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Making sense of capitulation: reciprocal primes

by

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1. Introduction. Let ℓ be a rational prime and K be a number field containing a primitive ℓth root of unity. Let $\mathfrak{c}$ denote an ideal class of K of order ℓ, $\mathfrak{p}$ a prime ideal contained in $\mathfrak{c}$, and L an abelian extension of K in which $\mathfrak{c}$ capitulates, i.e., $\mathfrak{p}$ becomes principal in L. General descriptions of which ideal classes capitulate in a given extension L of K do not presently exist; see [FLR, letter 31] for details on the problem's development and its eventual description by Artin as 'hopeless'. On the other hand, given an ideal class, there is an easy way of finding *an* extension of K in which $\mathfrak{c}$ capitulates: Let $a_{\mathfrak{p}}$ be any generator of the principal ideal $\mathfrak{p}^{\ell}$; then $\mathfrak{c}$ capitulates in the Kummer extension $K(\sqrt[\ell]{a_{\mathfrak{p}}})$. Letting $\mathfrak{q}$ denote a prime ideal of K whose corresponding Frobenius element generates $\mathrm{Gal}(K(\sqrt[\ell]{a_{\mathfrak{p}}})/K)$ for every choice of $a_{\mathfrak{p}}$, we show that $\mathfrak{p}$ becomes principal in L if and only if $\mathfrak{q}$ is not a norm from a specific ray class field.

2. Notation and conventions. Let ℓ denote a fixed rational prime, K a number field containing a primitive ℓth root of unity, $K^{\times}$ the set of nonzero elements of K, $\mathcal{O}_K$ its ring of integers, and $\mathcal{O}_K^{\times}$ the group of global units of K. As $\mathcal{O}_K^{\times}$ is a finitely generated group, the extension one gets by adjoining all of the ℓth roots of these units is a finite abelian extension of K. Ordinarily, we would denote this extension by $K(\sqrt[\ell]{\mathcal{O}_K^{\times}})$, but to ease the notational load later on, we will denote this field by F_K. Further, $\mathfrak{c}$ will denote an ideal class of K of order ℓ, $\mathfrak{p}$ a prime ideal in $\mathfrak{c}$, and $a_{\mathfrak{p}}$ a generator of the principal ideal $\mathfrak{p}^{\ell}$. Now, $a_{\mathfrak{p}}$ is unique only up to the multiplication of $a_{\mathfrak{p}}$ by some global unit $\eta \in \mathcal{O}_K^{\times}$, so that there are many different extensions of $K^{\times}$ one can get by adjoining the ℓth root of such an $a_{\mathfrak{p}}$. However, since all of the $a_{\mathfrak{p}}$'s differ by a global unit, there is only one extension of F_K one gets by adjoining the ℓth

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root of any such $a_{\mathfrak{p}}$. That is, the composite $F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$ is independent of the choice of $a_{\mathfrak{p}}$. Finally, L will denote an *abelian* extension of K with $\ell \mid [L : K]$.

3. Results

Lemma 3.1. *With the notation as above, the ideal class $\mathfrak{c}$ capitulates in L if and only if*

$$F_K(\sqrt[\ell]{a_{\mathfrak{p}}}) \subset F_L.$$

Proof. If $\mathfrak{c}$ capitulates in L, then $\mathfrak{p}\mathcal{O}_L = \alpha\mathcal{O}_L$ for some $\alpha \in L$, so that $\mathfrak{p}^\ell\mathcal{O}_L = \alpha^\ell\mathcal{O}_L = a_{\mathfrak{p}}\mathcal{O}_L$. Thus $a_{\mathfrak{p}} = \alpha^\ell \cdot \eta$ for some $\eta \in \mathcal{O}_L^\times$, $\sqrt[\ell]{a_{\mathfrak{p}}} = \alpha \cdot \sqrt[\ell]{\eta}$, and we conclude that $F_K(\sqrt[\ell]{a_{\mathfrak{p}}}) = F_K(\alpha \cdot \sqrt[\ell]{\eta}) \subset F_L$, since both α and $\sqrt[\ell]{\eta}$ are in F_L.

Conversely, assume $F_K(\sqrt[\ell]{a_{\mathfrak{p}}}) \subset F_L$. We consider two cases:

Case I: $L = L(\sqrt[\ell]{a_{\mathfrak{p}}})$. This implies that $\sqrt[\ell]{a_{\mathfrak{p}}} \in L$, so $\mathfrak{p}^\ell\mathcal{O}_L = (\sqrt[\ell]{a_{\mathfrak{p}}})^\ell\mathcal{O}_L$. By the unique factorization of ideals in $\mathcal{O}_L$, we get $\mathfrak{p}\mathcal{O}_L = (\sqrt[\ell]{a_{\mathfrak{p}}})\mathcal{O}_L$, i.e., $\mathfrak{p}$ is principal in L.

Case II: $L \subset L(\sqrt[\ell]{a_{\mathfrak{p}}}) \subset F_L$. This implies that $[L(\sqrt[\ell]{a_{\mathfrak{p}}}) : L] = \ell$. However, all of the subfields of F_L that are degree-ℓ extensions of L are of the form $L(\sqrt[\ell]{\eta})$, where $\eta \in \mathcal{O}_L^\times$. So, $L(\sqrt[\ell]{\eta}) = L(\sqrt[\ell]{a_{\mathfrak{p}}})$ for some such global unit η. Then by Kummer theory, $a_{\mathfrak{p}} = b^\ell \cdot \eta^m$ for $b \in \mathcal{O}_L$ and $(m, \ell) = 1$. Hence

$$\mathfrak{p}^\ell\mathcal{O}_L = (b^\ell \cdot \eta^m)\mathcal{O}_L \;\Rightarrow\; \mathfrak{p}\mathcal{O}_L = b\mathcal{O}_L,$$

by the unique factorization of ideals in L, i.e., $\mathfrak{p}$ is now principal, and $\mathfrak{c}$ has capitulated in L. ■

It will be convenient to recast this as

Corollary 3.2. *$\mathfrak{c}$ does not capitulate in L if and only if*

$$F_K = F_L \cap F_K(\sqrt[\ell]{a_{\mathfrak{p}}}).$$

Proof. Since F_L already contains F_K and since ℓ is prime, the only choices for $F_L \cap F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$ are that it equals F_K (which we know by Lemma 3.1 is equivalent to $\mathfrak{c}$ not capitulating) or that it equals $F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$ (which we know by Lemma 3.1 is equivalent to $\mathfrak{c}$ capitulating). ■

Let $\mathfrak{f}$ denote the conductor of F_L over K. The ideal versions of class field theory [J] then give us:

Fact 1. *Because F_L is abelian over L, mapping ideals of L that are prime to $\mathfrak{f}$ to $\mathrm{Gal}(F_L/F)$ via the Frobenius map (or rather its extension) yields an isomorphism*

$$\frac{I_L^{\mathfrak{f}}}{N_{F_L/L}(I_{F_L}^{\mathfrak{f}})i(L_{\mathfrak{f},1})} \xrightarrow{\sim} \mathrm{Gal}(F_L/L),$$

where $I_L^{\mathfrak{f}}$ denotes the ideals of L that are prime to $\mathfrak{f}$, and $i(L_{\mathfrak{f},1})$ is the set of principal ideals of L that have a generator that is congruent to 1 *modulo $\mathfrak{f}$.*

FACT 2. *The maximal abelian extension of K inside F_L corresponds to the F_L/K norm of $I_{F_L}^{\mathfrak{f}}$. More precisely,*

$$\frac{I_K^{\mathfrak{f}}}{N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})} \xrightarrow{\sim} \mathrm{Gal}((F_L \cap K^{\mathrm{ab}})/K),$$

so that the image of an ideal under this map acts trivially on $F_L \cap K^{\mathrm{ab}}$ if and only if it is contained in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$.

For the remainder of the paper, $\mathfrak{q}$ will denote a prime ideal not dividing ℓ whose Frobenius element $\left[\frac{F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/K}{\mathfrak{q}}\right]$ generates $\mathrm{Gal}(F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/F_K)$. That is, $\mathfrak{q}$ is to split completely in F_K and its Frobenius element is to generate $\mathrm{Gal}(F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/F_K)$. Such primes $\mathfrak{q}$ will be called *reciprocal primes* of $\mathfrak{p}$. Our theorem below can then be loosely paraphrased as 'The ideal $\mathfrak{p}$ of K becomes principal in L if and only if every one of its reciprocal primes $\mathfrak{q}$ is *not* a norm in a suitable ray class group':

THEOREM 3.3 (Reciprocity). *The prime ideal $\mathfrak{p}$ becomes principal in L if and only if* every *reciprocal prime $\mathfrak{q}$ of $\mathfrak{p}$ not dividing $\mathfrak{f}$ is* not *contained in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$.*

Proof. First, assume $\mathfrak{p}$ becomes principal in L. By Corollary 3.2, this implies that $F_K(\sqrt[\ell]{a_{\mathfrak{p}}}) \subset F_L$. Let $\mathfrak{q}$ be any reciprocal prime of $\mathfrak{p}$, i.e., a prime ideal of K whose Frobenius element $\left[\frac{F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/K}{\mathfrak{q}}\right]$ generates $\mathrm{Gal}(F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/F_K)$. With this as our definition of $\mathfrak{q}$, we see that $\left[\frac{F_L/K}{\mathfrak{q}}\right]$, which restricts to the nontrivial $\left[\frac{F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/K}{\mathfrak{q}}\right]$, cannot be trivial in $\mathrm{Gal}((F_L \cap K^{\mathrm{ab}})/K)$, hence by Fact 2, $\mathfrak{q}$ is *not* contained in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$.

Conversely, assume that $\mathfrak{p}$ does *not* become principal in L. By Corollary 3.2, this implies that $F_K(\sqrt[\ell]{a_{\mathfrak{p}}}) \not\subset F_L$. Since F_L and $F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$ are both Galois over K, so is their composite $F_L(\sqrt[\ell]{a_{\mathfrak{p}}})$. We would now like to apply the Chebotarev density theorem, using the same notation as in [L-S, Theorem 3]; with that in mind, let $G = \mathrm{Gal}(F_L(\sqrt[\ell]{a_{\mathfrak{p}}})/K)$, $H_1 = \langle 1 \rangle \subset H_2 = \mathrm{Gal}(F_L(\sqrt[\ell]{a_{\mathfrak{p}}})/F_L)$. The Chebotarev density theorem then guarantees the existence of an infinitude of prime ideals $\mathfrak{q}$ of K, hence at least one not dividing $\mathfrak{f}$, whose Frobenius element $\left[\frac{F_L(\sqrt[\ell]{a_{\mathfrak{p}}})/K}{\mathfrak{q}}\right]$ lies in $H_2 \setminus H_1$, i.e., generates H_2, since $|H_2| = \ell$ is a prime. This Frobenius element fixes F_L, hence fixes F_K; it also nontrivially permutes the different ℓth roots of $a_{\mathfrak{p}}$, all of which are contained in $F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$. Thus, its restriction to $F_K(\sqrt[\ell]{a_{\mathfrak{p}}})$ in fact generates $\mathrm{Gal}(F_K(\sqrt[\ell]{a_{\mathfrak{p}}})/F_K)$ (thus validating our calling this prime $\mathfrak{q}$). Finally, since $\left[\frac{F_L(\sqrt[\ell]{a_{\mathfrak{p}}})/K}{\mathfrak{q}}\right]$ *does* fix F_L, the isomorphism of Fact 2 assures us that $\mathfrak{q}$ does in fact lie in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$. ∎

In the examples of the next section, we will make use of the following, using the same notation and conventions as in the theorem:

COROLLARY 3.4. *Let $\mathfrak{p}$ be any nonprincipal prime ideal of K, and $\mathfrak{q}$ any one if its reciprocal primes that does not divide $\mathfrak{f}$.*

(a) *If $\mathfrak{q}$ does not split completely in L, then $\mathfrak{p}$ becomes principal in L.*
(b) *If $\mathfrak{q}$ splits completely in F_L, then $\mathfrak{p}$ does not become principal in L.*

Proof. To prove (a), note that this implies that $\mathfrak{q}$ does not split completely in $F_L \cap K^{\mathrm{ab}}$, hence does not lie in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$. The result then follows from the theorem. To prove (b), note that this implies that $\mathfrak{q}$ splits completely in $F_L \cap K^{\mathrm{ab}}$, hence does lie in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$. This result, too, then follows from the theorem. ■

REMARKS. 1. Note that if every set of reciprocal primes contains a nonprincipal ideal, we would have a quick proof of a poor man's version of the Principal Ideal Theorem, as every nonprincipal prime ideal is not a norm in the full Hilbert class field, yielding the result that every ideal class of order ℓ becomes principal there.

2. Note that even if L/K is unramified, the condition of $\mathfrak{p}$ becoming principal still requires the use of the conductor $\mathfrak{f}$ (which in that case is only divisible by primes of K which divide ℓ).

3. It is tempting to try and conclude that the capitulation kernel $V_{L/K}$ is isomorphic to the Galois group of $F_L \cap K^{\mathrm{ab}}/F_K$, but this latter group unfortunately includes ℓ-power cyclic extensions of L/K, which have nothing to do with the capitulation (and whose interference here would have to be filtered out.)

4. Examples. We would now like to illustrate how the idea of reciprocal primes can be used to tie the capitulation problem with much of the basics of class field theory—norms, splitting of primes, Artin reciprocity, even classical reciprocity laws—using some of the celebrated examples that have been used to demonstrate the capriciousness of the capitulation process. As described in [C, pp. 98–99], these are (i) the 'bad principalization' of the field $\mathbb{Q}(\sqrt{-21})$, where every ideal class becomes principal in every subfield of its Hilbert class field, and (ii) the 'good principalization' of the field $\mathbb{Q}(\sqrt{-195})$, where there appears to be a Galois-type correspondence between the ideal classes and the subfields of the Hilbert class field in which they capitulate. Nearly all of the numerical evidence cited below was calculated using SAGE.

EXAMPLE (Bad principalization). The first example, as described in [C, pp. 98–99], is that of 'bad principalization' where every ideal class becomes principal in every subfield of the Hilbert class field. In our notation, $K = \mathbb{Q}(\sqrt{-21})$, and its class group is of the form $C_2 \times C_2$, with prime ideal generators $\mathfrak{p}_5 = \langle 5, \sqrt{-21}+2\rangle$ (one of the two prime ideals of K lying over over the rational prime 5) and $\mathfrak{p}_2 = \langle 2, \sqrt{-21}+1\rangle$ (the lone prime ideal of K lying over the rational prime ideal (2), which ramifies in K). For each of these two generating ideal classes and each subfield L of the Hilbert class field, we will show that a very modest search yields a reciprocal prime $\mathfrak{q}$ that does not split in L, hence showing by Corollary 3.4(a) that each $\mathfrak{p}$ does become principal in L. For the first prime $\mathfrak{p}_5$ and subfield $L = K(\sqrt{7})$ we provide all the details. For each of the other prime ideal-subfield pairs we will simply cite the reciprocal prime $\mathfrak{q}$ that satisfies the corollary for that subfield.

(a) Since all of the nontrivial ideal classes of K have order 2, we know that $\mathfrak{p}_5^2$ must be principal, and in fact, $\mathfrak{p}_5^2 = (\sqrt{-21}+2)$. Since K is a complex quadratic field that contains no roots of unity, its only units are ± 1, hence $F_K = K(\sqrt{-1})$, and then

$$F_K(\sqrt{a_{\mathfrak{p}_5}}) = K(\sqrt{-1}, \sqrt{\sqrt{-21}+2}).$$

Theorem 3.3 asserts that $\mathfrak{p}_5$ becoming principal in L is equivalent to some reciprocal prime $\mathfrak{q}$ to $\mathfrak{p}_5$ not lying in $N_{F_L/K}(I_{F_L}^{\mathfrak{f}})i(K_{\mathfrak{f},1})$. We wish to apply Corollary 3.4(a), so search for a prime ideal of K which simultaneously (a) splits completely in F_K, (b) does not split completely in $F_K(\sqrt{a_{\mathfrak{p}_5}}) = K(\sqrt{-1}, \sqrt{\sqrt{-21}+2})$, and (c) does not split in L. Note that satisfying (a) and (b) qualify $\mathfrak{q}$ as a reciprocal prime and that (c) then shows that $\mathfrak{q}$ satisfies the hypothesis of Corollary 3.4(a). Consider the rational prime ideal (41): Since both Legendre symbols $\left(\frac{-1}{41}\right) = \left(\frac{-21}{41}\right) = 1$, (41) splits completely in F_K, factoring in K, say into $\mathfrak{p}_{41}\mathfrak{p}'_{41}$. Note further that as the minimal polynomial of $\sqrt{\sqrt{-21}+2}$ over $\mathbb{Q}$ is $x^4 - 4x^2 + 25$, that $F_K(\sqrt{\sqrt{-21}+2})$ is Galois both over K and over $\mathbb{Q}$, and that

$$x^4 - 4x^2 + 25 \equiv (x^2+13)(x^2+24) \pmod{41},$$

we see both primes $\mathfrak{p}_{41}$ and $\mathfrak{p}'_{41}$ have degree 2 from F_K to $F_K(\sqrt{\sqrt{-21}+2})$, i.e., either can serve as reciprocal prime to $\mathfrak{p}_5$. Let $\mathfrak{q} = \mathfrak{p}_{41}$. Finally, note that $\left(\frac{7}{41}\right) = -1$, hence the rational prime ideal (41) does not split in $\mathbb{Q}(\sqrt{7})$. Thus the reciprocal prime $\mathfrak{q}$ does not split in $L = K(\sqrt{7})$, implying that $\mathfrak{p}_5$ does become principal in L by Corollary 3.4(a).

(b) The capitulation of all the other ideal class generators follows along exactly the same lines as in (a); that is, in each case it was possible to find a reciprocal prime $\mathfrak{q}$ to the corresponding class's prime ideal generator which did not split in L. The following table summarizes this field's entire capitulation situation for unramified extensions of K; as in the calculation above, the reciprocal primes listed refer to either of the two prime ideals of K occurring in the factorization of the corresponding rational prime.

Subfield	Prime generator	Reciprocal prime
$K(\sqrt{7})$	$\mathfrak{p}_5$	$\mathfrak{p}_{41}$
$K(\sqrt{7})$	$\mathfrak{p}_2$	$\mathfrak{p}_5$
$K(\sqrt{3})$	$\mathfrak{p}_5$	$\mathfrak{p}_{17}$
$K(\sqrt{3})$	$\mathfrak{p}_2$	$\mathfrak{p}_5$
$K(\sqrt{-1})$	$\mathfrak{p}_5$	$\mathfrak{p}_5$
$K(\sqrt{-1})$	$\mathfrak{p}_2$	$\mathfrak{p}_{41}$

Example (Good principalization). The second example, also described in [C, pp. 98–99], is that of 'good principalization', where there appears to be a Galois-type correspondence between the ideal classes and the subfields of the Hilbert class field in which they capitulate, i.e., in each subfield we have prime ideals becoming principal while others do not. In this example, $K = \mathbb{Q}(\sqrt{-195})$, whose class group is again of the form $C_2 \times C_2$. Prime ideal generators for the class group of K are given by $\mathfrak{p}_7 = \langle 7, -\sqrt{-195}/2 + 1/2\rangle$ (one of the two primes of K lying over the rational prime ideal (7)) and $\mathfrak{p}_3 = \langle 3, \sqrt{-195}/2 + 3/2\rangle$ (the lone prime of K lying over the rational prime ideal (3), which ramifies in K). As our goal here is to give some concrete flavor as to how reciprocal primes and capitulation interact, we will look only at the details of what happens in $L = K(\sqrt{65}) = K(\sqrt{-3})$.

(a) The prime $\mathfrak{p}_3$ becomes principal in L; this can be shown by exhibiting a reciprocal prime $\mathfrak{q}$ which splits in $F_K = K(\sqrt{-1})$, but not in $\mathbb{Q}(\sqrt{-3})$. The rational prime 17 satisfies

$$\left(\frac{-195}{17}\right) = \left(\frac{-1}{17}\right) = 1, \qquad \left(\frac{-3}{17}\right) = -1,$$

so that either prime of K lying above the rational prime ideal (17) satisfies the criteria of being a reciprocal prime that does not split in $L = K(\sqrt{-3})$, hence $\mathfrak{p}_3$ does become principal in L by Corollary 3.4(a).

(b) The other prime ideal of K given above is $\mathfrak{p}_7 = \langle 7, -\sqrt{-195}/2 + 1/2\rangle$, which satisfies the relation $\mathfrak{p}_7^2 = (-\sqrt{-195}/2 + 1/2)$. In order to use Corollary 3.4(b) to show that the class of $\mathfrak{p}_7$ does not capitulate in L, we need to find a reciprocal prime that splits completely in F_L, so we have to assemble quite a bit more information:

(i) K is a complex quadratic field whose only units are ± 1, hence $F_K = K(\sqrt{-1})$, and then, following the notation of the theorem, $F_K(\sqrt{a_{\mathfrak{p}}}) = K\big(i, \sqrt{-\sqrt{-195}/2 + 1/2}\big)$, which is Galois over both K and $\mathbb{Q}$, and the minimal polynomial of $\sqrt{-\sqrt{-195}/2 + 1/2}$ over $\mathbb{Q}$ is $x^4 - x^2 + 49$.

(ii) The structure of the unit group of $L = K(\sqrt{65}) = K(\sqrt{-3})$ is that of $C_6 \times \mathbb{Z}$. That is, it contains a primitive sixth root of unity ζ_6 (which visibly comes from the presence of $\sqrt{-3}$ in the field) and one fundamental unit. In this case, it is possible to use the fundamental unit $\eta = \sqrt{65} + 8$ of $\mathbb{Q}(\sqrt{65})$ as L's fundamental unit. Since $\zeta_{12} = i \cdot \zeta_3$ and $\zeta_3 \in L$, we have $F_L = L(\sqrt{\zeta_6}, \sqrt{\eta}) = L(i, \sqrt{\eta})$. Finally, since the minimal polynomial of η over $\mathbb{Q}$ is $x^2 - 16x + 1$, the minimal polynomial of $\sqrt{\eta}$ over $\mathbb{Q}$ is $x^4 - 16x^2 + 1$.

In order to use Corollary 3.4(b), we will try to find a rational prime ideal (q) which splits completely in F_L (hence splits in $F_K \subset F_L$), but does not split completely in $F_K(\sqrt{a_{\mathfrak{p}_7}}) = K\big(i, \sqrt{-\sqrt{-195}/2 + 1/2}\big)$. Finding a rational prime number q that satisfies all of the following simultaneously is sufficient:

(1) $\left(\frac{-195}{q}\right) = \left(\frac{-3}{q}\right) = \left(\frac{-1}{q}\right) = 1$,
(2) $x^4 - 16x^2 + 1$ splits completely (mod q),
(3) $x^4 - x^2 + 49$ does *not* split completely (mod q).

Now, consider the prime $q = 193$. It is easily verified that this choice of q satisfies (1) above. Note that

$$x^4 - 16x^2 + 1 \equiv (x + 68)(x + 88)(x + 105)(x + 125) \pmod{193},$$

which shows that (2) is also satisfied. These two facts, together with what we know about F_L, imply that the rational prime ideal (193) splits completely in F_L, hence also splits in F_K, and K. Let $(193) = \mathfrak{p}_{193}\mathfrak{p}'_{193}$ be the factorization of (193) in K. Finally, noting that

$$x^4 - x^2 + 49 \equiv (x^2 + 79)(x^2 + 113) \pmod{193}$$

is a factorization into irreducibles, we see that both primes of K dividing 193 have degree 2 in $F_K(\sqrt{a_{\mathfrak{p}_7}}) = K\big(i, \sqrt{-\sqrt{-195}/2 + 1/2}\big)$, and can serve as reciprocal prime to $\mathfrak{p}_7$. Thus $\mathfrak{q} = \mathfrak{p}_{193}$ is a reciprocal prime to $\mathfrak{p}_7$, which we have already shown splits completely in F_L. By Corollary 3.4(b), then, $\mathfrak{p}_7$ does not become principal in L.

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