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On Baker type lower bounds for linear forms

by

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1. Introduction. We give a criterion for studying (explicit) Baker type lower bounds of linear forms in given numbers $\Theta_0, \ldots, \Theta_m \in \mathbb{C}^*$. Throughout this work, let $\mathbb{I}$ denote an imaginary quadratic field with $\mathbb{Z}_{\mathbb{I}}$ its ring of integers. By an *explicit Baker type lower bound* we mean any positive lower bound

$$(1.1) \qquad |\beta_0\Theta_0 + \cdots + \beta_m\Theta_m| > F(H_0, \ldots, H_m, m)$$

valid for all $\overline{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ with $\prod_{j=0}^m H_j \geq \hat{H} \geq 1$, $H_j \geq h_j = \max\{1, |\beta_j|\}$, where the dependence on each individual term $H_0, \ldots, H_m$, m and the numbers $\Theta_0, \ldots, \Theta_m$ is explicitly given in the functional dependence $F(H_0, \ldots, H_m, m)$ and the dependence on $\Theta_0, \ldots, \Theta_m, m$ is explicitly given in the constant $\hat{H} = \hat{H}(\Theta_0, \ldots, \Theta_m, m)$.

With the assumption that $\gamma_0, \ldots, \gamma_m \in \mathbb{Q}^*$ are distinct, Baker [1] proved that there exist positive constants δ_1, δ_2 and δ_3 such that

$$(1.2) \qquad |\beta_0 e^{\gamma_0} + \cdots + \beta_m e^{\gamma_m}| > \frac{\delta_1 M^{1-\delta(M)}}{\prod_{j=0}^m h_j}$$

for all $\overline{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}^m \setminus \{\overline{0}\}$, $h_j = \max\{1, |\beta_j|\}$, with

$$(1.3) \qquad \delta(M) \leq \frac{\delta_2}{\sqrt{\log\log M}}, \qquad M = \max_{0\leq j\leq m}\{|\beta_j|\} \geq \delta_3 > e.$$

Here we note that the constants $\delta_1, \delta_2, \delta_3$ in Baker's work [1] are not explicitly given. Mahler [9] made Baker's result completely explicit.

There are many subsequent works where the authors prove Baker type lower bounds for values of functions belonging usually to a class of Siegel's E- or G-functions or q-hypergeometric functions evaluated at rational points

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(see e.g. [5], [6], [13] and [14]). For a more comprehensive list of references, see [6]. In the above mentioned works Siegel's lemma is a standard tool for producing a first or second kind Padé-approximation construction of certain auxiliary functions. These constructions correspond to one linear form (one auxiliary function) or simultaneous linear forms (several auxiliary functions).

In this work we shall not do such constructions, but we are interested in the next step. Namely: how to use appropriate linear forms to prove Baker type lower bounds? We shall answer the above question by giving a criterion in the simultaneous linear forms case.

Let us describe our criterion in a nutshell. Fix $\Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$ and set $\overline{n} = (n_1, \ldots, n_m)^T$, $N = N(\overline{n}) = n_1 + \cdots + n_m$. Assume that we have a sequence of simultaneous linear forms

$$L_{k,j}(\overline{n}) = A_{k,0}(\overline{n})\Theta_j + A_{k,j}(\overline{n}), \qquad k = 0, \ldots, m,\ j = 1, \ldots, m,\ \overline{n} \in \mathbb{Z}^m_{\geq 1}, \tag{1.4}$$

where $A_{k,j} = A_{k,j}(\overline{n}) \in \mathbb{Z}_{\mathbb{I}}$ satisfy a certain determinant condition. Suppose also that

$$|A_{k,0}(\overline{n})| \leq e^{(aN+b\log N)g(N)+b_0N(\log N)^{1/2}+b_1N+b_2\log N+b_3}, \tag{1.5}$$

$$|L_{k,j}(\overline{n})| \leq e^{(dN-cn_j)g(N)+e_0N(\log N)^{1/2}+e_1N+e_2\log N+e_3}, \tag{1.6}$$

for $k, j = 0, 1, \ldots, m$, where a, b, c, d, b_i, e_i are non-negative parameters satisfying $a, c-dm > 0$. Then, in the cases $g(N) \in \{1, \log N, N\}$, we shall prove that there exist explicit positive constants F_l, G_l ($l \in \{1, 2, 3\}$) such that

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| > F_l\Big(\prod_{j=1}^{m}(2mH_j)\Big)^{-\frac{a}{c-dm}-\epsilon_l(H)} \tag{1.7}$$

for all $\overline{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ and $H = \prod_{j=1}^{m}(2mH_j) \geq G_l$, $H_j \geq h_j = \max\{1, |\beta_j|\}$ with an error term $\epsilon_l(H) \to 0$ as $H \to \infty$. The constants F_l, G_l and the error term will be given explicitly in terms of the parameters a, b, c, d, b_i, e_i and in particular of m.

The underlying idea behind our treatment is well known already from Baker's work [1]. Namely, the idea (see [1, formula (22)]) is to fix the parameter n_j with the corresponding individual height H_j (in our notation). In our work we shall express this phenomenon first in a nutshell (see (4.10)) and then in a refined form (see (4.14)).

An advantage of our treatment compared with existing treatments is that one can easily see if the contribution to the lower bound is coming from the Diophantine method itself or from the auxiliary construction. For example, apart from the condition $n_1 + \cdots + n_m = N$, we do not need any extra condition relating n_j and N. Of course, some extra conditions may be needed for good auxiliary constructions. In particular, this is the case when Siegel's lemma is involved. See e.g. [13, formula (14)], where the authors additionally assume that $n_j > \delta N$, $j = 1, \ldots, m$, for some $0 < \delta < 1/m$. In [10, formula (4) in Chapter III] the corresponding condition reads $n_j > 2N/\log N$, $j = 1, \ldots, m$. In [4], however, one can find a slightly different approach.

Our Theorems 3.2, 3.4 and 3.6 are designed to be applied in the following manner. Let $f(z)$ be a G-, E- or q-hypergeometric function and denote $\Theta_1 = f(\alpha_1), \ldots, \Theta_m = f(\alpha_m)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{I}^*$. Suppose that one can construct simultaneous linear forms of the type (1.4) satisfying the estimates (1.5) and (1.6) with a certain determinant condition; then our Theorem 3.2, 3.4 or 3.6 will give a corresponding Baker type lower bound (1.7). So far our results (Theorems 3.4 and 3.6) have been applied in [4] and [8].

In [4], Ernvall-Hytönen, Leppälä and Matala-aho constructed simultaneous linear forms of the type (1.4) (satisfying (1.5)–(1.6) with $g(N) = \log N$) for the exponential function values $e^{\alpha_0}, \ldots, e^{\alpha_m}$, where $\alpha_0, \ldots, \alpha_m \in \mathbb{I}$. (Note that the exponential function belongs to the class of Siegel's E-functions.) By applying Theorem 3.4 of the present paper the authors in [4] proved substantial improvements of the explicit versions (see Mahler [9] and Sankilampi [10]) of Baker's work [1] about exponential values at rational points. In particular, the dependence on m is improved. As an example from [4] we mention a new explicit Baker type lower bound

$$|\beta_0 + \beta_1 e + \beta_2 e^2 + \cdots + \beta_m e^m| > \frac{1}{h^{1+\hat{\epsilon}(h)}}, \qquad h = h_1 \cdots h_m,$$

valid for all $\overline{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^m \setminus \{\overline{0}\}$, $h_i = \max\{1, |\beta_i|\}$ with

$$\hat{\epsilon}(h) = \frac{(4 + 7m)\sqrt{\log(m+1)}}{\sqrt{\log\log h}},$$
$$\log h \geq m^2(41 \log(m+1) + 10)e^{m^2(81 \log(m+1)+20)}.$$

As far as we know, the published dependences on m in $\hat{\epsilon}(h)$ have been at least quadratic and in lower bounds of $\log\log h$ at least quartic.

The second application of our work is presented in Leinonen's paper [8]. In a pioneer work [13] Väänänen and Zudilin proved Baker type results for a class of q-hypergeometric series. Following [13], Leinonen [8] constructed simultaneous linear forms of the type (1.4) (satisfying (1.5)–(1.6) with $g(N) = N$) and proved some generalizations of the results in [13]. Moreover, she applied our Theorem 3.6 with her linear forms, and gave explicit Baker type lower bounds which sharpened her results as well as the results of Väänänen and Zudilin.

2. Background from metrical theory. From the general metrical theory (see [2], [3], [6], [11], [12]) we get the following well known results.

THEOREM 2.1. *Let $1, \Theta_1, \ldots, \Theta_m \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then there exist infinitely many primitive vectors $(\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}^{m+1} \setminus \{\overline{0}\}$ with $h_j := \max\{1, |\beta_j|\}$, $j = 1, \ldots, m$, satisfying*

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| < \frac{1}{\prod_{j=1}^{m} h_j}.$$

In the complex case, Shidlovskii [12] studies linear forms over the ring of rational integers and gives the following result.

THEOREM 2.2 ([12]). *Let $\Theta_0 = 1$, $\Theta_1, \ldots, \Theta_m \in \mathbb{C}$ and $H \in \mathbb{Z}_{\geq 1}$ be given. Then there exists a non-zero rational integer vector $(\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}^{m+1} \setminus \{\overline{0}\}$ with $|\beta_j| \leq H$, $j = 0, 1, \ldots, m$, satisfying*

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| \leq \frac{c}{H^{(m-1)/2}}, \qquad c = \sqrt{2}\sum_{j=0}^{m} |\Theta_j|.$$

We are interested in linear forms over the ring of integers $\mathbb{Z}_{\mathbb{I}}$ in an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{Z}_{\geq 1}$, $D \not\equiv 0 \pmod 4$. For that purpose we prove

THEOREM 2.3. *Let $\Theta_1, \ldots, \Theta_m \in \mathbb{C}$ and $H_1, \ldots, H_m \in \mathbb{Z}_{\geq 1}$ be given. Then there exists a non-zero integer vector $(\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ with $|\beta_j| \leq H_j$, $j = 1, \ldots, m$, satisfying*

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| \leq \left(\frac{2^{\tau} D^{1/4}}{\sqrt{\pi}}\right)^{m+1} \frac{1}{H_1 \cdots H_m}, \tag{2.1}$$

where $\tau = 1$ if $D \equiv 1$ or $2 \pmod 4$, and $\tau = 1/2$ if $D \equiv 3 \pmod 4$.

3. Results

3.1. A general target. Let $f(z)$ belong to one of the following classes of functions:

1. The class of Siegel's G-functions. Typical examples are logarithm and Gauss hypergeometric functions and more generally non-entire hypergeometric series.
2. The class of Siegel's E-functions. Typical examples are exponential and Bessel functions and more generally entire hypergeometric series. For definition of Siegel's E- and G-functions we refer to [6].
3. The q-hypergeometric series. Typical examples are

$$\sum_{n=0}^{\infty} q^{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} 1/\prod_{i=1}^{n}(1-q^i), \qquad |q| < 1.$$

Our Theorems 3.2, 3.4 and 3.6 are designed to be applied in the following manner. Denote $\Theta_1 = f(\alpha_1), \ldots, \Theta_m = f(\alpha_m)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{I}^*$. Suppose that one can construct simultaneous linear forms of the type (3.2) satisfying the conditions (3.4)–(3.7). Then Theorem 3.2, 3.4 or 3.6 will give a Baker type lower bound for the quantity

$$|\beta_0 + \beta_1 \Theta_1 + \cdots + \beta_m \Theta_m|. \tag{3.1}$$

It is a general phenomenon in the field of Diophantine approximations that Padé approximations and Siegel's lemma give estimates of the shape (3.6) and (3.7). However, it is often hard to find such bounds if also the condition (3.5) holds.

3.2. A criterion. Fix now $\Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$ and write

$$\overline{n} = (n_1, \ldots, n_m)^T, \qquad N = N(\overline{n}) = n_1 + \cdots + n_m.$$

Assume that we have a sequence of simultaneous linear forms

$$L_{k,j}(\overline{n}) = A_{k,0}(\overline{n})\Theta_j + A_{k,j}(\overline{n}), \qquad \overline{n} \in \mathbb{Z}^m_{\geq 1}, \tag{3.2}$$

$k = 0, 1, \ldots, m$, $j = 1, \ldots, m$, where

$$A_{k,j} = A_{k,j}(\overline{n}) \in \mathbb{Z}_{\mathbb{I}}, \qquad k, j = 0, 1, \ldots, m, \tag{3.3}$$

satisfy a determinant condition, say,

$$\Delta = \begin{vmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m,0} & A_{m,1} & \cdots & A_{m,m} \end{vmatrix} \neq 0. \tag{3.4}$$

Further, let $a, b, c, d, b_i, e_i \in \mathbb{R}_{\geq 0}$, $a > 0$, and suppose that

$$c, c - dm > 0, \tag{3.5}$$

$$|A_{k,0}(\overline{n})| \leq Q(\overline{n}) = e^{q(N)}, \tag{3.6}$$

$$|L_{k,j}(\overline{n})| \leq R_j(\overline{n}) = e^{-r_j(\overline{n})}, \tag{3.7}$$

where

$$q(N) = (aN + b\log N)g(N) + b_0 N(\log N)^{1/2} + b_1 N + b_2 \log N + b_3,$$
$$-r_j(\overline{n}) = (dN - cn_j)g(N) + e_0 N(\log N)^{1/2} + e_1 N + e_2 \log N + e_3$$

for all $k, j = 0, 1, \ldots, m$.

Let the above assumptions be valid for all $N \geq N_l$, $l = 1, 2, 3$ (where l refers to case number) in our cases:

Case 1:

$$\begin{cases} g(N) = g_1(N) := 1, \\ q(N) = q_1(N) := aN + b\log N, \\ -r_j(\overline{n}) = -r_{j,1}(\overline{n}) := dN - cn_j + e_2 \log N, \end{cases}$$

and all other b's and e's are zero.

Case 2:

$$\begin{cases} g(N) = g_2(N) := \log N, \quad b = 0, \\ q(N) = q_2(N) := aN\log N + b_0 N(\log N)^{1/2} + b_1 N + b_2 \log N + b_3, \\ -r_j(\overline{n}) = -r_{j,2}(\overline{n}) \\ \qquad\qquad := (dN - cn_j)\log N + e_0 N(\log N)^{1/2} + e_1 N + e_2 \log N + e_3. \end{cases}$$

Case 3:

$$\begin{cases} g(N) = g_3(N) := N, \\ q(N) = q_3(N) := aN^2 + b_1 N, \\ -r_j(\overline{n}) = -r_{j,3}(\overline{n}) := (dN - cn_j)N + e_1 N, \end{cases}$$

and all other b's and e's are zero.

The following theorem gives a unified result in the above three cases.

Theorem 3.1. *Under the above assumptions there exist explicit positive constants F_l and G_l not depending on H such that*

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| > F_l\Big(\prod_{j=1}^{m}(2mH_j)\Big)^{-\frac{a}{c-dm}-\epsilon_l(H)} \tag{3.8}$$

for all $\overline{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ and

$$H = \prod_{j=1}^{m}(2mH_j) \geq G_l, \quad H_j \geq h_j = \max\{1, |\beta_j|\}, \tag{3.9}$$

with an error term $\epsilon_l(H) \to 0$ as $H \to \infty$.

In Subsections 3.3–3.5 we consider the three cases more closely.

3.3. Case 1

Theorem 3.2. *Denote $f = 2/(c - dm)$ and*

$$A_1 = \frac{acm}{c-dm} + B_1\log(ef), \qquad B_1 = \frac{ae_2 m}{c-dm} + b.$$

Then

$$F_1^{-1} = 2e^{A_1}, \qquad \epsilon_1(H) = B_1\frac{\log\log H}{\log H}$$

and

$$G_1 = \max\{m, N_1, e^{x_1/f}\}, \qquad x_1 = \max\{S_1, 1\}, \tag{3.10}$$

where S_1 is the largest solution of the equation

$$S = f(e_2 m\log S + dm^2 + e_2 m). \tag{3.11}$$

3.4. Case 2. Before stating our results we introduce a function $z : \mathbb{R} \to \mathbb{R}$, the inverse function of the function $y(z) = z \log z$, $z \geq 1/e$, considered in [7].

LEMMA 3.3 ([7]). *The inverse function $z(y)$ of the function $y(z) = z \log z$, $z \geq 1/e$, is strictly increasing. Define $z_0(y) = y$ and $z_n(y) = y/\log z_{n-1}$ for $n \in \mathbb{Z}^+$. Suppose $y > e$. Then $z_1 < z_3 < \cdots < z < \cdots < z_2 < z_0$. Thus the inverse function may be given by the infinite nested logarithm fraction*

$$z(y) = \lim_{n\to\infty} z_n(y) = \frac{y}{\log \frac{y}{\log \frac{y}{\log \ldots}}}$$

for $y > e$. In particular,

$$z(y) < z_2(y) = \frac{y}{\log \frac{y}{\log y}} \tag{3.12}$$

for $y > e$.

THEOREM 3.4. *Denote $f = 2/(c - dm)$ and*

$$A_2 = b_0 + \frac{ae_0 m}{c-dm}, \qquad B_2 = a + b_0 + b_1 + \frac{ae_1 m}{c-dm},$$
$$C_2 = am + b_2 + \frac{a(dm^2 + e_2 m)}{c-dm}, \qquad D_2 = b_0 m + \frac{ae_0 m^2}{c-dm},$$
$$E_2 = (a + b_0 + b_1)m + b_2 + b_3 + \frac{a((2d + 2e_0 + e_1)m^2 + (e_2 + e_3)m)}{c-dm}.$$

Then $F_2^{-1} = 2e^{E_2}$ and

$$\begin{aligned} \epsilon_2(H) = \xi(z, H) := A_2 \left(f \frac{z(f \log H)}{\log H} \right)^{1/2} + B_2 \frac{z(f \log H)}{\log H} \\ + C_2 \frac{\log z(f \log H)}{\log H} + D_2 \frac{(\log z(f \log H))^{1/2}}{\log H} \end{aligned} \tag{3.13}$$

with

$$G_2 = \max\{m, N_2, e^{(x_2 \log x_2)/f}, e^{e/f}\}, \qquad x_2 = \max\{S_2, 1\}, \tag{3.14}$$

where S_2 is the largest solution of the equation

$$\begin{aligned} S \log S = f\big(e_0 m S (\log S)^{1/2} + e_1 m S + (dm^2 + e_2 m) \log S \\ + e_0 m^2 (\log S)^{1/2} + 2dm^2 + 2e_0 m^2 + e_1 m^2 + e_2 m + e_3 m\big). \end{aligned} \tag{3.15}$$

In this case the estimate corresponding to (3.8) may be written as follows:

$$\begin{aligned} &|\beta_0 + \beta_1 \Theta_1 + \cdots + \beta_m \Theta_m| \geq \\ &F_2 (z(f \log H))^{-C_2} H^{-\frac{a}{c-dm} - A_2 \left(f \frac{z(f \log H)}{\log H}\right)^{1/2} - B_2 \frac{z(f \log H)}{\log H} - D_2 \frac{(\log z(f \log H))^{1/2}}{\log H}}. \end{aligned} \tag{3.16}$$

Note that

$$z(f\log H) < z_2(f\log H) \tag{3.17}$$

for $f\log H > e$ by (3.12), and thus

$$\epsilon_2(H) = \xi(z,H) < \xi(z_2,H) \tag{3.18}$$

for $f\log H > e$. Write now

$$\rho_2(x) = \frac{\log x}{\log x - \log\log x}.$$

Then (3.17) may further be estimated by using

$$z_2(f\log H) \le \rho_2(x_0) f\left(1 - \frac{\log f}{\log(f\log H)}\right)\frac{\log H}{\log\log H} \tag{3.19}$$

valid for all

$$f\log H \ge x_0 \ge e^e, \qquad H > e. \tag{3.20}$$

Note that if $0 < c - dm \le 2$, then

$$z_2(f\log H) \le \rho_2(x_0) f\frac{\log H}{\log\log H}. \tag{3.21}$$

By using the estimate (3.21) we get the following corollary where the lower bound in (3.23) is a generalization of what we see in the works on E-functions.

COROLLARY 3.5. *Write* $\rho = \rho_2(x_0)$. *If* $0 < c - dm \le 2$, $H > e$ *and*

$$f\log H \ge x_0 := \max\{f\log m, f\log N_2, x_2\log x_2, e^e\}, \tag{3.22}$$

then

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| \ge \frac{1}{2e^{E_2}(f\rho)^{C_2}}\left(\frac{\log\log H}{\log H}\right)^{C_2} H^{-\frac{a}{c-dm} - \frac{A_2 f\sqrt{\rho}}{\sqrt{\log\log H}} - \frac{B_2 f\rho}{\log\log H} - \frac{D_2}{\log H}\sqrt{\log\left(\frac{f\rho\log H}{\log\log H}\right)}}. \tag{3.23}$$

In [4], $c = 1$ and $d = 0$, so Corollary 3.5 applies.

In most of the existing works only the terms corresponding to A_2 and C_2 are presented, and usually only a main term is given, while the other terms are included implicitly. Hence in such a situation explicit dependence on the parameters, say for example on m, may become invisible. Further, all the methods applied to E-functions seem to yield the situation where $A_2 \ne 0$. If we had $A_2 = 0$, then the terms with B_2 and C_2 would become more important. That would be the case if e.g. one could find appropriate explicit Padé type approximations instead of those produced by Siegel's lemma.

3.5. Case 3

THEOREM 3.6. *We have*

$$F_3^{-1} = 2e^{B_3}, \quad \epsilon_3(H) = A_3 \frac{1}{\sqrt{\log H}}, \qquad G_3 = \max\{m, N_3, e\},$$

where the general A_3 and B_3 are given in the proof section. In the particular case of $b_1 = e_1 = 0$, they read

$$A_3 = \frac{2acm}{(c-dm)^{3/2}}, \qquad B_3 = \frac{acm^2(c+dm+2\sqrt{cdm})}{(c-dm)^2}.$$

4. Proofs

4.1. Proof of Theorem 2.3. For $D \in \mathbb{Z}_{\geq 1}$, $D \not\equiv 0 \pmod 4$ the ring of integers may be given by $\mathbb{Z}_{\mathbb{I}} = \mathbb{Z} + \mathbb{Z}(h + l\sqrt{-D})$ with $h = 0$, $l = 1$ if $D \equiv 1$ or $2 \pmod 4$, and $h = l = 1/2$ if $D \equiv 3 \pmod 4$.

We start with a simple principle. First we define a lattice

$$\lambda = \mathbb{Z}(1,0) + \mathbb{Z}(h, l\sqrt{D}), \qquad \det \lambda = \sqrt{D}\, 2^{-2h}$$

and a complex disk

$$\mathcal{D}_R = \{x + y(h + l\sqrt{-D}) \in \mathbb{C} \mid x, y \in \mathbb{R},\ |x + y(h + l\sqrt{-D})| \leq R\}$$

with radius $R > 0$, and a corresponding real disk

$$\mathcal{C}_R = \{(v,w)^T \in \mathbb{R}^2 \mid v^2 + w^2 \leq R^2\}, \qquad \mathrm{Vol}\, \mathcal{C}_R = \pi R^2.$$

Then

$$x + y(h + l\sqrt{-D}) \in \mathcal{D}_R \cap \mathbb{Z}_{\mathbb{I}} \ \Leftrightarrow\ (x + yh, yl\sqrt{D})^T \in \mathcal{C}_R \cap \lambda. \tag{4.1}$$

Next we define a lattice

$$\Lambda = \mathbb{Z}\bar{l}_1 + \cdots + \mathbb{Z}\bar{l}_{2m+2} \subseteq \mathbb{R}^{2m+2} \tag{4.2}$$

generated by

$$\begin{cases} \bar{l}_1 = (1,0,0,0,\ldots,0,0)^T, & \bar{l}_2 = (h, l\sqrt{D}, 0, 0, \ldots, 0, 0)^T, \\ \bar{l}_3 = (0,0,1,0,\ldots,0,0)^T, & \bar{l}_4 = (0,0,h,l\sqrt{D},0,0,\ldots,0,0)^T, \\ \ldots & \\ \bar{l}_{2m+1} = (0,0,\ldots,0,0,1,0)^T, & \bar{l}_{2m+2} = (0,0,\ldots,0,0,h,l\sqrt{D})^T. \end{cases}$$

Immediately, $\det \Lambda = (\sqrt{D}\, 2^{-2h})^{m+1}$.

By using the notations

$$a + b(h + l\sqrt{-D}) = -(z_1\Theta_1 + \cdots + z_m\Theta_m), \qquad z_k = x_k + y_k(h + l\sqrt{-D}),$$
$$v_k = x_k + y_k h, \qquad w_k = y_k l\sqrt{D}, \qquad x_k, y_k \in \mathbb{R}, \qquad k = 0, 1, \ldots, m,$$

and

$$R_0 := \left(\frac{2^\tau D^{1/4}}{\sqrt{\pi}}\right)^{m+1} \frac{1}{H_1 \cdots H_m},$$

we define the sets

$$\mathcal{D} = \{(z_0, z_1, \ldots, z_m)^T \in \mathbb{C}^{m+1} \mid |z_0 - (a + b(h + l\sqrt{-D}))| \le R_0,$$
$$|z_k| \le H_k,\ k = 1, \ldots, m\},$$

$$\mathcal{C} = \{(v_0, w_0, v_1, w_1, \ldots, v_m, w_m)^T \in \mathbb{R}^{2m+2} \mid$$
$$(v_0 - (a + bh))^2 + (w_0 - bl\sqrt{D})^2 \le R_0^2,\ v_k^2 + w_k^2 \le H_k^2,\ k = 1, \ldots, m\}.$$

First we note that $\mathcal{C}$ is a symmetric convex body. For the volume of $\mathcal{C}$ we get

$$\begin{aligned}
\operatorname{Vol}\mathcal{C} &= \int\cdots\int \Big(\iint\limits_{(v_0-(a+bh))^2+(w_0-bl\sqrt{D})^2\le R_0^2} dv_0\, dw_0 \Big)\, dv_1\, dw_1 \cdots dv_m\, dw_m \\
&= \pi R_0^2 \int\cdots\int \Big(\iint\limits_{v_1^2+w_1^2\le H_1^2} dv_1\, dw_1 \Big)\, dv_2\, dw_2 \cdots dv_m\, dw_m \\
&= \cdots = \pi^{m+1} H_1^2 \cdots H_m^2 R_0^2 \\
&= \pi^{m+1} H_1^2 \cdots H_m^2 \left(\frac{2^{2\tau}\sqrt{D}}{\pi} \right)^{m+1} \frac{1}{H_1^2 \cdots H_m^2} = 2^{2m+2} \left(\frac{\sqrt{D}}{2^{2h}} \right)^{m+1} \\
&= 2^{2m+2} \det \Lambda.
\end{aligned}$$

Thus by Minkowski's convex body theorem (see [11]) there exists a non-zero lattice vector

$$(x_0 + y_0 h, y_0 l\sqrt{D}, \ldots, x_m + y_m h, y_m l\sqrt{D})^T \in \mathcal{C} \cap \Lambda \setminus \{\overline{0}\}. \tag{4.3}$$

Consequently, by the above principle (4.1), we get a non-zero integer vector

$$(\beta_0, \beta_1, \ldots, \beta_m)^T = (x_0 + y_0(h + l\sqrt{-D}), \ldots, x_m + y_m(h + l\sqrt{-D}))^T$$
$$\in \mathcal{D} \cap \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$$

with $|\beta_k| \le H_k,\ k = 1, \ldots, m$, satisfying

$$|\beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m| \le \left(\frac{2^\tau D^{1/4}}{\sqrt{\pi}} \right)^{m+1} \frac{1}{H_1 \cdots H_m}. \ \blacksquare \tag{4.4}$$

4.2. Proof of Theorems 3.1–3.6. Our proof starts in a classical manner, and then we give a rough description how to get Baker type estimates. Next we will introduce our tuning process, which allows us to continue from the classical startup.

4.2.1. *A classical start.* We use the notation

$$\Lambda := \beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m, \qquad \beta_j \in \mathbb{Z}_{\mathbb{I}},$$

for the linear form to be estimated. Using our simultaneous linear forms

$$L_{k,j}(\overline{n}) = A_{k,0}(\overline{n})\Theta_j + A_{k,j}(\overline{n})$$

from (3.2), we get

$$A_{k,0}\Lambda = \Omega_k + \beta_1 L_{k,1}(\overline{n}) + \cdots + \beta_m L_{k,m}(\overline{n}), \tag{4.5}$$

where

$$\Omega_k = \Omega_k(\overline{n}) = A_{k,0}(\overline{n})\beta_0 - \beta_1 A_{k,1}(\overline{n}) - \cdots - \beta_m A_{k,m}(\overline{n}) \in \mathbb{Z}_{\mathbb{I}}. \tag{4.6}$$

If now $\Omega_k \neq 0$, then by (3.6), (3.7), (3.9), (4.5) and (4.6) we get

$$\begin{aligned} 1 \le |\Omega_k| &= |A_{k,0}\Lambda - (\beta_1 L_{k,1} + \cdots + \beta_m L_{k,m})| \\ &\le |A_{k,0}|\,|\Lambda| + \sum_{j=1}^{m} |\beta_j|\,|L_{k,j}| \le Q(\overline{n})|\Lambda| + \sum_{j=1}^{m} H_j R_j(\overline{n}). \end{aligned} \tag{4.7}$$

Here we want to have, say,

$$\sum_{j=1}^{m} H_j R_j(\overline{n}) \le \frac{1}{2}, \tag{4.8}$$

in order to get a lower bound

$$1 \le 2|\Lambda|Q(\overline{n}) \tag{4.9}$$

for our linear form Λ.

4.2.2. *A rough version.* Here we outline a rough version of the proof by studying the case $b = b_0 = b_1 = b_2 = b_3 = e_0 = e_1 = e_2 = e_3 = 0$, for simplicity. It starts by fixing the remainders and heights:

$$\begin{aligned} H_j R_j(\overline{n}) = \frac{1}{2m} \ &\Leftrightarrow\ 2mH_j = e^{r_j(\overline{n})} = e^{(-dN+cn_j)g(N)} \\ &\Rightarrow\ e^{(-dmN+c\sum_{j=1}^{m} n_j)g(N)} = e^{(c-dm)Ng(N)} = \prod_{j=1}^{m}(2mH_j) \\ &\Rightarrow\ Q(\overline{n}) = e^{aNg(N)} = \Big(\prod_{j=1}^{m}(2mH_j)\Big)^{a/(c-dm)} \\ &\Rightarrow\ 1 \le 2|\Lambda|Q(\overline{n}) = 2|\Lambda|\Big(\prod_{j=1}^{m}(2mH_j)\Big)^{a/(c-dm)}. \end{aligned} \tag{4.10}$$

4.2.3. *Tuning.* Now a direct generalization of the second equality of (4.10) would be

$$r_j(\overline{n}) = \log(2mH_j), \tag{4.11}$$

where

$$r_j(\overline{n}) = (-dN + cn_j)g(N) - e_0 N(\log N)^{1/2} - e_1 N - e_2 \log N - e_3.$$

However, (4.11) will be too rough, and thus we tune it into right frequency by defining

$$B_j = \log(2mH_j) + dm\hat{g}_l(W) + e_0 m((\log W)^{1/2} + 2) + e_1 m + e_2, \tag{4.12}$$

where

$$\hat{g}_1(W) = 1, \quad \hat{g}_2(W) = \log W + 2, \quad \hat{g}_3(W) = 2W + m,$$

corresponding to our three cases. Now we state a new system of equations

$$\sum_{j=1}^{m} w_j = W, \tag{4.13}$$

$$r_j(\overline{w}) = B_j, \quad j = 1, \ldots, m. \tag{4.14}$$

Here (4.14) reads

$$\begin{aligned}(-dW + cw_j)g(W) - e_0 W(\log W)^{1/2} - e_1 W - e_2 \log W - e_3 \\ = \log(2mH_j) + dm\hat{g}_l(W) + e_0 m((\log W)^{1/2} + 2) + e_1 m + e_2,\end{aligned} \tag{4.15}$$

which by (4.13) gives

$$\begin{aligned}(c - dm)Wg(W) - dm^2\hat{g}_l(W) - e_0 mW(\log W)^{1/2} - e_1 mW \\ - e_2 m \log W - e_3 m - e_0 m^2((\log W)^{1/2} + 2) - e_1 m^2 - e_2 m = \log H.\end{aligned} \tag{4.16}$$

The equation (4.16) has a solution $W \geq m$ if H is large enough. Then we choose the largest W, say $S := W_L \geq m$. (Any solution $W \geq 1$ would be satisfactory but for technical reasons we choose $W \geq m$.) From our assumptions it follows that $m \geq 2$, $c > 0$, $g(S) \geq 1$, $g_l(S) \geq 1$ for $l = 1, 2, 3$, and $H_j \geq 1$ for $j = 1, \ldots, m$. Hence $B_j \geq \log 4$ for $j = 1, \ldots, m$, which by (4.15) implies

$$\begin{aligned}s_j := w_j &= \frac{B_j + e_0 S(\log S)^{1/2} + e_1 S + e_2 \log S + e_3 + dSg(S)}{cg(S)} \\ &> \frac{\log 4}{cg(S)} > 0.\end{aligned} \tag{4.17}$$

Consequently, also the estimate (4.17) is valid for H large enough (independently of each individual term H_j).

Write $\sigma_j = \lfloor s_j \rfloor$ and $\overline{\sigma} = (\sigma_1, \ldots, \sigma_m)^T$, $\overline{1} = (1, \ldots, 1)^T$. Then

$$\overline{\sigma} \leq \overline{s} < \overline{\sigma} + \overline{1}. \tag{4.18}$$

First we note that

$$T := N(\overline{\sigma} + \overline{1}) = N(\overline{\sigma}) + m \leq N(\overline{s}) + m = S + m, \quad S < T. \tag{4.19}$$

Next we give an estimate for the difference

$$(4.20)\qquad r_j(\overline{s}) - r_j(\overline{\sigma}+\overline{1})$$
$$= (-dN(\overline{s}) + cs_j)g(S) - e_0N(\overline{s})(\log N(\overline{s}))^{1/2} - e_1N(\overline{s}) - e_2\log N(\overline{s}) - e_3$$
$$- \big((-dN(\overline{\sigma}+\overline{1}) + c(\sigma_j+1))g(N(\overline{\sigma}+\overline{1})) - e_0N(\overline{\sigma}+\overline{1})(\log N(\overline{\sigma}+\overline{1}))^{1/2}$$
$$- e_1N(\overline{\sigma}+\overline{1}) - e_2\log N(\overline{\sigma}+\overline{1}) - e_3\big)$$
$$= d(Tg(T) - Sg(S)) + c(s_jg(S) - (\sigma_j+1)g(T))$$
$$+ e_0(T(\log T)^{1/2} - S(\log S)^{1/2}) + e_1(T-S) + e_2(\log T - \log S).$$

By $s_j < \sigma_j + 1$, the increasing property of $g(x)$ and the mean value theorem we get

$$(4.21)\qquad r_j(\overline{s}) - r_j(\overline{\sigma}+\overline{1})$$
$$\le d(Tg(T) - Sg(S)) + e_0m((\log S)^{1/2} + 2) + e_1m + e_2.$$

Hence

(4.22)
$$r_j(\overline{s}) < r_j(\overline{\sigma}+\overline{1}) + dm\hat{g}_l(S) + e_0m((\log S)^{1/2}+2) + e_1m + e_2, \qquad l \in \{1,2,3\},$$

which is the reason to define (4.12).

From the non-vanishing of the determinant (3.4) and the assumption $\overline{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^T \neq \overline{0}$ it follows that

$$(4.23)\qquad \Omega_k(\overline{\sigma}+\overline{1}) \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$$

with some integer $k \in [0, m]$. Now we are ready to prove the essential estimate

$$(4.24)\qquad \sum_{j=1}^{m} H_jR_j(\overline{\sigma}+\overline{1}) = \sum_{j=1}^{m} H_je^{-r_j(\overline{\sigma}+\overline{1})}$$
$$\overset{(4.22)}{<} \sum_{j=1}^{m} H_je^{-B_j + dm\hat{g}_l(S) + e_0m((\log S)^{1/2}+2) + e_1m + e_2} = \frac{1}{2}.$$

Hence by (4.7) we get

$$(4.25)\qquad 1 < 2|\Lambda|Q(\overline{\sigma}+\overline{1}) = 2|\Lambda|e^{q(N(\overline{\sigma}+\overline{1}))} \le 2|\Lambda|e^{q(S+m)},$$

where

$$q(S+m) = (a(S+m) + b\log(S+m))g(S+m)$$
$$+ b_0(S+m)(\log(S+m))^{1/2} + b_1(S+m) + b_2\log(S+m) + b_3.$$

Since $g(x)$ is increasing we get

$$(4.26)\qquad g(S+m) = g(S) + mV(S), \qquad V(S) = \max_{S \le x \le S+m}\{g'(x)\}.$$

Or, remembering the assumption $m \le S$, we may use the estimates

$$(4.27)\qquad \log(S+m) \le \log S + 1, \qquad (\log(S+m))^{1/2} \le (\log S)^{1/2} + 1.$$

Consequently,

$$q(S+m) \le aSg(S) + Y(S), \tag{4.28}$$

where

$$\begin{aligned} Y(S) &= amg(S) + amSV(S) + am^2V(S) + bg(S+m)\log(S+m) \\ &\quad + b_0(S+m)(\log(S+m))^{1/2} + b_1(S+m) + b_2\log(S+m) + b_3. \end{aligned}$$

From (4.16) we get

$$Sg(S) = \frac{\log H}{c-dm} + \frac{X(S)}{c-dm}, \tag{4.29}$$

where

$$\begin{aligned} X(S) &= dm^2\hat{g}_l(S) + e_0mS(\log S)^{1/2} + e_1mS + e_2m\log S \\ &\quad + e_0m^2((\log S)^{1/2}+2) + e_1m^2 + e_2m + e_3m. \end{aligned}$$

Hence

$$Q(\overline{\sigma}+\overline{1}) \le H^{\frac{a}{c-dm}+Z(S)}, \quad Z(S) = \frac{1}{\log H}\left(\frac{a}{c-dm}X(S) + Y(S)\right). \tag{4.30}$$

In the following we will consider S as a variable greater than W_L.

4.2.4. *Case 1.* We have $\hat{g}_1(S) = 1$ and thus

$$\begin{aligned} Z(S) &= \frac{1}{\log H}\left(\frac{a}{c-dm}(dm^2 + e_2m\log S + e_2m) + am + b\log(S+m)\right) \\ &\le \frac{1}{\log H}\left(\frac{a(dm^2+e_2m)}{c-dm} + am + b\right) + \frac{\log S}{\log H}\left(\frac{ae_2m}{c-dm} + b\right). \end{aligned}$$

Here (4.16) reads

$$(c-dm)W - dm^2 - e_2m\log W - e_2m = \log H. \tag{4.31}$$

Let W_1 denote the largest solution of the equation

$$(c-dm)W - dm^2 - e_2m\log W - e_2m = \tfrac{1}{2}(c-dm)W. \tag{4.32}$$

Hence

$$(c-dm)S - dm^2 - e_2m\log S - e_2m \ge \tfrac{1}{2}(c-dm)W_1 \tag{4.33}$$

for all $S \ge x_1 := \max\{W_1, W_L, m\}$. Further, we choose H such that

$$x_1 \le S \le f\log H, \qquad f = \frac{2}{c-dm}. \tag{4.34}$$

Thus

$$Z(S) \le A_1\frac{1}{\log H} + B_1\frac{\log\log H}{\log H}, \tag{4.35}$$

where

$$B_1 = \frac{ae_2m}{c-dm} + b,$$

$$A_1 = \frac{adm^2}{c-dm} + am + \frac{ae_2m}{c-dm} + b + B_1 \log f = \frac{acm}{c-dm} + B_1 \log(ef).$$

Hence

$$1 < 2|\Lambda|Q(\overline{\sigma}+\overline{1}) \le |\Lambda| 2e^{A_1} H^{\frac{a}{c-dm} + B_1 \frac{\log\log H}{\log H}}, \tag{4.36}$$

where $\Lambda = \beta_0 + \beta_1\Theta_1 + \cdots + \beta_m\Theta_m$ is our linear form. This proves Theorem 3.2.

4.2.5. *Case 2.* Here

$$\begin{aligned} q_2(S+m) &\le a(S+m)\log(S+m) + b_0(S+m)(\log(S+m))^{1/2} \\ &\quad + b_1(S+m) + b_2\log(S+m) + b_3 \\ &\overset{(4.27)}{\le} aS\log(S) + Y(S) \end{aligned}$$

and

$$\begin{aligned} Y(S) = b_0S(\log S)^{1/2} &+ (a+b_0+b_1)S + (am+b_2)\log S \\ &+ b_0m(\log S)^{1/2} + (a+b_0+b_1)m + b_2 + b_3. \end{aligned}$$

From (4.16) we get

$$S\log S = \frac{\log H}{c-dm} + \frac{X(S)}{c-dm}, \tag{4.37}$$

where

$$\begin{aligned} X(S) &= dm^2\hat{g}_2(S) + e_0mS(\log S)^{1/2} + e_1mS + e_2m\log S \\ &\quad + e_0m^2(\log S)^{1/2} + (2e_0+e_1)m^2 + e_2m + e_3m, \\ \hat{g}_2(S) &= \log S + 2. \end{aligned}$$

Hence, by (4.30),

$$Z(S) = \frac{1}{\log H}\big(A_2S(\log S)^{1/2} + B_2S + C_2\log S + D_2(\log S)^{1/2} + E\big),$$

where A_2, B_2, C_2, D_2, E_2 are as in the statement of Theorem 3.4. Here (4.16) has the form

$$\begin{aligned} (c-dm)W\log W - dm^2(\log W + 2) - e_0mW(\log W)^{1/2} - e_1mW \\ - e_2m\log W - e_0m^2((\log W)^{1/2}+2) - e_1m^2 - e_2m - e_3m = \log H. \end{aligned} \tag{4.38}$$

Let W_2 denote the largest solution of the equation

$$\begin{aligned} (c-dm)W\log W - dm^2(\log W + 2) - e_0mW(\log W)^{1/2} - e_1mW \\ - e_2m\log W - e_0m^2((\log W)^{1/2}+2) - e_1m^2 - e_2m - e_3m = \frac{c-dm}{2}W\log W. \end{aligned} \tag{4.39}$$

Assume then $S \geq x_2 := \max\{W_2, W_L, m\}$. Analogously to Case 1 we may choose H such that

$$S \log S \leq f \log H, \qquad f = \frac{2}{c-dm}. \tag{4.40}$$

By (3.12) we get

$$S \leq z(f \log H) \leq z_2(f \log H) = \frac{f \log H}{\log \frac{f \log H}{\log(f \log H)}} \tag{4.41}$$

valid for

$$f \log H > e. \tag{4.42}$$

Note the estimate

$$\frac{S(\log S)^{1/2}}{\log H} = \frac{S^{1/2}(S \log S)^{1/2}}{\log H} \leq \left(f \frac{z(f \log H)}{\log H}\right)^{1/2} \leq \left(f \frac{z_2(f \log H)}{\log H}\right)^{1/2},$$

too. By using the notation $\xi(z, H)$ given in (3.13) we have

$$Q(\overline{\sigma} + \overline{1}) \leq H^{\frac{a}{c-dm} + Z(S)} \leq e^{E_2} H^{\frac{a}{c-dm} + \xi(z,H)}, \tag{4.43}$$

where the error term satisfies

$$\xi(z, H) \leq \xi(z_2, H). \tag{4.44}$$

Note that

$$B_2 \frac{z(f \log H)}{\log H} = o\left(A_2 \left(f \frac{z(f \log H)}{\log H}\right)^{1/2}\right),$$

and similarly for the terms involving C_2 and D_2. Thus

$$A_2 \left(f \frac{z(f \log H)}{\log H}\right)^{1/2}$$

will be the main error term, for any H large enough, if $A_2 \neq 0$.

Further, we note that the estimate (4.43) may be written as follows:

$$\begin{aligned} &Q(\overline{\sigma} + \overline{1}) \leq \\ &\quad e^{E_2} (z(f \log H))^{C_2} H^{\frac{a}{c-dm} + A_2 \left(f \frac{z(f \log H)}{\log H}\right)^{1/2} + B_2 \frac{z(f \log H)}{\log H} + D_2 \frac{(\log z(f \log H))^{1/2}}{\log H}}, \end{aligned} \tag{4.45}$$

which by (4.44) implies

$$\begin{aligned} &Q(\overline{\sigma} + \overline{1}) \leq \\ &\quad e^{E_2} (z_2(f \log H))^{C_2} H^{\frac{a}{c-dm} + A_2 \left(f \frac{z_2(f \log H)}{\log H}\right)^{1/2} + B_2 \frac{z_2(f \log H)}{\log H} + D_2 \frac{(\log z_2(f \log H))^{1/2}}{\log H}}. \end{aligned} \tag{4.46}$$

Next we shall prove the estimates (3.19), (3.21) under the assumption (3.20). First we get

$$z_2(y) = \frac{y}{\log y - \log\log y} \le \frac{\log x_0}{\log x_0 - \log\log x_0}\,\frac{y}{\log y} = \rho_2(x_0)\frac{y}{\log y}$$

to be valid for all $y \ge x_0 \ge e^e$. Further, we have

$$z_2(fy) \le \rho_2(x_0) f \frac{y}{\log fy} = \rho_2(x_0) f\left(1 - \frac{\log f}{\log fy}\right)\frac{y}{\log y}$$

for $fy \ge x_0$. In particular,

$$\begin{aligned} z_2(f\log H) &\le \rho_2(x_0) f\left(1 - \frac{\log f}{\log(f\log H)}\right)\frac{\log H}{\log\log H} \\ &\le \rho_2(x_0) f \frac{\log H}{\log\log H} \end{aligned} \tag{4.47}$$

for all

$$f\log H \ge x_0 \ge e^e, \qquad H > e,$$

where the last inequality in (4.47) is valid with $0 < c - dm \le 2$. Hence

$$Q(\overline{\sigma} + \overline{1}) \le e^{E_2}\left(f\rho\frac{\log H}{\log\log H}\right)^{C_2} H^{\frac{a}{c-dm} + \frac{A_2 f\sqrt{\rho}}{\sqrt{\log\log H}} + \frac{B_2 f\rho}{\log\log H} + \frac{D_2}{\log H}\sqrt{\log\left(\frac{f\rho\log H}{\log\log H}\right)}}, \tag{4.48}$$

if $\rho \ge \rho_2(x_0)$, by (4.44) and (4.47). Now substitute (4.43), (4.45) and (4.48), respectively, into

$$1 < 2|\Lambda| Q(\overline{\sigma} + \overline{1}), \tag{4.49}$$

proving (3.13), (3.16) and (3.23). This ends the proof of Theorem 3.4 and Corollary 3.5.

4.2.6. *Case 3.* Here $\hat{g}_3(S) = 2S + m$, so (4.16) reads

$$(c - dm)W^2 - (2dm^2 + e_1 m)W - dm^3 - e_1 m^2 = \log H. \tag{4.50}$$

Now we simply choose the larger solution

$$S = \frac{2dm^2 + e_1 m + \sqrt{(2dm^2 + e_1 m)^2 + 4(dm^3 + e_1 m^2 + \log H)(c - dm)}}{2(c - dm)}. \tag{4.51}$$

For convenience, we will use the estimate

$$1 = S_3 \le S \le v_1 + v_2\sqrt{\log H} \tag{4.52}$$

with

$$v_1 = \frac{2dm^2 + e_1 m + \sqrt{e_1^2 m^2 + 4cdm^3 + 4ce_1 m^2}}{2(c - dm)}, \qquad v_2 = \frac{1}{\sqrt{c - dm}}.$$

Now, by (4.50) and (4.52), we get

$$\begin{aligned} q_3(S+m) &= a(S+m)^2 + b_1(S+m) \\ &= \frac{a}{c-dm}\log H + \left(\frac{a(2dm^2+e_1m)}{c-dm} + 2am + b_1\right)S \\ &\quad + \frac{a(dm^3+e_1m^2)}{c-dm} + am^2 + b_1m \\ &\le \frac{a}{c-dm}\log H + v_2w_1\sqrt{\log H} + v_1w_1 + w_2, \end{aligned}$$

where

$$w_1 = \frac{a(2dm^2+e_1m)}{c-dm} + 2am + b_1, \qquad w_2 = \frac{a(dm^3+e_1m^2)}{c-dm} + am^2 + b_1m.$$

Hence

$$Q(\overline{\sigma}+\overline{1}) \le H^{\frac{a}{c-dm}+\frac{B_3}{\log H}+\frac{A_3}{\sqrt{\log H}}} = e^{B_3}H^{\frac{a}{c-dm}+A_3\frac{1}{\sqrt{\log H}}},$$

where

$$A_3 = v_2w_1, \qquad B_3 = v_1w_1 + w_2.$$

In particular, if $b_1 = e_1 = 0$, then

$$A_3 = \frac{2acm}{(c-dm)^{3/2}}, \qquad B_3 = \frac{acm^2(c+dm+2\sqrt{cdm})}{(c-dm)^2}.$$

This proves Theorem 3.6.

4.2.7. *The term G_l.* Yet we need to determine terms G_l, $l = 1,2,3$. In each case, there are some assumptions imposed on H. The determinant condition (3.4) and the conditions $S \ge m$, (4.34) and (4.40) should be satisfied. So, if we define $f_1 = x_1/f$, $f_2 = (x_2\log x_2)/f$, $f_3 = S_3$ and suppose

$$H \ge G_l := \max\{m, N_l, e^{f_l}\}, \tag{4.53}$$

then Theorem 3.1 is proved. Finally we note that in Corollary 3.5 we also need the assumption (3.20). The condition (4.53) applied in Case 2 shows in particular that

$$f\log H \ge f\log G_2 \ge x_2\log x_2, \tag{4.54}$$

and thus in (3.22) we may choose

$$\rho = \frac{\log(x_0)}{\log(x_0) - \log\log(x_0)}, \qquad x_0 = \max\{f\log m, f\log N_2, x_2\log x_2, e^e\}. \ \blacksquare$$

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