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Borel chromatic number of closed graphs

by

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Abstract. We construct, for each countable ordinal ξ, a closed graph with Borel chromatic number 2 and Baire class ξ chromatic number $\aleph_0$.

1. Introduction. The study of the Borel chromatic number of analytic graphs on Polish spaces was initiated by Kechris et al. [KST]. In particular, they proved that the Borel chromatic number of the graph generated by a partial Borel function has to be in $\{1, 2, 3, \aleph_0\}$. They also provided a minimum graph $\mathcal{G}_0$ of uncountable Borel chromatic number. This last result had a lot of developments. For example, B. Miller [Mi] gave some other versions of it, which helped him to generalize a number of known dichotomy theorems in descriptive set theory. The first author [L2] generalized the $\mathcal{G}_0$-dichotomy to any dimension making sense in classical descriptive set theory, and also used versions of $\mathcal{G}_0$ to study the non-potentially closed subsets of a product of two Polish spaces (see [L1]).

A study of the $\mathbf{\Delta}^0_\xi$ chromatic number of analytic graphs on Polish spaces was initiated in [LZ1] and was motivated by the $\mathcal{G}_0$-dichotomy. More precisely, let B be a Borel binary relation, on a Polish space X, having a Borel countable coloring (i.e., a Borel map $c : X \to \omega$ such that $c(x) \neq c(y)$ if $(x, y) \in B$). Is there a relation between the Borel class of B and that of the coloring? In other words, is there a map $k : \omega_1 \setminus \{0\} \to \omega_1 \setminus \{0\}$ such that any $\mathbf{\Pi}^0_\xi$ binary relation having a Borel countable coloring has in fact a $\mathbf{\Delta}^0_{k(\xi)}$-measurable countable coloring, for each $\xi \in \omega_1 \setminus \{0\}$?

In [LZ2], the authors give a negative answer: for each countable ordinal $\xi \geq 1$, there is a partial injection with disjoint domain and range $i : \omega^\omega \to \omega^\omega$, whose graph

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- is $D_2(\mathbf{\Pi}^0_1)$ (i.e., the difference of two closed sets),
- has Borel chromatic number 2,
- has no $\mathbf{\Delta}^0_\xi$-measurable countable coloring.

On the other hand, they note that an open binary relation having a finite coloring c has also a $\mathbf{\Delta}^0_2$-measurable finite coloring (consider the differences of the $\overline{c^{-1}(\{n\})}$'s for n in the range of the coloring). Note that an irreflexive closed binary relation on a zero-dimensional space has a continuous countable coloring (this coloring is $\mathbf{\Delta}^0_2$-measurable in non-zero-dimensional spaces). So they wonder whether we can build, for each countable ordinal $\xi \geq 1$, a closed binary relation with a Borel finite coloring but no $\mathbf{\Delta}^0_\xi$-measurable finite coloring. This is indeed the case:

Main Theorem. *Let $\xi \geq 1$ be a countable ordinal. Then there exists a partial injection with disjoint domain and range $f : \omega^\omega \to \omega^\omega$ whose graph is closed* (*and thus has Borel chromatic number two*), *and has no $\mathbf{\Delta}^0_\xi$-measurable finite coloring* (*and thus has $\mathbf{\Delta}^0_\xi$ chromatic number $\aleph_0$*).

The previous discussion shows that this result is optimal. Its proof uses, among other things, the method used in [LZ2] improving [M, Theorem 4]. This method relates topological complexity and Baire category.

2. Mátrai sets. Before proving our main result, we recall some material from [LZ2].

Notation. The symbol τ denotes the usual product topology on the Baire space ω^ω.

Definition 2.1. We say that a partial map $f : \omega^\omega \to \omega^\omega$ is *nice* if its graph $\mathrm{Gr}(f)$ is a $(\tau \times \tau)$-closed subset of $\omega^\omega \times \omega^\omega$.

The construction of P_ξ and τ_ξ, and the verification of the properties (i)–(iii) from the next lemma (a corollary of [LZ2, Lemma 2.6]), can be found in [M], up to minor modifications.

Lemma 2.2. *Let $1 \leq \xi < \omega_1$. Then there are $P_\xi \subseteq \omega^\omega$ and a topology τ_ξ on ω^ω such that*

(i) *τ_ξ is zero-dimensional perfect Polish and $\tau \subseteq \tau_\xi \subseteq \mathbf{\Sigma}^0_\xi(\tau)$,*
(ii) *P_ξ is a non-empty τ_ξ-closed nowhere dense set,*
(iii) *if $S \in \mathbf{\Sigma}^0_\xi(\omega^\omega, \tau)$ is τ_ξ-non-meager in P_ξ, then S is τ_ξ-non-meager in ω^ω,*
(iv) *if V, W are non-empty τ_ξ-open subsets of ω^ω, then we can find a τ_ξ-dense G_δ subset H of $V \setminus P_\xi$, a τ_ξ-dense G_δ subset L of $W \setminus P_\xi$, and a nice (τ_ξ, τ_ξ)-homeomorphism from H onto L.*

The following lemma (a corollary of [LZ2, Lemma 2.7]) is a consequence of the previous one. It provides, among other things, a topology T_ξ that we will use.

LEMMA 2.3. *Let $1 \leq \xi < \omega_1$. Then there are a disjoint countable family $\mathcal{G}_\xi$ of subsets of ω^ω and a topology T_ξ on ω^ω such that*

(i) *T_ξ is zero-dimensional perfect Polish and $\tau \subseteq T_\xi \subseteq \mathbf{\Sigma}^0_\xi(\tau)$,*
(ii) *for any non-empty T_ξ-open sets V, V', there are distinct $G, G' \in \mathcal{G}_\xi$ with $G \subseteq V$, $G' \subseteq V'$, and there is a nice (T_ξ, T_ξ)-homeomorphism from G onto G',*

and, for every $G \in \mathcal{G}_\xi$,

(iii) *G is non-empty, T_ξ-nowhere dense, and in $\mathbf{\Pi}^0_2(T_\xi)$,*
(iv) *if $S \in \mathbf{\Sigma}^0_\xi(\omega^\omega, \tau)$ is T_ξ-non-meager in G, then S is T_ξ-non-meager in ω^ω.*

The construction of $\mathcal{G}_\xi$ and T_ξ ensures that T_ξ is $(\tau_\xi)^\omega$, where τ_ξ is as in Lemma 2.2. This topology is on $(\omega^\omega)^\omega$, identified with ω^ω. We will need the following consequence of the construction of $\mathcal{G}_\xi$ and T_ξ.

LEMMA 2.4. *Let $1 \leq \xi < \omega_1$, and V be a non-empty T_ξ-open set. Then $\overline{V}^\tau$ is not τ-compact.*

Proof. The fact that T_ξ is $(\tau_\xi)^\omega$ gives a finite sequence $U_0, \ldots, U_n$ of non-empty open subsets of $(\omega^\omega, \tau_\xi)$ with $U_0 \times \cdots \times U_n \times (\omega^\omega)^\omega \subseteq V$. Thus $\overline{V}^\tau$ contains the τ-closed set $\overline{U_0}^\tau \times \cdots \times \overline{U_n}^\tau \times (\omega^\omega)^\omega$, and it is enough to see that this last set is not τ-compact. This comes from the fact that the Baire space (ω^ω, τ) is not compact. ∎

3. Proof of the main result. We begin with an example giving the flavor of what follows. R. Zamora [Za] gave a Hurewicz-like test to see when two disjoint subsets A, B of a product $Y \times Z$ of Polish spaces can be separated by an open rectangle. We set

$$\begin{aligned} \mathbb{A} &:= \{(n^\infty, n^\infty) \mid n \in \omega\}, \\ \mathbb{B}_0 &:= \{(0^{m+1}(n+1)^\infty, (m+1)^{n+1}0^\infty) \mid m, n \in \omega\}, \\ \mathbb{B}_1 &:= \{((m+1)^{n+1}0^\infty, 0^{m+1}(n+1)^\infty) \mid m, n \in \omega\}. \end{aligned}$$

Then A is not separable from B by an open rectangle exactly when there are $\varepsilon \in 2$ and continuous maps $g : \omega^\omega \to Y$ and $h : \omega^\omega \to Z$ such that $\mathbb{A} \subseteq (g \times h)^{-1}(A)$ and $\mathbb{B}_\varepsilon \subseteq (g \times h)^{-1}(B)$.

EXAMPLE. Here we are looking for closed graphs with Borel chromatic number 2 and of arbitrarily high finite $\mathbf{\Delta}^0_\xi$ chromatic number n. There is an example with $\xi = 1$ and $n = 3$ where $\mathbb{B}_0$ is involved. We set

$$C := \{((2m)^\infty, (2m+1)^\infty) \mid m \in \omega\} \cup \mathbb{B}_0,$$
$$D := \{(2m)^\infty \mid m \in \omega\} \cup \{0^{m+1}(n+1)^\infty \mid m, n \in \omega\},$$
$$R := \{(2m+1)^\infty \mid m \in \omega\} \cup \{(m+1)^{n+1}0^\infty \mid m, n \in \omega\},$$

and $f((2m)^\infty) := (2m+1)^\infty$ and $f(0^{m+1}(n+1)^\infty) := (m+1)^{n+1}0^\infty$. This defines $f : D \to R$ whose graph is C. The first part of C is discrete, and thus closed. Assume that $(\alpha_k, \beta_k) := (0^{m_k+1}(n_k+1)^\infty, (m_k+1)^{n_k+1}0^\infty)$ is in $\mathbb{B}_0$ and converges to $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ as k goes to infinity. We may assume that (m_k) is constant, and (n_k) too, so that $(\alpha, \beta) \in \mathbb{B}_0$, which is therefore closed. This shows that C is closed. Note that D, R are disjoint and Borel, so that C has Borel chromatic number 2. Let Δ be a clopen subset of ω^ω. Let us prove that $C \cap \Delta^2$ or $C \cap (\neg\Delta)^2$ is not empty. We argue by contradiction. Then Δ or $\neg\Delta$ has to contain 0^∞. Assume that it is Δ, the other case being similar. Then $0^{m+1}(n+1)^\infty \in \Delta$ if m is large enough. Thus $(m+1)^{n+1}0^\infty$ is not in Δ if m is large enough. Therefore $(m+1)^\infty \notin \Delta$ if m is large enough. Thus $((2m)^\infty, (2m+1)^\infty) \in C \cap (\neg\Delta)^2$ if m is large enough, which is absurd.

We now turn to the general case. Our main lemma is as follows. We equip ω^m with the discrete topology τ_d, for each $m > 0$.

Main Lemma. *Let $\xi \geq 1$ be a countable ordinal, $n \geq 1$ be a natural number, and $X := \omega \times \omega^\omega$. Then we can find a partial injection $f : X \to X$ and a disjoint countable family $\mathcal{F}$ of subsets of X such that*

(i) *f has disjoint domain and range,*
(ii) $\mathrm{Gr}(f)$ *is $((\tau_d \times \tau) \times (\tau_d \times \tau))$-closed,*
(iii) *there is no sequence $(\Delta_i)_{i<n}$ of $\mathbf{\Delta}^0_\xi$ subsets of $(X, \tau_d \times \tau)$ such that*
  (a) $\mathrm{Gr}(f) \cap \Delta_i^2 = \emptyset$ *for all $i < n$,*
  (b) $\bigcup_{i<n} \Delta_i$ *is $(\tau_d \times T_\xi)$-comeager in X,*
(iv) *$\mathcal{F}$ has properties* (ii)–(iv) *in Lemma 2.3, where $\mathcal{G}_\xi$, ω^ω, T_ξ and τ are respectively replaced with $\mathcal{F}$, X, $\tau_d \times T_\xi$ and $\tau_d \times \tau$,*
(v) $(\bigcup \mathcal{F}) \cap (\mathrm{Domain}(f) \cup \mathrm{Range}(f)) = \emptyset$.

Proof. We argue by induction on n.

Base case $(n = 1)$. Let $\mathcal{G}_\xi$ be the family given by Lemma 2.3. We split $\mathcal{G}_\xi$ into disjoint subfamilies $\mathcal{G}^0_\xi$ and $\mathcal{G}^1_\xi$ having property (ii) in Lemma 2.3. This is possible since the elements of $\mathcal{G}_\xi$ are T_ξ-nowhere dense. Let $G_0, G_1 \in \mathcal{G}^0_\xi$ be distinct, and φ be a nice (T_ξ, T_ξ)-homeomorphism from G_0 onto G_1. We then set $f(0, \alpha) := (0, \varphi(\alpha))$ if $\alpha \in G_0$, and $\mathcal{F} := \{\{n\} \times G \mid n \in \omega \wedge G \in \mathcal{G}^1_\xi\}$. It remains to check that property (iii) is satisfied. We argue by contradiction, which gives $\Delta_0 \in \mathbf{\Delta}^0_\xi$. By property (iv) in Lemma 2.3, $\Delta_0 \cap (\{0\} \times G_\varepsilon)$ is

$(\tau_d \times T_\xi)$-comeager in $\{0\} \times G_\varepsilon$ for each $\varepsilon \in 2$. As f is a $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphism, $\Delta_0 \cap (\{0\} \times G_0) \cap f^{-1}(\Delta_0 \cap (\{0\} \times G_1))$ is $(\tau_d \times T_\xi)$-comeager in $\{0\} \times G_0$, which contradicts the fact that $\mathrm{Gr}(f) \cap \Delta_0^2 = \emptyset$.

Induction step $(n \to n+1)$. The induction assumption gives f and $\mathcal{F}$. Here again, we split $\mathcal{F}$ into two disjoint subfamilies $\mathcal{F}^0$ and $\mathcal{F}^1$ having property (ii) in Lemma 2.3, where $\mathcal{G}_\xi$, ω^ω, T_ξ and τ are respectively replaced with $\mathcal{F}^\varepsilon$, X, $\tau_d \times T_\xi$ and $\tau_d \times \tau$. Let (V_p) be a basis for the topology $\tau_d \times T_\xi$ made of non-empty sets. Fix $p \in \omega$. By Lemma 2.4, there is a countable family $(W_q^p)_{q \in \omega}$, with $(\tau_d \times \tau)$-closed union, and made of pairwise disjoint $(\tau_d \times \tau)$-clopen subsets of X intersecting V_p.

• Let $b : \omega \to \omega^2$ be a bijection. We construct, for $\vec{v} = (p,q) \in \omega^2$ and $\varepsilon \in 2$, and by induction on $b^{-1}(\vec{v})$,

– $G_\varepsilon^{\vec{v}} \in \mathcal{F}^0$,
– a nice $(\tau_d \times T_\xi, \tau_d \times T_\xi)$-homeomorphism $\varphi^{\vec{v}} : G_0^{\vec{v}} \to G_1^{\vec{v}}$.

We want these objects to satisfy the following:

– $G_0^{\vec{v}} \subseteq (V_p \cap W_q^p) \setminus \bigcup_{m<b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi}$,
– $G_1^{\vec{v}} \subseteq V_q \setminus \big(G_0^{\vec{v}} \cup \bigcup_{m<b^{-1}(\vec{v})} \overline{G_0^{b(m)} \cup G_1^{b(m)}}^{\tau_d \times T_\xi}\big)$.

• We now define the desired partial map $\tilde{f} : \omega \times \omega \times \omega^\omega \to \omega \times \omega \times \omega^\omega$, as well as $\tilde{\mathcal{F}} \subseteq 2^{\omega \times \omega \times \omega^\omega}$, as follows:

$$\tilde{f}(l,x) := \begin{cases} (p+1, \varphi^{p,q}(x)) & \text{if } l = 0 \wedge x \in G_0^{p,q}, \\ (l, f(x)) & \text{if } l > 0 \wedge x \in \mathrm{Domain}(f), \end{cases}$$

and $\tilde{\mathcal{F}} := \{\{l\} \times G \mid l \in \omega \wedge G \in \mathcal{F}^1\}$. Note that $\tilde{f}$ is well-defined and injective, by disjointness of the $(G_0^{\vec{v}} \cup G_1^{\vec{v}})$'s. Identifying X with $\omega \times \omega \times \omega^\omega$, we can consider $\tilde{f}$ as a partial map from X into itself and $\tilde{\mathcal{F}}$ as a family of subsets of X (this identification is based on the identification of ω with $\omega \times \omega$).

(i), (iv) and (v) are clearly satisfied.

(ii) Assume $((l_k, x_k), (m_k, y_k)) \in \mathrm{Gr}(\tilde{f})$ tends to $((l,x),(m,y)) \in (\omega \times X)^2$ as k goes to infinity. We may assume that (l_k) and (m_k) are constant.

If $l = 0$, then there is p such that $p+1 = m$ and $(x_k, y_k) \in G_0^{p,q_k} \times G_1^{p,q_k}$. As $G_0^{p,q_k} \subseteq W_{q_k}^p$, we may also assume that (q_k) is constant and equals q. As $\varphi^{p,q}$ is nice, $((l,x),(m,y)) \in \mathrm{Gr}(\tilde{f})$.

If $l > 0$, then $(x_k, y_k) \in \mathrm{Gr}(f)$. As $\mathrm{Gr}(f)$ is $((\tau_d \times \tau) \times (\tau_d \times \tau))$-closed, we have $((l,x),(m,y)) \in \mathrm{Gr}(\tilde{f})$.

(iii) We argue by contradiction, which gives $(\varDelta_i)_{i\leq n}$. We may assume, without loss of generality, that $(\{0\}\times\omega\times\omega^\omega)\cap\varDelta_n$ is not meager in

$$(\{0\}\times\omega\times\omega^\omega,\tau_d\times T_\xi).$$

This gives $p\in\omega$ such that $(\{0\}\times V_p)\cap\varDelta_n$ is $(\tau_d\times T_\xi)$-comeager in $V'_p:=\{0\}\times V_p$. As $V'_p\setminus\varDelta_n\in\mathbf{\Sigma}^0_\xi(\tau_d\times\tau)$, $(\{0\}\times G^{p,q}_0)\cap\varDelta_n$ is $(\tau_d\times T_\xi)$-comeager in $\{0\}\times G^{p,q}_0$ for each $q\in\omega$.

As $\mathrm{Gr}(\tilde{f})\cap\varDelta_n^2=\emptyset$ and the $\varphi^{\vec{v}}$'s are $(\tau_d\times T_\xi,\tau_d\times T_\xi)$-homeomorphisms, $(\{p+1\}\times G^{p,q}_1)\cap\varDelta_n$ is $(\tau_d\times T_\xi)$-meager in $\{p+1\}\times G^{p,q}_1$ for each q.

As $(\omega\times\omega\times\omega^\omega)\setminus\bigcup_{i\leq n}\varDelta_i$ is $(\tau_d\times T_\xi)$-meager in $\omega\times\omega\times\omega^\omega$ and $\mathbf{\Delta}^0_\xi(\tau_d\times\tau)$,

$$(\{p+1\}\times G^{p,q}_1)\setminus\bigcup_{i\leq n}\varDelta_i$$

is $(\tau_d\times T_\xi)$-meager in $\{p+1\}\times G^{p,q}_1$ for each q. Thus $(\{p+1\}\times G^{p,q}_1)\cap\bigcup_{i<n}\varDelta_i$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times G^{p,q}_1$ for each q.

Claim. The set $(\{p+1\}\times\omega\times\omega^\omega)\cap\bigcup_{i<n}\varDelta_i$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times\omega\times\omega^\omega$.

We argue by contradiction. This gives $W\in(\tau_d\times T_\xi)\setminus\{\emptyset\}$ such that

$$(\{p+1\}\times W)\cap\bigcup_{i<n}\varDelta_i$$

is $(\tau_d\times T_\xi)$-meager in $W':=\{p+1\}\times W$. Let $q\in\omega$ be such that $V_q\subseteq W$. Then $G^{p,q}_1\subseteq W$ and $\{p+1\}\times G^{p,q}_1\subseteq W'$. As $W'\cap\bigcup_{i<n}\varDelta_i\in\mathbf{\Sigma}^0_\xi(\tau_d\times\tau)$ and $(\{p+1\}\times G^{p,q}_1)\cap W'\cap\bigcup_{i<n}\varDelta_i$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times G^{p,q}_1$, $W'\cap\bigcup_{i<n}\varDelta_i$ is not $(\tau_d\times T_\xi)$-meager in W', which is absurd. ◇

Now we set $\varDelta'_i:=(\{p+1\}\times\omega\times\omega^\omega)\cap\varDelta_i$ if $i<n$. Note that

$$\varDelta'_i\in\mathbf{\Delta}^0_\xi(\{p+1\}\times\omega\times\omega^\omega,\tau_d\times\tau),$$

$\mathrm{Gr}(\tilde{f})\cap(\varDelta'_i)^2=\emptyset$, and $\bigcup_{i<n}\varDelta'_i$ is $(\tau_d\times T_\xi)$-comeager in $\{p+1\}\times\omega\times\omega^\omega$, which contradicts the induction assumption. ■

In order to get our main result, it is enough to apply the Main Lemma to each $n\geq 1$. This gives $f_n:\omega\times\omega^\omega\to\omega\times\omega^\omega$. It remains to define

$$f:\bigcup_{n\geq 1}(\{n\}\times\omega\times\omega^\omega)\to\bigcup_{n\geq 1}(\{n\}\times\omega\times\omega^\omega)$$

by $f(n,x):=f_n(x)$ (we identify $(\omega\setminus\{0\})\times\omega\times\omega^\omega$ with ω^ω).

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