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Spatiality of derivations of Fréchet GB*-algebras

by

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Abstract. We show that every continuous derivation of a countably dominated Fréchet GB*-algebra A is spatial whenever A is additionally an AO*-algebra.

1. Introduction. Bounded and unbounded derivations of C*-algebras are well understood. For example, all derivations $\delta : A \to A$ of a C*-algebra $A[\|\cdot\|]$ are continuous [22]. By a derivation of an algebra A, we mean a linear map $\delta : D(\delta) \to A$ satisfying $\delta(xy) = x\delta(y)+\delta(x)y$ for all $x, y \in D(\delta)$, where $D(\delta)$ denotes the domain of A. We recall that unbounded derivations of C*-algebras, in general, play an important role in mathematical physics in that they are, in some cases, generators of one-parameter automorphism groups of C*-algebras, which model the dynamics of the underlying quantum system of observables [22]. Since the observables are unbounded operators in a Hilbert space, one is therefore motivated to study derivations of unbounded operator algebras.

The first paper on derivations of unbounded operator algebras is [8]. A few more results appeared later in 1992, when R. Becker [7] proved, amongst other things, that every derivation of a pro-C*-algebra (an inverse limit of C*-algebras) is continuous. There are also some results on derivations of measurable operators affiliated with a von Neumann algebra, as can be found in [1] and [2].

An important class of locally convex $*$-algebras is that of generalized B*-algebras, or GB*-algebras for short, introduced by G. R. Allan [4]. As explained in Section 2, these algebras constitute a class of topological $*$-algebras $A[\tau]$ which contain a C*-algebra $A[B_0]$ as a dense $*$-subalgebra. Every GB*-algebra has a faithful representation as a $*$-algebra of unbounded operators on a Hilbert space [10], and therefore GB*-algebras can be re-

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garded as $*$-algebras consisting of unbounded operators. All of the above provides sufficient motivation for a general study of derivations of GB*-algebras.

Recall that every derivation $\delta : A \to A$ of a C*-algebra of bounded operators A on some Hilbert space H is spatial in the enveloping von Neumann algebra of A, which is identified with the bidual A^{**} of A. In this paper, we extend this result to countably dominated Fréchet GB*-algebras which are also AO*-algebras (see Proposition 3.19). As is the case for C*-algebras, we first show that the strong bidual $A^{**}[t_s]$ of a Fréchet GB*-algebra $A[\tau]$ is also a Fréchet GB*-algebra over the W*-algebra $A[B_0]^{**}$. Most of Section 3 is devoted to proving this result. In Lemma 3.2, we show that every continuous derivation $\delta : A \to A$ of a Fréchet GB*-algebra $A[\tau]$ can be extended to a derivation $\delta^{**} : A^{**}[t_s] \to A^{**}[t_s]$ of $A^{**}[t_s]$. In [27], we proved that every derivation $\delta : A \to A$ of a GB*-algebra $A[\tau]$, for which $A[B_0]$ is a W*-algebra, is inner. Therefore δ^{**} is inner, implying that every continuous derivation of a certain Fréchet GB*-algebra is spatial in its strong bidual.

Section 2 contains background on GB*-algebras, necessary to understand the main results of this paper. In Section 4, we give a nontrivial example of a countably dominated Fréchet GB*-algebra which is also an AO*-algebra.

2. Preliminaries. All vector spaces in this paper are over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed to be Hausdorff. Moreover, all algebras are assumed to have an identity element denoted by 1.

A *topological algebra* is an algebra which is also a topological vector space such that multiplication is separately continuous [12]. A *topological $*$-algebra* is a topological algebra endowed with a continuous involution. A topological $*$-algebra which is also a locally convex space is called a *locally convex $*$-algebra*. The symbol $A[\tau]$ will stand for a topological ($*$-)algebra A endowed with a given topology τ.

Definition 2.1 ([4]). Let $A[\tau]$ be a topological $*$-algebra and $\mathcal{B}^*$ a collection of subsets B of A with the following properties:

(i) B is absolutely convex, closed and bounded;
(ii) $1 \in B$, $B^2 \subset B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B, which is a normed algebra under the gauge function $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called (Allan) *bounded* if for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is bounded in A. We denote by A_0 the set of all bounded elements in A.

A topological $*$-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to A_0.

In [10], the collection $\mathcal{B}^*$ in the definition above is defined to be the same as above, except that $B \in \mathcal{B}^*$ is no longer assumed to be absolutely convex. The notion of a bounded element is a generalization of the concept of bounded operator on a Banach space, and was introduced by G. R. Allan [3] in order to develop a spectral theory for general locally convex $*$-algebras.

DEFINITION 2.2 ([4]). A symmetric pseudo-complete locally convex $*$-algebra $A[\tau]$ such that the collection $\mathcal{B}^*$ has a greatest member denoted by B_0, is called a *GB^*-algebra* over B_0.

Every sequentially complete locally convex algebra is pseudo-complete [3, Proposition 2.6]. In [10], P. G. Dixon extended the notion of GB*-algebras to include topological $*$-algebras which are not locally convex. In this definition, GB*-algebras are not assumed to be pseudo-complete, B_0 is the only element in $\mathcal{B}^*$ which is necessarily absolutely convex (see the paragraph before Definition 2.2), and only $A[B_0]$ is assumed to be complete with respect to the gauge function $\|\cdot\|_{B_0}$. For a survey on GB*-algebras, see [13].

Every C*-algebra is a GB*-algebra, but the Arens algebra $L^\omega[0,1]$ is a GB*-algebra over $L^\infty[0,1]$ which is not a C*-algebra. For further examples, see [4], [10].

PROPOSITION 2.3 ([4, Theorem 2.6]). *If $A[\tau]$ is a GB^*-algebra, then the Banach $*$-algebra $A[B_0]$ is a C^*-algebra which is sequentially dense in A, and $(1+x^*x)^{-1} \in A[B_0]$ for every $x \in A$. Furthermore, B_0 is the unit ball of $A[B_0]$.*

If A is commutative, then $A_0 = A[B_0]$ [4, p. 94]. In general, A_0 is not a $*$-subalgebra of A, and $A[B_0]$ contains all normal elements of A_0 [4, p. 94].

It is well known that every commutative C*-algebra is topologically and algebraically $*$-isomorphic to $C(X)$ for some compact Hausdorff space (in fact, X is the maximal ideal space of A). More generally, any commutative GB*-algebra is algebraically $*$-isomorphic to an algebra of functions on a compact Hausdorff space X, which are allowed to take the value infinity at most on a nowhere dense subset of X [4, Theorem 3.9]. This algebraic $*$-isomorphism extends the Gelfand isomorphism of $A[B_0]$ onto the corresponding $C(X)$.

Recall that every C*-algebra is topologically-algebraically $*$-isomorphic to a norm closed $*$-subalgebra of $B(H)$ for some Hilbert space H. In general, for every GB*-algebra $A[\tau]$, there exists a faithful $*$-representation $\pi : A \to \pi(A)$, which we shall call the *universal representation* of A, such that $\pi(A)$ is an algebra of closable and densely defined operators in a Hilbert space H with B_0 being identified with $\{x \in \pi(A) \cap B(H) : \|x\| \le 1\}$ [10, Theorem 7.6]. Therefore, for every $a \in A$, it follows that $\|(1+a^*a)^{-1}\|_{B_0} \le 1$ (see also [4, Theorem 2.6]) and $a(1+a^*a)^{-1} \in A[B_0]$.

The algebra $\pi(A)$, where π is the universal representation of A, acts on the invariant domain D which is the algebraic direct sum $\bigoplus_{f\in F} A/N_f$, where F denotes the set of all positive linear functionals on A, $N_f = \{a \in A : f(a^*a) = 0\}$ and A/N_f is an inner product space under the inner product $\langle a + N_f, b + N_f\rangle = f(b^*a)$, $a, b \in A$. The domain D is an inner product space under the inner product $\langle(\xi_f)_{f\in F}, (\eta_f)_{f\in F}\rangle = \sum_{f\in F}\langle\xi_f, \eta_f\rangle$, and the Hilbert space H related to the universal representation π is taken to be the norm completion of D. The representation π is defined by

$$\pi(a)((\xi_f)_{f\in F}) = (\pi_f(a)\xi_f)_{f\in F}, \qquad a \in A,\ (\xi_f)_{f\in F} \in D,$$

where

$$\pi_f(a)(b + N_f) = ab + N_f, \qquad a, b \in A,\ f \in F.$$

The domain D is also equipped with the graph topology $t_{\pi(A)}$, which is defined by the seminorms $\xi \in D \mapsto \|\pi(a)\xi\|$, $a \in A$. The algebra $\pi(A)$ can be viewed as being a $*$-subalgebra of

$$\mathcal{L}^\dagger(D) = \{T : D \to D \text{ is a closable linear map} : D \subset D(T^*),\ T^*(D) \subset D\},$$

where $D(T^*)$ is the domain of the adjoint T^* of the densely defined operator T. For a dense domain D in some Hilbert space H, the algebra $\mathcal{L}^\dagger(D)$ is a $*$-algebra of closable operators with involution given by $T^\dagger = T^*|_D$, and was introduced by G. Lassner [21]. A $*$-subalgebra U of $\mathcal{L}^\dagger(D)$ is said to be *closed* if $D = \bigcap_{a\in U} D(\overline{a})$, where $\overline{a}$ denotes the smallest closed extension of a.

A $*$-subalgebra of $\mathcal{L}^\dagger(D)$ containing the identity operator on D is called an *O^*-algebra* on D [21]. An O*-algebra B on D is endowed with the uniform topology τ_D [21], which is defined by the family of seminorms $p_{\mathcal{M}}(a) = \sup\{|\langle a\xi, \eta\rangle| : \xi, \eta \in \mathcal{M}\}$, for all subsets $\mathcal{M}$ of D which are bounded with respect to the graph topology t_B.

A locally convex $*$-algebra $A[\tau]$ is said to be an *AO^*-algebra* if it is algebraically and topologically $*$-isomorphic to an O*-algebra $B[\tau_D]$ which is complete.

For an O*-algebra A on D, an operator $a \in A$ is called *positive*, denoted by $a \geq 0$, if $\langle a\xi, \xi\rangle \geq 0$ for all $\xi \in D$. For such an operator $a \geq 0$, the following vector subspace of A is defined:

$$\eta_a = \{b \in A : \rho_a(b) < \infty\}, \quad \text{where} \quad \rho_a(b) = \sup_{\xi\in D} \frac{|\langle b\xi, \xi\rangle|}{\langle a\xi, \xi\rangle}$$

$\big(\frac{\lambda}{0} = \infty$ for $\lambda > 0\big)$. For every $a \in A^+ := \{b \in A : b \geq 0\}$, the space η_a is a normed space under the norm ρ_a, and the subspaces η_b, $b \in A^+$, form an inductive system of normed spaces. The locally convex inductive limit topology of the system $(\eta_a, \rho_a)_{a\in A^+}$ of normed spaces is denoted by ρ [18].

An O*-algebra A on a dense domain D in some Hilbert space H for which the topology ρ can be constructed by a sequence of subspaces η_{a_n}, $a_n \in A^+$, $n \in \mathbb{N}$, is called *countably dominated* [19, p. 756]. Countably dominated algebras occur frequently in analysis, as pointed out in [19]. Particular examples of countably dominated algebras are studied in [6, Section 2]. As noted in [19, p. 756], an O*-algebra A on D is countably dominated if and only if its positive cone admits a *cofinal* sequence for its natural order (i.e. there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A^+ such that for every $a \in A^+$ there is some $n \in \mathbb{N}$ with $a \leq a_n$), which is equivalent to the fact that the domain D is a metrizable space with respect to the graph topology t_A (for a proof of this fact, see Lemma 3.6). We recall that for a given vector topology τ on A, its positive cone A^+ is called *normal* if there exists a base of neighborhoods of 0 for the topology τ consisting of order convex sets. A subset V of A is *order convex* if $\{z \in A : x \leq z \leq y\} \subset V$ whenever $x, y \in V$ and $x \leq y$.

Recall that a *derivation* $\delta : D(\delta) \to A$ is a linear map satisfying $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in D(\delta)$. From here on we will only consider derivations whose domain is the entire algebra A, i.e. derivations $\delta : D(\delta) \to A$ with $D(\delta) = A$. If $\delta : A \to A$ is a derivation of an algebra A which is a subalgebra of an algebra B, then we say that δ is *spatial* if there exists an element $b \in B$ such that

$$\delta(x) = bx - xb \quad \text{for all } x \in A.$$

If this element b can be found in A, then we say that δ is an *inner* derivation.

3. Main results. Let $A[\tau]$ be a locally convex algebra and $\delta : A \to A$ a τ-τ continuous derivation of A. We denote by A^* the dual of A endowed with the *dual topology*, i.e. the topology of uniform convergence on τ-bounded subsets of A. Moreover A^{**} stands for the bidual of A endowed with the *bidual topology*, denoted by t_s, i.e. the topology of uniform convergence on bounded subsets of A^* with respect to the dual topology.

Since $\delta : A \to A$ is τ-τ continuous, the map $\delta^* : A^* \to A^*$, $\delta^*(f) = f \circ \delta$, is a well-defined linear map.

LEMMA 3.1. *For a locally convex algebra $A[\tau]$ and $\delta : A \to A$ a τ-τ continuous derivation, the map $\delta^* : A^* \to A^*$, $f \mapsto f \circ \delta$, is continuous with respect to the dual topology on A^*.*

Proof. Let $(f_i)_{i\in I} \subset A^*$ be such that $f_i \to 0$ with respect to the dual topology on A^*. Then $\sup\{|f_i(a)| : a \in B\} \to 0$ for every τ-bounded subset B of A. Hence $\sup\{|\delta^*(f_i)(a)| : a \in B\} = \sup\{|f_i(\delta(a))| : a \in B\} \to 0$, since $\delta(B)$ is τ-bounded because B is τ-bounded and δ is τ-τ continuous. ∎

We now consider the map $\delta^{**} : A^{**} \to A^{**}$, $\delta^{**}(x^{**})(f) = x^{**}(\delta^*(f))$, where $x^{**} \in A^{**}$ and $f \in A^*$. By similar arguments to those in the proof of the previous lemma, we easily find that δ^{**} is a well-defined t_s-t_s continuous linear map.

Suppose that $A[\tau]$ is a Fréchet locally convex algebra. Then $A[\tau]$ is barrelled and hence multiplication on A is hypocontinuous [25, p. 160]. Moreover, since $A[\tau]$ is metrizable, $A^{**}[t_s]$ is a Fréchet space [23, Corollary 2, p. 153].

As in [16, Lemma 3.4], the following multiplication is defined on A^{**}, which we will denote by $\Box$: for $x^{**}, y^{**} \in A^{**}$,

$$\begin{aligned} x^{**} \Box y^{**} \in A^{**}, &\quad \text{where} \quad (x^{**} \Box y^{**})(f) = x^{**}(y^{**} \cdot f), \quad f \in A^*;\\ y^{**} \cdot f \in A^*, &\quad \text{where} \quad (y^{**} \cdot f)(a) = y^{**}(f \cdot a), \quad a \in A;\\ f \cdot a \in A^*, &\quad \text{where} \quad (f \cdot a)(b) = f(ab), \quad b \in A. \end{aligned}$$

The map $\Box : (A^{**}, t_s) \times (A^{**}, t_s) \to (A^{**}, t_s)$ is separately continuous [16, Theorem 3.8], hence A^{**} endowed with the multiplication $\Box$ is a Fréchet topological algebra [16, Theorem 3.9].

LEMMA 3.2. *Let $A[\tau]$ be a Fréchet locally convex algebra and $\delta : A \to A$ a τ-τ continuous derivation. The map $\delta^{**} : A^{**} \to A^{**}$ is a derivation when A^{**} is endowed with the multiplication $\Box$.*

Proof. For $x^{**}, y^{**} \in A^{**}$, $f \in A^*$, we have

$$\delta^{**}(x^{**} \Box y^{**})(f) = (x^{**} \Box y^{**})(\delta^*(f)) = x^{**}(y^{**} \cdot \delta^*(f)).$$

Also,

$$(\delta^{**}(x^{**}) \Box y^{**} + x^{**} \Box \delta^{**}(y^{**}))(f) = \delta^{**}(x^{**})(y^{**} \cdot f) + x^{**}(\delta^{**}(y^{**}) \cdot f).$$

So it suffices to show that

$$\delta^*(y^{**} \cdot f) + \delta^{**}(y^{**}) \cdot f = y^{**} \cdot \delta^*(f).$$

On the one hand, for $a \in A$, we have

$$\begin{aligned} (\delta^*(y^{**} \cdot f) + \delta^{**}(y^{**}) \cdot f)(a) &= (y^{**} \cdot f)(\delta(a)) + \delta^{**}(y^{**})(f \cdot a)\\ &= y^{**}(f \cdot \delta(a) + \delta^*(f \cdot a)). \end{aligned}$$

Moreover, $(f \cdot \delta(a) + \delta^*(f \cdot a))(b) = f(\delta(a)b) + (f \cdot a)(\delta(b)) = f(\delta(ab))$ for all $b \in A$.

On the other hand, $(y^{**} \cdot \delta^*(f))(a) = y^{**}(\delta^*(f) \cdot a)$, where, for $b \in A$,

$$(\delta^*(f) \cdot a)(b) = \delta^*(f)(ab) = f(\delta(ab)),$$

hence we have the result. ∎

Note that δ^{**} is an extension of δ, since for $a \in A$ and $f \in A^*$, we have

$$\begin{aligned} \delta^{**}(a)(f) &= \widehat{a}(\delta^*(f)) = \delta^*(f)(a)\\ &= f(\delta(a)) = \widehat{\delta(a)}(f), \quad \text{so} \quad \delta^{**}_{|A} = \delta, \end{aligned}$$

where $\widehat{\ }: A \to A^{**}$ denotes the canonical embedding of A into A^{**}.

Let now $A[\tau]$ be a GB*-algebra. We consider A as being faithfully represented, via the universal representation π (see Section 2), as a $*$-subalgebra of $\mathcal{L}^\dagger(D)$ for a domain D dense in some Hilbert space H. *Throughout what follows, we refer to π as the universal representation of a GB*-algebra.* The *weak topology*, w, on $\pi(A)$ is the topology induced by the family of the seminorms

$$p_{\xi,\eta}(\pi(a)) = |\langle \pi(a)\xi, \eta\rangle|, \quad a \in A,\ \xi, \eta \in D$$

[17, p. 101]. The *σ-weak topology*, σw, on $\pi(A)$ is the topology induced by the seminorms

$$p_{(\xi_n)_n,\,(\eta_n)_n}(\pi(a)) = \Big|\sum_{n=1}^{\infty} \langle \pi(a)\xi_n, \eta_n\rangle\Big|,$$

where $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ are sequences in D such that $\sum_{n=1}^{\infty} \|\pi(a)\xi_n\|^2 < \infty$ for every $a \in A$, and similarly for $(\eta_n)_n$ [17, p. 101].

Since A is a GB*-algebra, $\pi(A)$ is a closed symmetric $*$-algebra [10, Theorem 7.11]. Therefore, from [17, Theorem 3], we see that $\pi(A)^{cc} = [(\pi(A))_b]^w = [(\pi(A))_b]^{\sigma w}$, where $[\ ]^w$ (resp. $[\ ]^{\sigma w}$) stands for the weak (resp. the σ-weak) closure of $\pi(A)_b$ in $\mathcal{L}^\dagger(D)$, and $\pi(A)_b$ is the bounded part of $\pi(A)$, i.e. $\pi(A)_b = \{x \in \pi(A) : \overline{x} \in B(H)\}$. Furthermore,

$$\pi(A)^c = \{S \in \mathcal{L}^\dagger(D) : S\pi(a) = \pi(a)S \text{ for all } a \in A\},$$
$$\pi(A)^{cc} = \{S \in \mathcal{L}^\dagger(D) : ST = TS \text{ for all } T \in \pi(A)^c\}$$

are the commutant and bicommutant of $\pi(A)$ respectively [17, p. 98].

Lemma 3.3. *Let $A[\tau]$ be a GB*-algebra and π the universal representation of A. Then:*

(1) $\pi(A)_b = \pi(A[B_0])$.
(2) $[\pi(A[B_0])]^w = [\pi(A)]^w = [\pi(A)]^{\sigma w}$.

Proof. (1) B_0 is the unit ball of $A[B_0]$ (see Proposition 2.3) and $\pi(B_0) = \{x \in \pi(A) \cap B(H) : \|x\| \le 1\}$ [10, Theorem 7.6]. Therefore $\pi(B_0) = (\pi(A)_b)_1$, where $(\ )_1$ stands for the unit ball of the space in brackets. Since π is faithful, we get the result.

(2) On the one hand, $\pi(A[B_0]) = \pi(A)_b \subset \pi(A)$, which implies that $[\pi(A[B_0])]^w \subset [\pi(A)]^w$. On the other hand, $\pi(A) \subset \pi(A)^{cc} = [\pi(A[B_0])]^w$, which implies that $[\pi(A)]^w \subset [\pi(A[B_0])]^w$. Similarly, we show that $[\pi(A)]^{\sigma w} = [\pi(A)_b]^{\sigma w} = \pi(A)^{cc} = [\pi(A)]^w$. ■

Remark 3.4. From [17, Proposition 1], $\pi(A)^{cc}$ is a symmetric closed *-algebra on D whose bounded part is the von Neumann algebra

$$\pi(A[B_0])'' = \{S \in B(H) : SX = XS \text{ for all } X \in \pi(A[B_0])'\}.$$

Since $\pi(A[B_0])$ is a C*-algebra, $(\pi(A[B_0]))''$ is its enveloping von Neumann algebra of $\pi(A[B_0])$, i.e.

$$(\pi(A[B_0]))'' = [\pi(A[B_0])]^{wot} = [\pi(A[B_0])]^{sot},$$

where $[\ ]^{wot}$ (resp. $[\ ]^{sot}$) denotes the closure of the set in brackets with respect to the weak (resp. strong) operator topology on $B(H)$. Therefore, as we have seen above,

$$[\pi(A[B_0])]^{wot} = (\pi(A[B_0]))'' \subset \pi(A)^{cc} = [\pi(A[B_0])]^{w} = [\pi(A)]^{w}.$$

Let us now assume that A is a GB*-algebra whose image $\pi(A)$ through its universal representation π is a countably dominated algebra. Then there exists a cofinal sequence, say $(\pi(a_n))_{n\in\mathbb{N}}$, in $\pi(A)^+$ such that $\pi(A) = \bigcup_{n\in\mathbb{N}} \eta_{\pi(a_n)}$. Note that $\pi(a_n) \in \pi(A)^+$ implies that $a_n \in A^+$ for all $n \in \mathbb{N}$. Indeed, first, a_n is self-adjoint for all $n \in \mathbb{N}$, as can be easily seen from the faithfulness of π. Furthermore, since $\langle \pi(a_n)\xi, \xi\rangle \geq 0$ for all $\xi \in D$ and from the way π is constructed, we see that $f(a_n) \geq 0$ for every positive linear functional f on A. Therefore $a_n \in A^+$ from [10, Theorem 6.7]. Now since $\pi(a_n) \leq \pi(a_n + 1)$, $n \in \mathbb{N}$, we can assume without loss of generality that $\pi(a_n) \geq 1$ for all $n \in \mathbb{N}$. Then, from [5, Lemma 4.1], we have $\pi(a_n)^2 \geq \pi(a_n)$, $n \in \mathbb{N}$, so $\pi(A) = \bigcup_{n\in\mathbb{N}} \eta_{(\pi(1+a_n^2))}$. Since for all $n \in \mathbb{N}$, $(1 + a_n^2)^{-1}$ exists and belongs to $A[B_0]$, we find that $\pi((1 + a_n^2)^{-1}) = (\pi(1 + a_n^2))^{-1}$ exists and belongs to $\pi(A[B_0])$. Therefore $\pi(1 + a_n^2)D = D$. Thus the positive cone $\pi(A)^+$ is normal with respect to the ρ-topology [5, Theorem 1]. This yields the following result.

Corollary 3.5. *Let $A[\tau]$ be a GB*-algebra such that $\pi(A)$ is countably dominated, where π is the universal representation of A. Then every ρ-continuous linear functional on $\pi(A)$ is σw-continuous.*

Proof. Let f be a ρ-continuous linear functional on $\pi(A)$. Since $\pi(A)^+$ is normal with respect to the ρ-topology, there exist positive and ρ-continuous linear functionals f_1 and f_2 on $\pi(A)$ such that $f = f_1 - f_2$ [23, Chapter 5, §3.3, Corollary 1]. From the way the representation π is constructed (see Section 2), there exist $\xi_1, \xi_2 \in D$ such that $f_i(\pi(a)) = \langle \pi(a)\xi_i, \xi_i\rangle$, $a \in A$, $i = 1, 2$. Therefore f_1 and f_2 are weakly continuous, and hence σw-continuous, and so f is σw-continuous. ∎

The following simple lemma can be found in [19, p. 756] (without proof).

Lemma 3.6. *Let $A \subseteq \mathcal{L}^\dagger(D)$ be a GB*-algebra, for a domain D dense in a Hilbert space H. The positive cone A^+ admits a countable cofinal subset, hence A is countably dominated, if and only if D is a metrizable space under the graph topology t_A, defined by the seminorms $\xi \in D \mapsto \|a\xi\|$, $a \in A$.*

Proof. For the reverse implication, assume that (D, t_A) is a metrizable space. Then it has a countable basis of 0-neighborhoods, say $\{V_n\}_{n\in\mathbb{N}}$. We can suppose that there exists $a_n \in A$ such that $V_n = \{\xi \in D : \|a_n\xi\| \le 1\}$, $n \in \mathbb{N}$. Observe that

$$\|a_n\xi\|^2 = \langle a_n^* a_n \xi, \xi\rangle \le \|(1 + a_n^* a_n)^{1/2}\xi\|^2.$$

Therefore if

$$\Omega_n = \{\xi \in D : \|(1 + a_n^* a_n)^{1/2}\xi\| \le 1\},$$

then $\Omega_n \subset V_n$, hence $\{\Omega_n\}_{n\in\mathbb{N}}$ is a basis of 0-neighborhoods of D for the topology t_A, and from functional calculus for GB*-algebras we know that $(1 + a_n^* a_n)^{1/2} \in A^+$ [10, Theorem 4.12 and Proposition 5.1]. For brevity let us denote $(1 + a_n^* a_n)^{1/2}$ by b_n for all $n \in \mathbb{N}$. Note that for every $\xi \in D$, $\xi \neq 0$, we have $\|b_n\xi\| \neq 0$ for all $n \in \mathbb{N}$.

Now let $a \in A^+$. Then for $V = \{\xi \in D : \|a\xi\| \le 1\}$, there exists $n \in \mathbb{N}$ such that $\Omega_n \subset V$. So for $\xi \in D$, we get $\|a(\xi/\|b_n(\xi)\|)\| \le 1$ and thus $\|a\xi\| \le \|b_n\xi\|$. Hence for every $\xi \in D$, from the geometric mean inequality we have

$$\begin{aligned}\langle a\xi, \xi\rangle &\le \langle a\xi, a\xi\rangle^{1/2}\langle \xi, \xi\rangle^{1/2}\\ &\le \frac{1}{2}\langle (a^*a + 1)\xi, \xi\rangle \le \frac{1}{2}\langle (b_n^2 + 1)\xi, \xi\rangle.\end{aligned}$$

Hence we deduce that $a \le \frac{1}{2}(b_n^2 + 1)$, which implies that A^+ has a countable cofinal subset, namely the set $\{\frac{1}{2}(b_n^2 + 1) : n \in \mathbb{N}\} = \{\frac{1}{2}a_n^* a_n + 1 : n \in \mathbb{N}\}$.

For the forward implication, suppose that A^+ has a cofinal sequence, say $\{a_n : n \in \mathbb{N}\}$. Let V be a 0-neighborhood in D, say

$$V \equiv V_{\epsilon,a} = \{\xi \in D : \|a\xi\| \le \epsilon\},$$

where $\epsilon > 0$ and $a \in A$. Then there exists $n \in \mathbb{N}$ such that $a^*a \le a_n$, hence $\|a\xi\|^2 = \langle a^*a\xi, \xi\rangle \le \langle a_n\xi, \xi\rangle \le \|a_n\xi\|\,\|\xi\|$. Now we can assume that $1/n < \epsilon$, where ϵ is as above, for otherwise there exists $m > n$ such that $1/m < \epsilon$ and $a_n \le a_m$, if we suppose without loss of generality that $(a_n)_n$ is increasing since it is cofinal. So, if

$$V_n = \{\xi \in D : \|\xi\| \le 1/n,\ \|a_n\xi\| \le 1/n\},$$

then $\|a\xi\|^2 \le (1/n)\|\xi\| < \epsilon\|\xi\| < \epsilon^2$ for all $\xi \in V_n$. Thus $\|a\xi\| < \epsilon$, so $V_n \subset V_{\epsilon,a}$. Hence $\{V_n\}_{n\in\mathbb{N}}$ is a countable basis of 0-neighborhoods for D with respect to t_A, i.e. (D, t_A) is a metrizable space. ∎

Now and in what follows, we shall make the assumption that A is a GB-algebra such that $\pi(A)$ is countably dominated. For brevity we will refer to such a GB*-algebra as a* countably dominated GB*-algebra.

We now consider the map $j : D \times D \to \pi(A)^*$, $(\xi, \eta) \mapsto \omega_{\xi,\eta}$, where $\omega_{\xi,\eta}(\pi(a)) = \langle \pi(a)\xi, \eta\rangle$ for all $a \in A$, and $\pi(A)^*$ denotes the set of all ρ-continuous linear functionals on $\pi(A)$. Since $\omega_{\xi,\eta}$ is weakly continuous on $\pi(A)$, it is ρ-continuous (see [19, p. 761]) and thus j is well-defined. Since A is assumed to be countably dominated and say $(\pi(a_n))_{n\in\mathbb{N}}$ is the cofinal sequence in $\pi(A)^+$, we can easily deduce that the graph topology on D is equivalently described by the seminorms $\|\cdot\|_{\pi(a_n)}$, $n \in \mathbb{N}$, where $\|\xi\|_{\pi(a_n)} = \|\pi(a_n)\xi\|$ for every $\xi \in D$. Since for all $n \in \mathbb{N}$, $\pi(a_n) : D \to D$ is $t_{\pi(A)}$-$\|\cdot\|$ continuous, all $\pi(a_n)$ extend to the completion $\tilde{D}$ of D with respect to the graph topology $t_{\pi(A)}$. Therefore the extensions of the seminorms $\|\cdot\|_{\pi(a_n)}$, $n \in \mathbb{N}$, to $\tilde{D}$ define the $t_{\pi(A)}$ topology on $\tilde{D}$. Hence, without loss of generality, we can suppose that the metrizable space D is $t_{\pi(A)}$-complete, i.e. a Fréchet space.

LEMMA 3.7. *Let $A[\tau]$ be a countably dominated GB^*-algebra acting on a domain D via its universal representation π. The map $j : D \times D \to \pi(A)^*$, $(\xi,\eta) \mapsto \omega_{\xi,\eta}$, is continuous when D is endowed with the graph topology $t_{\pi(A)}$ and $\pi(A)^*$ is endowed with the dual topology.*

Proof. Let $\eta_0 \in D$ and $(\xi_n)_{n\in\mathbb{N}} \subset D$ be such that $\xi_n \to 0$ with respect to $t_{\pi(A)}$. Let W be a ρ-bounded subset of $\pi(A)$. From [6, Proposition 1.2], there is an $a_m \in A^+$, $m \in \mathbb{N}$, such that $|\langle T\xi,\xi\rangle| \le \langle \pi(a_m)\xi,\xi\rangle$ for all $\xi \in D$ and $T \in W$. Hence $T \in \eta_{\pi(a_m)}$, and so, as is implied in [18, p. 471], there exists $M < \infty$ such that

$$|\langle T\xi,\eta\rangle| \le M\langle \pi(a_m)\xi,\xi\rangle\langle \pi(a_m)\eta,\eta\rangle, \qquad \xi,\eta \in D,\ T \in W.$$

Therefore

$$\begin{aligned}\sup\{|j(\xi_n,\eta_0)(T)| : T \in W\} &= \sup\{|\langle T\xi_n,\eta_0\rangle| : T \in W\}\\ &\le M\langle \pi(a_m)\xi_n,\xi_n\rangle\, \langle \pi(a_m)\eta_0,\eta_0\rangle\\ &\le M\|\pi(a_m)\xi_n\|\, \|\xi_n\|\langle \pi(a_m)\eta_0,\eta_0\rangle \to 0\end{aligned}$$

as $n \to \infty$. Similarly it can be shown that $j(\eta_0,\xi_n) \to 0$, with respect to the dual topology in $\pi(A)^*$. Hence j is separately continuous, therefore jointly continuous since D is assumed to be a Fréchet space. ∎

REMARK 3.8. (1) From the previous lemma, j extends to a continuous linear map from $D \hat{\otimes} D$ into $\pi(A)^*$, for which we retain the same symbol j. The space $D \hat{\otimes} D$ is the completion of the projective tensor product $D \underset{\pi}{\otimes} D$ when D is equipped with the graph topology $t_{\pi(A)}$. Every σw-continuous linear functional f on $\pi(A)$ is of the form $f(T) = \sum_{n=1}^{\infty} \lambda_n \langle T\xi_n,\eta_n\rangle$ for a unique element $u = \sum_{n=1}^{\infty} \lambda_n \xi_n \otimes \eta_n \in D \hat{\otimes} D$, where $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ are sequences in D converging to zero with respect to $t_{\pi(A)}$, and $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ [5, p. 1017]. Then, from Corollary 3.5, the map j is onto, hence $\pi(A)^*$ is vectorially isomorphic to $D \hat{\otimes} D/\ker j$, via the map induced from j, for which we keep the same symbol.

(2) Since $\pi(A)$ is countably dominated, its dual $\pi(A)^*$ is a Fréchet space (see [19, p. 756]). Therefore j is an injective continuous map from the Fréchet space $D \hat{\otimes} D/\ker j$ onto the Fréchet space $\pi(A)^*$. Then from the open map-

ping theorem for Fréchet spaces we see that j is a topological isomorphism. Therefore $\pi(A)^{**}$ is topologically and vectorially isomorphic, via the transpose map j^* of j, to the set $\{f \in (D \hat{\otimes} D)^* : f|_{\ker j} = 0\}$, which, up to topological vector space isomorphism, is equal to $\pi(A)^{\circ\circ}$, the bipolar of $\pi(A)$ with respect to the duality $(D \hat{\otimes} D, B(D, D))$ (for this duality see [19, Corollary 1]). The symbol $B(D, D)$ stands for the set of all continuous sesquilinear forms on $D \times D$, and $\pi(A)$ is viewed as a subset of $B(D, D)$ via the relation $\pi(a)(\xi, \eta) = \langle \pi(a)\xi, \eta \rangle$, $a \in A$, $\xi, \eta \in D$. The bipolar of $\pi(A)$ equals $[\pi(A)]^{\sigma w} = [\pi(A)]^w$ by the bipolar theorem.

In the proposition that follows, j^* denotes the transpose of j, and π^*, π^{**} denote the transpose and the bi-transpose maps of π respectively. With regard to the above mentioned topological vector space isomorphism of $(D \hat{\otimes} D/\ker j)^*$ with $[\pi(A)]^w$, we are going to view an element $(j^* \circ \pi^{**})(x^{**})$, $x^{**} \in A^{**}$, interchangeably as an element of these two spaces, via the following equality, which holds up to topological vector space isomorphism:

$$\langle (j^* \circ \pi^{**})(x^{**})\xi, \eta \rangle = j^*(\pi^{**}(x^{**}))(\xi \otimes \eta + \ker j), \qquad x^{**} \in A^{**},\ \xi, \eta \in D.$$

PROPOSITION 3.9. *Let $A[\tau]$ be a countably dominated Fréchet GB^*-algebra. The map $j^* \circ \pi^{**} : A^{**} \to [\pi(A)]^w$ is a $\sigma(A^{**}, A^*)$-weak continuous algebraic morphism.*

Proof. Consider a net $(x_i^{**})_{i \in I}$ in A^{**} such that $x_i^{**} \to 0$ with respect to $\sigma(A^{**}, A^*)$. Then, for every $\xi \in D$, we have

$$\begin{aligned} \langle (j^* \circ \pi^{**})(x_i^{**})\xi, \xi \rangle &= j^*(\pi^{**}(x_i^{**}))(\xi \otimes \xi + \ker j) \\ &= \pi^{**}(x_i^{**})(j(\xi \otimes \xi + \ker j)) = \pi^{**}(x_i^{**})(\omega_{\xi,\xi}) \\ &= x_i^{**}(\pi^*(\omega_{\xi,\xi})) \to 0, \end{aligned}$$

since $\pi^*(\omega_{\xi,\xi}) \in A^*$. Therefore $(j^* \circ \pi^{**})(x_i^{**}) \to 0$ weakly. Hence $j^* \circ \pi^{**}$ is $\sigma(A^{**}, A^*)$-weak continuous.

Also, j^* is an algebraic morphism when restricted to $\pi(A)$: indeed, if $c \in A$, then $\pi(c)$ induces a continuous linear map on $D \hat{\otimes} D/\ker j$ given by

$$\pi(c)\Big(\sum_{k=1}^{n} \xi_k \otimes \eta_k + \ker j\Big) = \sum_{k=1}^{n} \langle \pi(c)\xi_k, \eta_k \rangle, \qquad (\xi_k)_{k=1}^n, (\eta_k)_{k=1}^n \subset D.$$

Then

$$\begin{aligned} j^*(\pi(c))\Big(\sum_{k=1}^{n} \xi_k \otimes \eta_k + \ker j\Big) &= \pi(c)\Big(j\Big(\sum_{k=1}^{n} \xi_k \otimes \eta_k + \ker j\Big)\Big) \\ &= \pi(c)\Big(\sum_{k=1}^{n} \omega_{\xi_k,\eta_k}\Big) = \sum_{k=1}^{n} \langle \pi(c)\xi_k, \eta_k \rangle \\ &= \pi(c)\Big(\sum_{k=1}^{n} \xi_k \otimes \eta_k + \ker j\Big). \end{aligned}$$

Therefore $j^*|_{\pi(A)}$ can be identified with the representation π, hence $j^*\circ\pi^{**}|_A$ is an algebraic morphism. For $a \in A$, the map $A^{**} \to A^{**}$, $x^{**} \mapsto \widehat{a} \,\square\, x^{**}$, is $\sigma(A^{**}, A^*)$-$\sigma(A^{**}, A^*)$ continuous [16, Lemma 3.6]. Therefore, if $a \in A$, $b^{**} \in A^{**}$ and $(b_i)_{i\in I} \subset A$ is a net such that $b_i \to b^{**}$ with respect to $\sigma(A^{**}, A^*)$, then

$$\begin{aligned}(j^* \circ \pi^{**})(\widehat{a} \,\square\, b^{**}) &= (j^* \circ \pi^{**})\Big(\lim_{\sigma(A^{**},A^*)} ab_i\Big) \\ &= \lim_{w}\,(j^* \circ \pi^{**})(ab_i) = j^*(\pi^{**}(a))(j^* \circ \pi^{**})\Big(\lim_{\sigma(A^{**},A^*)} b_i\Big) \\ &= (j^* \circ \pi^{**})(a)(j^* \circ \pi^{**})(b^{**}).\end{aligned}$$

Also, for $b^{**} \in A^{**}$, the map $A^{**} \to A^{**}$, $a^{**} \to a^{**} \,\square\, b^{**}$, is $\sigma(A^{**}, A^*)$-$\sigma(A^{**}, A^*)$ continuous [16, Lemma 3.4]. So, if $a^{**}, b^{**} \in A^{**}$ and $(a_i)_{i\in I} \subset A$ with $a_i \to a^{**}$ with respect to $\sigma(A^{**}, A^*)$, we get

$$\begin{aligned}(j^* \circ \pi^{**})(a^{**} \,\square\, b^{**}) &= (j^* \circ \pi^{**})\Big(\lim_{\sigma(A^{**},A^*)} \widehat{a}_i \,\square\, b^{**}\Big) = \lim_{w}\,(j^* \circ \pi^{**})(\widehat{a}_i \,\square\, b^{**}) \\ &= \Big(\lim_{w}(j^* \circ \pi^{**})(a_i)\Big)(j^* \circ \pi^{**})(b^{**}) \\ &= (j^* \circ \pi^{**})(a^{**})(j^* \circ \pi^{**})(b^{**}).\end{aligned}$$

So, $j^* \circ \pi^{**}$ is an algebraic morphism. ∎

LEMMA 3.10. *Let $A[\tau]$ be a countably dominated Fréchet GB^*-algebra. There exists a t_s-t_s continuous vector space involution, say $\flat$, on A^{**}, such that*

$$(j^* \circ \pi^{**})((x^{**})^\flat) = ((j^* \circ \pi^{**})((x^{**})))^\dagger, \qquad x^{**} \in A^{**},$$

where $\dagger$ stands for the involution of $\mathcal{L}^\dagger(D)$.

Proof. For $x^{**} \in A^{**}$, let

$$(x^{**})^\flat(f) := \overline{x^{**}(f^\sharp)},\ f \in A^*, \quad \text{where} \quad f^\sharp(a) := \overline{f(a^*)},\ a \in A.$$

Observe that $f^\sharp \in A^*$, due to the continuity of the involution $*$ on A. The map $\flat$ is well-defined, i.e. $(x^{**})^\flat \in A^{**}$. Indeed, if $(f_i)_{i\in I} \subset A^*$ is such that $f_i \to 0$ with respect to the dual topology on A^*, then $f_i^\sharp \to 0$ with respect to the dual topology: for every τ-bounded subset V of A, we have $\sup\{|f_i^\sharp(a)| : a \in V\} = \sup\{|f_i(a^*)| : a^* \in C\} \to 0$, since $C = \{a^* : a \in V\}$ is a bounded subset of A due to the continuity of the involution on A. Therefore $x^{**}(f_i^\sharp) \to 0$, hence $(x^{**})^\flat \in A^{**}$. Similarly, it can easily be shown that $\flat$ is a t_s-t_s continuous map which defines a vector space involution on A^{**}.

Clearly, for every $a \in A$, $\widehat{a}^\flat$ is identified with a^*, the adjoint element of a in A, since

$$\widehat{a}^\flat(f) = \overline{\widehat{a}(f^\sharp)} = \overline{f^\sharp(a)} = f(a^*) = \widehat{a^*}(f), \qquad f \in A^*.$$

Let $x^{**} \in A^{**}$ and $(x_i)_{i\in I} \subset A$ be such that $x_i \to x^{**}$ with respect to $\sigma(A^{**}, A^*)$. Then, for every $f \in A^*$, we have $(\widehat{x_i})^\flat(f) = \overline{\widehat{x_i}(f^\sharp)} \to \overline{x^{**}(f^\sharp)} = (x^{**})^\flat(f)$, which implies that $\widehat{x_i}^\flat \to (x^{**})^\flat$ with respect to $\sigma(A^{**}, A^*)$. Therefore

$$(j^* \circ \pi^{**})((x^{**})^\flat) = (j^* \circ \pi^{**})\Big(\lim_{\sigma(A^{**},A^*)} (x_i)^*\Big) = \lim_w j^*((\pi(x_i)^*))$$

$$= \Big(\lim_w j^*((\pi(x_i)))\Big)^\dagger = ((j^* \circ \pi^{**})((x^{**})))^\dagger. \ \blacksquare$$

For a countably dominated Fréchet GB*-algebra, let us now consider a second product on A^{**} denoted by $\diamondsuit$ and defined as follows. For $x^{**}, y^{**} \in A^{**}$,

$$(x^{**} \diamondsuit y^{**})(f) = y^{**}(f \cdot x^{**}), \qquad f \in A^*,$$

where $f \cdot x^{**} \in A^*$ such that $(f \cdot x^{**})(a) = x^{**}(a \cdot f)$, $a \in A$, and $a \cdot f \in A^*$ with $(a \cdot f)(b) = f(ba)$, $b \in A$. Since multiplication on A is hypocontinuous, the well-definedness of all of these actions can be seen by using exactly the same arguments as those applied in [16, p. 75].

As noted in the proof of Proposition 3.9, for every $a \in A$, $b^{**} \in A^{**}$, the maps $x^{**} \mapsto \widehat{a}\square x^{**}$ and $x^{**} \to x^{**}\square b^{**}$ are $\sigma(A^{**}, A^*)$-$\sigma(A^{**}, A^*)$ continuous. Therefore, if $x^{**}, y^{**} \in A^{**}$ and $(x_i)_{i\in I}, (y_j)_{j\in J} \subset A$ are such that $x_i \to x^{**}$ and $y_j \to y^{**}$ with respect to the $\sigma(A^{**}, A^*)$-topology, then we get

$$x^{**} \square y^{**} = \lim_i \lim_j \widehat{x_i y_j}$$

in the $\sigma(A^{**}, A^*)$-topology (for a statement of this fact in the normed case, see [9, p. 824]). Similarly, since the maps $x^{**} \mapsto x^{**} \diamondsuit \widehat{a}$ and $x^{**} \mapsto b^{**} \diamondsuit x^{**}$ are $\sigma(A^{**}, A^*)$-$\sigma(A^{**}, A^*)$ continuous,

$$x^{**} \diamondsuit y^{**} = \lim_j \lim_i \widehat{x_i y_j}$$

with respect to the $\sigma(A^{**}, A^*)$-topology.

Proposition 3.11. *Let $A[\tau]$ be a countably dominated Fréchet GB*-algebra which is also an AO*-algebra. Then the two products, $\square, \diamondsuit$, on A^{**} coincide.*

Proof. Let $x^{**}, y^{**} \in A^{**}$ and $(x_i)_{i\in I}, (y_j)_{j\in J} \subset A$ be such that $x_i \to x^{**}$ and $y_j \to y^{**}$ with respect to the $\sigma(A^{**}, A^*)$-topology. Also let f be a positive linear functional on A. From the construction of the universal representation π of A, there exists $\xi_f \in D$ such that $f(a) = \langle \pi(a)\xi_f, \xi_f\rangle$, $a \in A$. Then, by Proposition 3.9, Lemma 3.10 and the comments which follow it, we have

$$\begin{aligned}(x^{**} \square y^{**})(f) &= \lim_i \lim_j \langle \pi(x_i y_j)\xi_f, \xi_f\rangle \\ &= \lim_i \lim_j \langle (j^* \circ \pi^{**})(y_j)\xi_f, (j^* \circ \pi^{**})(x_i^*)\xi_f\rangle \\ &= \langle (j^* \circ \pi^{**})(y^{**})\xi_f, ((j^* \circ \pi^{**})(x^{**}))^\dagger \xi_f\rangle\end{aligned}$$

$$= \lim_j \lim_i \langle \pi(x_i y_j)\xi_f, \xi_f \rangle = \lim_j \lim_i \widehat{x_i y_j}(f)$$
$$= (x^{**} \diamondsuit y^{**})(f).$$

Based on the previous equality and on the fact that on A, as an AO*-algebra, every continuous linear functional is a linear combination of continuous positive linear functionals [24, Corollary 4.4], we get the result. ■

Based on the previous proposition, we derive the following result.

PROPOSITION 3.12. *Let $A[\tau]$ be a countably dominated Fréchet GB*-algebra which is also an AO*-algebra. Then $A^{**}[t_s]$, endowed with the involution $\flat$, is a Fréchet locally convex $*$-algebra.*

Proof. As noted in the remarks before Lemma 3.2, $A^{**}[t_s]$, endowed with the multiplication $\square$, is a Fréchet topological algebra. Also the map $\flat$ is a t_s-continuous vector involution on A^{**} (see Lemma 3.10). So it suffices to show that $\flat$ is an algebraic involution. Let $a \in A$, $y^{**} \in A^{**}$, $f \in A^*$. We have

$$(\widehat{a} \square y^{**})^\flat(f) = \overline{(\widehat{a} \square y^{**})(f^\sharp)} = \overline{(y^{**} \cdot f^\sharp)(a)} = \overline{y^{**}(f^\sharp \cdot a)}$$
$$= \overline{y^{**}((a^* \cdot f)^\sharp)} = (y^{**})^\flat(a^* \cdot f) = ((y^{**})^\flat \square \widehat{a^*})(f)$$
$$= ((y^{**})^\flat \square \widehat{a}^\flat)(f).$$

Let $x^{**}, y^{**} \in A^{**}$ and $(x_i)_{i \in I} \subset A$ be such that $x_i \to x^{**}$ with respect to the $\sigma(A^{**}, A^*)$-topology. Then, from the t_s-continuity of $\flat$ and from Proposition 3.11, we get

$$(x^{**} \square y^{**})^\flat = \lim_{\sigma(A^{**},A^*)} (\widehat{x_i} \square y^{**})^\flat = \lim_{\sigma(A^{**},A^*)} (y^{**})^\flat \square \widehat{x_i}^\flat$$
$$= \lim_{\sigma(A^{**},A^*)} (y^{**})^\flat \diamondsuit \widehat{x_i}^\flat = (y^{**})^\flat \diamondsuit (x^{**})^\flat$$
$$= (y^{**})^\flat \square (x^{**})^\flat. \ ■$$

PROPOSITION 3.13. *Let $A[\tau]$ be a countably dominated Fréchet GB*-algebra. The universal map $\pi : A \to \pi(A) \subset \mathcal{L}^\dagger(D)$ is τ-ρ continuous.*

Proof. Following the argument in [19, p. 771, last paragraph of §2], we are going to prove that every equicontinuous subset of $\pi(A)^*$ corresponds to a $t_{\pi(A)}$-bounded subset of the domain D. Let Ω be an equicontinuous subset of $\pi(A)^*$. Since $\pi(A)^+$ is normal, we can focus on Ω consisting of positive linear functionals (see [23, Corollary 1, pp. 219–220]). Then for every $f \in \Omega$, there exists $\xi^f \in D$ such that $f(\pi(a)) = \langle \pi(a)\xi^f, \xi^f \rangle$ for all $a \in A$. Hence for every $a \in A$, we get

$$\sup_{f \in \Omega} \|\xi^f\|_{\pi(a)} = \sup_{f \in \Omega} \|\pi(a)\xi^f\| = \sup_{f \in \Omega} \langle \pi(a^*a)\xi^f, \xi^f \rangle^{1/2}$$
$$= \sup_{f \in \Omega} f(\pi(a^*a))^{1/2} < \infty,$$

since Ω is equicontinuous. Therefore the set $\{\xi^f : f \in \Omega\}$, to which Ω corresponds, is a bounded subset of D with respect to the graph topology $t_{\pi(A)}$.

Let now $(a_n)_{n\in\mathbb{N}} \subset A$ be such that $a_n \to_\tau 0$. Then, from [21, Theorem 4.2], we see that $\pi(a_n) \to 0$ with respect to the uniform topology τ_D on $\mathcal{L}^\dagger(D)$, i.e. $\sup_{\xi,\eta\in\mathcal{M}} |\langle\pi(a_n)\xi,\eta\rangle| \to_n 0$, where $\mathcal{M}$ runs through the bounded subsets of D with respect to the graph topology $t_{\pi(A)}$. So if Ω is an equicontinuous subset of $\pi(A)^*$ consisting of positive linear functionals, and $B_\Omega = \{\xi_f : f \in \Omega\}$, the $t_{\pi(A)}$-bounded subset of D to which Ω corresponds, we get $\sup\{|\widehat{\pi(a_i)}(f)| : f \in \Omega\} = \sup\{|\langle\pi(a_i)\xi^f,\xi^f\rangle| : \xi^f \in B_\Omega\} \to 0$. Hence, $\pi(a_i) \to 0$ with respect to the topology of uniform convergence on equicontinuous subsets of $\pi(A)^*$ (see [23, Corollary 3, p. 220]). Therefore $\pi(a_i) \to 0$ with respect to the ρ-topology and thus π is τ-ρ continuous. ∎

LEMMA 3.14. *Let $A[\tau]$ be a countably dominated Fréchet GB*-algebra such that $\pi(A)[\rho]$ is a Fréchet space, where π is the universal representation of A. Then the map $j^* \circ \pi^{**} : A^{**} \to (D \hat{\otimes} D/\ker j)^* = [\pi(A)]^w \subset \mathcal{L}^\dagger(D)$ is an algebraic and topological isomorphism from A^{**}, endowed with the bidual topology, onto $(D \hat{\otimes} D/\ker j)^*$, endowed with the dual topology.*

Proof. The fact that $j^* \circ \pi^{**}$ is an algebraic morphism was shown in Proposition 3.9. From Proposition 3.13 and the hypothesis, the universal map $\pi : A \to \pi(A)$ is a τ-ρ continuous, injective and surjective linear map between the Fréchet spaces $A[\tau]$ and $\pi(A)[\rho]$. Thus, from the open mapping theorem for Fréchet spaces, π is a topological isomorphism. Hence π^{**} is a topological isomorphism. Moreover, from Remark 3.8(2), we find that the map $j : D \hat{\otimes} D/\ker j \to \pi(A)^*$ is a topological isomorphism, so its transpose j^* is a topological isomorphism too. From the above facts, the result follows. ∎

REMARK 3.15. (1) In the proof of Proposition 3.13, we have seen that for a countably dominated Fréchet GB*-algebra A, the uniform topology τ_D is stronger than the ρ-topology on $\pi(A)$. Let us now assume that for a GB*-algebra A, $\pi(A)$ is ρ-closed, i.e. $\pi(A)$ is countably dominated, say $\pi(A) = \bigcup_{n\in\mathbb{N}} \eta_{\pi(a_n)}$, and $(\eta_{\pi(a_n)}, \rho_{\pi(a_n)})$ is a Banach space for every $n \in \mathbb{N}$ [6, Definition 1.2]. Consider a $t_{\pi(A)}$-bounded subset B of D and for every $\xi \in B$, let $f_\xi : \pi(A) \to \mathbb{C}$ be defined by

$$f_\xi(\pi(a)) = \langle\pi(a)\xi,\xi\rangle, \quad a \in A.$$

Since f_ξ is bounded on $\pi(A)$ (as can be easily seen) and $(\pi(A),\rho)$ is bornological [18, Theorem 1(1)], we see that $f_\xi \in \pi(A)^*$. Let $\Omega_B = \{f_\xi : \xi \in B\}$. Then Ω_B is a simply bounded subset of $\pi(A)^*$ (i.e. bounded with respect to the topology of uniform convergence on finite subsets of $\pi(A)$). From our assumption of $\pi(A)$ being ρ-closed, $\pi(A)[\rho]$ is a barrelled space, and hence, from [23, Theorem 4.2, p. 83], we conclude that Ω_B is an equicontinuous subset of $\pi(A)^*$.

Let now $(\pi(a_i))_{i\in I} \subset \pi(A)$ be such that $\pi(a_i) \to 0$ with respect to ρ. Then $\pi(a_i) \to 0$ with respect to the topology of uniform convergence on equicontinuous subsets of $\pi(A)^*$ [23, Corollary 4, p. 127]. Therefore for every $t_{\pi(A)}$-bounded subset B of D,

$$\begin{aligned}\sup\{|\langle \pi(a_i)\xi,\xi\rangle| : \xi \in B\} &= \sup\{|f_\xi(\pi(a_i))| : f_\xi \in \Omega_B\}\\ &= \sup\{|\widehat{\pi(a_i)}(f_\xi)| : f_\xi \in \Omega_B\} \to 0.\end{aligned}$$

Hence $\pi(a_i) \to 0$ with respect to τ_D. Therefore, for a GB*-algebra A such that $\pi(A)$ is ρ-closed, the ρ-topology is stronger than the uniform topology τ_D on $\pi(A)$.

(2) If a Fréchet GB*-algebra A is such that $\pi(A) = \bigcup_{n\in\mathbb{N}} \eta_{\pi(a_n)}$ is ρ-closed, then, from [15, Theorem 1, p. 147], there is an $m \in \mathbb{N}$ such that $\pi(A) = \eta_{\pi(a_m)}$. Consequently, by Proposition 3.13 and the open mapping theorem, π is a topological isomorphism.

(3) Let $A[\tau]$ be a countably dominated GB*-algebra. Then, from the paragraph just before Corollary 3.5, the positive cone $\pi(A)^+$ is normal with respect to the ρ-topology. If, in addition, the countably dominated GB*-algebra A is assumed to be Fréchet and such that $\pi(A)[\rho]$ is a Fréchet locally convex algebra, then from the open mapping theorem for Fréchet spaces, $A[\tau]$ is topologically isomorphic to $\pi(A)[\rho]$. Hence, from the aforementioned isomorphism and a useful characterization of normality given in [23, §3.1, (a)⇔(c), p. 215] we can easily conclude that A^+ is normal with respect to τ. Therefore, from [24, Theorem 5.1(II)], $A[\tau]$ is an AO*-algebra.

We are now able to prove the following result.

PROPOSITION 3.16. *Let $A[\tau]$ be a countably dominated Fréchet GB*-algebra. The following statements are equivalent:*

(i) *$A[\tau]$ is an AO*-algebra.*
(ii) *$\pi(A)[\rho]$ is a Fréchet locally convex $*$-algebra.*

Proof. (ii)⇒(i). This is Remark 3.15(3).

(i)⇒(ii). Suppose that $A[\tau]$ is an AO*-algebra. We first define a topology τ' on $\pi(A)$. Let (p_n) be a family of $*$-seminorms on A defining the topology τ. Then we define a family (q_n) of $*$-seminorms on $\pi(A)$ by $q_n(\pi(x)) = p_n(x)$ for all $x \in A$ and $n \in \mathbb{N}$. The topology τ' on $\pi(A)$ defined by the family (q_n) has the property that π is a topological $*$-isomorphism of $A[\tau]$ onto $\pi(A)[\tau']$ and $\pi(A)[\tau']$ is complete.

Since $A[\tau]$ is an AO*-algebra, it follows that A^+ is τ-normal, and hence $\pi(A)^+$ is τ'-normal. By [23, Corollary 1, pp. 219–220], τ' is the topology of uniform convergence on equicontinuous subsets of $(\pi(A)^+)'$, the set of all τ'-continuous positive linear functionals on $\pi(A)$. Recall that since A is countably dominated, $\pi(A)^+$ is ρ-normal. Therefore, the topology ρ on $\pi(A)$ is the topology of uniform convergence of equicontinuous subsets of $(\pi(A)^+)^*$, the set of all ρ-continuous positive linear functionals on $\pi(A)$. By [18, Theorem 1(3)], every positive linear functional on $\pi(A)$ is ρ-continuous. Since $\pi(A)[\tau']$ is complete, every positive linear functional on $\pi(A)$ is τ'-continuous [10, Corollary 8.2]. It is now immediate from the above that the topologies τ' and ρ of $\pi(A)$ coincide. Therefore $\pi(A)[\rho]$ is a Fréchet locally convex $*$-algebra. ∎

The following theorem is the main result of this article.

THEOREM 3.17. *Let $A[\tau]$ be a Fréchet countably dominated GB*-algebra which is also an AO*-algebra. Then $A^{**}[t_s]$ endowed with the multiplication $\square$ is a Fréchet GB*-algebra.*

Proof. By Propositions 3.16 and 3.12, $\pi(A)[\rho]$ and $A^{**}[t_s]$ are Fréchet locally convex $*$-algebras. To show that $A^{**}[t_s]$ is a GB*-algebra, by [20] it suffices to show that the following three conditions hold:

(1) A^{**} contains a $*$-subalgebra B that is a C*-algebra with respect to some norm.
(2) $(1 + (x^{**})^\flat \square x^{**})^{-1} \in B$ for all $x^{**} \in A^{**}$.
(3) The unit ball of B is t_s-bounded.

(1) Clearly A^{**} contains $A[B_0]^{**}$, the bidual of $A[B_0]$ with respect to the norm topology on $A[B_0]$. The latter is isomorphic as a Banach space to the enveloping von Neumann algebra of $A[B_0]$, namely to $\pi(A[B_0])''$. If we define a norm $\|\cdot\|_0$ on $A[B_0]^{**}$ by $\|x^{**}\|_0 := \|j^* \circ \pi^{**}(x^{**})\|$, $x^{**} \in A[B_0]^{**}$, then $A[B_0]^{**}$ endowed with the multiplication $\square$ is a C*-algebra. In fact, $(j^* \circ \pi^{**})(A[B_0]^{**}) = \pi(A[B_0])''$.

(2) Let $x^{**} \in A^{**}$. Since $[\pi(A)]^w$ is a closed symmetric $*$-algebra over $\pi(A[B_0])'' = (j^* \circ \pi^{**})(A[B_0]^{**})$ by [17, Proposition 1], we deduce that $(1 + (j^* \circ \pi^{**})(x^{**})^\dagger (j^* \circ \pi^{**})(x^{**}))^{-1}$ exists in $\pi(A[B_0])''$. Therefore, since $j^* \circ \pi^{**}$ is injective and onto (see Lemma 3.14), we have $(1 + (x^{**})^\flat \square x^{**})^{-1} \in A[B_0]^{**}$.

(3) The unit ball of $(j^* \circ \pi^{**})((A[B_0])^{**})$ coincides with the unit ball of $(\pi(A[B_0]))^{**}$, where the former set is $\|\cdot\|$-bounded. Hence, for every T in the unit ball of $(\pi(A[B_0]))^{**}$ and for all $\xi \in D$,

$$\langle T\xi, \xi \rangle \le \|T\| \langle 1\xi, \xi \rangle \le \langle 1\xi, \xi \rangle,$$

where the identity operator, 1, belongs to $[\pi(A)]^w$. Thus the unit ball of $(\pi(A[B_0]))^{**}$ is ρ-bounded. Now $(D \,\hat{\otimes}\, D/\ker j)^*$ is bornological since, by Lemma 3.14, it is topologically isomorphic to $A^{**}[t_s]$ and the latter space is bornological since it is a Fréchet space provided that A is Fréchet. Therefore, from the proof of [19, Proposition 7], the ρ-bounded subsets of $[\pi(A)]^w$ and the bounded subsets of $(D \hat{\otimes} D/\ker j)^*$ with respect to the dual topology coincide. Hence, the unit ball $(\pi(A[B_0]))^{**}$, when considered as a subset of $(D \hat{\otimes} D/\ker j)^*$, is bounded with respect to the dual topology. So, by Lemma 3.14 the unit ball of $A[B_0]^{**}$ is t_s-bounded: for this, recall that A^{**} contains $A[B_0]^{**}$. ■

In [27], we have proved the following result.

THEOREM 3.18. *Every derivation of a Fréchet GB*-algebra $A[\tau]$ such that $A[B_0]$ is a W^*-algebra is inner, and hence continuous.*

From Theorems 3.17 and 3.18, we obtain the following result.

COROLLARY 3.19. *Let $A[\tau]$ be a Fréchet countably dominated GB*-algebra which is also an AO*-algebra. Then every τ-τ continuous derivation $\delta : A \to A$ is spatial and implemented by an element of A^{**}.*

It is an open question whether every derivation of a Fréchet GB*-algebra is continuous, as is the case for C*-algebras. In addition to Theorem 3.18, we recently proved in [26] that every derivation of a smooth Fréchet nuclear GB*-algebra is continuous. By a *smooth Fréchet nuclear GB*-algebra $A[\tau]$*, we mean a Fréchet GB*-algebra $A[\tau]$ for which the C*-algebra $A[B_0]$ is nuclear, and for which there is a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms defining the topology τ on A such that for every $\lambda \in \Lambda$, there exists $\mu \in \Lambda$ such that $p_\lambda(ab) \le p_\mu(a)p_\lambda(b)$ for all $a, b \in A$. Nuclear GB*-algebras are introduced in [14], and some characterizations and examples can be found therein.

4. An example of a countably dominated Fréchet GB*-algebra which is an AO*-algebra. In this section, we give examples of countably dominated Fréchet GB*-algebras which are also AO*-algebras. All C*-algebras are trivial examples of such algebras. This section is therefore devoted to an example of a countably dominated Fréchet GB*-algebra which is an AO*-algebra, but not necessarily a C*-algebra.

We begin by sketching an example of a GB*-algebra given in [11].

DEFINITION 4.1 ([11, Definition 1.1]). A set $\mathcal{R}$ of bounded self-adjoint linear operators on a Hilbert space H is called a *generating family* if it satisfies the following conditions:

(i) $0 \le a \le 1$ for all $a \in \mathcal{R}$,
(ii) $ab = ba$ for all $a, b \in \mathcal{R}$,
(iii) for all $a, b \in \mathcal{R}$, there exists $c \in \mathcal{R}$ such that $a \le c$ and $b \le c$,
(iv) for every $a \in \mathcal{R}$, there exists $b \in \mathcal{R}$ such that $a \le b^2$.

Observe that the identity operator 1 need not be in $\mathcal{R}$. In what follows, we equip the set $\mathcal{L}_\mathcal{R} = \bigcup_{a\in\mathcal{R}} aH$ with the inductive limit topology

[11, Definition 1.3]. Let x be a densely defined linear operator on H whose domain contains $\mathcal{L}_{\mathcal{R}}$. Then x is called $\mathcal{R}$-*bounded* if xa is a bounded linear operator for every $a \in \mathcal{R}$ [11, Definition 2.1]. We use $\mathcal{R}B(H)$ to denote the vector space of all $\mathcal{R}$-bounded linear operators. For every $x \in \mathcal{R}B(H)$, the restriction of x to $\mathcal{L}_{\mathcal{R}}$ is a continuous linear operator into H, and conversely, every continuous linear operator from $\mathcal{L}_{\mathcal{R}}$ into H is $\mathcal{R}$-bounded [11, remark, p. 112]. It is clear that $\mathcal{R} \subset \mathcal{R}B(H)$.

On $\mathcal{R}B(H)$, we define seminorms p_a, $a \in \mathcal{R}$, by $p_a(x) = \|xa\|$ for every $x \in \mathcal{R}B(H)$. It follows from the previous paragraph that if $p_a(x) = 0$ for all $a \in \mathcal{R}$, then $x = 0$ [11, p. 112], implying that $\mathcal{R}B(H)$ is a Hausdorff locally convex space.

If $A \subset \mathcal{R}B(H)$, then we define $A^c \subset \mathcal{R}B(H)$ to be the set [11, Definition 2.4]

$$\{y \in \mathcal{R}B(H) : yx \in \mathcal{R}B(H), xy \in \mathcal{R}B(H) \text{ and } yxa = xya \text{ for all } x \in A \text{ and } a \in \mathcal{R}\}.$$

Furthermore, we define A^{cc} to be $(A^c)^c$. Observe that $(A^c)^c$ is a subset of $\mathcal{R}B(H)$.

The map $x \mapsto x^+ := x^*|_{\mathcal{L}_{\mathcal{R}}}$ defines an involution on $\mathcal{R}^{cc}$ [11, Definition 3.3 and Lemma 3.4], and $p_a(x) = p_a(x^+)$ for every $a \in \mathcal{R}$ and $x \in \mathcal{R}^{cc}$ [11, Lemma 3.6(ii)]. For this, one also uses the facts that $\mathcal{R} \subset \mathcal{R}^c$, and therefore $\mathcal{R}^{cc} \subset \mathcal{R}^c$.

THEOREM 4.2 ([11, Theorem 3.10, Lemma 3.6(i) and Corollary 3.8]). *Let $\mathcal{R}$ be a generating family of bounded linear operators on H. Then $\mathcal{R}^{cc}$ is a sequentially complete GB*-algebra with respect to the locally convex topology defined by the family of seminorms p_a, $a \in \mathcal{R}$, restricted to $\mathcal{R}^{cc}$, and with respect to the involution defined above.*

The following lemma is trivial, and therefore we omit the proof.

LEMMA 4.3. *Let $0 \le a \in B(H)$, where H is a Hilbert space. Then there exists $n \in \mathbb{N}$ such that $a \le n1$.*

Now let $\mathcal{R}$ be a countable generating family for which there exists $a_0 \in \mathcal{R}$ invertible in $B(H)$ with $a_0^{-1} \in \mathcal{R}^{cc}$. From Theorem 4.2, $\mathcal{R}^{cc}$ is a commutative Fréchet GB*-algebra. Let $x \in \mathcal{R}^{cc}$ with $x \ge 0$. Then xa is bounded for all $a \in \mathcal{R}$. Thus xa_0 is bounded. Since $\mathcal{R}^{cc}$ is commutative, we get $xa_0 = a_0x$, and therefore $xa_0 \ge 0$ (since also $a_0 \ge 0$, as $a_0 \in \mathcal{R}$). By Lemma 4.3, there exists $n \in \mathbb{N}$ such that $xa_0 \le n1$. Now

$$x = xa_0a_0^{-1} = xa_0^{-1}a_0 = a_0^{-1}xa_0,$$

since $\mathcal{R}^{cc}$ is commutative. Observe that we have $n1 - xa_0 \ge 0$, $a_0^{-1} \ge 0$ and

$a_0^{-1}(n1 - xa_0) = (n1 - xa_0)a_0^{-1}$. It follows that

$$na_0^{-1} - x = a_0^{-1}(n1 - xa_0) \geq 0,$$

i.e. $x \leq na_0^{-1}$. Hence $\{na_0^{-1} : n \in \mathbb{N}\}$ is a countable cofinal sequence for $(\mathcal{R}^{cc})^+$, and therefore $\mathcal{R}^{cc}$ is a countably dominated Fréchet GB*-algebra. Since every commutative Fréchet GB*-algebra is an AO*-algebra ([4, Theorem 4.3] and [24, Corollary 6.1]), it follows that $\mathcal{R}^{cc}$ is an AO*-algebra.

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