point spectrum of a symmetrizable operator coincides with the point spectrum of its extension (see [30, Thm. 2-3]). Therefore, $0\notin\sigma_{\mathrm{p},\mathcal{E}}(C_{\mathcal{S}_1,\mathcal{T}_1})$. Hence $C_{\mathcal{S}_1,\mathcal{T}_1}$ is invertible on $\mathcal{E}$, and thus $\mathcal{S}_1$ and $\mathcal{T}_1$ are compatible subspaces. ∎

The following result will be crucial to constructing proper subspaces which are non-compatible. In the statement, we use the notion of proper operators in two different spaces (see Remark 2.4).

THEOREM 4.11. *Let $\mathcal{S}$ be a compatible subspace of $\mathcal{E}$. Consider an operator $z\in\mathfrak{P}(\mathcal{S}^\perp\cap\mathcal{E},\mathcal{S})$ and the proper projection with range $\mathcal{S}$ whose matrix* *with respect to the decomposition* $\mathcal{E} = \mathcal{S} \dot{+} (\mathcal{S}^\perp \cap \mathcal{E})$ *is given by*

$$Q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}.$$

Then the proper subspace $\mathcal{T} = N(Q)$ *is compatible if and only if* $-1 \notin \sigma_{\mathcal{S}}(zz^+)$.

REMARK 4.12. Clearly, we have $R(Q) = \mathcal{S}$. We also note that the operator z is chosen to be proper in order that Q be a proper projection. Moreover, it is straightforward to check that

$$Q^+ = \begin{pmatrix} 1 & 0 \\ z^+ & 0 \end{pmatrix}.$$

In addition, we observe that $\mathcal{T} = N(Q)$ is a proper subspace since it is the range of the proper projection $I - Q$.

Proof of Theorem 4.11. Recall that $\mathcal{T}$ is compatible exactly when $\mathcal{E} = \mathcal{T} \dot{+} (\mathcal{T}^\perp \cap \mathcal{E})$ by Lemma 4.1. According to Theorem 4.8 we have $R(C_{\mathcal{S},\mathcal{T}}) = \mathcal{T} \dot{+} (\mathcal{T}^\perp \cap \mathcal{E})$, since we have assumed that the subspace $\mathcal{S}$ is compatible. Therefore $\mathcal{T}$ is compatible if and only if $C_{\mathcal{S},\mathcal{T}}$ is surjective. Note that

$$C_{\mathcal{S},\mathcal{T}} = Q + Q^+ - I = \begin{pmatrix} 1 & z \\ z^+ & -1 \end{pmatrix}.$$

Thus, the operator $C_{\mathcal{S},\mathcal{T}}$ is surjective if and only if

$$\begin{pmatrix} 1 & z \\ z^+ & -1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

can be solved with $f_1 \in \mathcal{S}$ and $f_2 \in \mathcal{S}^\perp \cap \mathcal{E}$ for any choice of $h_1 \in \mathcal{S}$ and $h_2 \in \mathcal{S}^\perp \cap \mathcal{E}$. This can be rewritten as the following equations:

$$f_1 + zf_2 = h_1, \qquad z^+ f_1 - f_2 = h_2.$$

Since $f_2 = z^+ f_1 - h_2$, we only need to solve

$$(1 + zz^+) f_1 = h_1 + zh_2.$$

Thus, we find that $C_{\mathcal{S},\mathcal{T}}$ is surjective if and only if the symmetrizable operator $1 + zz^+ : \mathcal{S} \to \mathcal{S}$ is surjective. Now we note that the extension $1 + \bar{z}\bar{z}^*$ to $\bar{\mathcal{S}}$ is a positive operator, and hence it is invertible on $\bar{\mathcal{S}}$. In particular, this implies that $1 + zz^+$ is always injective. Therefore $1 + zz^+ : \mathcal{S} \to \mathcal{S}$ is surjective if and only if $-1 \notin \sigma_{\mathcal{S}}(zz^+)$. ■

We construct proper subspaces which are non-compatible in $\mathcal{E} \times \mathcal{E}$ using the existence of symmetrizable operators with non-real points in the spectrum in $\mathcal{E}$. To our knowledge, the first such example was constructed in [22]. Other examples were given in [25, 11, 4], and they rely on a fundamental result by Krein on the spectrum of Toeplitz matrices (see [28, Thm. 13.2]).

We shall consider the Banach space $\mathcal{E} \times \mathcal{E}$ endowed with the norm $\|(f_1, f_2)\|_{\mathcal{E}\times\mathcal{E}} = (\|f_1\|_{\mathcal{E}}^2 + \|f_2\|_{\mathcal{E}}^2)^{1/2}$. Apparently, $\mathcal{E} \times \mathcal{E}$ is densely included in the Hilbert space $\mathcal{L} \times \mathcal{L}$ and $\|(f_1, f_2)\|_{\mathcal{L}\times\mathcal{L}} \le \|(f_1, f_2)\|_{\mathcal{E}\times\mathcal{E}}$ for all $f_1, f_2 \in \mathcal{E}$.

COROLLARY 4.13. *Given a symmetrizable operator $X \in \mathcal{B}(\mathcal{E})$ with a non-real point λ in the spectrum $\sigma_{\mathcal{E}}(X)$, consider the following projection in terms of the decomposition $\mathcal{E} \times \mathcal{E}$:*

$$Q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix},$$

where $z = \frac{1}{\mathrm{Im}(\lambda)}(X - \mathrm{Re}(\lambda)I)$. Then $N(Q)$ is a proper subspace of $\mathcal{E} \times \mathcal{E}$ which is non-compatible.

Proof. Clearly, the subspace $\mathcal{E} \simeq \mathcal{E} \times \{0\}$ is compatible in $\mathcal{E} \times \mathcal{E}$ and the proper projection Q has range equal to $\mathcal{E}$. Note that the symmetrizable operator z satisfies $i \in \sigma_{\mathcal{E}}(z)$. Hence $-1 \in \sigma_{\mathcal{E}}(z^2)$, and by Theorem 4.11, the proper subspace $N(Q) \subseteq \mathcal{E} \times \mathcal{E}$ is non-compatible. ∎

Conversely, any non-compatible proper subspace gives rise to a symmetrizable operator with non-real points in the spectrum as an operator in $\mathcal{E}$.

COROLLARY 4.14. *Let $\mathcal{S}$ be a proper subspace of $\mathcal{E}$ which is not a compatible subspace. Let Q be a proper projection with range $\mathcal{S}$. Then $X = VV^+$, where $V = 2Q - I$, is a symmetrizable operator with non-real points in the spectrum $\sigma_{\mathcal{E}}(X)$.*

Proof. If the proper subspace $\mathcal{S}$ is non-compatible, and Q is a proper projection with range $\mathcal{S}$, then by Theorem 4.9 the operator $Q + Q^+ - I$ is not invertible in $\mathcal{E}$. Equivalently, $V + V^+$ is not invertible, where $V = 2Q - I$. Since $V^2 = I$, we deduce that $-1 \in \sigma_{\mathcal{E}}(VV^+)$. Now consider the continuous unital monomorphism given by

$$\varphi : \mathfrak{P} \to \mathcal{B}(\mathcal{L}), \qquad \varphi(X) = \bar{X}.$$

Since φ is a unital morphism, it follows that $\sigma_{\mathcal{L}}(\bar{X}) \subseteq \sigma_{\mathfrak{P}}(X)$ (see also Theorem 2.3). Moreover, each connected component Δ of $\sigma_{\mathfrak{P}}(X)$ satisfies $\Delta \cap \sigma_{\mathcal{L}}(\bar{X}) \neq \emptyset$ (see [10, Thm. 4.5]). If we apply this result to $X = VV^+$, and take into account that $\sigma_{\mathcal{L}}(\bar{X}) \subseteq (0, \infty)$ and $0 \notin \sigma_{\mathfrak{P}}(X)$, then we find that there is some $z \in \Delta$ with non-trivial imaginary part, where Δ is the connected component that contains -1. Thus, $\sigma_{\mathfrak{P}}(X)$ has non-real points. Hence $\sigma_{\mathcal{E}}(X)$ also has non-real points. ∎

5. Examples

5.1. Non-compatible proper subspaces. We give examples of proper subspaces which are non-compatible.

Example 5.1. Let $\mathcal{E}$ be the space of all sequences (x_n) of complex numbers for which $\sum_n 2^n|x_n| < \infty$ with the norm

$$\|(x_n)\| = \sum_{n=1}^{\infty} 2^n |x_n|.$$

Take $\mathcal{L} = \ell^2$, the Hilbert space of square-summable sequences with its usual inner product. We consider the unilateral shift, i.e. $V : \mathcal{E} \to \mathcal{E}$, $V(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, and the backward shift $V^{(-1)}(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. It is straightforward to see that both are proper operators and $V^+ = V^{(-1)}$. It was proved in [25, Section 3.1] that the symmetrizable operator

$$W_1 = V^{(-1)} + 2I + V = (V^+ + I)(V + I)$$

has spectrum in $\mathcal{E}$ given by all the points inside and on the ellipse

$$\frac{(\mathrm{Re}(\lambda) - 2)^2}{(5/2)^2} + \frac{(\mathrm{Im}(\lambda))^2}{(3/2)^2} = 1.$$

On the other hand, we note that $\sigma_{\mathcal{L}}(W_1) = [0, 4]$. Now consider the proper projection acting on $\mathcal{E} \times \mathcal{E}$ defined by

$$Q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}.$$

We apply Corollary 4.13 with $\lambda = \frac{9}{10} i \in \sigma_{\mathcal{E}}(W_1)$ and $z = \frac{10}{9} W_1$ to conclude that $N(Q) \subseteq \mathcal{E} \times \mathcal{E}$ is a proper subspace which is non-compatible.

Example 5.2. In the previous example, we set $\mathcal{S} = \mathcal{E} \simeq \mathcal{E} \times \{0\}$ and $\mathcal{T} = N(Q)$. Note that $\mathcal{S}$ is a compatible subspace in $\mathcal{E} \times \mathcal{E}$ and $\mathcal{T}$ is a proper companion which is non-compatible. It is not difficult to exhibit other proper companions of $\mathcal{S}$ which are compatible. For instance, we may take

$$Q_z = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix},$$

where z is any symmetrizable operator such that $-1 \notin \sigma_{\mathcal{E}}(z^2)$. By Theorem 4.11 we know $N(Q_z)$ is a proper companion of $\mathcal{S}$ which is a compatible subspace of $\mathcal{E} \times \mathcal{E}$.

Example 5.3. Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\partial\Omega$ is smooth. Let $\mathcal{E} = H_0^1(\Omega)$ be the Sobolev space given by the completion of $C_0^\infty(\Omega)$ with the inner product norm

$$\|f\|_1 = \Big(\int_\Omega (|f(x)|^2 + |(\nabla f)(x)|^2)\, dx \Big)^{1/2}.$$

Let $\mathcal{L} = L^2(\Omega)$ be the Lebesgue space of square-integrable functions endowed with its usual inner product. We will write for short $H_0^1 = H_0^1(\Omega)$ and $L^2 = L^2(\Omega)$. This example is based on [4, Example 4.4], where the reader can find the proofs of all the results used below.

Since Ω is bounded with smooth boundary, there is an orthonormal basis (e_k) of L^2 consisting of eigenfunctions of the Laplacian. These eigenfunctions belong to H_0^1, and $s_k = e_k/(1+\mu_k)^{1/2}$, where (μ_k) are the eigenvalues of the Laplacian, form an orthonormal basis of H_0^1.

Set $\gamma_k = (1+\mu_k)^{1/2}$. Define the following bounded operator on H_0^1:

$$T(s_k) = \begin{cases} \dfrac{\gamma_{2k}}{\gamma_k} s_{2k} & \text{for } k \text{ odd}, \\ \dfrac{\gamma_{2k}}{\gamma_k} s_{2k} + \dfrac{\gamma_{k/2}}{\gamma_k} s_{k/2} & \text{for } k \text{ even}. \end{cases}$$

Note that T is symmetrizable: $\bar{T} = B + S$, where the operators B and S have the following expressions in the basis (e_k):

$$S : L^2 \to L^2, \quad S(e_k) = e_{2k},$$
$$B : L^2 \to L^2, \quad B(e_k) = \begin{cases} 0 & \text{for } k \text{ odd}, \\ e_{k/2} & \text{for } k \text{ even}. \end{cases}$$

In particular, $B = S^*$, and the operator $\bar{T} = B + S$ is self-adjoint in L^2. This gives $\sigma_{L^2}(T) \subseteq \mathbb{R}$. On the other hand, $\sigma_{H_0^1}(T)$ consists of all the points inside and on the ellipse

$$\frac{(\mathrm{Re}(\lambda))^2}{(\sqrt[n]{2} + 1/\sqrt[n]{2})^2} + \frac{(\mathrm{Im}(\lambda))^2}{(\sqrt[n]{2} - 1/\sqrt[n]{2})^2} = 1.$$

If we use Corollary 4.13 with $\lambda = i(\sqrt[n]{2} - 1/\sqrt[n]{2})$, $z = (\sqrt[n]{2} - 1/\sqrt[n]{2})^{-1} T$ and

$$Q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix},$$

we find that $N(Q)$ is a non-compatible proper subspace in $H_0^1 \times H_0^1$.

Example 5.4. One can find a non-compatible proper subspace of H_0^1 using the previous example. If we take the subspace $\mathcal{S} = \overline{\mathrm{span}}\{s_{2k} : k \geq 1\}^{\| \|_1}$, then $\mathcal{S}^\perp \cap H_0^1 = \overline{\mathrm{span}}\{s_{2k+1} : k \geq 0\}^{\| \|_1}$. It follows that $\mathcal{S}$ is a compatible subspace of H_0^1.

We define the following proper operators: $U : H_0^1 \to \mathcal{S}$, $Ue_k = e_{2k}$ and $V : \mathcal{S}^\perp \cap H_0^1 \to H_0^1$, $Ve_k = e_{(k+1)/2}$. Note that $U^+U = VV^+ = I$. Take z as at the end of the previous example. Then

$$\sigma_{H_0^1}((UzV)(UzV)^+) \setminus \{0\} = \sigma_{H_0^1}((Uzz^+U^+)) \setminus \{0\} = \sigma_{H_0^1}(zz^+) \setminus \{0\}.$$

Hence we have $-1 \in \sigma_{H_0^1}((UzV)(UzV)^+)$. Using the decomposition $H_0^1 = \mathcal{S} \dot{+} (\mathcal{S}^\perp \cap H_0^1)$ we define the projection

$$Q = \begin{pmatrix} 1 & UzV \\ 0 & 0 \end{pmatrix}.$$

By Theorem 4.11, the proper subspace $N(Q)$ is non-compatible.

5.2. Trace class and Hilbert–Schmidt operators. In the examples of this subsection, we take $\mathcal{E} = (\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ and $\mathcal{L} = (\mathcal{B}_2(\mathcal{H}), \|\cdot\|_2)$ the spaces of trace class operators and Hilbert–Schmidt operators on a Hilbert space $\mathcal{H}$, respectively. Recall that $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space with inner product given by $\langle x, y\rangle = \mathrm{Tr}(xy^*)$, where Tr denotes the usual trace and $x, y \in \mathcal{B}_2(\mathcal{H})$.

Example 5.5. A projection q acting on $\mathcal{H}$ gives rise to a projection on $\mathcal{B}_1(\mathcal{H})$ by left multiplication, i.e.

$$L_q : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H}), \quad L_q(x) = qx.$$

We note that $\langle L_q(x), y\rangle = \langle x, L_{q^*}(y)\rangle$ for all $x, y \in \mathcal{B}_1(\mathcal{H})$. Thus, L_q is a proper projection, and $L_q^+ = L_{q^*}$. Then, the range of L_q, that is,

$$\mathcal{S} = \{qx : x \in \mathcal{B}_1(\mathcal{H})\},$$

is a proper subspace. Now we prove that $\mathcal{S}$ is a compatible subspace. Let $\sigma(L_x)$ denote that spectrum of L_x in $\mathcal{B}(\mathcal{B}_1(\mathcal{H}))$ and $\sigma(x)$ denote the spectrum of x in $\mathcal{B}(\mathcal{H})$. If $\lambda \notin \sigma(x)$, then there exists $y \in \mathcal{B}(\mathcal{H})$ such that $(x-\lambda)y = y(x-\lambda) = 1$. This implies $(L_x - \lambda)L_y = L_y(L_x - \lambda) = I$, so that $\sigma(L_x) \subseteq \sigma(x)$. Using this fact with $x = q - q^*$, we see that $\sigma(L_q - L_{q^*}) \subseteq \sigma(q - q^*) \subseteq i\,\mathbb{R}$. Also note that $(L_q + L_q^+ - I)^2 = I - (L_q - L_q^+)^2$, so $\sigma((L_q + L_q^+ - I)^2) = \sigma(I - (L_q - L_{q^*})^2) \subseteq [1, \infty]$. We conclude that $L_q + L_q^+ - I$ is invertible on $\mathcal{B}_1(\mathcal{H})$, and by Theorem 4.9, it follows that $\mathcal{S}$ is a compatible subspace.

Example 5.6. Let q be a projection in $\mathcal{B}(\mathcal{H})$. Assume that $R(q) = \mathcal{K}$ is an infinite-dimensional subspace of $\mathcal{H}$ satisfying $\mathcal{K} \oplus \mathcal{K} = \mathcal{H}$ (orthogonal sum). Denote by C_q the operator

$$C_q : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H}), \quad C_q(x) = qxq.$$

Clearly, C_q is a continuous projection. It is easily seen that $C_q^+ = C_{q^*}$, so C_q is a proper projection and its range

$$\mathcal{S} = \{qxq : x \in \mathcal{B}_1(\mathcal{H})\}$$

is a proper subspace. We shall see below that this subspace is compatible.

We write q as a matrix with respect to the above decomposition of $\mathcal{H}$ as

$$q = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}.$$

We consider the matrix representation of arbitrary operators $x, y \in \mathcal{B}(\mathcal{H})$ with respect to the decomposition $\mathcal{K} \oplus \mathcal{K} = \mathcal{H}$:

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Then

$$qxq = \begin{pmatrix} x_{11} + zx_{21} & (x_{11} + zx_{21})z \\ 0 & 0 \end{pmatrix}, \tag{5.1}$$

and

$$qxqy^* = \begin{pmatrix} (x_{11} + zx_{21})y_{11}^* + (x_{11} + zx_{21})zy_{12}^* & * \\ 0 & 0 \end{pmatrix}.$$

Thus, y is orthogonal to $\mathcal{S}$ if and only if $\mathrm{Tr}((x_{11} + zx_{21})(y_{11}^* + zy_{12}^*)) = 0$ for all $x \in \mathcal{B}_1(\mathcal{H})$. Therefore,

$$\mathcal{S}^\perp \cap \mathcal{B}_1(\mathcal{H}) = \{y \in \mathcal{B}_1(\mathcal{H}) : y_{11} + y_{12}z^* = 0\}.$$

The subspace $\mathcal{S}$ is compatible if and only if $\mathcal{S} \dot{+} (\mathcal{S}^\perp \cap \mathcal{B}_1(\mathcal{H})) = \mathcal{B}_1(\mathcal{H})$. This means that any operator $w \in \mathcal{B}_1(\mathcal{H})$ can be written as

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + zx_{21} - y_{12}z^* & (x_{11} + zx_{21})z + y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

The only non-trivial equations to solve are

$$w_{11} = x_{11} + zx_{21} - y_{12}z^*, \qquad w_{12} = (x_{11} + zx_{21})z + y_{12}.$$

Set $a = x_{11} + zx_{21}$, $b = y_{12}$, $x = w_{11}$ and $y = w_{12}$. Then $\mathcal{S}$ is a compatible subspace if and only if the system

$$x = a - bz^*, \qquad y = az + b$$

has a solution $a, b \in \mathcal{B}_1(\mathcal{K})$ for each pair $x, y \in \mathcal{B}_1(\mathcal{K})$. By the second equation, $b = y - az$, so we have to solve $x + yz^* = a + azz^*$. As $1 + zz^*$ is positive, we find

$$a = (x + yz^*)(1 + zz^*)^{-1}, \qquad b = y - (x + yz^*)(1 + zz^*)^{-1}z.$$

Hence $\mathcal{S}$ is a compatible subspace.

5.3. Proper invertible operators. Proper operators have three different notions of inverses (see Remark 2.5). In this subsection we study proper invertible operators.

EXAMPLE 5.7. We consider invertible operators in $\mathcal{E}$ which are isometric for the $\mathcal{L}$ inner product. We shall call them *unitarizable operators*. In the special case when $\mathcal{E} = \mathcal{H}$ is a Hilbert space, these were studied in [4] and [3]. They can be obtained, for instance, as exponentials $A = e^{iX}$ with X a symmetrizable operator. But not every $\mathcal{L}$-isometric operator is an exponential (see [4, Example 4.9]).

EXAMPLE 5.8. A special case of the above example occurs if we take $\mathcal{E} = \mathcal{B}_1(\mathcal{H})$ and $\mathcal{L} = \mathcal{B}_2(\mathcal{H})$. Let u, v be unitary operators in $\mathcal{H}$, and denote by x^t the transpose of $x \in \mathcal{B}(\mathcal{H})$ with respect to a given orthonormal basis of $\mathcal{H}$. Then the operators

$$\mu_{u,v}, \theta_{u,v} \in \mathcal{B}(\mathcal{B}_1(\mathcal{H})), \quad \mu_{u,v}(x) = uxv, \quad \theta_{u,v}(x) = ux^t v,$$

are isometric for the norms $\|\cdot\|_p$ for any $1 \le p \le \infty$ (for $p \ne 2$, any isometry for the p norm is of this type [7]). Thus $\mu_{u,v}$ and $\theta_{u,v}$ are invertible in $\mathfrak{P}$ (in fact they are unitarizable). If one replaces the unitaries u, v by invertible operators g, h in $\mathcal{H}$, then $\mu_{g,h}$ and $\theta_{g,h}$ are proper and invertible operators in $\mathcal{B}_1(\mathcal{H})$ with inverses $\mu_{g^{-1},h^{-1}}$ and $\theta_{g^{-1},h^{-1}}$ which are also proper operators.

Example 5.9. Let $\mathcal{E} = H_0^1(\Omega)$ be the Sobolev space of the domain $\Omega \subset \mathbb{R}^n$. Let $\mathcal{L} = L^2(\Omega)$. Pick a function φ which is C^1 in Ω, continuous and bounded in $\bar{\Omega}$ and satisfies $|\varphi(x)| \ge r > 0$ for $x \in \Omega$. In addition, assume that $\nabla\varphi$ is also bounded in Ω. Then the multiplication operator

$$M_\varphi f = \varphi f$$

preserves $\mathcal{E}$. Its adjoint in $\mathcal{L}$, which is $M_{\bar{\varphi}}$, also preserves $\mathcal{E}$. Thus M_φ is a proper operator. Its inverse $M_{1/\varphi}$ also belongs to $\mathfrak{P}$. Thus, $M_\varphi \in \mathfrak{P}^\times$. Moreover, apparently

$$\sigma_{\mathcal{L}}(M_\varphi) = \sigma_E(M_\varphi) = \sigma_{\mathfrak{P}}(M_\varphi) = \varphi(\bar{\Omega}).$$

There is another situation in which the three spectra coincide.

Proposition 5.10. *Let $G \in \mathfrak{P}$ be such that $G - I$ and $G^+ - I$ are compact operators on $\mathcal{E}$. Assume that $\bar{G}$ is invertible on $\mathcal{L}$. Then $G \in \mathfrak{P}^\times$.*

Proof. The set of invertible operators G in $\mathcal{E}$ such that $G - I$ is compact forms a closed subgroup of the invertible group of $\mathcal{E}$ (it is sometimes called the *Fredholm group* of $\mathcal{E}$). Let $G = I + K$ for some K compact in $\mathcal{E}$. The operator K is proper, and therefore $\bar{K}$ is compact in $\mathcal{L}$, with the same (non-zero) eigenvalues as K. Furthermore, the multiplicity of each non-zero eigenvalue is the same over $\mathcal{E}$ and $\mathcal{L}$ (see Theorem 2.3). Thus 0 does not belong to the spectrum of G. Since $K^+ = \bar{K}^*|_{\mathcal{E}}$ is also compact on $\mathcal{E}$, $\bar{K}^*$ is compact on $\mathcal{L}$, and its eigenvalues are the conjugates of the eigenvalues of K. It follows that $G^+ = I + K^+$ has trivial kernel, and thus, by the Fredholm alternative, it is invertible in $\mathcal{E}$. ∎

Remark 5.11. Unitarizable operators preserve compatible subspaces: if G is unitarizable and $\mathcal{S}$ is compatible, then $G(\mathcal{S})$ is also compatible. This allows one to produce more examples of proper subspaces which are non-compatible. Namely, if $\mathcal{S}$ is a proper non-compatible subspace and G is unitarizable, then $G(\mathcal{S})$ is a proper non-compatible subspace.

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Esteban Andruchow, María Eugenia Di Iorio y Lucero
Instituto de Ciencias
Universidad Nacional de Gral. Sarmiento
J.M. Gutierrez 1150
1613 Los Polvorines, Argentina
and
Instituto Argentino de Matemática, 'Alberto P. Calderón'
CONICET, Saavedra 15 3er. piso
1083 Buenos Aires, Argentina
E-mail: eandruch@ungs.edu.ar
mdiiorio@ungs.edu.ar

Eduardo Chiumiento
Departamento de Matemática, FCE-UNLP
Calles 50 y 115
1900 La Plata, Argentina
and
Instituto Argentino de Matemática, 'Alberto P. Calderón'
CONICET, Saavedra 15 3er. piso
1083 Buenos Aires, Argentina
E-mail: eduardo@mate.unlp.edu.ar