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On non-realization results and conjectures of N. Kuhn

by

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Abstract. We discuss two extensions of results conjectured by Nick Kuhn about the non-realization of unstable algebras as the mod-p singular cohomology of a space, for p a prime. The first extends and refines earlier work of the second and fourth authors, using Lannes' mapping space theorem. The second (for the prime 2) is based on an analysis of the -1 and -2 columns of the Eilenberg–Moore spectral sequence, and of the associated extension.

In both cases, the statements and proofs use the relationship between the categories of unstable modules and functors between $\mathbb{F}_p$-vector spaces. The second result in particular exhibits the power of the functorial approach.

1. Introduction. Let p be a prime number, $\mathscr{U}$ denote the category of unstable modules and $\mathcal{K}$ the category of unstable algebras over the mod-p Steenrod algebra $\mathcal{A}_p$ [Sch94]. The mod-p singular cohomology of a space X is denoted H^*X.

In the first part of the paper, the topological spaces X considered are p-complete and connected. We assume that H^*X is of finite type (finite-dimensional in each degree) and, moreover, that Jean Lannes' functor T_V acts nicely on H^*X, in the sense that T_VH^*X is of finite type for all V. In order to apply Lannes' theory [Lan92], we also suppose that the spaces considered are 1-connected and that T_VH^*X is 1-connected for all V. For the current arguments, the connectivity hypothesis is not a significant restriction, since it is always possible to collapse the 1-skeleton; the finiteness hypotheses can be relaxed using methods of Fabien Morel, as explained by François-Xavier Dehon and Gérald Gaudens [DG03].

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In an earlier paper, the second and fourth authors gave a proof of the following result, Nick Kuhn's realization conjecture [Kuh95b]:

THEOREM 1.1 ([GS13]). *Let X be a space such that H^*X is finitely generated as an $\mathcal{A}_p$-module. Then H^*X is finite.*

This is a consequence of the following result, Kuhn's strong realization conjecture [Kuh95b], which uses the Krull filtration

$$\mathscr{U}_0 \subset \mathscr{U}_1 \subset \mathscr{U}_2 \subset \cdots \subset \mathscr{U}$$

of the category $\mathscr{U}$ (see Section 2); in particular, $\mathscr{U}_0$ is the full subcategory of locally finite unstable modules.

THEOREM 1.2 ([GS13]). *Let X be a space such that $H^*X \in \mathscr{U}_n$ for some $n \in \mathbb{N}$. Then $H^*X \in \mathscr{U}_0$.*

The aim of this paper is to extend these results in two directions. The first exploits the nilpotent filtration (see Section 3)

$$\mathscr{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \cdots \supset \mathcal{N}il_s \supset \cdots$$

of the category of unstable modules. Here $\mathcal{N}il_1$ is the full subcategory of nilpotent unstable modules. For $p = 2$, an unstable module is nilpotent if the operator $\mathrm{Sq}_0 \colon x \mapsto \mathrm{Sq}^{|x|} x$ acts nilpotently on any element (a similar definition applies for p odd); in particular, the augmentation ideal of a connected, noetherian unstable algebra is nilpotent if and only if it is nilpotent as an unstable module.

Recall that an unstable module is *reduced* if it contains no non-trivial nilpotent submodule. The following result explains how unstable modules are built from (suspensions of) reduced unstable modules:

PROPOSITION 1.3 ([Sch94, Kuh95b]). *An unstable module M has a natural, convergent decreasing filtration $\{\mathrm{nil}_s M\}_{s\geq 0}$ with $\mathrm{nil}_s M \in \mathcal{N}il_s$ and $\mathrm{nil}_s M/\mathrm{nil}_{s+1} M \cong \Sigma^s R_s M$, where $R_s M$ is a reduced unstable module.*

A reduced unstable module M is said to have *degree* $n \in \mathbb{N}$ (written $\deg(M) = n$) if $M \in \mathscr{U}_n \setminus \mathscr{U}_{n-1}$; otherwise $\deg(M) = \infty$. Following Kuhn, define the profile function $w_M \colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ of an unstable module M by

$$w_M(i) := \deg(R_i M).$$

Recall that the module of indecomposable elements

$$QH^*X := \tilde{H}^*X/(\tilde{H}^*X)^2$$

of the cohomology of a pointed space X is an unstable module (here $\tilde{H}^*X$ is the augmentation ideal). Set $w_X := w_{H^*X}$ and $q_X := w_{QH^*X}$.

The following theorem provides a generalization of Theorem 1.2 and stresses the relationship between the profile functions w_X and q_X (which is examined in greater detail in Section 4). A version of the theorem was announced with a sketch proof in [GS12]; the current statement strengthens and unifies existing results.

THEOREM 1.4. *Let X be a space such that $\tilde{H}^*X$ is nilpotent. The following conditions are equivalent:*

(1) $H^*X \in \mathscr{U}_0$*;*
(2) $w_X = 0$*;*
(3) $q_X = 0$*;*
(4) $w_X \leq \mathrm{Id}$*;*
(5) $q_X \leq \mathrm{Id}$*;*
(6) $w_X - \mathrm{Id}$ *is bounded.*

To see that this result implies Theorem 1.2, let X be a space such that $H^*(X) \in \mathscr{U}_n$. This condition implies easily (see Sections 2 and 3) that $w_{\Sigma X} - \mathrm{Id}$ is bounded, hence $H^*(\Sigma X)$ is locally finite, by Theorem 1.4, thus H^*X also.

Theorem 1.4 provides evidence for the following:

CONJECTURE 1. Let X be a space such that $\tilde{H}^*X$ is nilpotent. If q_X is bounded, then $H^*X \in \mathscr{U}_0$.

This should be compared with the following stronger conjecture, which is equivalent to the unbounded strong realization conjecture of [Kuh95b]:

CONJECTURE 2. Let X be a space such that $\tilde{H}^*X$ is nilpotent. If q_X takes finite values, then $H^*X \in \mathscr{U}_0$.

The second generalization concerns the first and second layers of the nilpotent filtration. The method of proof is of independent interest and can be applied in other situations (see [Pow15]); a generalization of this approach may lead to a proof of Conjecture 1.

The fact that the argument is based upon the Eilenberg–Moore spectral sequence for computing $H^*\Omega X$ from H^*X means that the restrictions upon the space X can be relaxed in the following theorem, in which the prime is taken to be 2.

THEOREM 1.5. *Let X be a* 1*-connected space such that $\tilde{H}^*X$ is of finite type and nilpotent. If* $\deg(R_1H^*X) = d \in \mathbb{N}$ *then* $\deg(R_2H^*X) \geq 2d$*.*

The result proved is slightly stronger, giving a precise statement on the cup product on H^*X (see Remark 6.13).

The strategy of proof for Theorem 1.5 is different from that of the previous results and is related to that of [Sch98]. It depends on an analysis of the second stage of the Eilenberg–Moore filtration of $H^*\Omega X$ and uses results on triviality and non-triviality of certain extension groups $\mathrm{Ext}^1_{\mathscr{F}}(-,-)$, where $\mathscr{F}$ is the category of functors on $\mathbb{F}_2$-vector spaces (see Section 2).

The paper is organized as follows. The Krull filtration is reviewed in Section 2 and the nilpotent filtration in Section 3; using this material, the profile functions are considered in Section 4. Theorem 1.4 is proved in Section 5, based on Lannes' theory, which is reviewed briefly. Section 6 is devoted to the proof of Theorem 1.5, using the Eilenberg–Moore spectral sequence.

Remark 1.6. A first version of this work was made available by three of the authors [CGS14]. The third-named author of the present paper proposed the current approach to Section 6.

2. The Krull filtration of $\mathscr{U}$. Gabriel [Gab62] introduced the Krull filtration of an abelian category; for the category of unstable modules, $\mathscr{U}$, this gives the filtration

$$\mathscr{U}_0 \subset \mathscr{U}_1 \subset \mathscr{U}_2 \subset \cdots \subset \mathscr{U}$$

by thick subcategories stable under colimits, which is described in [Sch94].

For an abelian category $\mathcal{C}$, the category $\mathcal{C}_0$ is the largest thick subcategory generated by the simple objects and stable under colimits; $\mathscr{U}_0$ identifies with the subcategory of locally finite modules ($M \in \mathscr{U}$ is locally finite if $\mathcal{A}_p x$ is finite for any $x \in M$). The categories $\mathscr{U}_n$ are then defined recursively as follows. Having defined $\mathscr{U}_n$, form the quotient category $\mathscr{U}/\mathscr{U}_n$; then $\mathscr{U}_{n+1}$ is the pre-image under the canonical projection $\mathscr{U} \to \mathscr{U}/\mathscr{U}_n$ of the subcategory $(\mathscr{U}/\mathscr{U}_n)_0 \subset (\mathscr{U}/\mathscr{U}_n)$.

Proposition 2.1 ([Sch94, Kuh95b]). *If M is a finitely generated unstable module then $M \in \mathscr{U}_d$ for some $d \in \mathbb{N}$.*

Example 2.2. For $k, n \in \mathbb{N}$, we have $\Sigma^k F(n) \in \mathscr{U}_n \setminus \mathscr{U}_{n-1}$, where $F(n)$ is the free unstable module on a generator of degree n. (In the terminology of the Introduction, $\deg F(n) = n$.)

For M an unstable module and $n \in \mathbb{N}$, write $k_n M$ for the largest submodule of M that is in $\mathscr{U}_n$.

Proposition 2.3 ([Kuh14]). *For $M \in \mathscr{U}$, $M = \bigcup_n k_n M$.*

There is a characterization of the Krull filtration in terms of Lannes' T-functor. The functor T_V (for V an elementary abelian p-group) is left adjoint to $M \mapsto H^*BV \otimes M$; $T_{\mathbb{F}_p}$ is denoted simply T. Since $H^*B\mathbb{Z}/p$ splits in $\mathscr{U}$ as $\mathbb{F}_p \oplus \tilde{H}^*B\mathbb{Z}/p$, the functor T is naturally equivalent to $\mathrm{Id} \oplus \bar{T}$, where $\bar{T}$ is left adjoint to $M \mapsto \tilde{H}^*B\mathbb{Z}/p \otimes M$.

Theorem 2.4 ([Lan92, Sch94]). *The functor T_V is exact and commutes with colimits. Moreover, there is a canonical isomorphism*

$$T_V(M_1 \otimes M_2) \cong T_V(M_1) \otimes T_V(M_2);$$

in particular, $T_V(\Sigma M) \cong \Sigma T_V(M)$.

Theorem 2.5 ([Sch94, Kuh14]). *The following are equivalent:*

(1) $M \in \mathscr{U}_n$;
(2) $\bar{T}^{n+1}M = 0$.

Corollary 2.6. *If* $M \in \mathscr{U}_m$ *and* $N \in \mathscr{U}_n$ *then* $M \otimes N \in \mathscr{U}_{m+n}$.

Let $\mathscr{F}$ be the category of functors from finite-dimensional $\mathbb{F}_p$-vector spaces to $\mathbb{F}_p$-vector spaces. There is an exact functor $f\colon \mathscr{U} \to \mathscr{F}$ defined by

$$f(M) : V \mapsto \mathrm{Hom}_{\mathscr{U}}(M, H^*(BV))',$$

where $(-)'$ denotes the continuous dual with respect to the profinite topology provided by considering M as the colimit of its finitely generated submodules [HLS93]. By definition of T_V, there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{U}}(M, H^*(BV))' \cong T_V(M)^0,$$

so that $f(M)$ corresponds to the degree zero part of the T-functor. The functor f induces an embedding of $\mathscr{U}/\mathscr{N}il_1$ into $\mathscr{F}$.

The reduced T-functor $\bar{T}$ corresponds to the difference functor Δ : $\mathscr{F} \to \mathscr{F}$ which is defined on $F \in \mathscr{F}$ by

$$\Delta(F)(V) := \mathrm{Ker}(F(V \oplus \mathbb{F}_p) \to F(V)).$$

Namely, $\Delta(fM) \cong f(\bar{T}M)$ for $M \in \mathscr{U}$.

Let $\mathscr{F}_n \subset \mathscr{F}$ be the subcategory of polynomial functors of degree at most n, defined as the full subcategory of functors F such that $\Delta^{n+1}(F) = 0$. The polynomial degree of a functor F is written $\deg F \in \mathbb{N} \cup \{\infty\}$.

By Theorem 2.5, the following holds:

Proposition 2.7. *For* $n \in \mathbb{N}$, *the functor* $f : \mathscr{U} \to \mathscr{F}$ *restricts to an exact functor* $f : \mathscr{U}_n \to \mathscr{F}_n$.

Corollary 2.8 ([Sch94]). *For* M *a reduced unstable module and* $n \in \mathbb{N}$, *the following are equivalent:*

(1) $M \in \mathscr{U}_n$;
(2) $\deg(fM) \leq n$.

There is also a combinatorial characterization of modules in $\mathscr{U}_n$. This is stated here for $p = 2$; there are analogous results for odd primes. Denote by $\alpha(k)$ the sum of the digits in the binary expansion of k.

Theorem 2.9 ([FS90]). *For* M *a reduced unstable* $\mathcal{A}_2$*-module and* $n \in \mathbb{N}$, $M \in \mathscr{U}_n$ *if and only if* $M^j = 0$ *for* $\alpha(j) > n$.

Proof. The proof given in [FS90] (see also [Sch94, Sections 5.5 and 5.6]) relies on an understanding of reduced objects which are simple in $\mathscr{U}_n \backslash \mathscr{U}_{n-1}$. An alternative argument is sketched here, using Theorem 2.5.

If M is reduced, there is an exact sequence

$$0 \to \Phi M \to M \to \Sigma \Omega M \to 0 \tag{2.1}$$

[Sch94, Section 1.7], where Φ is the Frobenius functor and Ω is the algebraic loop functor. Using standard properties of T, Theorem 2.5 implies for M reduced that $M \in \mathscr{U}_n \setminus \mathscr{U}_{n-1}$ if and only if $\Omega M \in \mathscr{U}_{n-1} \setminus \mathscr{U}_{n-2}$.

To prove the result we show by induction on n that, if $M \in \mathscr{U}_n$ is reduced, then $\sup\{\alpha(j) \mid M^j \neq 0\} = n$. The case $n = 0$ is clear. At the inductive step, it is straightforward to reduce to the case where M is nilclosed (see [Sch94] for this notion). For M nilclosed, [Zar90, Proposition 2.4.2.1] implies that ΩM is reduced, hence the inductive hypothesis applies to ΩM. The proof of the inductive step is then straightforward using (2.1). ∎

Remark 2.10. This result has a generalization, stated below as Theorem 3.5, which exploits the nilpotent filtration reviewed in the following section.

3. The nilpotent filtration. The main results of the paper concern spaces X such that the positive degree elements of the cohomology H^*X are nilpotent; by the restriction axiom for unstable algebras, this corresponds to $\tilde{H}^*X$ being nilpotent as an unstable module.

For $p = 2$, the following definition applies, where Sq_0 is the operator $x \mapsto \mathrm{Sq}^{|x|} x$. (A similar characterization exists for p odd.)

Definition 3.1. An unstable $\mathcal{A}_2$-module M is *nilpotent* if, for any $x \in M$, there exists k such that $\mathrm{Sq}_0^k x = 0$.

The archetypal example of a nilpotent unstable module is a suspension, and in general one has:

Proposition 3.2 ([Sch94]). *An unstable module is nilpotent if and only if it is the colimit of unstable modules which have a finite filtration whose quotients are suspensions.*

The full subcategory of nilpotent unstable modules is denoted $\mathcal{N}il_1 \subset \mathscr{U}$, and an unstable module is said to be *reduced* if it contains no non-trivial subobject which lies in $\mathcal{N}il_1$ (this is equivalent to containing no non-trivial suspension).

More generally the category $\mathscr{U}$ is filtered by thick subcategories $\mathcal{N}il_s$, $s \geq 0$, where $\mathcal{N}il_s$ is the smallest thick subcategory stable under colimits and containing all s-fold suspensions:

$$\mathscr{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \cdots \supset \mathcal{N}il_s \supset \cdots .$$

Proposition 3.3 ([Sch94, Kuh95b, Kuh14]). *The inclusion $\mathcal{N}il_s \hookrightarrow \mathscr{U}$ admits a right adjoint $\mathrm{nil}_s : \mathscr{U} \to \mathcal{N}il_s$ so that $M \in \mathscr{U}$ has a convergent decreasing filtration*

$$\cdots \subset \mathrm{nil}_{s+1} M \subset \mathrm{nil}_s M \subset \cdots \subset M$$

and $\mathrm{nil}_s M/\mathrm{nil}_{s+1} M \cong \Sigma^s R_s M$, where $R_s M$ is a reduced unstable module.

The convergence statement is a consequence of the fact that, for any unstable module M, $\mathrm{nil}_s M$ is $(s-1)$-connected.

Proposition 3.4 ([Sch94, Kuh95b, Kuh14]).

(1) *The T-functor restricts to $T\colon \mathcal{N}il_s \to \mathcal{N}il_s$ and $T \circ \mathrm{nil}_s \cong \mathrm{nil}_s \circ T$.*
(2) *The tensor product restricts to $\otimes\colon \mathcal{N}il_s \otimes \mathcal{N}il_t \to \mathcal{N}il_{s+t}$.*
(3) *For M a finitely generated unstable module, the nilpotent filtration is finite (i.e. $\mathrm{nil}_s M = 0$ for $s \gg 0$) and each R_sM is finitely generated.*
(4) *An unstable module M lies in $\mathscr{U}_n$ if and only if $R_sM \in \mathscr{U}_n$ for all $s \in \mathbb{N}$.*

Proof. The result follows from the commutation of T with suspension and the respective definitions. For part (4), since T does not commute with projective limits in general, in addition one uses the fact that the category $\mathscr{U}$ is locally noetherian [Sch94, Section 1.8] and the connectivity of objects of $\mathcal{N}il_s$. ■

The following result is the promised generalization of Theorem 2.9. By convention $\alpha(j)$ is taken to be ∞ if $j < 0$.

Theorem 3.5. *For M an unstable $\mathcal{A}_2$-module such that $\mathrm{nil}_s M = 0$ for $s \gg 0$ and $n \in \mathbb{N}$, the following conditions are equivalent:*

(1) *$M \in \mathscr{U}_n$;*
(2) *if $\alpha(d-i) > n$ for each $0 \le i \le \inf\{t \mid \mathrm{nil}_{t+1} M = 0\}$, then $M^d = 0$;*
(3) *there exists $k \in \mathbb{N}$ such that, if $\alpha(d-i) > n$ for each $0 \le i \le k$, then $M^d = 0$.*

Proof. (1)⇒(2). Suppose that $M \in \mathscr{U}_n$; the hypothesis that M has a finite nilpotent filtration implies that this has associated graded $\bigoplus_{i=0}^{k} \Sigma^i R_i M$ for $k = \inf\{t \mid \mathrm{nil}_{t+1} M = 0\}$. Each R_iM lies in $\mathscr{U}_n$ by Proposition 3.4 and is reduced, hence Theorem 2.9 implies that $(R_iM)^{d-i} = 0$ if $\alpha(d-i) > n$ for each $0 \le i \le k$. It follows that $M^d = 0$ if $\alpha(d-i) > n$ for each $0 \le i \le k$, as required.

(2)⇒(3) is immediate.

(3)⇒(1). Suppose that $M \notin \mathscr{U}_n$. Then Proposition 3.4 implies that $R_jM \notin \mathscr{U}_n$ for some $j \in \mathbb{N}$. Since R_jM is reduced, Theorem 2.9 provides $0 < c \in \mathbb{N}$ such that $\alpha(c) > n$ and $(R_jM)^{2^l c} \neq 0$ for all $l \in \mathbb{N}$. Hence $M^{2^l c + j} \neq 0$ for all $l \in \mathbb{N}$. Choose $k \in \mathbb{N}$; it is straightforward to see that there exists an integer $l(k) \gg 0$ such that $\alpha((2^{l(k)}c + j) - i) > n$ for each $0 \le i \le k$. However, by construction $M^{2^{l(k)}c+j} \neq 0$. Thus condition (3) is not satisfied for any k. ■

Remark 3.6. (1) The hypothesis that $\mathrm{nil}_s M = 0$ for $s \gg 0$ is satisfied, for example, if M is a finitely generated unstable module, by Proposition 3.4.

(2) A hypothesis on the nilpotent filtration is necessary for (1)⇒(3). For example, consider $M := \bigoplus_{s\geq 0} \Sigma^s \mathbb{F}_2$, so that M is in $\mathscr{U}_0$ and is non-trivial in every degree $d \in \mathbb{N}$. To see that (3) does not hold, for $k \in \mathbb{N}$, take $d = k+1$.

(3) If M is reduced, the equivalence (1)⇔(2) is Theorem 2.9.

NOTATION 3.7. For $\varepsilon : K \to \mathbb{F}_p$ an augmented unstable algebra, denote by $\bar{K}$ the augmentation ideal $\ker(\varepsilon)$ and by $QK \in \mathscr{U}$ the indecomposables: $QK := \bar{K}/(\bar{K})^2$.

PROPOSITION 3.8 ([Sch94, Section 6.4]). *For K an unstable algebra, $QK \in \mathcal{N}il_1$. If $p = 2$ then QK is a suspension.*

Recall (see [Sch94, Section 6.1]) that, for $s \in \mathbb{N}$, $\Omega^s : \mathscr{U} \to \mathscr{U}$ is the left adjoint to the iterated suspension functor Σ^s (hence is right exact) and restricts to a functor

$$\Omega^s : \mathcal{N}il_{k+s} \to \mathcal{N}il_k$$

for $k \in \mathbb{N}$. The following is applied in Section 6.

LEMMA 3.9. *For $M \in \mathcal{N}il_s$ ($s \in \mathbb{N}$), the natural surjection $M \twoheadrightarrow \Sigma^s R_s M$ induces an isomorphism $f(\Omega^s M) \xrightarrow{\cong} f(R_s M)$. If $N \subset M$ is a submodule such that $N \in \mathcal{N}il_{s+1}$, then the surjection $M \twoheadrightarrow M/N$ induces an isomorphism $f(\Omega^s M) \xrightarrow{\cong} f(\Omega^s(M/N))$; in particular $f(\Omega^s(M/N)) \cong f(R_s M)$.*

Proof. By hypothesis, there is a short exact sequence of unstable modules

$$0 \to \mathrm{nil}_{s+1} M \to M \to \Sigma^s R_s M \to 0,$$

so that applying Ω^s gives the exact sequence

$$\Omega^s \, \mathrm{nil}_{s+1} M \to \Omega^s M \to R_s M \to 0,$$

where $\Omega^s \, \mathrm{nil}_{s+1} M \in \mathcal{N}il_1$ (see [Sch94, Section 6.1]). Applying the exact functor f gives the isomorphism $f(\Omega^s M) \cong f(R_s M)$.

The proof of the second statement is similar. ∎

The following technical result is used in the proof of Theorem 1.5 in Section 6; for simplicity only the case $p = 2$ is considered.

PROPOSITION 3.10. *Let $\psi : M \to N$ be a morphism of unstable $\mathcal{A}_2$-modules such that*

(1) *$M \in \mathcal{N}il_d$ and $N \in \mathcal{N}il_{d+1}$ for some $d \in \mathbb{N}$;*
(2) *$R_d M$ is finitely generated.*

Then there exists a finitely generated submodule $U \subset M$ such that

(3) *the restriction $\psi|_U$ is trivial;*
(4) *the monomorphism $R_d\psi : R_d U \to R_d M$ has nilpotent cokernel (equivalently $f R_d \psi$ is an isomorphism).*

Proof. It is straightforward to reduce to the case where M is finitely generated. Moreover, without loss of generality, we may assume that ψ is surjective.

For $k \gg 0$ (an explicit bound can be supplied), consider the diagram

$$\begin{array}{ccc} & & M \xrightarrow{\ \psi\ } N \\ & \overset{\sigma_k}{\nearrow} & \downarrow \\ \Sigma^d \Phi^k R_d M & \hookrightarrow & \Sigma^d R_d M \end{array}$$

where Φ denotes the Frobenius functor (see [Sch94, Section 1.7]) and the bottom arrow is the canonical inclusion (recall that $R_d M$ is reduced). The dashed arrow denotes a choice of linear section σ_k (not $\mathcal{A}_2$-linear in general).

By hypothesis, M is finitely generated, hence so is N; thus there exists $h \in \mathbb{N}$ such that $R_s N = 0$ if $s \notin [d+1, h]$. Moreover, since each $R_s N$ is finitely generated, Theorem 3.5 implies that N is concentrated in degrees of the form $\ell + t$ with $\ell \in [d+1, h]$ and $t \in \mathbb{N}$ such that $\alpha(t) \leq D$ for some $D \in \mathbb{N}$.

Elements of $\Sigma^d \Phi^k R_d M$ lie in degrees of the form $d + 2^k v$ for $v \in \mathbb{N}$; hence a non-zero element in the image of $\psi\sigma_k$ lies in a degree of the form

$$d + 2^k v = \ell + t,$$

so that $t = 2^k v - (\ell - d)$, where $\alpha(t) \leq D$ and $\ell - d \in [1, h-d]$, with the values of D and h independent of k. If k is sufficiently large, this leads to a contradiction; thus, for $k \gg 0$, we have $\psi\sigma_k = 0$.

Consider the submodule U of M generated by the image under σ_k of a (finite-dimensional) space of generators of the unstable module $\Sigma^d \Phi^k R_d M$, which is finitely generated, since $R_d M$ is.

By construction, $\psi\sigma_k = 0$, hence $U \subset \ker \psi$. The functor R_d preserves monomorphisms, hence $R_d U \subset R_d M$; moreover, from the construction, it is clear that the cokernel is nilpotent. ∎

4. Kuhn's profile functions. The profile function of an unstable module is defined using ideas of Kuhn [Kuh95b]:

Definition 4.1. For $M \in \mathscr{U}$, the *profile function* $w_M : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ is defined by

$$w_M(i) := \deg f(R_i M) = \deg R_i M.$$

Remark 4.2. For M an unstable module, $w_M = 0$ if and only if M is locally finite (i.e. $M \in \mathscr{U}_0$).

Notation 4.3. For functions $f, g : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ we write $f \leq g$ if $f(i) \leq g(i)$ for all $i \in \mathbb{N}$. The inclusion $\mathbb{N} \hookrightarrow \mathbb{N} \cup \{\infty\}$ is denoted Id.

The following operations on functions $\mathbb{N} \to \mathbb{N} \cup \{\infty\}$ are useful:

Definition 4.4. For $f, g : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, define:

(1) $f \bullet g(i) := \sup_k \{f(k) + g(i-k)\}$;
(2) $f \circ g(i) := \sup_{0<k<i} \{f(k) + g(i-k)\}$ for $i > 1$ and $:= 0$ otherwise;
(3) $\sup\{f, g\}(i) := \sup\{f(i), g(i)\}$ (likewise for arbitrary sets of functions);
(4) $\partial f(i) := \sup\{0, f(i) - 1\}$;
(5) $[f](i) := \sup\{f(j) \mid 0 \leq j \leq i\}$.

The following states some evident properties:

Lemma 4.5. *For $f_i, g_i : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, $i \in \{1, 2\}$, such that $f_i \leq g_i$:*

(1) $[f_i] \leq [g_i]$;
(2) $\partial f_i \leq \partial g_i$;
(3) $f_i \circ g_i \leq f_i \bullet g_i$;
(4) $\sup\{f_1, g_1\} \leq \sup\{f_2, g_2\}$;
(5) $f_1 \circ g_1 \leq f_2 \circ g_2$ *and* $f_1 \bullet g_1 \leq f_2 \bullet g_2$.

Proposition 4.6. *For M, N unstable modules:*

(1) $w_{M \otimes N} \leq w_M \bullet w_N$ *and, if M, N are both nilpotent,* $w_{M \otimes N} \leq w_M \circ w_N$;
(2) $w_{\bar{T}M} = \partial w_M$.

For a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of unstable modules, the following hold:

(3) $w_{M_1} \leq w_{M_2}$;
(4) $w_{M_2} \leq \sup\{w_{M_1}, w_{M_3}\}$;
(5) $w_{M_3} \leq [w_{M_2}]$.

Proof. For tensor products, the statement holds by the behaviour of the functor R_i with respect to tensor products [Kuh95b, Proposition 2.5]. When M and N are both nilpotent, then $R_0 M = 0 = R_0 N$, so that the terms $R_0 M \otimes R_k N$ and $R_k M \otimes R_0 N$ do not contribute.

The statement for the reduced T-functor is a consequence of the compatibility of T with the nilpotent filtration and the definition of polynomial degree (see Proposition 3.4 and [Sch94, Kuh95b]).

For the short exact sequence, the first two properties follow from the left exactness of the composite functor $f \circ R_i$ [Kuh07, Corollary 3.2] and the fact that $\mathscr{F}_n$ is thick.

For the final point, it suffices to show that $\Sigma^t R_t M_3$ lies in $\mathscr{U}_d$, where

$$d = [w_{M_2}](t) = \sup\{w_{M_2}(i) \mid 0 \leq i \leq t\}.$$

Now $\Sigma^t R_t M_3$ is a subquotient of $M_2/\mathrm{nil}_{t+1} M_2$ and the latter lies in $\mathscr{U}_d$, since each $\Sigma^i R_i M_2$ does, for $0 \leq i \leq t$, by definition of d. ∎

Remark 4.7. The final statement on w_{M_3} can be strengthened slightly; the above is sufficient for the current purposes.

EXAMPLE 4.8. Let M, N be unstable modules such that $w_M \leq \mathrm{Id}$ and $w_N \leq \mathrm{Id}$. Then $w_{M\otimes N} \leq \mathrm{Id}$. Hence, if $\mathbb{T}(M) := \bigoplus_i M^{\otimes i}$ denotes the tensor algebra on M then $w_{\mathbb{T}(M)} \leq \mathrm{Id}$.

For K an unstable algebra, Proposition 4.6 immediately provides a bound for w_{QK} in terms of w_K; the following provides a converse.

PROPOSITION 4.9. *For K a connected unstable algebra with $\bar{K} \in \mathcal{N}il_1$,*

$$w_K \leq \sup\{[w_{QK}]^{\circ t} \mid t \in \mathbb{N}\}.$$

Proof. Since $K \cong \bar{K} \oplus \mathbb{F}_p$ as unstable modules, $w_K = w_{\bar{K}}$, hence it suffices to consider the latter.

The result follows from an analysis of the short exact sequence

$$0 \to (\bar{K})^2 \to \bar{K} \to QK \to 0$$

together with the surjection $\bar{K} \otimes \bar{K} \twoheadrightarrow (\bar{K})^2$. Proposition 4.6 provides the inequalities $w_{(\bar{K})^2} \leq [w_{\bar{K}\otimes\bar{K}}] \leq [w_{\bar{K}}] \circ [w_{\bar{K}}]$, so $w_{\bar{K}} \leq \sup\{w_{QK}, [w_{\bar{K}}] \circ [w_{\bar{K}}]\}$, since $\bar{K}$ is nilpotent. It follows from Lemma 4.5 that

$$[w_{\bar{K}}] \leq \sup\{[w_{QK}], [w_{\bar{K}}] \circ [w_{\bar{K}}]\}. \tag{4.1}$$

The assertion follows by induction on i showing that

$$[w_{\bar{K}}](i) \leq \sup_t\{[w_{QK}]^{\circ t}(i)\}.$$

The cases $i \in \{0, 1\}$ are clear and the inductive step uses (4.1). ∎

The following is clear:

COROLLARY 4.10. *For K a connected unstable algebra with $\bar{K} \in \mathcal{N}il_1$, $\bar{K}$ is locally finite if and only if QK is locally finite; equivalently $w_K = 0$ if and only if $w_{QK} = 0$.*

COROLLARY 4.11. *For K a connected unstable algebra such that $\bar{K} \in \mathcal{N}il_1$, $w_K \leq \mathrm{Id}$ if and only if $w_{QK} \leq \mathrm{Id}$.*

Proof. The hypothesis $w_K \leq \mathrm{Id}$ implies that $[w_K] \leq [\mathrm{Id}] = \mathrm{Id}$. The result follows from the observation that $\mathrm{Id} \bullet \mathrm{Id} = \mathrm{Id}$. ∎

The profile function of an unstable module M gives information about the connectivity of the iterates $\bar{T}^n M$ of the reduced T-functor applied to M. The following results are used in the proof of Theorem 1.4 in Section 5.

LEMMA 4.12. *For M an unstable module such that $w_M \leq \sup\{\mathrm{Id} + d, 0\}$ ($d \in \mathbb{Z}$) and $0 < n \in \mathbb{N}$, $\bar{T}^n M$ is $(n - d - 1)$-connected.*

Proof. By definition of the profile function and by hypothesis, $R_i M \in \mathscr{U}_{\sup\{\mathrm{Id}+d,0\}}$. Hence, by Theorem 2.5, $\bar{T}^n R_i M = 0$ for $n > i + d$, with the first non-trivial value for $i = n - d$ (here it is essential that $n > 0$). By compatibility of the action of the T-functor with the nilpotent filtration (see Proposition 3.4), it follows that

$$\bar{T}^n M = \mathrm{nil}_{n-d}\,\bar{T}^n M.$$

This implies, in particular, that $\bar{T}^n M$ is $(n-d-1)$-connected. ∎

The above result can be applied in the study of unstable modules M for which $w_M - \mathrm{Id}$ is bounded:

LEMMA 4.13. *Let $M \in \mathscr{U}$ be an unstable module such that $w_M \neq 0$ and $w_M - \mathrm{Id}$ is bounded, and set*

$$d := \sup\{w_M(i) - i \mid i \in \mathscr{I}\}, \qquad s := \inf\{i \in \mathscr{I} \mid w_M(i) - i = d\},$$

where $\mathscr{I}$ is the support $\{i \in \mathbb{N} \mid w_M(i) > 0\}$ of w_M. Then $s + d \geq 1$ and

$$R_i \bar{T}^{s+d-1} M \in \begin{cases} \mathscr{U}_0 & \text{if } i < s, \\ \mathscr{U}_1 \setminus \mathscr{U}_0 & \text{if } i = s, \\ \mathscr{U}_{i-s+1} & \text{if } i > s. \end{cases}$$

In particular, $w_{\bar{T}^{s+d-1}M} \leq \sup\{\mathrm{Id} + 1 - s, 0\}$. Moreover, if $s + d > 1$ then $\bar{T}^{s+d-1} M \in \mathscr{N}il_s$ (equivalently $R_i \bar{T}^{s+d-1} M = 0$ for $i < s$).

Hence, for $0 < n \in \mathbb{N}$, $\bar{T}^{n+s+d-1} M \in \mathscr{N}il_{n+s-1}$, thus $\bar{T}^{n+s+d-1} M$ is $(n+s-2)$-connected.

Proof. The definition of d, s ensures that $s + d \geq 1$, hence $\bar{T}^{s+d-1} M$ is defined. Proposition 4.6 implies that

$$w_{\bar{T}^{s+d-1}M} = \partial^{s+d-1} w_M = \sup\{0, w_M + 1 - (s+d)\}.$$

If $s + d > 1$ then $w_M(i) \leq \sup\{d + i, 0\}$, with $w_M(s) = s + d$ and strict inequality for $i < s$ if $d + i > 0$. It follows that

$$\begin{aligned} w_M(i) + 1 - (s+d) &< 0 && \text{for } i < s, \\ w_M(s) + 1 - (s+d) &= 1, && \\ w_M(i) + 1 - (s+d) &\leq 1 - s + i && \text{for } i > s. \end{aligned}$$

The fact that $w_M(i) + 1 - (s+d)$ is negative for $i < s$ implies that $R_i \bar{T}^{s+d-1} M = 0$ for $i < s$; the statements for $i \geq s$ follow from the definition of $w_{\bar{T}^{s+d-1}M}$.

The case $s + d = 1$ (so that $\bar{T}^{s+d-1} M = M$) follows from the definition of d and s, which implies in particular that $w_M(i) = 0$ for $i < s$.

In both cases $w_{\bar{T}^{s+d-1}M} \leq \sup\{\mathrm{Id} + 1 - s, 0\}$. Finally, Lemma 4.12 implies that $\bar{T}^{n+s+d-1} M \cong \bar{T}^n \bar{T}^{s+d-1} M$ is $(n-(1-s)-1) = (n+s-2)$-connected. ∎

5. Proof of Theorem 1.4. We commence by a rapid review of Lannes' theory, which is the main ingredient in the proof of Theorem 1.4 and is also the reason for the restrictions imposed on the topological spaces considered.

The evaluation map

$$B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, X) \to X$$

induces a map in cohomology $H^*X \to H^*B\mathbb{Z}/p \otimes H^*\mathrm{map}(B\mathbb{Z}/p, X)$, and hence, by adjunction,

$$TH^*X \xrightarrow{\lambda} H^*\mathrm{map}(B\mathbb{Z}/p, X).$$

THEOREM 5.1 ([Lan92]). *For X a p-complete, 1-connected space such that TH^*X is of finite type and 1-connected, the natural map*

$$TH^*X \xrightarrow{\lambda} H^*\mathrm{map}(B\mathbb{Z}/p, X)$$

is an isomorphism of unstable algebras.

NOTATION 5.2 ([Kuh95b]). Denote by ΔX the homotopy cofibre of the map $X \to \mathrm{map}(B\mathbb{Z}/p, X)$ induced by $B\mathbb{Z}/p \to *$.

Theorem 5.1 yields the following:

PROPOSITION 5.3. *Under the hypotheses of Theorem 5.1 on X, we have $H^*\Delta X \cong \bar{T}H^*X$, hence*

(1) $w_{\Delta X} = \partial w_X$;
(2) *if $H^*X \in \mathscr{U}_n$ then $H^*\Delta X \in \mathscr{U}_{n-1}$.*

In the proof of Theorem 1.4, it is necessary to work with pointed mapping spaces. Here having an H-space structure is useful (cf. [CCS07]), since there is a homotopy equivalence:

$$\mathrm{map}(B\mathbb{Z}/p, Z) \simeq Z \times \mathrm{map}_*(B\mathbb{Z}/p, Z) \tag{5.1}$$

when Z is an H-space.

This is an essential ingredient in the proof of the following connectivity result, which is applied in the proof of Theorem 1.4 below.

PROPOSITION 5.4. *Let Z be a p-complete H-space and $n \in \mathbb{N}$ such that:*

(1) *T^nH^*Z is of finite type;*
(2) *$\bar{T}^iH^*Z$ is 1-connected for $0 \le i < n$;*
(3) *$\bar{T}^nH^*Z$ is k-connected.*

Then $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is k-connected.

The proof is based upon:

THEOREM 5.5 ([Sch94, Theorem 8.6.1]). *For X a connected, nilpotent space such that H^*X is of finite type and π_1X is finite, the following conditions are equivalent:*

(1) *$\bar{T}H^*X$ is k-connected;*
(2) *$\mathrm{map}_*(B\mathbb{Z}/p, X)$ is k-connected.*

Lemma 5.6. *Let X be a pointed space such that:*

(1) *the natural transformation $TH^*X \xrightarrow{\lambda} H^*\mathrm{map}(B\mathbb{Z}/p, X)$ is an isomorphism;*
(2) *the map $\tilde{H}^*\mathrm{map}(B\mathbb{Z}/p, X) \to \tilde{H}^*\,\mathrm{map}_*(B\mathbb{Z}/p, X)$ admits a section in $\mathscr{U}$.*

Then $\tilde{H}^*\,\mathrm{map}_*(B\mathbb{Z}/p, X)$ is a direct summand of $\bar{T}\tilde{H}^*X$ in $\mathscr{U}$. In particular, the result applies for X an H-space such that TH^*X is of finite type.

Proof. It is straightforward to check that restricting the evaluation map to $B\mathbb{Z}/p \times \mathrm{map}_*(B\mathbb{Z}/p, X) \subset B\mathbb{Z}/p \times \mathrm{map}(B\mathbb{Z}/p, X)$ leads to a natural commutative diagram

$$\begin{array}{ccc} T\tilde{H}^*X & \xrightarrow{\tilde{\lambda}} & \tilde{H}^*\mathrm{map}(B\mathbb{Z}/p, X) \\ \downarrow & & \downarrow \; \vdots\!\uparrow \\ \bar{T}\tilde{H}^*X & \longrightarrow & \tilde{H}^*\,\mathrm{map}_*(B\mathbb{Z}/p, X) \end{array}$$

where the left hand vertical arrow is the natural projection and $\tilde{\lambda}$ is obtained similarly to λ by restricting the evaluation map to $\tilde{H}^*X$.

If λ is an isomorphism, so is $\tilde{\lambda}$. The section (indicated by the dotted arrow) therefore gives the required splitting.

Finally, if X is an H-space, the decomposition (5.1) leads to the required section and [CCS07, Proposition 1.1] implies that λ is an isomorphism. ∎

Proof of Proposition 5.4. The result follows by applying Theorem 5.5 to the spaces $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge i}, Z)$ recursively.

If X is an H-space, $\mathrm{map}_*(B\mathbb{Z}/p, X)$ and its p-completion are H-spaces. Moreover, $\mathrm{map}_*(B\mathbb{Z}/p, X) \simeq \mathrm{map}_*(B\mathbb{Z}/p, X^{\wedge p})$, by [Mil84, Theorem 1.5]. This allows us to work with p-completed H-spaces.

Lemma 5.6 implies that the finite type hypothesis in Theorem 5.5 is satisfied at each stage. Similarly, the connectivity hypothesis for $\bar{T}$ implies that the mapping spaces are all 1-connected, in particular the condition on the fundamental group in Theorem 5.5 holds.

At the final step, the hypothesis on $\bar{T}^n H^*Z$ implies that $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is k-connected, as required. ∎

Notation 5.7. For X a connected space, write $w_X := w_{H^*X}$ and $q_X := w_{QH^*X}$.

Proof of Theorem 1.4. Proposition 3.4 implies that an unstable module M is locally finite if and only if $w_M = 0$, which implies (1)⇔(2). Corollary 4.10 gives (2)⇔(3), and Corollary 4.11 yields (4)⇔(5). The implications (2)⇒(4) and (4)⇒(6) are clear, hence it suffices to establish

- (6)⇒(2): if $w_X - \mathrm{Id}$ is bounded then $w_X = 0$.

Suppose that there exists a space X (satisfying the global hypotheses) such that $0 \neq w_X$, $w_X - \mathrm{Id}$ is bounded and $\tilde{H}^*X \in \mathcal{N}il_1$; *reductio ad absurdum.*

The first step is analogous to Kuhn's reduction [Kuh95b]. As in Lemma 4.13, set

$$d = \sup\{w_X(i) - i \mid i \in \mathscr{I}\}, \qquad s = \inf\{i \in \mathscr{I} \mid w_X(i) - i = d\},$$

where $\mathscr{I}$ is the support $\{i \in \mathbb{N} \mid w_X(i) > 0\}$ of w_X. This ensures $s + d > 0$ and $s > 0$, since $\tilde{H}^*X$ is nilpotent. Replacing X by ΣX if necessary, we may assume that $s \geq 2$, since $w_{\Sigma X}(i) = w_X(i-1)$ for $i > 0$ and $w_{\Sigma X}(0) = 0$, so that ΣX also satisfies the hypotheses.

Proceeding as in Lemma 4.13, set $Y := (\Delta^{s+d-1}X)^{\wedge}_p$; then Proposition 5.3 gives

$$\begin{aligned} w_Y(i) &= 0, && i < s,\\ w_Y(s) &= 1, && \\ w_Y(i) &\leq i - s + 1, && i > s. \end{aligned}$$

Moreover, by collapsing down a low-dimensional skeleton and p-completing, one can arrange that $R_i H^*Y = 0$ for $0 < i < s$.

In order to work with pointed mapping spaces, an H-space structure is introduced by considering $Z := \Omega(\Sigma Y)^{\wedge}_p$. By the Bott–Samelson theorem, $H^*Z \cong \mathbb{T}(\tilde{H}^*Y)$ as an unstable module, hence the global hypotheses are satisfied by Z. Moreover, by Proposition 4.6, one has $R_i H^*Z = 0$ for $0 < i < s$ and $R_i H^*Z \in \mathscr{U}_{i-s+1}$ for $i \geq s$. Lemma 4.12 therefore implies that $\bar{T}^n H^*Z$ is $(n+s-2)$-connected for $n > 0$ (and $\tilde{H}^*Z$ is $(s-1)$-connected).

In particular, since we have arranged that $s \geq 2$, $\bar{T}^n \tilde{H}^*Z$ is 1-connected for all $n \in \mathbb{N}$. Proposition 5.4 therefore implies that $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is $(n+s-2)$-connected.

By construction, $R_s H^*Z$ is a reduced unstable module in $\mathscr{U}_1$, hence (by [Kuh95b, Proposition 0.6], for example) there exists a non-trivial morphism $R_s H^*Z \to F(1)$, where $F(1)$ is the free unstable module on a generator of degree 1. Composing with the canonical inclusion $F(1) \hookrightarrow \tilde{H}^*B\mathbb{Z}/p$ gives

$$\begin{array}{ccccc} H^*Z & \twoheadrightarrow & H^*Z/\mathrm{nil}_{s+1} H^*Z \cong \mathbb{F}_p \oplus \Sigma^s R_s H^*Z & \longrightarrow & \mathbb{F}_p \oplus \Sigma^s F(1) \\ & \searrow^{\varphi_s^*} & & & \big\downarrow \\ & & & & \mathbb{F}_p \oplus \Sigma^s \tilde{H}^* B\mathbb{Z}/p \end{array}$$

that is, a morphism of unstable algebras, by compatibility of the nilpotent filtration with multiplicative structures.

By Lannes' theory [Lan92], the morphism φ_s^* can be realized as the cohomology of a map $\varphi_s : \Sigma^s B\mathbb{Z}/p \to Z$, by applying Theorem 5.1 together with the Hurewicz theorem, since $\mathrm{map}_*(B\mathbb{Z}/p, Z)$ is $(s-1)$-connected.

Thus, consider the extension problem

$$\begin{array}{ccc} \Sigma^s B\mathbb{Z}/p & \xrightarrow{\varphi_s} & Z \\ \downarrow & \nearrow & \\ \Sigma^{s-1}K(\mathbb{Z}/p,2) & & \end{array}$$

where the vertical map is the $(s-1)$-fold suspension of the canonical map $\Sigma B\mathbb{Z}/p \to K(\mathbb{Z}/p, 2)$.

Algebraically it is clear that no such extension can exist, since φ_s^* is non-trivial in positive degrees (by construction), whereas

$$\mathrm{Hom}_{\mathscr{U}}(\tilde{H}^* Z, \tilde{H}^* \Sigma^{s-1} K(\mathbb{Z}/p, 2)) = 0$$

since $\tilde{H}^* Z \in \mathcal{N}il_s$ and $\tilde{H}^* \Sigma^{s-1} K(\mathbb{Z}/p, 2)$ is the $(s-1)$-fold suspension of a reduced module ($H^* K(\mathbb{Z}/p, 2)$ is reduced by Proposition 5.8 below). However, obstruction theory shows that one can construct such a factorization, as follows.

Recall that $K(\mathbb{Z}/p, 2)$ has the homotopy type of the iterated bar construction $BB\mathbb{Z}/p$. The filtration of the bar construction gives a filtration $* = C_0 \subset C_1 = \Sigma B\mathbb{Z}/p \subset C_2 \subset \cdots \subset \bigcup_n C_n = K(\mathbb{Z}/p, 2)$ with associated (homotopy) cofibre sequences

$$\Sigma^{n-1} B\mathbb{Z}/p^{\wedge n} \to C_{n-1} \to C_n. \tag{5.2}$$

This cofibre sequence can be constructed by using Milnor's description [Mil56] of $BB\mathbb{Z}/p$, taking an abelian topological group model for $B\mathbb{Z}/p$. This exhibits C_{n-1} as a quotient of the iterated join $(B\mathbb{Z}/p)^{*n}$, which has the homotopy type of $\Sigma^{n-1} B\mathbb{Z}/p^{\wedge n}$. The first map in (5.2) is the quotient map, and the identification of the homotopy cofibre follows from the proof of [Mil56, Lemma 4.1].

The associated obstructions to extending $\varphi_s \colon \Sigma^s B\mathbb{Z}/p = \Sigma^{s-1} C_1 \to Z$ lie in the pointed homotopy groups

$$[\Sigma^{n+s-2}(B\mathbb{Z}/p)^{\wedge n}, Z] = \pi_{n+s-2}\, \mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$$

for $n \geq 2$.

The groups $\pi_{n+s-2}\, \mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ are trivial, since $\mathrm{map}_*(B\mathbb{Z}/p^{\wedge n}, Z)$ is $(n+s-2)$-connected, by the application of Proposition 5.4 given above. It follows that an extension exists, which is a contradiction to the existence of such a space Z, completing the proof of Theorem 1.4. ■

Proposition 5.8. *The unstable module $H^* K(\mathbb{Z}/p, 2)$ is reduced.*

Proof. This result is well known to the experts, and holds for $H^* K(\mathbb{Z}/p, n)$ for any $n \in \mathbb{N}$. For $p = 2$ the result is established in [Kuh98]; a proof is sketched here for p odd, since we do not know a convenient reference.

The cohomology $H^*K(\mathbb{Z}/p,2)$ is isomorphic (with the usual notation) to

$$\mathbb{F}_p[x,\beta P^{I_h}\beta(x)]\otimes E(P^{I_\ell}\beta(x))\cong \mathbb{F}_p[x,x_h\,|\,h\geq 1]\otimes E(y_\ell\,|\,\ell\geq 0)$$

with $|x|=2$, $I_h=(p^{h-1},p^{h-2},\ldots,p,1)$, $h\geq 1$, and $\beta(x)=y_0$; in particular, $\beta(y_h)=x_h$, $h\geq 1$. It is enough to show that, for any non-zero element $z\in\tilde{H}^*K(\mathbb{Z}/p,2)$, there exists an operation θ such that $\theta(z)\in\mathbb{F}_p[x,x_h]$ and $\theta(z)\neq 0$.

Any element z can be written as a sum $\sum_{0\leq i\leq t}\sum_j P_{i,j}(x,x_h)\otimes L_{i,j}(y_\ell)$. If z has degree $2n$ and there is a non-trivial term with exterior part of degree 0, then a straightforward application of the Cartan formula shows that $\theta=P^n$ suffices, since the reduced powers act trivially on the exterior generators.

For a general element, one reduces to such elements by applying operations which decrease (non-trivially) the length of the exterior factors that occur. Consider amongst the exterior factors a term $L_{i,j}$ of minimal length, and the minimal ℓ for which y_ℓ occurs in it. Let Q_i be the usual Milnor derivation. Then $[\beta,Q_i]$ is also a derivation; it acts trivially on the x_h, sends x to y_{i+1}, and y_ℓ to $x_{i-\ell+1}^{p^\ell}$. Using a standard lexicographic order argument, one can see that this operation does the job for i large enough. ∎

6. Using the Eilenberg–Moore spectral sequence. In this section, the prime p is taken to be 2, and the space X is 1-connected such that $\tilde{H}^*X$ is nilpotent and of finite type. The purpose of this section is to prove the following, which is equivalent to Theorem 1.5 of the Introduction.

THEOREM 6.1. *Let X be a space such that $\tilde{H}^*X$ is of finite type and is nilpotent. If $w_X(1)=d\in\mathbb{N}$ then $w_X(2)\geq 2d$.*

The interest of the result is to give some control on R_2H^*X, starting from information about R_1H^*X. See Remark 6.13 for a slightly refined version of this theorem.

REMARK 6.2. The theorem is stated only for $p=2$; the difficulties that occur in the odd prime case in [Sch98, Sch10] also arise here, but look more manageable.

The method was originally suggested by the following observation:

PROPOSITION 6.3. *Let M be a connected, reduced unstable module such that* $\deg(fM)=d\in\mathbb{N}$. *If $d>0$, then the unstable module $\mathbb{T}(M)=\bigoplus_i M^{\otimes i}$ does not carry the structure of an unstable algebra.*

REMARK 6.4. This result is a special case of a general structure result for reduced unstable algebras. From the viewpoint of this paper, heuristically the idea is that cup products of classes in M should appear in $M\otimes M$, whereas the restriction axiom for unstable algebras implies that cup squares occur in M. Thus, the triviality of the extension between M and $M \otimes M$ is incompatible with an unstable algebra structure.

The proof of Theorem 6.1 is based on the analysis of an $\mathrm{Ext}^1_{\mathscr{F}}$ group, playing off the following non-splitting result against a vanishing criterion.

Recall from Section 2 that $\mathscr{F}$ denotes the category of functors on $\mathbb{F}_2$-vector spaces; a functor of $\mathscr{F}$ is finite if it has a finite composition series. As usual, S^n denotes the nth symmetric power functor and Λ^n the nth exterior power functor.

LEMMA 6.5. *For $F \in \mathscr{F}$ a non-constant finite functor, post-composition with the short exact sequence $0 \to S^1 \to S^2 \to \Lambda^2 \to 0$, where $S^1 \to S^2$ is the Frobenius mapping $x \mapsto x^2$, induces an exact sequence*

$$0 \to F \to S^2(F) \to \Lambda^2(F) \to 0$$

which does not split.

Proof. This follows from [Kuh95a, Theorem 4.8], since the Frobenius mapping fits into the non-split short exact sequence

$$0 \to S^1 \to S^2 \to \Lambda^2 \to 0. \blacksquare$$

NOTATION 6.6. For $F \in \mathscr{F}$ a finite functor of polynomial degree d, set $\overline{F} := \ker\{F \twoheadrightarrow q_{d-1}F\}$, where $q_{d-1}F$ is the largest quotient of degree $\leq d-1$.

Lemma 6.5 will be played off against the vanishing result for $\mathrm{Ext}^1_{\mathscr{F}}$ in the following statement.

LEMMA 6.7. *Let F be a finite functor of polynomial degree d, $G_{<d}$ of degree $< d$, $G_{<2d}$ of degree $< 2d$ and $G_{\leq d}$ of degree $\leq d$. Then:*

(1) $\mathrm{Hom}_{\mathscr{F}}(\overline{F}, G_{<d}) = 0$;
(2) $\mathrm{Hom}_{\mathscr{F}}(\overline{F} \otimes \overline{F}, G_{<2d}) = 0 = \mathrm{Hom}_{\mathscr{F}}(\Lambda^2(\overline{F}), G_{<2d})$;
(3) $\mathrm{Ext}^1_{\mathscr{F}}(\overline{F} \otimes \overline{F}, G_{\leq d}) = 0$.

Proof. The first statement is a consequence of the definition of $\overline{F}$.

The second is similar and follows from the compatibility of the polynomial filtration of $\mathscr{F}$ with tensor products, which implies that $\overline{F} \otimes \overline{F}$ has no quotient of polynomial degree $< 2d$. The second equality follows from the fact that the functor $\Lambda^2(\overline{F})$ is a quotient of $\overline{F} \otimes \overline{F}$.

The result for $\mathrm{Ext}^1_{\mathscr{F}}$ is proved as follows. Using *dévissage* it is straightforward to reduce to the case where $G_{\leq d}$ is simple. Since a simple functor of polynomial degree n embeds in the nth tensor functor $T^n \colon V \mapsto V^{\otimes n}$, which is finite and has polynomial degree n, using the previous statement for $\mathrm{Hom}_{\mathscr{F}}$ it suffices to show that $\mathrm{Ext}^1_{\mathscr{F}}(\overline{F} \otimes \overline{F}, T^n) = 0$ for $n \leq d$. This follows by the methods introduced in [FLS94], in particular in the proof of

[FLS94, Lemme 0.4], by exploiting the tensor product on the left hand side and the exponential properties of T^n. ∎

The proof of Theorem 6.1 uses these results in conjunction with the Eilenberg–Moore spectral sequence. For relevant details (and further references) on the Eilenberg–Moore spectral sequence computing $H^*\Omega X$ from H^*X, see [Sch94, Section 8.7]. Note that the hypothesis that X is simply connected ensures strong convergence of the spectral sequence.

Recall that the (-2)-layer of the Eilenberg–Moore filtration $F_\infty^{-2,*}$ on $H^*\Omega X$ is an extension in unstable modules between the column $E_\infty^{-1,*}$ desuspended once and the column $E_\infty^{-2,*}$ desuspended twice:

$$0 \to \Sigma^{-1} E_\infty^{-1,*} \to F_\infty^{-2,*} \to \Sigma^{-2} E_\infty^{-2,*} \to 0.$$

The term $E_\infty^{-1,*}$ is a quotient of $QH^*X \cong E_2^{-1,*}$ by a submodule in $\mathcal{N}il_2$, by [Sch94, Theorem 8.7.1]. Similarly, $E_\infty^{-2,*}$ is a quotient of $\mathrm{Tor}_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2)$ (which belongs to $\mathcal{N}il_2$ by [Sch94, Theorem 6.4.1]) by a submodule in $\mathcal{N}il_3$ (see [Sch94, Proposition 8.7.7]). (In the reference, $\overline{\mathcal{N}il_3}$ is used, where $\overline{\mathcal{N}il_3}$ is generated by $\mathcal{N}il_3$ and $\mathscr{U}_0$ [Sch94, Section 6.2], however here the situation reduces to $\mathcal{N}il_3$.)

From Lemma 3.9, it follows that applying the exact functor $f\colon \mathscr{U} \to \mathscr{F}$ yields the short exact sequence

$$0 \to fR_1QH^*X \to fF_\infty^{-2,*} \to fR_2 \,\mathrm{Tor}_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2) \to 0. \tag{6.1}$$

Moreover, the surjection $\tilde{H}^*X \twoheadrightarrow QH^*X$ induces an isomorphism $fR_1\tilde{H}^*X \cong fR_1QH^*X$. This allows arguments to be carried out in the category $\mathscr{F}$ of functors.

Notation 6.8. Write

$$F_1 := fR_1\tilde{H}^*X \cong f(\Sigma^{-1}E_\infty^{-1,*}), \qquad F_2 := f(F_\infty^{-2,*}),$$

so that $F_2/F_1 \cong fR_2 \,\mathrm{Tor}_{H^*X}^{-2}(\mathbb{F}_2, \mathbb{F}_2)$ by (6.1).

Remark 6.9. With this notation, the short exact sequence (6.1) represents a class $[F_2] \in \mathrm{Ext}^1_{\mathscr{F}}(F_2/F_1, F_1)$.

The compatibility of the cup product with the Eilenberg–Moore spectral sequence gives a morphism $S^2(F_1) \to F_2$ and hence a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & S^2(F_1) & \longrightarrow & \Lambda^2(F_1) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_2/F_1 & \longrightarrow & 0 \end{array}$$

so that the right hand square is a pullback.

By hypothesis, the functor F_1 is of degree d; the inclusion $\overline{F_1} \subset F_1$ induces the inclusions $S^2(\overline{F_1}) \subset S^2(F_1)$ and $\Lambda^2(\overline{F_1}) \subset \Lambda^2(F_1)$, which fit into the following diagram of morphisms of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{F_1} & \longrightarrow & S^2(\overline{F_1}) & \longrightarrow & \Lambda^2(\overline{F_1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F_1 & \longrightarrow & E & \longrightarrow & \Lambda^2(\overline{F_1}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & S^2(F_1) & \longrightarrow & \Lambda^2(F_1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_2/F_1 & \longrightarrow & 0
\end{array}
\qquad (6.2)
$$

in which the second row can be viewed either as the pushout of the top row or the pullback of the third. Moreover, the first and third rows are not split, by Lemma 6.5, hence the bottom row is not split.

LEMMA 6.10. *If F_1 is non-constant then the short exact sequence*

$$0 \to F_1 \to E \to \Lambda^2(\overline{F_1}) \to 0$$

does not split.

Proof. Consider the long exact sequence for $\mathrm{Ext}^*_{\mathscr{F}}$ induced by the defining short exact sequence $\overline{F_1} \to F_1 \to q_{d-1}F_1$, which gives the exact sequence

$$\mathrm{Hom}(\Lambda^2(\overline{F_1}), q_{d-1}F_1) \to \mathrm{Ext}^1_{\mathscr{F}}(\Lambda^2(\overline{F_1}), \overline{F_1}) \to \mathrm{Ext}^1_{\mathscr{F}}(\Lambda^2(\overline{F_1}), F_1),$$

in which the first term is zero, by Lemma 6.7. Thus the second morphism is injective.

By Lemma 6.5, the top row in diagram (6.2) represents a non-trivial class in $\mathrm{Ext}^1_{\mathscr{F}}(\Lambda^2(\overline{F_1}), \overline{F_1})$, hence pushes out to a non-split short exact sequence represented by a non-zero class in $\mathrm{Ext}^1_{\mathscr{F}}(\Lambda^2(\overline{F_1}), F_1)$. ∎

PROPOSITION 6.11. *The morphism $\Lambda^2\overline{F_1} \to F_2/F_1$ does not factor over $\overline{F_1} \otimes \overline{F_1}$:*

$$
\begin{array}{ccc}
\Lambda^2\overline{F_1} & \longrightarrow & \overline{F_1} \otimes \overline{F_1} \\
\downarrow & \swarrow \; \nexists & \\
F_2/F_1 & &
\end{array}
$$

Proof. The non-trivial extension in Lemma 6.10 is the image of the class $[F_2] \in \mathrm{Ext}^1_{\mathscr{F}}(F_2/F_1, F_1)$ under the morphism induced by $\Lambda^2(\bar{F}_1) \to F_2/F_1$.

Hence there can be no factorization across the group $\mathrm{Ext}^1_{\mathscr{F}}(\overline{F_1} \otimes \overline{F_1}, F_1)$, which is trivial by Lemma 6.7. ∎

PROPOSITION 6.12. *Assume that $\tilde{H}^*X$ is nilpotent and of finite type. If* $\deg(fR_1H^*X) = d \in \mathbb{N}$ *and* $\deg(fR_2H^*X) < 2d$ *then*

(1) *there is an inclusion* $\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2 \,\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2) \cong F_2/F_1$ *with cokernel of degree* $< 2d$*;*
(2) *the morphism* $\Lambda^2(\overline{F_1}) \to F_2/F_1$ *factors through*

$$\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2 \,\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2).$$

Proof. The cup product of H^*X induces a morphism $\tilde{H}^*X \otimes \tilde{H}^*X \to \mathrm{nil}_2 H^*X$, since $\tilde{H}^*X \in \mathcal{N}il_1$, hence in $\mathscr{F}$ we have

$$F_1 \otimes F_1 = fR_1H^*X \otimes fR_1H^*X \to fR_2H^*X.$$

The restriction of this morphism to $\overline{F_1} \otimes \overline{F_1} \subset F_1 \otimes F_1$ is trivial, by Lemma 6.7, since $\deg(fR_2H^*X) < 2d$ by hypothesis.

Lift $\overline{F_1}$ to a submodule M of $\tilde{H}^*X$ (so that fR_1M corresponds to $\overline{F_1}$) and consider the restriction of the product. By construction this gives

$$M \otimes M \to \mathrm{nil}_3 H^*X$$

with $M \otimes M \in \mathcal{N}il_2$. Moreover, it is easily checked that the finiteness hypothesis required to apply Proposition 3.10 is satisfied, hence there exists a finitely generated submodule $U \subset M \subset \tilde{H}^*X$ such that $fR_1U = fR_1M = \overline{F_1}$ and the cup product restricts to a trivial map $U \otimes U \to \tilde{H}^*X$. (This is a slight extension of Proposition 3.10 using the fact that the choices in the proof can be taken to be compatible with the tensor product $M \otimes M$.)

Using the bar resolution shows that $\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$ is the homology in the middle of the bottom sequence of

$$\begin{array}{ccccc} & & U \otimes U & & \\ & & \downarrow & \searrow^{0} & \\ (\tilde{H}^*X)^{\otimes 3} & \longrightarrow & (\tilde{H}^*X)^{\otimes 2} & \xrightarrow{\mu} & \tilde{H}^*X \end{array}$$

where μ is induced by the cup product and $(\tilde{H}^*X)^{\otimes 3} \in \mathcal{N}il_3$. The vertical map is induced by $U \subset \tilde{H}^*X$, hence the indicated composite is zero and there is an induced morphism in $\mathcal{N}il_2$,

$$U \otimes U \to \mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2),$$

with kernel in $\mathcal{N}il_3$. Applying the functor fR_2 to this gives an injection and $fR_2(U \otimes U) \cong \overline{F_1} \otimes \overline{F_1}$, by construction of U.

This gives the inclusion $\overline{F_1} \otimes \overline{F_1} \hookrightarrow fR_2 \,\mathrm{Tor}^{-2}_{H^*X}(\mathbb{F}_2, \mathbb{F}_2)$. Moreover, by the definition of $\overline{F_1} \subset F_1$, it is clear that the cokernel has degree $< 2d$.

Finally, again by Lemma 6.7, there is no non-trivial map from $\Lambda^2(\overline{F_1})$ to a functor of degree $< 2d$, which gives the factorization statement. ∎

Proof of Theorem 6.1. Suppose that $\deg(fR_2H^*X) < 2d$. Then Proposition 6.12 provides a factorization

$$\Lambda^2(\overline{F_1}) \to \overline{F_1} \otimes \overline{F_1} \to F_2/F_1.$$

This contradicts Proposition 6.11. ∎

REMARK 6.13. The result proved is slightly stronger, without supposing $w_X(2) < 2d$. Namely the argument shows that the composite

$$\overline{F_1} \otimes \overline{F_1} \subset F_1 \otimes F_1 \to fR_2H^*X,$$

where the second morphism is induced by the cup product of H^*X, is necessarily non-trivial.

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