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# A classification of small homotopy functors from spectra to spectra

by

**Boris Chorny** (Haifa)

**Abstract.** We show that every small homotopy functor from spectra to spectra is weakly equivalent to a filtered colimit of representable functors represented in cofibrant spectra. Moreover, we present this classification as a Quillen equivalence of the category of small functors from spectra to spectra equipped with the homotopy model structure and the opposite of the pro-category of spectra with the strict model structure.

**1. Introduction.** Let Sp denote the closed symmetric monoidal model category of spectra, which is also combinatorial. Either symmetric spectra [17] or Lydakis' linear functors from finite pointed simplicial sets to simplicial sets [21] may serve as a model.

In this work, we suggest a classification of small homotopy functors from spectra to spectra. Namely, we show that, up to a weak equivalence, every small homotopy functor is a filtered colimit of representable functors represented in cofibrant spectra.

Our interest in this question stems from classification problems related to Goodwillie's calculus of homotopy functors. Finitary linear (more generally, homogeneous) functors from spaces to spaces or spectra were classified by T. Goodwillie [15]. Finitary polynomial functors were classified by W. G. Dwyer and C. Rezk (unpublished) and, independently, by G. Arone and M. Ching [1]. Small functors are rather like finitary functors, except that they commute with filtered colimits starting from a certain non-fixed cardinality instead of commuting with all filtered colimits as finitary functors do. It is a natural question whether these classifications extend to more general small functors.

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In this work, we present a classification of small linear functors from spectra to spectra. Since homotopy pushouts are also homotopy pullbacks in Sp, every representable functor is linear (if it is represented by a cofibrant spectrum, and we look at its values only in fibrant spectra), and so are filtered colimits of representable functors. The purpose of this work is to show that these are all small linear functors.

It turns out that a small homotopy functor is linear, since small functors are continuous with respect to the spectral enrichment. This fact is a topological counterpart of a well-known algebraic phenomenon: any additive functor preserving quasi-isomorphisms of chain complexes gives rise to a triangulated functor of derived categories, the total derived functor. Even though its proof seems to be missing from the literature, it is well-known to the experts. We are grateful to Michael Ching who brought this fact to our attention.

Our result may be viewed as a higher version of a well-known statement about homology functors defined on the homotopy category of spectra: every homology functor is a filtered colimit of representables [18, 4.19]. From this point of view, the current work continues to transfer the representability theorems into the enriched realm, which was initiated in [2], [11], [20].

Of course, the most convenient way to formulate our classification result is to exhibit it as a Quillen equivalence of certain model categories. Indeed, we define a new model structure on the category of small functors from spectra to spectra, localizing the fibrant-projective model structure [2]. In the new model structure the fibrant objects are the homotopy functors, therefore we call it the homotopy model structure. Next, we construct a Quillen pair

$$O\colon \mathrm{Sp}^{\mathrm{Sp}} \rightleftarrows (\text{pro-Sp})^{\mathrm{op}} : P,$$

where the right adjoint $P$ is the restriction of Yoneda embedding, sending every pro-space into a filtered colimit of representable functors. The classification of homotopy functors may be performed without using much of the model categories technique, and therefore we postpone the proof that this Quillen adjunction is a Quillen equivalence to the end of the paper.

It is interesting to compare our classification with Goodwillie's classification of finitary linear functors, according to which every finitary linear functor is equivalent to $-\wedge E$ for some spectrum $E$, so that the homotopy category of finitary linear functors is equivalent to the homotopy category of spectra. See [3] for the model-categorical formulation of this classification. However, every spectrum is a filtered colimit of compact spectra, say $E = \mathrm{colim}_i E_i$. Hence, $-\wedge E = \mathrm{colim}(-\wedge E_i)$ with $E_i$ compact for every $i$. A version of Spanier–Whitehead duality [2, 7.1] ensures that $-\wedge E_i = R^{DE_i}(-)$, and hence $-\wedge E = \mathrm{colim}_i R^{DE_i}$, which fits our descrip-

tion. The embedding of finitary functors into all small functors corresponds to the embedding of spectra into the opposite category of pro-spectra as filtered colimits of compact spectra, whose category is self-dual.

However, not every linear functor is small. For example, consider the functor $F(-) = \hom(\hom(-, A), B)$ for two fibrant spectra $A$ and $B$. It is not equivalent to a filtered colimit of representable functors, since it is not small (does not commute with filtered colimits of any size).

Another example, suggested by the anonymous referee, of a non-small functor is the following: $F(Y) = \tilde{Y} \wedge \tilde{Y}$. The reason in this case is different: this functor is not continuous with respect to the spectral enrichment, i.e. not a *spectral* functor. Suppose for contradiction that $F$ is a small spectral functor. Since $F$ is a homotopy functor, it may be approximated by a $C$-cellular functor $\tilde{F} \xrightarrow{\sim} F$ in the fibrant-projective model structure [2, 5.8] where

$$C = \left\{ R^X \wedge K \hookrightarrow R^X \wedge L \;\middle|\; \begin{array}{l} X \text{ fibrant and cofibrant spectrum;} \\ K \hookrightarrow L \text{ generating cofibration in Sp} \end{array} \right\}.$$

Enriched version of the Spanier–Whitehead duality [2, Lemma 7.2] implies, by induction on the skeleton of $\tilde{F}$, that for any compact cofibrant spectrum $A$, $F(A \wedge Y) \simeq A \wedge F(Y)$. On the other hand, $F(A \wedge Y) = (A \wedge Y)_{\mathrm{cof}} \wedge (A \wedge Y)_{\mathrm{cof}} \simeq (\tilde{A} \wedge \tilde{A}) \wedge (\tilde{Y} \wedge \tilde{Y})$, leading to a contradiction.

This example demonstrates a failed attempt to define a quadratic functor, which is not linear. Indeed, as we mentioned before, we will show in Proposition 4.1 that all small homotopy functors are linear, so there is no calculus of functors for the spectral functors, just linear algebra.

The situation is entirely different for simplicial functors from spectra to spectra [6]. The classification of simplicial linear functors demands the development of additional technical tools and will appear in a separate paper [7].

The paper is organized as follows. Section 2 is devoted to the construction of a left adjoint to $P$, which embeds the opposite of the category of pro-spectra as a full subcategory of pro-representable functors in $\mathrm{Sp}^{\mathrm{Sp}}$. Beginning with the fibrant-projective model structure [2] on the category of small functors, we show that this adjunction is a Quillen pair if pro-spectra are equipped with the strict model structure [19].

In Section 3, we localize the fibrant-projective model structure on the category of small functors with respect to a proper class of maps, ensuring that the local objects are precisely the fibrant homotopy functors. Therefore, we name it the homotopy model structure.

Sections 4 and 5 are the technical heart of the paper, where the classification of small homotopy and linear functors is performed, except for the model-categorical reformulation. Section 4 contains the proof that ev-

ery homotopy functor is linear, filling the gap in the literature. Section 5 is devoted to the proof that every linear functor is weakly equivalent to a filtered colimit of representables in the fibrant-projective model structure. These sections rely on a minimal model-categorical technique, and hopefully may be read by people not interested in model categories.

It is not immediate to show that the constructed Quillen pair is a Quillen equivalence again. In order to do so, we give an alternative localization construction in Section 6, which is expressed in terms of the adjoint functors with which we are working, and which may be described as a derived unit of this adjunction. Using our classification, we show that this adjunction coincides, up to homotopy, with the adjunction constructed in Section 3.

Finally, in Section 7, we prove our main result, that our homotopy model structure on the category of small functors is Quillen equivalent to the opposite of the category of pro-spectra. It is formulated as Theorem 5.4.

We would like to conclude this introduction with the notice that, unlike in [8, 9], the localization with respect to a proper class of maps appearing in this paper is not functorial. We do not know whether it is possible to find a localization functor inverting the same class of maps, but we have developed, with Georg Biedermann, an extension to the Bousfield–Friedlander localization machinery which is suitable for work with some non-functorial localization constructions [2, Appendix]. We apply this machinery to the localization construction from Section 3, whereas we were not able to apply it to the localization construction from Section 6.

**2. Preliminaries on pro-spectra.** The goal of this preliminary section is to show that the opposite of the category of pro-spectra is equivalent to a reflective subcategory of small functors from spectra to spectra. If we choose to work with the fibrant-projective model structure on the category of small functors, this adjunction carries over to the level of homotopy categories.

The objects of the category of pro-spectra are cofiltered diagrams of spectra, i.e., for every filtering $I$, any functor $X\colon I^{\mathrm{op}} \to \mathrm{Sp}$ is a pro-spectrum. We denote this pro-object as $X_\bullet = \{X_i\}_{i\in I}$.

The morphisms between two pro-objects $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ are ladders that commute after a composition with the bonding maps. Formally,

$$\hom_{\text{pro-Sp}}(\{X_i\},\{Y_j\}) = \lim_{j\in J} \operatorname*{colim}_{i\in I} \hom_{\mathrm{Sp}}(X_i, Y_j).$$

The category of pro-spectra is enriched over the category of spectra with the internal $\hom_{\text{pro-Sp}}(-,-)$ calculated by the above rule, while taking $\hom_{\mathrm{Sp}}(-,-)$ to be the internal hom-functor in the closed symmetric monoidal category of spectra.

The category $\mathrm{Sp}^{\mathrm{Sp}}$ of small functors from spectra to spectra consists of small functors as objects and natural transformations as morphisms. We remind the reader that a functor $F\colon \mathrm{Sp} \to \mathrm{Sp}$ is *small* if it is a left Kan extension of its restriction to some small subcategory; equivalently, small functors are small weighted colimits of representable functors.

The restriction of the Yoneda embedding on the category of spectra is a functor $P\colon (\text{pro-Sp})^{\mathrm{op}} \to \mathrm{Sp}^{\mathrm{Sp}}$ that sends every pro-spectrum $X_\bullet$ to the pro-representable functor $\hom_{\text{pro-Sp}}(X_\bullet, -)\colon \mathrm{Sp} \to \mathrm{Sp}$. By the definition of morphisms in the category of pro-spectra, the pro-representable functor $\hom_{\text{pro-Sp}}(\{X_i\}, -) = \operatorname{colim}_{i\in I} \hom_{\mathrm{Sp}}(X_i, -)$ is a filtered colimit of representable functors $R^{X_i}$ over $I$. In particular, every pro-representable functor is small.

Now, we show that the functor $Y$ has a left adjoint. The argument we give below works for every locally presentable, closed symmetric monoidal category and not just spectra.

Proposition 2.1. *The functor* $P\colon (\text{pro-Sp})^{\mathrm{op}} \to \mathrm{Sp}^{\mathrm{Sp}}$ *has a left adjoint* $O\colon \mathrm{Sp}^{\mathrm{Sp}} \to (\text{pro-Sp})^{\mathrm{op}}$.

*Proof.* We shall use the adjunction constructed in [2]:

$$Y\colon \mathrm{Sp}^{\mathrm{op}} \rightleftarrows \mathrm{Sp}^{\mathrm{Sp}} : Z,$$

and the fact that the category of small functors from spectra to spectra is class-finitely presentable [12].

Every small functor is a filtered colimit of finite weighted colimits of representable functors. Let $\mathrm{Sp}^{\mathrm{Sp}} \ni F = \operatorname{colim}_{i\in I} C_i$, where $C_i = A_k \star_{k\in K} R^{B_k}$ with all $A_k$ finite spectra. Then

$$\begin{aligned}
\hom_{\mathrm{Sp}^{\mathrm{Sp}}}(F, PX_\bullet) &= \hom_{\mathrm{Sp}^{\mathrm{Sp}}}\Big(\operatorname*{colim}_{i\in I} C_i, \operatorname*{colim}_{j\in J} R^{X_j}\Big) \\
&= \lim_{i\in I} \hom_{\mathrm{Sp}^{\mathrm{Sp}}}\Big(C_i, \operatorname*{colim}_{j\in J} R^{X_j}\Big) \\
&= \lim_{i\in I} \operatorname*{colim}_{j\in J} \hom_{\mathrm{Sp}^{\mathrm{Sp}}}(C_i, R^{X_j}) \\
&= \lim_{i\in I} \operatorname*{colim}_{j\in J} \hom_{\mathrm{Sp}^{\mathrm{op}}}(ZC_i, X_j) \\
&= \lim_{i\in I} \operatorname*{colim}_{j\in J} \hom_{\mathrm{Sp}}(X_j, ZC_i) \\
&= \hom_{\text{pro-Sp}}(\{X_j\}, \{ZC_i\}) = \hom_{(\text{pro-Sp})^{\mathrm{op}}}(\{ZC_i\}, \{X_j\}).
\end{aligned}$$

Of course, the representation of $F$ as a filtered colimit of compact objects is not unique, but if we can take any representation of the kind $F = \operatorname{colim}_{i\in I} C_i$, then the map $f\colon F \to \operatorname{colim}_{i\in I} R^{ZC_i} = P\{ZC_i\}$ serves as a solution set, since, according to the computation above, every map $F \to PX_\bullet$ factors through $f$. Freyd's adjoint functor theorem implies the existence of the left adjoint for $P$, and we can compute its value, up to an isomorphism,

by choosing a representation for $F$ as a filtered colimit of compact objects and declaring $OF = \{ZC_i\}_{i\in I}$. ∎

The category of small functors from spectra to spectra carries the fibrant-projective model structure constructed in [2]. Fibrant-projective weak equivalences and fibrations are the natural transformations of functors inducing levelwise weak equivalences or fibrations between their values in fibrant objects. We conclude the categorical preliminaries by the following proposition that states, essentially, that the opposite of the homotopy category of pro-spectra is a coreflective subcategory of the homotopy category of small functors.

PROPOSITION 2.2. *The pair of adjoint functors*

$$P\colon (\text{pro-Sp})^{\text{op}} \rightleftarrows \text{Sp}^{\text{Sp}} : O$$

*constructed in Proposition* 2.1 *is a Quillen pair if we equip the category of small functors with the fibrant-projective model structure, and the category of pro-spectra with the strict model structure.*

*Proof.* It suffices to show that the right adjoint $P$ preserves fibrations and trivial fibrations of pro-spectra.

Consider a trivial fibration or a fibration $f^{\text{op}}\colon Y_\bullet \to X_\bullet$ in the opposite category of pro-spectra, i.e., $f\colon X_\bullet \to Y_\bullet$ is a trivial cofibration or a cofibration in the strict model structure, which means $f$ is an essentially levelwise trivial cofibration or an essentially levelwise cofibration, where the word 'essentially' means 'up to reindexing'.

Let $f_i\colon X_i \to Y_i$, $i \in I$, be a levelwise trivial cofibration or a levelwise cofibration representing $f$. Recall that $PX_\bullet = \text{colim}_{i\in I} R^{X_i}$ and $PY_\bullet = \text{colim}_{i\in I} R^{Y_i}$. Then $Pf\colon \text{colim}_{i\in I} R^{X_i} \to \text{colim}_{i\in I} R^{Y_i}$ is a trivial fibration or a fibration, respectively, in the fibrant-projective model structure, since each $f_i$ induces a trivial fibration or a fibration of representable functors in the fibrant-projective model structure, and filtered colimits preserve levelwise trivial fibrations and fibrations. ∎

**3. Homotopy model structure.** The main objective of our work is to classify homotopy functors from spectra to spectra, up to homotopy. The most convenient way to provide such a classification is to define a model category structure on small functors with fibrant objects being exactly the fibrant homotopy functors, and to find a more familiar Quillen equivalent model.

In this section, we define the homotopy model structure using the extension of the Bousfield–Friedlander localization technique [4, Appendix A] to non-functorial localization constructions [2, Appendix].

We start from the fibrant-projective model structure on the category of small functors (i.e., weak equivalences are levelwise in fibrant objects). Homotopy functors are small functors that preserve weak equivalences of fibrant objects. If we precompose a homotopy functor with a fibrant replacement in Sp, we obtain a homotopy functor in the classical sense (preserving all weak equivalences), which is fibrant-projective equivalent to the original functor.

Fibrant homotopy functors are the local objects with respect to the following class of maps:

$$\mathcal{H} = \{R^B \to R^A \mid A \xrightarrow{\sim} B \text{ weak equivalence of fibrant objects in Sp}\}.$$

Recall that the class of generating trivial cofibrations for the fibrant-projective model structure is

$$\mathcal{J} = \{R^A \otimes K \hookrightarrow R^A \otimes L \mid A \in \text{Sp fibrant}, \\ K \xhookrightarrow{\sim} L \text{ generating trivial cofibration in Sp}\}.$$

**3.1. Construction of homotopy localization.** We formulate our construction and argumentation in such a way that it will be evident that the category of spectra may be replaced by any closed symmetric monoidal combinatorial model category.

If we were able to localize with respect to the proper class of maps $\mathcal{H}$, we would be done, since $\mathcal{H}$-local functors are exactly the homotopy functors. Instead, for each particular functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$ we choose a cardinal $\lambda_F$ which is the maximum between the accessibility rank of the small (i.e., accessible) functor $F$ and the degree of accessibility of the subcategory of weak equivalences in spectra. Then, we localize this particular functor $F$ with respect to a *set* of maps $\mathcal{H}_{\lambda_F} \subset \mathcal{H}$, and argue that, for this specific functor $F$, it is enough to invert the set $\mathcal{H}_{\lambda_F}$. Of course, we do not obtain a functorial localization construction in this way. However, the (non-functorial) localization we do obtain has enough good properties to ensure the existence of the localized model structure. The detailed construction follows.

Definition 3.1. Let $F \in \mathrm{Sp}^{\mathrm{Sp}}$ be a small functor of accessibility rank $\mu$ and let Sp be a $\kappa$-combinatorial closed symmetric monoidal model for spectra. In particular, the domains and the codomains of the generating (trivial) cofibrations are $\kappa$-presentable, and the class of weak equivalences is a $\kappa$-accessible subcategory of the category of maps of spectra. Let $\lambda_F = \max\{\kappa, \mu\}^+ \rhd \max\{\kappa, \mu\}$ (the $+$ is essential to ensure that the subcategory of weak equivalences in Sp is still $\lambda_F$-accessible), and $\mathrm{Sp}_{\lambda_F} \subset \mathrm{Sp}$ be the subcategory of $\lambda_F$-presentable objects. Then we define

$$\mathcal{H}_{\lambda_F} = \{R^B \to R^A \mid A \xrightarrow{\sim} B \text{ weak equivalence of fibrant objects in } \mathrm{Sp}_{\lambda_F}\}$$

and

$$\mathcal{J}_{\lambda_F} = \{R^A \otimes K \hookrightarrow R^A \otimes L \mid A \in \mathrm{Sp}_{\lambda_F} \text{ fibrant}, \\ K \overset{\sim}{\hookrightarrow} L \text{ generating trivial cofibration in Sp}\}.$$

As usual, we say that a map $f\colon F \to G$ is an $\mathcal{H}_{\lambda_F}$-*equivalence* if for every cofibrant replacement $\tilde{f} \to f$ and every $\mathcal{H}_{\lambda_F}$-local functor $W$ the induced map $\hom(\tilde{f}, W)$ is a weak equivalence of simplicial sets.

Remark 3.2. (1) $\mathcal{H}_{\lambda_F}$ and $\mathcal{J}_{\lambda_F}$ are sets of maps rather than proper classes, and hence it is possible to apply the small object argument.

(2) Every $\mathcal{H}$-local functor is also $\mathcal{H}_{\lambda_F}$-local, and hence every $\mathcal{H}_{\lambda_F}$-equivalence is also an $\mathcal{H}$-equivalence.

(3) Every $\lambda_F$-accessible functor taking fibrant values in $\lambda_F$-presentable fibrant objects (i.e., satisfying the right lifting property with respect to $\mathcal{J}_{\lambda_F}$) is fibrant-projectively fibrant.

(4) Every $\mathcal{H}_{\lambda_F}$-local functor that is also $\lambda_F$-accessible is $\mathcal{H}$-local.

(5) Every $\mathcal{H}_\lambda$-equivalence of $\lambda$-accessible functors is an $\mathcal{H}$-equivalence.

We form the set of horns on $\mathcal{H}_{\lambda_F}$ by first replacing every map in $\mathcal{H}_{\lambda_F}$ with a cofibration, obtaining the set $\tilde{\mathcal{H}}_{\lambda_F}$, and then forming a box product with every generating cofibration in Sp:

$$\mathrm{Hor}(\mathcal{H}_{\lambda_F}) = \Big\{ A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L \;\Big|\; (A \hookrightarrow B) \in \tilde{\mathcal{H}}_{\lambda_F} \text{ and} \\ K \hookrightarrow L \text{ a generating cofibration in Sp} \Big\}.$$

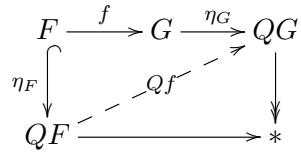
It is well-known (see, e.g., [16]) that if a fibration $X \to *$ has the right lifting property with respect to $\mathrm{Hor}(\mathcal{H}_{\lambda_F})$ then $X$ is $\mathcal{H}_{\lambda_F}$-local, and therefore, in order to construct a localization of an $F \in \mathrm{Sp}^{\mathrm{Sp}}$ with respect to $\mathcal{H}_{\lambda_F}$, it suffices to apply the small object argument for the map $F \to *$ with respect to the set $\mathcal{L} = \mathrm{Hor}(\mathcal{H}_{\lambda_F}) \cup \mathcal{J}_{\lambda_F}$. We obtain a factorization $F \hookrightarrow Q(F) \twoheadrightarrow *$, where the cofibration is an $\mathcal{L}$-cellular map and the fibration has the right lifting property with respect to $\mathcal{K}$.

We omit the standard verification, based on the left properness of $\mathrm{Sp}^{\mathrm{Sp}}$ (see [2, Section 4]), that the cofibration $F \hookrightarrow QF$ is an $\mathcal{H}_{\lambda_F}$-equivalence, and conclude that $QF$ is a homotopy localization of $F$ with respect to $\mathcal{H}_{\lambda_F}$.

Notice that $QF$ is obtained as a colimit of $\lambda_F$-accessible functors, and therefore $QF$ is itself a $\lambda_F$-accessible functor. However, the class of weak equivalences in spectra is $\lambda_F$-accessible, and hence every weak equivalence is a $\lambda_F$-filtered colimit of weak equivalences between $\lambda_F$-presentable objects. Fibrant objects in spectra are closed under $\lambda_F$-filtered colimits, and every spectrum is a $\lambda_F$-filtered colimit of $\lambda_F$-presentable spectra. Combining these facts with the $\lambda_F$-accessibility of $QF$, we conclude that $QF$ is a fibrant homotopy functor in the fibrant-projective model structure. Moreover, the

cofibration $\eta_F\colon F \hookrightarrow QF$ is an $\mathcal{H}$-equivalence, since any $\mathcal{H}_{\lambda_F}$-equivalence is an $\mathcal{H}$-equivalence. In other words, we have constructed a homotopy localization of $F$ with respect to the class $\mathcal{H}$ of maps. The only disadvantage of our construction is the lack of functoriality, since it depends on the choice of the cardinal $\lambda_F$ specific for each $F$.

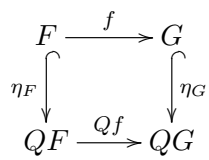
Since $Q$ is not a functor, we have to define separately its action on maps. Given a natural transformation of functors $f\colon F \to G$, we define $Qf$ as a lifting in the diagram




The lift exists since the left vertical map is $\mathcal{L}$-cellular and the right vertical map is $\mathcal{L}$-injective by construction.

Of course, we will have to choose $Qf$ out of many maps that are homotopic to each other, but the important property satisfied by any of these choices is the commutativity of the square

(1)




PROPOSITION 3.3. *Let $f\colon F \to G$ be a map of two functors. Then $Qf$ is a weak equivalence iff $f$ is an $\mathcal{H}$-equivalence.*

*Proof.* The 'if' direction follows by the '2-out-of-3' property for $\mathcal{H}$-equivalences applied to a commutative square (1) and the $\mathcal{H}$-local Whitehead theorem (cf. [16, 3.2.13]): an $\mathcal{H}$-local equivalence of $\mathcal{H}$-local functors is a weak equivalence.

The 'only if' direction follows since, if $Qf$ is a weak equivalence, $f$ is an $\mathcal{H}_{\max\{\lambda_F,\lambda_G\}}$-equivalence by the '2-out-of-3' property for $\mathcal{H}_{\max\{\lambda_F,\lambda_G\}}$-equivalences; but an $\mathcal{H}_{\max\{\lambda_F,\lambda_G\}}$-equivalence of $\max\{\lambda_F,\lambda_G\}$-accessible functors is an $\mathcal{H}$-equivalence. ∎

**3.2. Localization of the model structure.** The lack of functoriality of the homotopy localization $Q$ does not allow us to apply Bousfield and Friedlander's localization machinery [4, Appendix A]. Instead, we will use the generalization of their localization theorem developed in [2, Appendix].

In order to apply this generalization of the Bousfield–Friedlander machinery, we need to verify a number of properties of the localization construction $Q$.

The property [2, A.2] requires precisely the commutativity of the diagram (1), which we obtained by construction.

The properties [2, A.3, A.4] are satisfied because the class of weak equivalences is defined as a map that, after a cofibrant replacement, induces a weak equivalence on the mapping spaces into every $\mathcal{H}$-local object $W$, since mapping out of a retract diagram produces a retract diagram, and also any commutative triangular diagram gives rise to a commutative triangular diagram, which allows us to verify the '2-out-of-3' property.

In order to verify [2, A.5], for every commutative square

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow \\ F_3 & \longrightarrow & F_4 \end{array} \tag{2}$$

let $\lambda = \max\{\lambda_{F_i}\}^+_{1\le i\le 4} \rhd \max\{\lambda_{F_i}\}_{1\le i\le 4}$ be a cardinal, and construct $Q'F_i$ exactly as $QF_i$ using the cardinal $\lambda$ instead of $\lambda_{F_i}$ for each $1 \le i \le 4$. Then, for all $1 \le i \le 4$, there exists a factorization of the coaugmentation map $\eta'_{F_i} : F_i \to Q'F_i$ as follows:

$$\begin{array}{ccccc} F_i & \overset{\eta_{F_i}}{\hookrightarrow} & QF_i & \overset{\sim}{\longrightarrow} & Q'F_i \\ & \searrow & & \nearrow_{\eta'_{F_i}} & \end{array}$$

Moreover, since the only obstruction to naturality of the construction $Q$ is the choice of a different cardinal $\lambda_F$ for each functor $F$, here this obstruction is removed, and we obtain a natural map of the diagram (2) to the commutative square

$$\begin{array}{ccc} QF_1 & \longrightarrow & QF_2 \\ \downarrow & & \downarrow \\ QF_3 & \longrightarrow & QF_4 \end{array}$$

giving rise to a commutative cube.

An additional verification is required in order for [2, A.5] to be satisfied: $Q'f$ must be a weak equivalence iff $Qf$ is. By Proposition 3.3, it suffices to show that $Q'f$ is a weak equivalence iff $f$ is an $\mathcal{H}$-equivalence. Similar to the commutative square (1), we have a commutative square

$$\begin{array}{ccc} F & \overset{f}{\longrightarrow} & G \\ {\scriptstyle \eta'_F}\downarrow & & \downarrow{\scriptstyle \eta'_G} \\ Q'F & \overset{Q'f}{\longrightarrow} & Q'G \end{array}$$

The vertical arrows are $\mathcal{H}$-equivalences by construction, and hence the '2-out-of-3' property for $\mathcal{H}$-equivalences implies that $f$ is an $\mathcal{H}$-equivalence if and only if $Q'f$ is.

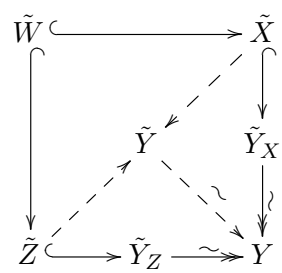
The last property of the homotopical localization $Q$ that requires verification in order to conclude that there exists a $Q$-localized model structure on $\mathrm{Sp}^{\mathrm{Sp}}$ is [2, A.6]: for all pullback squares

$$\begin{array}{ccc} W & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{h} & Z \end{array} \tag{3}$$

where $h$ is an $\mathcal{H}$-fibration (i.e., it has the right lifting property with respect to all $\mathcal{H}$-equivalences which are also cofibrations) and $f$ is an $\mathcal{H}$-equivalence; also, $g$ is an $\mathcal{H}$-equivalence.

Unfortunately, we do not have a simple description of $\mathcal{H}$-equivalences (apart from the fact that they coincide with the $Q$-equivalences, i.e., with the maps converted to weak equivalences by $Q$-construction), and therefore we will use the properties of the stable model category satisfied by $\mathrm{Sp}^{\mathrm{Sp}}$. Namely, we will use the fact that every homotopy pullback is a homotopy pushout in the fibrant-projective model structure.

We start by replacing the commutative square (3) with a weakly equivalent commutative square of cofibrant functors. If we start from $\tilde{W} \xrightarrow{\sim} W$ and continue to factor maps $\tilde{W} \xrightarrow{\sim} W \to Z$ and $\tilde{W} \xrightarrow{\sim} W \to X$ to obtain $\tilde{Z}$ and $\tilde{X}$, respectively, there are two possible ways to replace $Y$ by factoring $\tilde{Z} \xrightarrow{\sim} Z \twoheadrightarrow Y$ or $\tilde{X} \xrightarrow{\sim} X \to Y$ to obtain two different approximations $\tilde{Y}_Z \xrightarrow{\sim} Y$ and $\tilde{Y}_X \xrightarrow{\sim} Y$, respectively. Since the original square (3) is a levelwise homotopy pullback square for values of the functor in each fibrant spectrum, the outer square




is a fibrant-levelwise homotopy pullback and homotopy pushout square. Set $\tilde{Y} = \tilde{X} \coprod_{\tilde{W}} \tilde{Z} \xrightarrow{\sim} Y$ to obtain a cofibrant approximation of the original commutative square.

In order to verify whether $g$ is an $\mathcal{H}$-equivalence for each $\mathcal{H}$-local (i.e., fibrant homotopy) functor $H$, we form a commutative square of mapping

spectra

$$\begin{array}{ccc} H^{\tilde{Y}} & \twoheadrightarrow & H^{\tilde{Z}} \\ \wr\downarrow & & \downarrow \\ H^{\tilde{X}} & \twoheadrightarrow & H^{\tilde{W}} \end{array}$$

which is a homotopy pullback of spectra, and therefore also a homotopy pushout of spectra. Hence, the left properness of the category of spectra implies that the right-hand vertical map is a weak equivalence of spectra. Therefore, the original map $g\colon W \to Z$ is an $\mathcal{H}$-equivalence.

We conclude that by [2, Theorem A.8] there exists a $Q$-localization of the model structure, i.e., this is a localization with respect to $\mathcal{H}$.

We finish this section with an extension of Proposition 2.2 to the localized model structure.

PROPOSITION 3.4. *The adjunction constructed in Proposition* 2.1 *is still a Quillen pair after the localization, i.e., if we consider the $Q$-local model structure on* $\mathrm{Sp}^{\mathrm{Sp}}$ *and the strict model structure on* pro-Sp *then the adjunction $(O, P)$ is a Quillen adjunction.*

*Proof.* By Dugger's lemma [16, 8.5.4], it suffices to check that the right adjoint $P$ preserves fibrations between fibrant objects and all trivial fibrations.

Trivial fibrations did not change after the localization, and therefore it suffices to show that $P$ preserves fibrations of fibrant objects. Let $f^{\mathrm{op}}\colon X_\bullet \to Y_\bullet$ be a fibration of fibrant objects in $(\text{pro-Sp})^{\mathrm{op}}$. Then $f\colon Y_\bullet \to X_\bullet$ is an essentially levelwise cofibration of essentially levelwise cofibrant objects (i.e., up to reindexing). Choose a representative for $f$, which is a commutative diagram of cofibrations between cofibrant spectra, and apply $P$. For such a representative, $Pf$ is a filtered colimit of fibrations of functors represented in cofibrant spectra, i.e., a filtered colimit of projectively fibrant functors preserving weak equivalences of fibrant objects. In other words, $Pf$ is a fibration of homotopy functors. Homotopy functors are precisely the $Q$-local functors, i.e., $Pf$ is a $Q$-fibration by [2, Lemma A.10]. ∎

The rest of the paper is devoted to the proof that this Quillen map is indeed a Quillen equivalence. In other words, we will show that for every cofibrant functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$, every fibrant $X_\bullet \in (\text{pro-Sp})^{\mathrm{op}}$, and every map $f\colon F \to PX_\bullet$ in $\mathrm{Sp}^{\mathrm{Sp}}$, the map $f$ is a $Q$-equivalence if and only if the corresponding map $f^\sharp\colon OF \to X_\bullet$ is a weak equivalence in $(\text{pro-Sp})^{\mathrm{op}}$.

**4. All homotopy functors are linear.** The first reduction in our classification problem is to show that every small homotopy functor is linear,

i.e., that they take homotopy pushouts to homotopy pullbacks. We shall classify the linear functors in the next section.

PROPOSITION 4.1. *Every small functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$ taking weak equivalences to weak equivalences also takes homotopy pushouts to homotopy pullbacks.*

REMARK 4.2. This phenomenon appears only for functors enriched over spectra, such as the small functors, which are colimits of representables. There is a fully featured calculus theory for *simplicial* functors from spectra to spectra developed by Michael Ching [6] where the $n$-excisive functors appear for every $n$.

*Proof of Proposition 4.1.* Given a small homotopy functor $F$, consider its cofibrant replacement $\tilde{F}$, which is a cellular functor and has the filtration

$$0 = F_0 \hookrightarrow \cdots \hookrightarrow F_n \hookrightarrow F_{n+1} \hookrightarrow \cdots \hookrightarrow F_\lambda = \tilde{F}$$

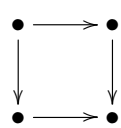
with $F_{n+1}$ obtained from $F_n$ by attaching a cell:

$$\begin{array}{ccc} R^A \wedge K & \longrightarrow & F_n \\ \downarrow & & \downarrow \\ R^A \wedge L & \longrightarrow & F_{n+1} \end{array} \tag{4}$$

where $A$ may be chosen to be a cofibrant and fibrant spectrum [2, Prop. 5.3], and $K \hookrightarrow L$ a generating cofibration in spectra.

Our first goal is to show that if $F_n$ in the diagram (4) preserves homotopy pullbacks (which coincide with homotopy pushouts) of fibrant spectra then $F_{n+1}$ also preserves homotopy pullbacks of fibrant spectra. This will give the inductive step.

Notice that all three functors in the commutative square (4) preserve homotopy pullbacks of fibrant spectra. We will show that $F_{n+1}$ preserves homotopy pushouts, too. Since homotopy pullbacks are also homotopy pushouts, this is a rather intuitive statement of the kind 'a homotopy pushout of homotopy pushouts is a homotopy pushout again'. The formal argument will say that homotopy colimits commute with homotopy colimits, and hence if we apply the functors in (4) to a homotopy pushout square, we obtain a diagram over the category $\mathcal{K} \times \mathcal{K}$, where $\mathcal{K}$ is the category



and conclude that the application of $F_{n+1}$ on any homotopy pullback of fibrant spectra is a homotopy pullback again.

Furthermore, $\tilde{F}$ is a sequential homotopy colimit of functors preserving homotopy pullbacks of fibrant objects, and therefore $\tilde{F}$ also preserves homotopy pullbacks of fibrant objects. However, in addition, $\tilde{F}$ is a homotopy functor, and hence it is a linear functor, which is fibrant-projective equivalent to the original small homotopy functor $F$. ∎

**5. Classification of small linear functors.** In this section, we present a classification of small linear functors. These are the small functors taking homotopy pushouts (= homotopy pullbacks) to homotopy pullbacks. Note that, since every small functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$ is a weighted colimit of representable functors, it preserves the zero spectrum up to homotopy. Note also that every linear functor is a homotopy functor.

Let $\mathcal{F}$ be the class of maps ensuring that $\mathcal{F}$-local objects are precisely the fibrant linear functors. Namely,

$$\mathcal{F} = \left\{ \mathrm{hocolim} \begin{pmatrix} R^D \to R^B \\ \downarrow \\ R^C \end{pmatrix} \to R^A \;\middle|\; \begin{array}{c} A \to B \\ \downarrow \quad \downarrow \\ C \to D \end{array} \text{ homotopy pullback in } \mathrm{Sp} \right\}.$$

Our goal is to show that every linear functor is (fibrant-projectively) weakly equivalent to a filtered colimit of functors represented in cofibrant objects, i.e., to an image of a cofibrant pro-spectrum under the restricted Yoneda embedding $P$ constructed in Section 2. We begin with the lemma stating that these functors are closed under filtered colimits. In other words, filtered colimits of filtered colimits of representable functors are again filtered colimits.

LEMMA 5.1. *The full subcategory generated by the filtered colimits of representable functors is closed under filtered colimits. Moreover, the subcategory of filtered colimits of functors represented in cofibrant objects is also closed under filtered colimits.*

*Proof.* Let $\mathcal{I}$ and $\mathcal{J}_i$, $i \in \mathcal{I}$, be filtered categories. Suppose that

$$F_i = \operatorname*{colim}_{j \in \mathcal{J}_i} R^{X_{i,j}} \text{ for some } X_{i,j} \in \mathrm{Sp}, \qquad F = \operatorname*{colim}_{i \in \mathcal{I}} F_i.$$

We need to show that $F$ may be represented as a filtered colimit of representable functors.

Applying the left adjoint $O$ to the functor $F$, we obtain a pro-object $\{X_j\}_{j \in \mathcal{J}}$ for some filtered category $\mathcal{J}$. Consequently,

$$\{X_\bullet\} = O(F) = O\Big(\operatorname*{colim}_{i \in \mathcal{I}} F_i\Big) = \operatorname*{colim}^{(\text{pro-Sp})^{\mathrm{op}}}_{i \in \mathcal{I}} O(F_i) = \lim^{\text{pro-Sp}}_{i \in \mathcal{I}} \{X_{i,\bullet}\}.$$

However, if we apply $P$ on $\{X_\bullet\}$, we recover $F$ again:

$$\begin{aligned} P(\{X_\bullet\})(-) &= \hom_{\text{pro-Sp}}(\{X_\bullet\}, -) \\ &= \hom_{\text{pro-Sp}}\Big(\lim_{i\in\mathcal{I}}{}^{\text{pro-Sp}}\{X_{i,\bullet}\}, -\Big) \quad \text{(constant pro-spaces are cosmall)} \\ &= \operatorname*{colim}_{i\in\mathcal{I}}\Big(\operatorname*{colim}_{j\in\mathcal{J}_i} R^{X_{i,j}}\Big) = \operatorname*{colim}_{i\in\mathcal{I}} F_i = F. \end{aligned}$$

Therefore, $F = P(\{X_\bullet\}) = \operatorname{colim}_{j\in\mathcal{J}} R^{X_j}$ is a filtered colimit of representable functors.

Suppose now that all $X_{i,j} \in \mathrm{Sp}$, $i \in \mathcal{I}$, $j \in \mathcal{J}_i$, are cofibrant spectra. Then $O(F) = \{X_\bullet\}$ is a cofibrant pro-spectrum as a cofiltered inverse limit of cofibrant pro-spectra $\{X_{i,\bullet}\}$ in the class-fibrantly generated strict model structure on pro-spectra [10]. In other words, $\{X_\bullet\}$ is an essentially levelwise cofibrant pro-spectrum, and hence $P(\{X_\bullet\})$ is a filtered colimit of functors represented in cofibrant spectra. ∎

Proposition 5.2. *Let $F \in \mathrm{Sp}^{\mathrm{Sp}}$ be a linear functor. Then there exists a filtered diagram $J$ and a functor $G = \operatorname{colim}_{j\in J} R^{X_j}$ with cofibrant $X_j \in \mathrm{Sp}$ for all $j \in J$, and a weak equivalence $f\colon \tilde{F} \to G$ for some cellular approximation $\tilde{F} \xrightarrow{\sim} F$ in the fibrant-projective model structure.*

*Proof.* Since $F$ is a linear functor, it is also a homotopy functor, and hence there exists a cellular approximation $\tilde{F} \xrightarrow{\sim} F$ such that for some cardinal $\lambda$ there is a transfinite sequence of functors $\tilde{F} = \operatorname{colim}_{i\le\lambda} F_i$, and $F_i$ is obtained from $F_{i-1}$ by attaching a generating cofibration of the form $A \wedge R^{\hat{X}} \hookrightarrow B \wedge R^{\hat{X}}$ for every successor cardinal $i \le \lambda$, and $F_i = \operatorname{colim}_{a<i} F_a$ for every limit ordinal $i \le \lambda$. The cofibration $A \hookrightarrow B$ is a generating cofibration in Sp, and therefore $A$ and $B$ are compact spectra. Moreover, the representing object $\hat{X}$ may be chosen to be cofibrant, since $\tilde{F}$ is a homotopy functor by [2, Prop. 5.3].

By [11, Lemma 3.3], there exists a countable sequence $\{F'_k\}_{k<\omega}$ such that $F'_0 = 0$, $F = \operatorname{colim}_{k<\omega} F'_k$, and for each $k > 0$ there is a pushout square

$$\begin{array}{ccc} \coprod_{s\in S_{k-1}} A_s \wedge R^{\hat{X}_s} & \longrightarrow & F'_{k-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{s\in S_{k-1}} B_s \wedge R^{\hat{X}_s} & \longrightarrow & F'_k \end{array}$$

where the coproduct is indexed by the subset $S_{k-1} \subset \lambda$ corresponding to the cells coming from various stages of the original sequence $\{F_i\}_{i\le\lambda}$, such that their attachment maps factor through the $(k-1)$st stage of the previously constructed sequence.

The coproduct of maps in the commutative square above is a filtered colimit of finite coproducts over the filtering $J_{k-1}$ of the finite subsets of $S_{k-1}$.

Let us think of the constant object $F'_{k-1}$ as a filtered colimit of the constant diagrams over the same filtering $J_{k-1}$. However, colimits over $J_{k-1}$ commute with pushouts, and hence we obtain the representation of $F'_k$ as a filtered colimit of pushouts of the following form:

$$\begin{array}{ccc} \coprod_{s\in S_{k-1,j}} A_s \wedge R^{\hat{X}_s} & \xrightarrow{\varphi_{k-1,j}} & F'_{k-1} \\ \downarrow & & \downarrow \\ \coprod_{s\in S_{k-1,j}} B_s \wedge R^{\hat{X}_s} & \longrightarrow & F_{k,j} \end{array} \tag{5}$$

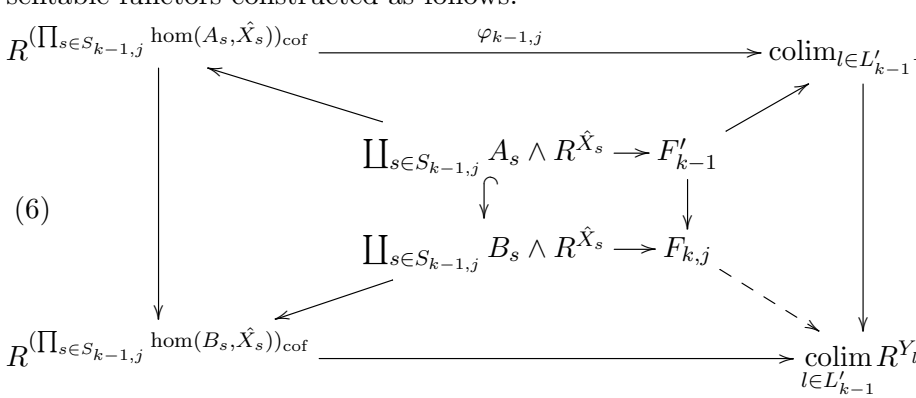
where $S_{k-1,j} \subset S_{k-1}$ is a finite subset corresponding to $j \in J_{k-1}$.

Now, by [2, Lemma 7.1], there are weak equivalences in the fibrant-projective model category: $A_s \wedge R^{\hat{X}_s} \simeq R^{\hat{X}_s \wedge DA_s}$ and $B_s \wedge R^{\hat{X}_s} \simeq R^{\hat{X}_s \wedge DB_s}$. Moreover, any finite coproduct of representable functors is $\mathcal{F}$-equivalent to a representable functor by an inductive argument on the number of terms that begins with the observation that a coproduct $R^{\hat{U}} \sqcup R^{\hat{V}}$ of two representables is $\mathcal{F}$-equivalent to $R^{\hat{U}\times\hat{V}}$, since the map $R^{\hat{U}} \sqcup R^{\hat{V}} \simeq \operatorname{hocolim}(R^{\hat{U}} \leftarrow R^0 \rightarrow R^{\hat{V}}) \rightarrow R^{\hat{U}\times\hat{V}}$ is an element in $\mathcal{F}$ corresponding to the homotopy pullback square

$$\begin{array}{ccc} \hat{U}\times\hat{V} & \longrightarrow & \hat{U} \\ \downarrow & & \downarrow \\ \hat{V} & \longrightarrow & 0 \end{array}$$

In other words, the entries on the left-hand side of the pushout square (5) are $\mathcal{F}$-equivalent to representable functors with fibrant and cofibrant spectra as representing objects.

Suppose for the induction that there is an $\mathcal{F}$-equivalence $F'_{k-1} \rightarrow \operatorname{colim}_{l\in L_{k-1}} R^{Y_l}$, where $L_{k-1}$ is a filtered category and the representable functors have fibrant and cofibrant spectra as representing objects. Then we obtain a morphism of the pushout diagram (5) into a commutative square (which is also a homotopy pushout) composed of filtered colimits of representable functors constructed as follows:




(6)

The diagonal maps on the left are obtained as compositions of the unit of the adjunction (see Proposition 2.1) with a map induced by the cofibrant approximations in pro-Sp:

$$\Big(\prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s)\Big)_{\mathrm{cof}} \overset{\sim}{\twoheadrightarrow} \prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s),$$
$$\Big(\prod_{s\in S_{k-1,j}} \hom(B_s,\hat{X}_s)\Big)_{\mathrm{cof}} \overset{\sim}{\twoheadrightarrow} \prod_{s\in S_{k-1,j}} \hom(B_s,\hat{X}_s).$$

The universal property of the unit of adjunction guarantees the existence of a natural map

$$R^{\prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s)} \to \operatorname*{colim}_{l\in L_{k-1}} R^{Y_l}.$$

The corresponding map in the pro-category has a lift to the cofibrant replacement of the constant pro-spectrum, since the pro-spectrum $\{Y_l\}_{l\in L_{k-1}}$ is (levelwise) cofibrant:

$$\begin{array}{ccc} & & \Big(\prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s)\Big)_{\mathrm{cof}} \\ & \nearrow & \big\downarrow \wr \\ \{Y_l\}_{l\in L_{k-1}} & \longrightarrow & \prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s) \end{array}$$

The source of the dashed map in the diagram above may be replaced by an isomorphic pro-object $\{Y_l\}_{l\in L'_{k-1}}$ with a final indexing subcategory $L'_{k-1}\subset L_{k-1}$, so that the resulting map is reindexed into a natural transformation of contravariant $L'_{k-1}$-diagrams with a constant diagram in the target. The induced map in the category of functors is denoted by $\varphi_{k-1,j}$, and it factors through every stage of the colimit. Thus, the outer pushout diagram in (6) may be viewed as a filtered colimit of pushout diagrams indexed by $L'_{k-1}$.

Let

$$Y'_l = Y_l \times_{(\prod_{s\in S_{k-1,j}} \hom(A_s,\hat{X}_s))_{\mathrm{cof}}} \Big(\prod_{s\in S_{k-1,j}} \hom(B_s,\hat{X}_s)\Big)_{\mathrm{cof}}.$$

This is a homotopy pullback of spectra, and hence $R^{Y'_l}$ is $\mathcal{F}$-equivalent to the homotopy pushout of the corresponding representable functors in the fibrant-projective model structure on the category of small functors from spectra to spectra.

Taking the filtered colimit of these commutative squares indexed by $L'_{k-1}$, we obtain the outer square of (6), and since filtered colimits preserve both $\mathcal{F}$-equivalences by [5, Lemma 1.2] and homotopy pushouts, we conclude that $\operatorname{colim}_{l\in L'_{k-1}} R^{Y'_l}$ is $\mathcal{F}$ to the homotopy pushout of the outer square of (6). Consequently, the dashed arrow in (6) is an $\mathcal{F}$-equivalence.

In other words, $F_{k,j}$ is $\mathcal{F}$-equivalent to a filtered colimit of representable functors.

Therefore, $F'_k = \operatorname{colim}_{j \in J_{k-1}} F_{k,j}$ is a filtered colimit of functors $\mathcal{F}$-equivalent to filtered colimits of representable functors, which, in turn, are $\mathcal{F}$-equivalent to filtered colimits of representable functors by Lemma 5.1.

Finally, $F = \operatorname{colim}_{k<\omega} F'_k$ is a countable sequential colimit of filtered colimits of functors $\mathcal{F}$-equivalent to representable functors, which may be reindexed into a single filtered colimit of functors $\mathcal{F}$-equivalent to representable functors by Lemma 5.1. ∎

COROLLARY 5.3. *Every small homotopy functor from spectra to spectra is fibrant-projective equivalent to a filtered colimit of representable functors represented in cofibrant objects.*

*Proof.* Every small homotopy functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$ is linear by Proposition 4.1. Therefore, $F$ is $\mathcal{F}$-local and, by Proposition 5.2, is $\mathcal{F}$-equivalent to a filtered colimit of representable functors represented in cofibrant objects. However, $\mathcal{F}$-equivalence of $\mathcal{F}$-local functors is fibrant-projective equivalence. ∎

So far, we have shown that the fibrant objects in the homotopy model structure constructed in Section 3 are fibrant-projective equivalent to filtered colimits of representable functors represented in cofibrant objects, i.e., they correspond to cofibrant pro-objects. Of course, a more elegant way to state this classification result is to show that the Quillen adjunction of Proposition 2.2 is actually a Quillen equivalence.

THEOREM 5.4. *The Quillen adjunction* $O : \mathrm{Sp}^{\mathrm{Sp}} \rightleftarrows \text{pro-Sp} : P$ *is a Quillen equivalence if* $\mathrm{Sp}^{\mathrm{Sp}}$ *is equipped with the homotopy model structure and* pro-Sp *is equipped with the strict model structure.*

The rest of the paper is devoted to the proof of this theorem.

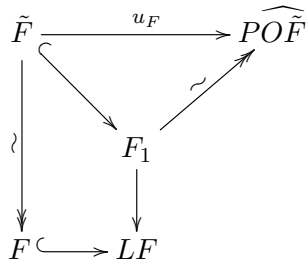
**6. Alternative localization construction.** In this section, we give an alternative localization of the fibrant-projective model structure on $\mathrm{Sp}^{\mathrm{Sp}}$, which produces homotopy approximations of small functors. It is better suited for establishing that the Quillen map constructed in Proposition 2.1 is a Quillen equivalence.

**6.1. The localization construction.** Given an arbitrary small functor $F \in \mathrm{Sp}^{\mathrm{Sp}}$, consider its (non-functorial) cofibrant replacement in the fibrant-projective model structure $\tilde{F} \xrightarrow{\sim} F$. Then the derived unit of the adjunction constructed in Proposition 2.1 has the right homotopy type of the localization we are constructing: $u \colon \tilde{F} \to \widehat{P(O(\tilde{F}))}$. However, the localization construction involves a coaugmentation map for every functor

$\eta\colon F \to LF$. Now, we factor $u$ into a cofibration followed by a trivial fibration $\tilde{F} \hookrightarrow F_1 \xrightarrow{\sim} \widehat{PO\tilde{F}}$, and declare $LF = F \times_{\tilde{F}} F_1$.

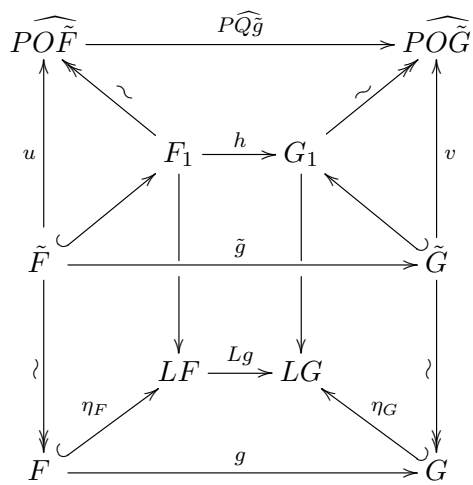
We summarize our localization construction in the following diagram:

(7)




The construction of $LF$ depends on the choice of a cofibrant replacement for $F$ and a factorization for $u_F$. We fix these choices once and for all. Since the procedure described above is homotopy meaningful, the homotopy type of $LF$ does not depend on the choices we make.

The localization construction $L$ is defined also on morphisms. Given a natural transformation $g\colon F \to G$ of small functors from spectra to spectra, we proceed through the stages of the definition of $L$, constructing at each stage a map corresponding to $g$, making a choice in the non-functorial parts of the definition of $L$. Namely, to obtain a map of cofibrant replacements and the factorizations, we use the lifting axiom. Such a choice is not functorial, and it is unique only up to homotopy. Nevertheless, we have the following commutative diagram:




This diagram defines a map $Lg$ for every natural transformation $g$ and a morphism of maps $\eta_g\colon g \mapsto Lg$.

We summarize this discussion in the following proposition.

PROPOSITION 6.1. *For every natural transformation $g\colon F \to G$ of small functors, the map $Lg\colon LF \to LG$ is defined and depends on the choices required at various stages of its construction. Moreover, there exist maps $\eta_F\colon F \to LF$ and $\eta_G\colon G \to LG$ depending on the same choices and no others, such that the square*

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & LF \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Lg} \\ G & \xrightarrow{\eta_G} & LG \end{array}$$

*is commutative.*

Our goal is to compare the localization construction $L$ with the non-functorial localization $Q$ previously constructed in Section 3. However, first we need to prove that $L$ is a homotopy localization construction in accordance with [2, Definition A.1], and to verify the conditions [2, Definitions A.2–A.6]. Proposition 6.1 above satisfies [2, Definition A.2].

### 6.2. Verification of homotopy idempotency

PROPOSITION 6.2. *For all $F \in \mathrm{Sp}^{\mathrm{Sp}}$, the maps $\eta_{LF}, L\eta_F\colon LF \to LLF$ are weak equivalences.*

We begin with a technical lemma about class-combinatorial model categories generalizing similar results for combinatorial model categories: weak equivalences are closed under $\lambda$-filtered colimits [14, 7.3].

LEMMA 6.3. *Let $\mathcal{M}$ be a class-cofibrantly generated model category with $\lambda$-presentable domains and codomains of generating (trivial) cofibrations. Then $\lambda$-filtered colimits of objects in $\mathcal{M}$ are homotopy colimits. In other words, every levelwise weak equivalence of $\lambda$-filtered diagrams in $\mathcal{M}$ induces a weak equivalence between their colimits.*

*Proof.* Let $\mathcal{A}$ be a $\lambda$-filtered category, let $\underset{\sim}{X}, \underset{\sim}{Y}\colon \mathcal{A} \to \mathcal{M}$ be two diagrams, and let $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ be a levelwise weak equivalence. Consider the projective model structure on the category $\mathcal{M}^{\mathcal{A}}$. It may be constructed by a straightforward generalization of [16, 11.6]. Now, we apply a cofibrant replacement in the projective model structure to the map $f$:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle \wr}\downarrow & & \downarrow{\scriptstyle \wr} \\ X & \xrightarrow{f} & Y \end{array}$$

The functor $\mathrm{colim}\colon \mathcal{M}^{\mathcal{A}} \to \mathcal{M}$ is a left Quillen functor if the domain category is equipped with the projective model structure, and hence it preserves

weak equivalences of cofibrant objects. Moreover, this functor preserves trivial fibrations, since the category $\mathcal{M}$ is class-cofibrantly generated. Therefore, applying a colimit to the commutative square above, we conclude, by the '2-out-of-3' property for weak equivalences, that $\operatorname{colim} f$ is a weak equivalence. ∎

LEMMA 6.4. *Let $X_\bullet \in$ pro-Sp be a cofibrant pro-spectrum. Then $PX_\bullet \in \mathrm{Sp}^{\mathrm{Sp}}$ is a filtered colimit of representable functors and not necessarily cofibrant. Consider a cofibrant replacement $p\colon \widetilde{PX_\bullet} \xrightarrow{\sim} PX_\bullet$. Then the left adjoint $O$ preserves this weak equivalence: $Op\colon O\widetilde{PX_\bullet} \xrightarrow{\sim} OPX_\bullet$.*

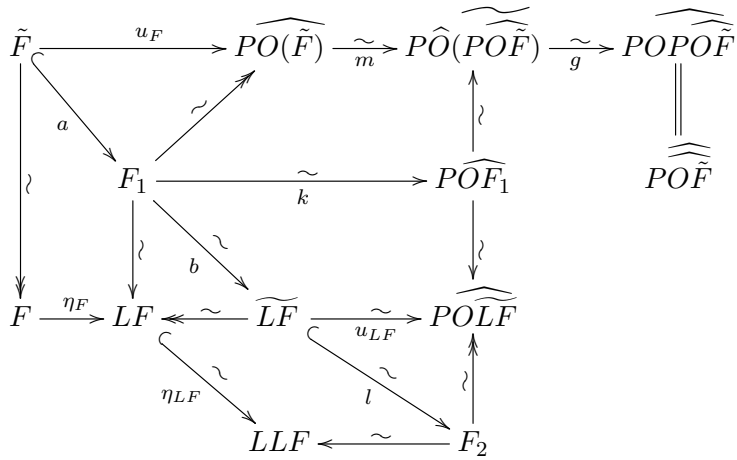
*Proof.* As the left Quillen functor $O$ preserves weak equivalences between cofibrant objects, it suffices to prove that $O$ takes to a weak equivalence some cofibrant approximation of $P\{X_i\} = \operatorname{colim}_i R^{X_i}$. Consider the cofibrant approximation $q\colon \operatorname{hocolim}_i R^{\hat{X}_i} = \overline{PX_\bullet} \to PX_\bullet$, where $q$ is induced by the fibrant-projective cofibrant approximations $R^{\hat{X}_i} \xrightarrow{\sim} R^{X_i}$, while the maps $X_i \xhookrightarrow{\sim} \hat{X}_i$ are the functorial fibrant approximations in Sp. Then $\overline{PX_\bullet}$ is cofibrant as a homotopy colimit of a diagram with cofibrant entries (we assume here that a homotopy colimit is defined as a coend with a projectively cofibrant, contractible diagram of spaces, i.e., a left Quillen functor preserving cofibrant objects). By Lemma 6.3, the map $q$ is a weak equivalence, since filtered colimits in the class-cofibrantly generated fibrant-projective model structure on $\mathrm{Sp}^{\mathrm{Sp}}$ are homotopy colimits.

The functor $O$ preserves colimits and homotopy colimits as a left Quillen functor, and hence the map $Oq\colon O\overline{PX_\bullet} \to OPX_\bullet$ is essentially the map $Oq\colon \operatorname{hocolim} OR^{\hat{X}_i} \to \operatorname{colim} OR^{X_i}$, or just $Oq\colon \operatorname{hocolim} \hat{X}_i \to \operatorname{colim} X_i$ in the opposite of the strict model structure on $(\text{pro-Sp})^{\mathrm{op}}$. However, the strict model structure on pro-Sp is class-fibrantly generated [10], and therefore the dual model structure is class-cofibrantly generated and the map $Oq$ is a weak equivalence by Lemma 6.3. Therefore, $Op$ is also a weak equivalence. ∎

LEMMA 6.5. *The map $O\widetilde{\eta_F}\colon O\widetilde{F} \xrightarrow{\sim} O\widetilde{LF}$ is a weak equivalence for all $F \in \mathrm{Sp}^{\mathrm{Sp}}$.*

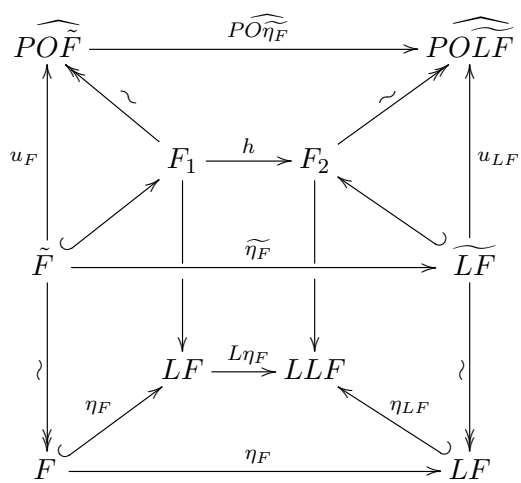
*Proof.* In the commutative diagram (7), the object $F_1$ may serve as a cofibrant replacement for both $LF$ and $\widetilde{PO\tilde{F}}$. Therefore, applying the functor $O$ to (7), we conclude that $O\widetilde{\eta_F}\colon O\widetilde{F} \xrightarrow{\sim} O\widetilde{LF}$ is a weak equivalence: the trivial fibration $F_1 \xrightarrow{\sim} \widetilde{PO\tilde{F}}$ remains a weak equivalence after application of $O$ by Lemma 6.4, and the derived unit of the $(O, P)$-adjunction $u_F$ is also turned by $O$ into a weak equivalence $O\widetilde{F} \to OPO\widetilde{F} \cong \widetilde{O\tilde{F}}$, and hence $\widetilde{F} \hookrightarrow F_1$ is turned by an application of $O$ into a weak equivalence; the map $O\widetilde{\eta_F}$ is then a weak equivalence by the '2-out-of-3' property. ∎

*Proof of Proposition 6.2.* Consider the construction of $\eta_{LF}$ first. In the commutative diagram




$g$ is a weak equivalence by Lemma 6.4, and therefore $m$ is also a weak equivalence. Applying the '2-out-of-3' property, we find that $k$, $u_{LF}$, and $l$ are weak equivalences. Therefore, $\eta_{LF}$ is a weak equivalence by the '2-out-of-3' property again.

Now, consider the construction of $L\eta_F$:




The map $\widehat{PO\widetilde{\eta_F}}$ is a weak equivalence as an application of the right Quillen functor $P$ on the weak equivalence, by Lemma 6.5, between fibrant objects $\widehat{O\widetilde{\eta_F}}$.

The '2-out-of-3' property then implies that $h$ and $L\eta_F$ are weak equivalences as well. ∎

In addition, we notice that every stage in the construction of $L$ preserves weak equivalences, and therefore we readily obtain the following

PROPOSITION 6.6. *The localization construction $L$ preserves weak equivalences.*

We are now ready to compare the two non-functorial localization constructions, and prove that $QF$ is weakly equivalent to $LF$ for all $F \in \mathrm{Sp}^{\mathrm{Sp}}$. We already know that the classes of $Q$-local objects and $L$-local objects coincide: these are fibrant functors weakly equivalent to filtered colimits of representable functors with cofibrant representing objects. Ideologically, this should imply the equivalence of localization constructions immediately. However, the proof of a general statement of this kind is involved and requires plenty of additional structure on the localization constructions, which does not exist in our case (cf. [5], [13]). Therefore, we shall carry out the proof in this particular situation.

LEMMA 6.7. *The derived unit map $u_F\colon \tilde{F} \to \widehat{P\tilde{O}\tilde{F}}$ is a $Q$-equivalence for all $F \in \mathrm{Sp}^{\mathrm{Sp}}$.*

*Proof.* By Proposition 3.3, it suffices to check whether $u_F$ is an $\mathcal{H}$-equivalence, i.e., it suffices to verify that $\hom(\widetilde{u_F}, W)$ for any $Q$-local functor $W$. By Corollary 5.3, $W$ is weakly equivalent to a filtered colimit of representable functors represented in cofibrant spectra, and hence $W \simeq PX_\bullet$ for some cofibrant pro-spectrum $X_\bullet$.

By adjunction, the map $\hom(\widetilde{u_F}, PX_\bullet)$ is naturally isomorphic to the map

$$\hom(\widetilde{O\widehat{P\tilde{O}\tilde{F}}}, X_\bullet) \to \hom(O\tilde{F}, X_\bullet).$$

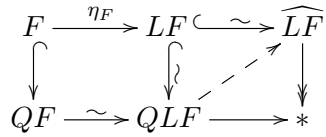
By Lemma 6.4, $\widetilde{O\widehat{P\tilde{O}\tilde{F}}} \simeq O\widehat{P\tilde{O}\tilde{F}} = \widehat{O\tilde{F}}$, showing that the last map is a weak equivalence. ∎

PROPOSITION 6.8. *For all $F \in \mathrm{Sp}^{\mathrm{Sp}}$ there is a weak equivalence $QF \simeq LF$.*

*Proof.* First, we notice that the coaugmentation map $\eta_F\colon F \to LF$ is a $Q$-equivalence.

Given $F$, similarly to the verification of [2, A.5] in 3.2, we choose a cardinal $\lambda$ big enough that all entries of the commutative diagram (7) are $\lambda$-accessible. Next, we apply a modification of $Q$ which is functorial and on this particular diagram provides results weakly equivalent to the application of $Q$, and conclude that $\eta_F\colon F \hookrightarrow LF$ is a $Q$-equivalence if and only if the derived unit map $u_F\colon \tilde{F} \to \widehat{P\tilde{O}\tilde{F}}$ is a $Q$-equivalence. Hence, by Lemma 6.7, $\eta_F\colon F \to LF$ is a $Q$-equivalence.

Consider now the following commutative diagram obtained by application of the construction $Q$ to the coaugmentation:



Since $\eta_F$ is a $Q$-equivalence, $Q\eta_F$ is a weak equivalence in the fibrant-projective model structure, and hence we obtain a zig-zag weak equivalence $QF \simeq LF$ for all $F \in \mathrm{Sp}^{\mathrm{Sp}}$. ∎

**7. Proof of Theorem 5.4.** In Proposition 2.2, we have shown that the adjunction $O\colon \mathrm{Sp}^{\mathrm{Sp}} \rightleftarrows (\text{pro-Sp})^{\mathrm{op}}\colon P$ is a Quillen pair. We need to prove that for every cofibrant $F \in \mathrm{Sp}^{\mathrm{Sp}}$ and every fibrant $X_\bullet \in (\text{pro-Sp})^{\mathrm{op}}$ the map $f\colon O(F) \to X_\bullet$ is a (strict) weak equivalence of pro-spectra if and only if the map $g\colon F \to PX_\bullet$ is a weak equivalence in the homotopy model structure on $\mathrm{Sp}^{\mathrm{Sp}}$, i.e., it is a $Q$-equivalence of small functors.

Suppose that $f\colon O(F) \to X_\bullet$ is a weak equivalence. Applying a fibrant replacement to $O(F)$, we obtain a trivial cofibration $j\colon OF \xhookrightarrow{\sim} \widehat{OF}$ and a factorization of $f$ as $f = \hat{f}j$, where the lifting $\hat{f}$ exists since $X_\bullet$ is fibrant (in pro-$\mathrm{Sp}^{\mathrm{op}}$). Moreover, $\hat{f}$ is a weak equivalence of fibrant objects by the '2-out-of-3' property. The adjoint map $g$ factors as a unit of the adjunction $u\colon F \to POF$ composed with $Pf$: $g = P(f)u$, but $Pf = P(\hat{f}j) = P(\hat{f})Pj$, and hence $g = P(\hat{f})(P(j)u)$. Now, $P(\hat{f})$ is a weak equivalence, since $P$ is a right Quillen functor and preserves weak equivalences of fibrant objects. The composed map $P(j)u$ is an $L$-equivalence by Proposition 6.2 and it is a $Q$-equivalence by Lemma 6.7, which applies since $F$ is cofibrant.

Conversely, suppose that $g\colon F \to PX_\bullet$ is a weak equivalence. Consider a cofibrant replacement $p\colon \widetilde{PX_\bullet} \xrightarrow{\sim} PX_\bullet$. Then there exists a lift $\tilde{g}\colon F \to \widetilde{PX_\bullet}$ in the homotopy model structure. Note that $\tilde{g}$ is a weak equivalence of cofibrant objects by the '2-out-of-3' property, since $g = p\tilde{g}$. The adjoint map $f\colon OF \to X_\bullet$ factors as $Og$ followed by the counit $c\colon OPX_\bullet \to X_\bullet$, which is a natural isomorphism for all $X_\bullet$. However, $Og = OpO\tilde{g}$, where $Op$ is a weak equivalence by Lemma 6.4 and $O\tilde{g}$ is a weak equivalence, since $O$ is a left Quillen functor. Hence, $f$ is a weak equivalence. ∎

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Boris Chorny
Department of Mathematics, Physics and Computer Science
University of Haifa at Oranim
Tivon 36006, Israel
E-mail: chorny@math.haifa.ac.il