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ON NEAR-PERFECT NUMBERS

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Abstract. For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n. We call n a near-perfect number if $\sigma(n) = 2n + d$ where d is a proper divisor of n. We show that the only odd near-perfect number with four distinct prime divisors is $3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

1. Introduction. For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n. In 2012, Pollack and Shevelev [PS] introduced the concept of near-perfect number. A positive number n is called *near-perfect* if it is the sum of all of its proper divisors except one of them. The missing divisor is called *redundant*. Near-perfect numbers are special cases of pseudoperfect numbers (equal to the sum of some subset of proper divisors). In particular, we call n a *quasiperfect* number if $\sigma(n) = 2n + 1$. It is not known whether there are infinitely many near-perfect numbers. Pollack and Shevelev presented an upper bound on the count of near-perfect numbers and constructed three types of near-perfect numbers. In 2013, Ren and Chen [RC] determined all near-perfect numbers with two distinct prime factors, and one sees from this classification that all such numbers are even. We know that any odd near-perfect number is a square. The first author of this paper, Ren and Li [TRL] proved that there is no odd near-perfect number with three distinct prime divisors. The only odd near-perfect number up to 1.4×10^{19} is $173369889 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$ (see [Slo, A181595]). For related problems, see [LL], [APP], [SMC], [TF].

In this paper, we obtain the following result:

THEOREM 1.1. *The only odd near-perfect number with four distinct prime divisors is* $173369889 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

Throughout this paper, let m be a positive integer and a be any integer relatively prime to m. If h is the least positive integer such that $a^h \equiv 1 \pmod{m}$, then h is called the *order* of a *modulo* m, denoted by $\mathrm{ord}_m(a)$.

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2. Lemmas

LEMMA 2.1. *Assume that* $n = 3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}p_4^{\alpha_4}$ *is an odd near-perfect number. Then:*

(i) $\alpha_1 \geq 4$;
(ii) *if* $p_4 \geq 71$, *then* $\alpha_1 \geq 6$;
(iii) *if* $p_4 \geq 59$, *then* $\alpha_2 \geq 4$;
(iv) *if* $p_4 \geq 47$, *then* $(\alpha_1, \alpha_2) \neq (4, 2)$;
(v) *if* $p_4 \geq 53$, *then* $(\alpha_1, \alpha_2, \alpha_3) \neq (6, 2, 2)$;
(vi) *if* $p_4 \geq 67$, *then* $(\alpha_1, \alpha_2, \alpha_3) \neq (4, 4, 2)$;
(vii) *if* $p_4 \geq 89$, *then* $(\alpha_1, \alpha_2, \alpha_3) \neq (6, 4, 2)$.

Proof. (i) Since $p_4 \geq 23$, we have $\alpha_1 \geq 4$, as otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2} \cdot \frac{5}{4} \cdot \frac{19}{18} \cdot \frac{23}{22} < 2,$$

a contradiction. Similarly we deduce (ii)–(vii). ∎

LEMMA 2.2. *There is no odd near-perfect number* n *of the form* $n = 3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}p_4^{\alpha_4}$.

Proof. Assume that $n = 3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}p_4^{\alpha_4}$ is an odd near-perfect number with redundant divisor $d = 3^{\beta_1}5^{\beta_2}19^{\beta_3}p_4^{\beta_4}$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}p_4^{\alpha_4}) = 2 \cdot 3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}p_4^{\alpha_4} + 3^{\beta_1}5^{\beta_2}19^{\beta_3}p_4^{\beta_4}, \tag{2.1}$$

where $\beta_1 + \beta_2 + \beta_3 + \beta_4 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\beta_i \leq \alpha_i$, $i = 1, 2, 3, 4$, and α_i's are even. Let

$$\begin{aligned} f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ = \left(1 - \frac{1}{3^{\alpha_1+1}}\right)\left(1 - \frac{1}{5^{\alpha_2+1}}\right)\left(1 - \frac{1}{19^{\alpha_3+1}}\right)\left(1 - \frac{1}{p_4^{\alpha_4+1}}\right), \end{aligned} \tag{2.2}$$

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2^5 \cdot 3 \cdot (p_4 - 1)}{5 \cdot 19 \cdot p_4} + \frac{2^4 \cdot 3 \cdot (p_4 - 1)}{D}, \tag{2.3}$$

where $D = 3^{\alpha_1 - \beta_1} \cdot 5^{\alpha_2 - \beta_2 + 1} \cdot 19^{\alpha_3 - \beta_3 + 1} \cdot p_4^{\alpha_4 - \beta_4 + 1}$. Thus $D > 5 \cdot 19 \cdot p_4$.

By (2.1)–(2.3), we have $f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < 1$.

If $p_4 \geq 97$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{19}{18} \cdot \frac{97}{96} < 2,$$

which is absurd. Thus $p_4 = 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83$ or 89.

By Lemma 2.1(i) we have

$$(2.4)\qquad f_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{19^3}\right)\left(1-\frac{1}{23^3}\right) = 0.98769\ldots,$$

Case 1: $p_4 = 23$. By (2.1) we know that $\beta_2 = 0$. By (2.3) we have

$$g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^6\cdot 11\cdot 3}{5\cdot 19\cdot 23} + \frac{2^5\cdot 11\cdot 3}{5^3\cdot 19\cdot 23} = 0.98592\ldots,$$

which contradicts (2.4).

Case 2: $p_4 = 31$. If $D \geq 3\cdot 5\cdot 19^2\cdot 31$, then by (2.3) we deduce that $g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.9865\cdots$, which contradicts (2.4). If $D \leq 3\cdot 5^2\cdot 19\cdot 31$, then by (2.3) we have $g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq 1.01\ldots > 1$, a contradiction. If $D = 5\cdot 19^2\cdot 31$, $3^3\cdot 5\cdot 19\cdot 31$, $5\cdot 19\cdot 31^2$ or $3^2\cdot 5^2\cdot 19\cdot 31$, then $\sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}31^{\alpha_4}) = 13\cdot 3^{\alpha_1+1}5^{\alpha_2}19^{\alpha_3-1}31^{\alpha_4}$, $11\cdot 3^{\alpha_1-3}5^{\alpha_2+1}19^{\alpha_3}31^{\alpha_4}$, $7\cdot 3^{\alpha_1+2}5^{\alpha_2}19^{\alpha_3}31^{\alpha_4-1}$, $7\cdot 13\cdot 3^{\alpha_1-2}5^{\alpha_2-1}19^{\alpha_3}31^{\alpha_4}$, respectively. We have $5 \mid \sigma(31^{\alpha_4})$, thus $\alpha_4+1 \equiv 5 \pmod{10}$, hence $17351 \mid \sigma(31^{\alpha_4})$, and the above relations cannot hold. If $D = 5^3\cdot 19\cdot 31$, then $\sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}31^{\alpha_4}) = 17\cdot 3^{\alpha_1+1}5^{\alpha_2-2}19^{\alpha_3}31^{\alpha_4}$. We have $17 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}31^{\alpha_4})$, a contradiction.

Case 3: $p_4 \in \{41, 61\}$. By (2.1) we know that $\beta_4 = 0$.

If $p_4 = 41$, then by (2.3) we have

$$g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^8\cdot 3}{19\cdot 41} + \frac{2^7\cdot 3}{19\cdot 41^3} = 0.98617\ldots,$$

which contradicts (2.4).

If $p_4 = 61$, then by Lemma 2.1(i), (iii), we have $\alpha_1, \alpha_2 \geq 4$. Thus

$$\begin{aligned} f_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{19^3}\right)\left(1-\frac{1}{61^3}\right) \\ &= 0.99541\ldots, \\ g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\leq \frac{2^5\cdot 3\cdot 60}{5\cdot 19\cdot 61} + \frac{2^4\cdot 3\cdot 60}{5\cdot 19\cdot 61^3} = 0.99409\ldots, \end{aligned}$$

which is impossible.

Case 4: $p_4 = 71$. By Lemma 2.1(ii), (iii), we have $\alpha_1 \geq 6$, $\alpha_2 \geq 4$. Thus

$$f_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{19^3}\right)\left(1-\frac{1}{71^3}\right) = 0.99907\ldots.$$

If $D \geq 3\cdot 5\cdot 19\cdot 71^2$, then by (2.3) we have $g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.99863\ldots$, a contradiction.

If $D \leq 5^4\cdot 19\cdot 71$, then by (2.3) we have $g_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4) > 1$, a contradiction.

If $D = 3^2 \cdot 5 \cdot 19^2 \cdot 71$, then $\sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}71^{\alpha_4}) = 7^3 \cdot 3^{\alpha_1-2}5^{\alpha_2}19^{\alpha_3-1}71^{\alpha_4}$. Thus $7 \mid \sigma(71^{\alpha_4})$, $\alpha_4 + 1 \equiv 7 \pmod{14}$, hence $883 \mid \sigma(71^{\alpha_4})$, a contradiction.

If $D = 3^3 \cdot 5^2 \cdot 19 \cdot 71$, then $\sigma(3^{\alpha_1}5^{\alpha_2}19^{\alpha_3}71^{\alpha_4}) = 271 \cdot 3^{\alpha_1-3}5^{\alpha_2-1}19^{\alpha_3}71^{\alpha_4}$. Thus $3 \mid \sigma(19^{\alpha_3})$, $\alpha_3 + 1 \equiv 3 \pmod 6$, hence $127 \mid \sigma(19^{\alpha_3})$, a contradiction.

Case 5: $p_4 \in \{29, 37, 43, 53, 73, 89\}$. By (2.1) we know that $\beta_2 = \beta_4 = 0$.

If $p_4 \le 43$, then by (2.3) we have

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{2^5 \cdot 3 \cdot 42}{5 \cdot 19 \cdot 43} + \frac{2^4 \cdot 3 \cdot 42}{5^3 \cdot 19 \cdot 43^3} = 0.98703\ldots,$$

which contradicts (2.4).

If $p_4 = 53$, then by (2.3) we have

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{2^5 \cdot 3 \cdot 52}{5 \cdot 19 \cdot 53} + \frac{2^4 \cdot 3 \cdot 52}{5^3 \cdot 19 \cdot 53^3} = 0.99146\ldots. \tag{2.5}$$

By Lemma 2.1(i), (iv), (v), we have $\alpha_1 \ge 4$, $(\alpha_1, \alpha_2) \ne (4, 2)$, $(\alpha_1, \alpha_2, \alpha_3) \ne (6, 2, 2)$. Thus $\alpha_1 \ge 8$, or $\alpha_1 = 4$, $\alpha_2 \ge 4$, or $\alpha_1 = 6$, $\alpha_2 \ge 4$, or $\alpha_1 = 6$, $\alpha_2 = 2$, $\alpha_3 \ge 4$. Then

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{19^5}\right)\left(1 - \frac{1}{53^3}\right) = 0.99153\ldots,$$

which contradicts (2.5).

If $p_4 = 73$, then by Lemma 2.1(ii), (iii), we have $\alpha_1 \ge 6$, $\alpha_2 \ge 4$. Thus

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{73^3}\right) = 0.99907\ldots,$$

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{2^5 \cdot 3 \cdot 72}{5 \cdot 19 \cdot 73} + \frac{2^4 \cdot 3 \cdot 72}{5^3 \cdot 19 \cdot 73^3} = 0.99668\ldots,$$

a contradiction.

If $p_4 = 89$, then by (2.3),

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{2^5 \cdot 3 \cdot 88}{5 \cdot 19 \cdot 89} + \frac{2^4 \cdot 3 \cdot 88}{5^3 \cdot 19 \cdot 89^3} = 0.99917\ldots. \tag{2.6}$$

By Lemma 2.1(ii), (iii), (vii), we have $\alpha_1 \ge 6$, $\alpha_2 \ge 4$ and $(\alpha_1, \alpha_2, \alpha_3) \ne (6, 4, 2)$. Thus $\alpha_1 \ge 8$, or $\alpha_1 = 6$, $\alpha_2 \ge 6$, or $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_3 \ge 4$. Then

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^5}\right)\left(1 - \frac{1}{89^3}\right) = 0.99922\ldots,$$

which contradicts (2.6).

Case 6: $p_4 \in \{47, 59, 67, 79, 83\}$. By (2.1) we know that $\beta_2 = 0$.

If $p_4 = 47$, then by Lemma 2.1(i), (iv), we have $\alpha_1 \ge 4$ and $(\alpha_1, \alpha_2) \ne (4, 2)$. Thus $\alpha_1 \ge 6$, or $\alpha_1 = 4, \alpha_2 \ge 4$. Therefore

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{47^3}\right) = 0.99139\ldots.$$

If $D \geq 3^2 \cdot 5^3 \cdot 19 \cdot 47$, then by (2.3) we have $g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99122\ldots$, a contradiction. If $D = 5^3 \cdot 19 \cdot 47$, then by (2.3) we have $g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 1$, a contradiction. If $D = 3 \cdot 5^3 \cdot 19 \cdot 47$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 19^{\alpha_3} 47^{\alpha_4}) = 151 \cdot 3^{\alpha_1 - 1} 19^{\alpha_3} 47^{\alpha_4}$. Thus $3 \mid \sigma(19^{\alpha_3})$, $\alpha_3 + 1 \equiv 3 \pmod 6$, and so $127 \mid \sigma(19^{\alpha_3})$, a contradiction.

If $p_4 = 59$, then by Lemma 2.1(i), (iii), we have $\alpha_1, \alpha_2 \geq 4$. Thus

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{59^3}\right) = 0.99541\ldots,$$

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^5 \cdot 3 \cdot 58}{5 \cdot 19 \cdot 59} + \frac{2^4 \cdot 3 \cdot 58}{5^5 \cdot 19 \cdot 59} = 0.99419\ldots,$$

a contradiction.

If $p_4 = 67$, then by Lemma 2.1(i), (iii), (vi), we have $\alpha_1, \alpha_2 \geq 4$ and $(\alpha_1, \alpha_2, \alpha_3) \neq (4, 4, 2)$. Thus $\alpha_1 \geq 6$, $\alpha_2 \geq 4$, or $\alpha_1 = \alpha_2 = 4$, $\alpha_3 \geq 4$, or $\alpha_1 = 4$, $\alpha_2 \geq 6$. Then

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^5}\right)\left(1 - \frac{1}{67^3}\right) = 0.99556\ldots,$$

By (2.3) and $\beta_2 = 0$, we have $D \geq 5^5 \cdot 19 \cdot 67$. If $D \geq 3^2 \cdot 5^5 \cdot 19 \cdot 67$, then by (2.3) we have $g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99553\ldots$, a contradiction. If $D = 5^5 \cdot 19 \cdot 67$ or $3 \cdot 5^5 \cdot 19 \cdot 67$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 19^{\alpha_3} 67^{\alpha_4}) = 139 \cdot 3^{\alpha_1 + 2} 19^{\alpha_3} 67^{\alpha_4}$, $11^2 \cdot 31 \cdot 3^{\alpha_1 - 1} 19^{\alpha_3} 67^{\alpha_4}$, respectively. We have $19 \mid \sigma(5^{\alpha_2})$, thus $\alpha_2 + 1 \equiv 9 \pmod{18}$. Since $\mathrm{ord}_{31}(5) = 3$, we have $31 \mid \sigma(5^{\alpha_2})$, and the above relations cannot hold.

If $p_4 = 79$, then by Lemma 2.1(ii), (iii), we have $\alpha_1 \geq 6$, $\alpha_2 \geq 4$. Thus

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{79^3}\right) = 0.99907\ldots,$$

$$g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^5 \cdot 3 \cdot 78}{5 \cdot 19 \cdot 79} + \frac{2^4 \cdot 3 \cdot 78}{5^5 \cdot 79 \cdot 59} = 0.99853\ldots,$$

a contradiction.

If $p_4 = 83$, then by Lemma 2.1(ii), (iii), we have $\alpha_1 \geq 6$, $\alpha_2 \geq 4$. Thus

$$f_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{19^3}\right)\left(1 - \frac{1}{83^3}\right) = 0.99907\ldots.$$

By (2.3) and $\beta_2 = 0$, we have $D \geq 5^5 \cdot 19 \cdot 83$. If $D \geq 3 \cdot 5^5 \cdot 19 \cdot 83$, then by (2.3) we have $g_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99861\ldots$, a contradiction. If $D = 5^5 \cdot 19 \cdot 83$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 19^{\alpha_3} 83^{\alpha_4}) = 139 \cdot 3^{\alpha_1 + 2} 19^{\alpha_3} 83^{\alpha_4}$. We have $3 \mid \sigma(19^{\alpha_3})$, thus $\alpha_3 + 1 \equiv 3 \pmod 6$, hence $127 \mid \sigma(19^{\alpha_3})$, a contradiction.

This completes the proof of Lemma 2.2. ∎

LEMMA 2.3. *Assume that* $n = 3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} p_4^{\alpha_4}$ *is an odd near-perfect number. Then:*

(i) *if* $p_4 \geq 23$, *then* $(\alpha_1, \alpha_2) \neq (2, 2)$;
(ii) *if* $p_4 \geq 29$, *then* $\alpha_1 \geq 4$;
(iii) *if* $p_4 \geq 89$, *then* $\alpha_2 \geq 4$;
(iv) *if* $p_4 \geq 67$, *then* $(\alpha_1, \alpha_2) \neq (4, 2)$; *if* $p_4 \geq 83$, *then* $(\alpha_1, \alpha_2) \neq (6, 2)$, $(\alpha_2, \alpha_3) \neq (2, 2)$;
(v) *if* $p_4 \geq 127$, *then* $\alpha_1 \geq 6$;
(vi) *if* $p_4 \geq 233$, *then* $\alpha_1 \geq 8$, $(\alpha_2, \alpha_3) \neq (4, 2)$;
(vii) *if* $p_4 \geq 239$, *then* $\alpha_2 \geq 6$;
(viii) *if* $p_4 \geq 251$, *then* $\alpha_3 \geq 4$;
(ix) *if* $p_4 \geq 211$, *then* $(\alpha_1, \alpha_2, \alpha_3) \neq (6, 4, 2)$,
(x) *if* $p_4 \geq 223$, *then* $(\alpha_1, \alpha_2) \neq (6, 4)$, $(\alpha_1, \alpha_3) \neq (6, 2)$, $(\alpha_1, \alpha_2, \alpha_3) \neq (6, 6, 2)$;
(xi) *if* $p_4 \geq 227$, *then* $(\alpha_2, \alpha_3) \neq (4, 2)$;
(xii) *if* $p_4 \geq 229$, *then* $(\alpha_1, \alpha_2) \neq (6, 6)$, $(\alpha_1, \alpha_2, \alpha_3) \neq (8, 4, 2), (10, 4, 2)$;
(xiii) *if* $p_4 \geq 241$, *then* $(\alpha_1, \alpha_3) \neq (8, 2)$.

Proof. (i) If $p_4 \geq 23$, then $(\alpha_1, \alpha_2) \neq (2, 2)$, as otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2 5^2)}{3^2 5^2} \cdot \frac{17}{16} \cdot \frac{23}{22} = 1.98955\ldots < 2,$$

a contradiction. Similarly we prove (ii)–(xiii). ∎

LEMMA 2.4. *There is no odd near-perfect number n of the form $n = 3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} p_4^{\alpha_4}$.*

Proof. Assume that $n = 3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} p_4^{\alpha_4}$ is an odd near-perfect number with redundant divisor $d = 3^{\beta_1} 5^{\beta_2} 17^{\beta_3} p_4^{\beta_4}$. Then

$$\sigma(3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} p_4^{\alpha_4}) = 2 \cdot 3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} p_4^{\alpha_4} + 3^{\beta_1} 5^{\beta_2} 17^{\beta_3} p_4^{\beta_4}, \tag{2.7}$$

where $\beta_1 + \beta_2 + \beta_3 + \beta_4 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\beta_i \leq \alpha_i$, $i = 1, 2, 3, 4$, and α_i's are even. Let

$$\begin{aligned} f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ = \left(1 - \frac{1}{3^{\alpha_1+1}}\right)\left(1 - \frac{1}{5^{\alpha_2+1}}\right)\left(1 - \frac{1}{17^{\alpha_3+1}}\right)\left(1 - \frac{1}{p_4^{\alpha_4+1}}\right), \end{aligned} \tag{2.8}$$

$$g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2^8 \cdot (p_4 - 1)}{3 \cdot 5 \cdot 17 \cdot p_4} + \frac{2^7 \cdot (p_4 - 1)}{D}, \tag{2.9}$$

where $D = 3^{\alpha_1-\beta_1+1} \cdot 5^{\alpha_2-\beta_2+1} \cdot 17^{\alpha_3-\beta_3+1} \cdot p_4^{\alpha_4-\beta_4+1}$. Thus $D > 3 \cdot 5 \cdot 17 \cdot p_4$. By (2.7)–(2.9), we have $f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < 1$.

If $p_4 \geq 257$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16} \cdot \frac{257}{256} < 2,$$

which is a contradiction. Thus $p_4 = 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157,$

163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241 or 251.

CASE 1: $p_4 = 19$. By (2.7) we know that $\beta_2 = \beta_3 = 0$. We have

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ge \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{19^3}\right) = 0.95492\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \le \frac{2^9\cdot 3}{5\cdot 17\cdot 19} + \frac{2^8\cdot 3}{5^3\cdot 17^3\cdot 19} = 0.95114\ldots,$$

a contradiction.

CASE 2: $p_4 \in \{23, 71, 131, 191, 251\}$. By (2.7) we know that $\beta_1 = \beta_3 = 0$. If $p_4 = 23$, then by Lemma 2.3(i), we know that $(\alpha_1,\alpha_2) \neq (2,2)$. Thus $\alpha_1 \ge 4$, or $\alpha_1 = 2$, $\alpha_2 \ge 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ge \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{23^3}\right) = 0.96238\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \le \frac{2^9\cdot 11}{3\cdot 5\cdot 17\cdot 23} + \frac{2^8\cdot 11}{3^3\cdot 5\cdot 17^3\cdot 23} = 0.96045\ldots,$$

a contradiction. If $p_4 = 71$, then by Lemma 2.3(ii), (iv), we know that $\alpha_1 \ge 4$ and $(\alpha_1,\alpha_2) \neq (4,2)$. Thus $\alpha_1 \ge 6$, or $\alpha_1 = 4$, $\alpha_2 \ge 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ge \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{71^3}\right) = 0.99134\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \le \frac{2^9\cdot 7}{3\cdot 17\cdot 71} + \frac{2^8\cdot 7}{3^5\cdot 17^3\cdot 71} = 0.98980\ldots,$$

a contradiction. If $p_4 = 131, 191$, then by Lemma 2.3(iii), (v), we have

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ge \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{131^3}\right) = 0.99901\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \le \frac{2^9\cdot 19}{3\cdot 17\cdot 191} + \frac{2^8\cdot 19}{3^7\cdot 17^3\cdot 191} = 0.99866\ldots,$$

a contradiction. If $p_4 = 251$, then by Lemma 2.3(vi)–(viii), we have

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ge \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^7}\right)\left(1-\frac{1}{17^5}\right)\left(1-\frac{1}{251^3}\right) = 0.99993\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \le \frac{2^9\cdot 5^2}{3\cdot 17\cdot 251} + \frac{2^8\cdot 5^2}{3^9\cdot 17^5\cdot 251} = 0.99992\ldots,$$

a contradiction.

CASE 3: $p_4 \in \{29, 53, 89, 113, 173, 197\}$. By (2.7) we know that $\beta_i = 0$, $i = 1, 2, 3, 4$. If n is near-perfect, then n is quasiperfect. By the result of Hagis and Cohen (see [HC, Theorem 3]) we know that there is no such integer n.

CASE 4: $p_4 \in \{31, 151, 181, 211\}$. By (2.7) we know that $\beta_3 = 0$. If $p_4 = 31$, then

$$f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{17^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.98768\ldots,$$

$$g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^9}{17 \cdot 31} + \frac{2^8}{17^3 \cdot 31} = 0.97321\ldots,$$

a contradiction. If $p_4 = 151$, then by Lemma 2.3(v), (iii),

$$f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{17^3}\right)\left(1 - \frac{1}{151^3}\right) = 0.99901\ldots,$$

$$g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^9 \cdot 5}{17 \cdot 151} + \frac{2^8 \cdot 5}{17^3 \cdot 151} = 0.99899\ldots,$$

a contradiction. If $p_4 = 181$, then by Lemma 2.3(v), (iii), we know that $\alpha_1 \geq 6$, $\alpha_2 \geq 4$. Thus

$$f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)(1 - \frac{1}{17^3}\Big)\left(1 - \frac{1}{181^3}\right) = 0.99901\ldots.$$

By (2.9) we know that if $D \geq 3^2 \cdot 5 \cdot 17^3 \cdot 181$, then $g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99895\ldots$, a contradiction. If $D = 3 \cdot 5 \cdot 17^3 \cdot 181$, then $g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 1$, a contradiction.

If $p_4 = 211$, then by Lemma 2.3(v), (iii), (ix) we know that $\alpha_1 \geq 6$, $\alpha_2 \geq 4$ and $(\alpha_1, \alpha_2, \alpha_3) \neq (6, 4, 2)$. Thus $\alpha_1 \geq 8$; or $\alpha_1 = 6$, $\alpha_2 \geq 6$; or $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_3 \geq 4$. Then

$$\begin{aligned} f_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &\geq f_2(6, 4, 4, 2) \\ &= \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{17^5}\right)\left(1 - \frac{1}{211^3}\right) \\ &= 0.99922\ldots. \end{aligned}$$

If $D \geq 3^2 \cdot 5^3 \cdot 17^3 \cdot 211$, then by (2.9) we have $g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99918\ldots$, a contradiction. If $D = 3 \cdot 5 \cdot 17^3 \cdot 211$, then $g_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 1$, which is impossible. If $D = 3^2 \cdot 5 \cdot 17^3 \cdot 211$, $3 \cdot 5^2 \cdot 17^3 \cdot 211$, $3^3 \cdot 5 \cdot 17^3 \cdot 211$, $3^2 \cdot 5^2 \cdot 17^3 \cdot 211$, $3 \cdot 5^3 \cdot 17^3 \cdot 211$, $3^4 \cdot 5 \cdot 17^3 \cdot 211$ or $3^3 \cdot 5^2 \cdot 17^3 \cdot 211$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 17^{\alpha_3} 211^{\alpha_4}) = 347 \cdot 3^{\alpha_1 - 1} 5^{\alpha_2 + 1} 211^{\alpha_4}$, $7^2 \cdot 59 \cdot 3^{\alpha_1} 5^{\alpha_2 - 1} 211^{\alpha_4}$, $11^2 \cdot 43 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} 211^{\alpha_4}$, $3^{\alpha_1 - 1} 5^{\alpha_2 - 1} 211^{\alpha_4} \cdot 13 \cdot 23 \cdot 29$, $4817 \cdot 3^{\alpha_1 + 1} 5^{\alpha_2 - 2} 211^{\alpha_4}$, $15607 \cdot 3^{\alpha_1 - 3} 5^{\alpha_2} 211^{\alpha_4}$, $19 \cdot 37^2 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2 - 1} 211^{\alpha_4}$, respectively. We have $3 \mid \sigma(211^{\alpha_4})$, thus $\alpha_4 + 1 \equiv 3 \pmod 6$, hence $31 \mid \sigma(211^{\alpha_4})$, which is impossible.

CASE 5: $p_4 \in \{37, 73, 97, 163, 193, 233\}$. By (2.7) we know that $\beta_i = 0$, $i = 2, 3, 4$.

If $p_4 = 37$, then by Lemma 2.3(ii) we know that $\alpha_1 \geq 4$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{37^3}\right) = 0.98769\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{10}\cdot 3}{5\cdot 17\cdot 37} + \frac{2^9\cdot 3}{5^3\cdot 17^3\cdot 37^3} = 0.97678\ldots,$$

a contradiction. If $p_4 = 73$, then by Lemma 2.3(ii), (iv), we know that $\alpha_1 \geq 4$ and $(\alpha_1,\alpha_1) \neq (4,2)$. Thus $\alpha_1 \geq 6$, or $\alpha_1 = 4, \alpha_2 \geq 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{73^3}\right) = 0.99134\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{11}\cdot 3}{5\cdot 17\cdot 73} + \frac{2^{10}\cdot 3}{5^3\cdot 17^3\cdot 73^3} = 0.99016\ldots,$$

a contradiction. If $p_4 = 97$, then by Lemma 2.3(ii), (iii), we have

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{97^3}\right) = 0.99536\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{13}}{5\cdot 17\cdot 97} + \frac{2^{12}}{5^5\cdot 17^3\cdot 97^3} = 0.99357\ldots,$$

a contradiction. If $p_4 = 163, 193$, then by Lemma 2.3(iii), (v), we have

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{163^3}\right) = 0.99901\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{14}}{5\cdot 17\cdot 193} + \frac{2^{13}}{5^5\cdot 17^3\cdot 193^3} = 0.99872\ldots,$$

a contradiction. If $p_4 = 233$, then by Lemma 2.3(iii), (vi), we know that $\alpha_1 \geq 8$, $\alpha_2 \geq 4$ and $(\alpha_2,\alpha_3) \neq (4,2)$. Thus $\alpha_1 = 8$, $\alpha_2 \geq 6$; or $\alpha_1 = 8$, $\alpha_2 = 4$, $\alpha_3 \geq 4$; or $\alpha_1 \geq 10, \alpha_2 = 4$, $\alpha_3 \geq 4$; or $\alpha_1 \geq 10$, $\alpha_2 \geq 6$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^5}\right)\left(1-\frac{1}{233^3}\right) = 0.99962\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{11}\cdot 29}{3\cdot 5\cdot 17\cdot 233} + \frac{2^{10}\cdot 29}{5^5\cdot 17^3\cdot 233^3} = 0.99961\ldots,$$

a contradiction.

Case 6: $p_4 \in \{41, 101\}$. By (2.7) we know that $\beta_1 = \beta_3 = \beta_4 = 0$.

If $p_4 = 41$, then by Lemma 2.3(ii), we have $\alpha_1 \geq 4$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{41^3}\right) = 0.98770\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{11}}{3\cdot 17\cdot 41} + \frac{2^{10}}{3^5\cdot 17^3\cdot 41^3} = 0.97943\ldots,$$

a contradiction. If $p_4 = 101$, then by Lemma 2.3(ii), (iii), we have $\alpha_1, \alpha_2 \geq 4$.

Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{101^3}\right) = 0.99536\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{10}\cdot 5}{3\cdot 17\cdot 101} + \frac{2^9\cdot 5}{3^5\cdot 17^3\cdot 101^3} = 0.99398\ldots,$$

a contradiction.

CASE 7. $p_4 \in \{43, 67, 79, 109, 127, 139, 157, 199, 223, 229\}$. By (2.7) we know that $\beta_2 = \beta_3 = 0$. If $p_4 = 43$, then by Lemma 2.3(ii) we have $\alpha_1 \geq 4$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{43^3}\right) = 0.98770\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 7}{5\cdot 17\cdot 43} + \frac{2^8\cdot 7}{5^3\cdot 17^3\cdot 43} = 0.98064\ldots,$$

a contradiction. If $p_4 = 67, 79$, then by Lemma 2.3(ii), (iv), we know that $\alpha_1 \geq 4$ and $(\alpha_1, \alpha_2) \neq (4, 2)$. Thus $\alpha_1 \geq 6$ or $\alpha_1 = 4$, $\alpha_2 \geq 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{67^3}\right) = 0.99134\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 13}{5\cdot 17\cdot 79} + \frac{2^8\cdot 13}{5^3\cdot 17^3\cdot 79} = 0.99128\ldots,$$

a contradiction. If $p_4 = 109$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{109^3}\right) = 0.99536\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{10}\cdot 3^2}{5\cdot 17\cdot 109} + \frac{2^9\cdot 3^2}{5^5\cdot 17^3\cdot 109} = 0.99471\ldots,$$

a contradiction. If $p_4 \in \{127, 139, 157, 199\}$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{127^3}\right) = 0.99901\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 11\cdot 3}{5\cdot 17\cdot 199} + \frac{2^8\cdot 11\cdot 3}{5^5\cdot 17^3\cdot 199} = 0.99887\ldots,$$

a contradiction. If $p_4 = 223$, then by Lemma 2.3(v), (iii), (x), we know that $\alpha_1 \geq 6$, $\alpha_2 \geq 4$, $(\alpha_1, \alpha_2) \neq (6, 4)$, $(\alpha_1, \alpha_3) \neq (6, 2)$. Thus $\alpha_1 \geq 8$, or $\alpha_1 = 6$, $\alpha_2 \geq 6$, $\alpha_3 \geq 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{223^3}\right) = 0.999425\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 37}{5\cdot 17\cdot 223} + \frac{2^8\cdot 37}{5^5\cdot 17^3\cdot 223} = 0.999422\ldots,$$

a contradiction. If $p_4 = 229$, then by Lemma 2.3(v), (iii), (xi), (xii), we know that $\alpha_1 \geq 6$, $\alpha_2 \geq 4$, $(\alpha_1, \alpha_3) \neq (6, 2)$, $(\alpha_2, \alpha_3) \neq (4, 2)$, $(\alpha_1, \alpha_2) \neq$

$(6,4),(6,6)$, $(\alpha_1,\alpha_2,\alpha_3)\neq(8,4,2)$. Thus $\alpha_1=6$, $\alpha_2\geq 8$, $\alpha_3\geq 4$; or $\alpha_1=8$, $\alpha_2=4,\alpha_3\geq 4$; or $\alpha_1=8$, $\alpha_2\geq 6$; or $\alpha_1\geq 10$, $\alpha_2=4$, $\alpha_3\geq 4$; or $\alpha_1\geq 10,\alpha_2\geq 6$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geq\left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^9}\right)\left(1-\frac{1}{17^5}\right)\left(1-\frac{1}{229^3}\right)=0.999541\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leq\frac{2^{10}\cdot 19}{5\cdot 17\cdot 229}+\frac{2^9\cdot 19}{5^5\cdot 17^3\cdot 229}=0.999540\ldots,$$

a contradiction.

Case 8: $p_4\in\{47,59,83,107,149,167,179,227\}$. By (2.7) we know that $\beta_i=0$, $i=1,2,3$. If $p_4=47,59$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geq\left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{47^3}\right)=0.98770\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leq\frac{2^9\cdot 29}{3\cdot 5\cdot 17\cdot 59}+\frac{2^8\cdot 29}{3^5\cdot 5^3\cdot 17^3\cdot 59}=0.98690\ldots,$$

a contradiction. If $p_4=83$, then by Lemma 2.3(ii), (iv), we know that $\alpha_1\geq 4$ and $(\alpha_1,\alpha_2)\neq(4,2),(6,2)$, $(\alpha_2,\alpha_3)\neq(2,2)$. Thus $\alpha_2\geq 4$, or $\alpha_1\geq 8$, $\alpha_2=2$, $\alpha_3\geq 4$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geq\left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^5}\right)\left(1-\frac{1}{83^3}\right)=0.99194\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leq\frac{2^9\cdot 41}{3\cdot 5\cdot 17\cdot 83}+\frac{2^8\cdot 41}{3^5\cdot 5^5\cdot 17^3\cdot 83}=0.99182\ldots,$$

a contradiction. If $p_4=107$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geq\left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{107^3}\right)=0.99536\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leq\frac{2^9\cdot 53}{3\cdot 5\cdot 17\cdot 107}+\frac{2^8\cdot 53}{3^5\cdot 5^5\cdot 17^3\cdot 107}=0.99453\ldots,$$

a contradiction. If $p_4\in\{149,167,179\}$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geq\left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{149^3}\right)=0.999019\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\leq\frac{2^9\cdot 89}{3\cdot 5\cdot 17\cdot 179}+\frac{2^8\cdot 89}{3^7\cdot 5^5\cdot 17^3\cdot 179}=0.99831\ldots,$$

a contradiction. If $p_4=227$, then by Lemma 2.3(v), (iii), (ix), (xi), we know that $\alpha_1\geq 6$, $\alpha_2\geq 4$, $(\alpha_1,\alpha_2)\neq(6,4)$, $(\alpha_2,\alpha_3)\neq(4,2)$ and $(\alpha_1,\alpha_3)\neq(6,2)$. Thus $\alpha_1=6$, $\alpha_2\geq 6$, $\alpha_3\geq 4$; or $\alpha_1\geq 8$, $\alpha_2=4$, $\alpha_3\geq 4$; or $\alpha_1\geq 8$, $\alpha_2\geq 6$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^7}\right)\left(1-\frac{1}{17^5}\right)\left(1-\frac{1}{227^3}\right) = 0.99952\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 113}{3\cdot 5\cdot 17\cdot 227} + \frac{2^8\cdot 113}{3^7\cdot 5^5\cdot 17^3\cdot 227} = 0.99949\ldots,$$

a contradiction.

CASE 9: $p_4 \in \{61, 241\}$. By (2.7) we know that $\beta_3 = \beta_4 = 0$. If $p_4 = 61$, then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{61^3}\right) = 0.987712\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{10}}{17\cdot 61} + \frac{2^9}{17^3\cdot 61^3} = 0.98746\ldots,$$

a contradiction. If $p_4 = 241$, then by Lemma 2.3(vi), (vii), we know that $\alpha_1 \geq 8$, $\alpha_2 \geq 6$ and $(\alpha_1, \alpha_3) \neq (8, 2)$. Thus $\alpha_1 = 8$, $\alpha_2 \geq 6$, $\alpha_3 \geq 4$; or $\alpha_1 \geq 10$, $\alpha_2 \geq 6$. Then

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^{11}}\right)\left(1-\frac{1}{5^7}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{241^3}\right) = 0.99977\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{12}}{17\cdot 241} + \frac{2^{11}}{17^3\cdot 241^3} = 0.99975\ldots,$$

a contradiction.

CASE 10: $p_4 = 103$. By (2.7) we know that $\beta_2 = 0$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{103^3}\right) = 0.99536\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9}{5\cdot 103} + \frac{2^8}{5^5\cdot 103} = 0.99497\ldots,$$

a contradiction.

CASE 11. $p_4 = 137$. By (2.7) we know that $\beta_1 = \beta_2 = \beta_4 = 0$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^7}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{137^3}\right) = 0.99901\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^{11}}{3\cdot 5\cdot 137} + \frac{2^{10}}{3^7\cdot 5^5\cdot 137^3} = 0.99659\ldots,$$

a contradiction.

CASE 12. $p_4 = 239$. By (2.7) we know that $\beta_1 = \beta_2 = 0$. Thus

$$f_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{5^7}\right)\left(1-\frac{1}{17^3}\right)\left(1-\frac{1}{239^3}\right) = 0.99973\ldots,$$

$$g_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 7}{3\cdot 5\cdot 239} + \frac{2^8\cdot 7}{3^9\cdot 5^7\cdot 239} = 0.99972\ldots,$$

a contradiction. This completes the proof of Lemma 2.4. ∎

LEMMA 2.5. *There is no odd near-perfect number n of the form $n = 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}$.*

Proof. Assume that $n = 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}$ is an odd near-perfect number with redundant divisor $d = 3^{\beta_1}5^{\beta_2}11^{\beta_3}p_4^{\beta_4}$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}) = 2\cdot 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4} + 3^{\beta_1}5^{\beta_2}11^{\beta_3}p_4^{\beta_4}, \tag{2.10}$$

where $\beta_1+\beta_2+\beta_3+\beta_4 < \alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\beta_i \le \alpha_i$, $i = 1,2,3,4$, and α_i's are even. Let

$$\begin{aligned} &f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\ &\qquad = \left(1-\frac{1}{3^{\alpha_1+1}}\right)\left(1-\frac{1}{5^{\alpha_2+1}}\right)\left(1-\frac{1}{11^{\alpha_3+1}}\right)\left(1-\frac{1}{p_4^{\alpha_4+1}}\right), \end{aligned} \tag{2.11}$$

$$g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\frac{32}{33}+\frac{2^4}{D}\right)\left(1-\frac{1}{p_4}\right), \tag{2.12}$$

where $D = 3^{\alpha_1-\beta_1+1}5^{\alpha_2-\beta_2}11^{\alpha_3-\beta_3+1}p_4^{\alpha_4-\beta_4}$. Then by (2.10)–(2.12) we have $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < 1$.

CASE 1: $\alpha_1 = 2$. If $p_4 \ge 149$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2}\cdot\frac{5}{4}\cdot\frac{11}{10}\cdot\frac{149}{148} < 2,$$

which is absurd. Thus $p_4 = 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137$ or 139. Note that if $p_4 \ge 71$ and $\alpha_1 = 2$, then $\alpha_2 \ge 4$, since otherwise

$$2 < \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2\cdot 5^2)}{3^2\cdot 5^2}\cdot\frac{11}{10}\cdot\frac{71}{70} < 2,$$

a contradiction. Similarly, if $p_4 \ge 67$, then $(\alpha_1,\alpha_2,\alpha_3) \ne (2,2,2)$; if $p_4 \ge 127$, then $(\alpha_1,\alpha_2,\alpha_3) \ne (2,4,2)$; if $p_4 \ge 131$ and $\alpha_1 = 2$, then $\alpha_3 \ge 4$; if $p_4 \ge 139$ and $\alpha_1 = 2$, then $\alpha_2 \ge 6$. Thus if $13 \le p_4 \le 67$, then

$$\begin{aligned} f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\ge \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{13^3}\right) \\ &= 0.95410\ldots. \end{aligned} \tag{2.13}$$

If $p_4 \ge 71$, then

$$\begin{aligned} f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\ge \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{71^3}\right) \\ &= 0.96192\ldots. \end{aligned} \tag{2.14}$$

If $p_4 \geq 131$, then

$$f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{11^5}\right)\left(1-\frac{1}{131^3}\right) = 0.96264\ldots. \tag{2.15}$$

Case 1.1: $p_4 = 13$. Then $\beta_4 = 1$. We observe that $D \geq 3 \cdot 11 \cdot 13$, thus by (2.12) we have $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq (\frac{32}{33} + \frac{2^4}{3\cdot 11\cdot 13}) \cdot \frac{12}{13} = 0.92953\ldots$, which contradicts (2.13).

Case 1.2: $p_4 = 23, 47, 59, 71, 83, 101, 107$ or 131. Then $\beta_1 = 0$ and we have $D \geq 3^3 \cdot 11$. If $D = 3^3 \cdot 11$, then by (2.12) we see that

$$\begin{aligned} g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\geq \left(\frac{32}{33} + \frac{2^4}{3^3\cdot 11}\right)\cdot\frac{22}{23} = 0.97906\ldots > 1 - \frac{1}{3^3} \\ &> f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \end{aligned}$$

a contradiction. Thus $D \geq 3^3 \cdot 5 \cdot 11$. Similarly, we can show that if $p_4 \geq 59$, then $D \geq 3^3 \cdot 11^2$; if $p_4 \geq 101$, then $D \geq 3^3 \cdot 5^2 \cdot 11$.

If $p_4 = 23$, then $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.93784\ldots$, which contradicts (2.13).

If $p_4 = 47$, then for $D \geq 3^3 \cdot 11^2$, we have $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.95385\ldots$, which contradicts (2.13); if $D = 3^3 \cdot 5 \cdot 11$, then we obtain $\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 47^{\alpha_4}) = 5^{\alpha_2} 11^{\alpha_3} 47^{\alpha_4} \cdot 13 \cdot 7$. We have $5 \mid \sigma(11^{\alpha_3})$, thus $\alpha_3 + 1 \equiv 5 \pmod{10}$, hence $3221 \mid \sigma(11^{\alpha_3})$, which is impossible.

If $p_4 = 59$, then for $D \geq 3^3 \cdot 5 \cdot 11^2$, we have

$$\begin{aligned} 0.95453\ldots = f_3(2,2,2,2) &\leq f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\ &\leq 0.95344\ldots, \end{aligned}$$

a contradiction; for $D = 3^3 \cdot 11^2$, we have $\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 59^{\alpha_4}) = 5^{\alpha_2} 11^{\alpha_3 - 1} 59^{\alpha_4} \cdot 199$, which is impossible; for $D = 3^3 \cdot 5^2 \cdot 11$, we have $\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 59^{\alpha_4}) = 5^{\alpha_2 - 2} 11^{\alpha_3 + 1} 59^{\alpha_4} \cdot 41$. If $\alpha_2 = 2$, then $31 \mid \sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 59^{\alpha_4})$, which is impossible. If $\alpha_2 \geq 4$, then this is also impossible.

If $p_4 = 71$, then for $D \geq 3^3 \cdot 11^2$, we have $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.96086\ldots$, which contradicts (2.14).

If $p_4 = 83$, then for $D \geq 3^3 \cdot 5^2 \cdot 11$, we have $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.96014\ldots$, contrary to (2.14); if $D = 3^3 \cdot 11^2$, then $\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 83^{\alpha_4}) = 5^{\alpha_2} 11^{\alpha_3 - 1} 83^{\alpha_4} \cdot 199$, which is impossible.

If $p_4 = 101, 107$, then $\alpha_2 \geq 4$. For $D \geq 3^3 \cdot 11 \cdot p_4$, we have $g_3 \leq 0.96113\ldots$, which contradicts (2.14); for $D = 3^3 \cdot 5^2 \cdot 11$; $3^3 \cdot 5 \cdot 11^2$, we have

$$\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} p_4^{\alpha_4}) = 5^{\alpha_2 - 2} 11^{\alpha_3 + 1} p_4^{\alpha_4} \cdot 41 \text{ or } 5^{\alpha_2} 11^{\alpha_3} p_4^{\alpha_4 - 1} \cdot 1063.$$

If $p_4 = 107$, then $5 \mid \sigma(11^{\alpha_3})$, hence $3221 \mid \sigma(11^{\alpha_3})$, which is impossible. If $p_4 = 101$, then $101 \mid \sigma(5^{\alpha_2})$, thus $71 \mid \sigma(5^{\alpha_2})$, which is also impossible.

If $p_4 = 131$, then $\alpha_2, \alpha_3 \geq 4$. If $\alpha_2 = 4$, then $\beta_3 = 1$, thus $D \geq 3^3 \cdot 11^4$. We have

$$0.96264\ldots = f_3(2,4,4,2) \leq g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.96233\ldots,$$

a contradiction; if $\alpha_2 \geq 6$ and $D \geq 3^3 \cdot 11^3$, then

$$\begin{aligned} 0.96294\ldots = f_3(2,6,2,2) \leq f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &\leq 0.96273\ldots, \end{aligned}$$

a contradiction; if $\alpha_2 \geq 6$ and $D \leq 3^3 \cdot 5 \cdot 11^2$, then

$$g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 0.96326\ldots > 1 - 1/3^3 > f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

a contradiction.

Case 1.3: $p_4 = 37, 61, 67, 73, 97$ or 103. Then $\beta_4 = 0$. We have $D \geq 3 \cdot 11 \cdot p_4^2$.

If $p_4 = 37, 61$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \left(\frac{32}{33} + \frac{2^4}{3 \cdot 11 \cdot 61^2}\right) \cdot \frac{60}{61} = 0.95392\ldots$, which contradicts (2.13).

If $p_4 = 67$, then either $\alpha_1 = \alpha_2 = 2$, $\alpha_3 \geq 4$, or $\alpha_1 = 2$, $\alpha_2 \geq 4$. Thus

$$\begin{aligned} f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &\geq \left(1 - \frac{1}{3^3}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{11^5}\right)\left(1 - \frac{1}{67^3}\right) \\ &= 0.95525\ldots. \end{aligned} \tag{2.16}$$

If $D \geq 5 \cdot 3 \cdot 11 \cdot 67^2$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.95524\ldots$, which contradicts (2.16). If $D = 3^2 \cdot 11 \cdot 67^2$ or $3 \cdot 11 \cdot 67^2$, then $\sigma(3^2 5^{\alpha_2} 11^{\alpha_3} 67^2) = 3^{\alpha_1 - 1} 5^{\alpha_2 + 1} 11^{\alpha_3} \cdot 5387$ or $3^{\alpha_1 + 1} 5^{\alpha_2} 11^{\alpha_3} \cdot 41 \cdot 73$. We have $5 \mid \sigma(11^{\alpha_3})$, thus $3221 \mid \sigma(11^{\alpha_3})$, which is impossible.

If $p_4 = 73, 97, 103$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \left(\frac{32}{33} + \frac{2^4}{3 \cdot 11 \cdot 103^2}\right) \cdot \frac{102}{103} = 0.96032\ldots$, which contradicts (2.14).

Case 1.4: $p_4 = 17, 29, 41, 53, 89, 113, 137$. Then $\beta_1 = \beta_4 = 0$, thus $D \geq 3^3 \cdot 11 \cdot p_4^2$.

If $p_4 = 17, 29, 41, 53$, then (2.12) gives $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.95142\ldots$, which contradicts (2.13). If $p_4 = 89, 113$, then by (2.12) we see that $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.96112\ldots$, which contradicts (2.14). If $p_4 = 137$, then by (2.12) we have $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.96262\ldots$, which contradicts (2.15).

Case 1.5: $p_4 = 19$. If $D \geq 3^2 \cdot 5 \cdot 11$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.94928\ldots$, which contradicts (2.13). If $D \leq 3 \cdot 5 \cdot 11$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 1$, a contradiction. If $D = 3^3 \cdot 11$ or $3 \cdot 11^2$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4}) = 29 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4}$ or $23 \cdot 3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3 - 1} 19^{\alpha_4}$, but $29 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4})$ or $23 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4})$, a contradiction.

Case 1.6: $p_4 = 31$. If $\alpha_2 = 2$, then $\beta_4 = 1$. Thus $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.95355\ldots$, which contradicts (2.13). If $\alpha_2 \ge 4$, then

$$f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^3}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.96189\ldots. \tag{2.17}$$

If $D \ge 3 \cdot 5^2 \cdot 11$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.95718\ldots$, which contradicts (2.17). If $D \le 3 \cdot 5 \cdot 11$, then $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 1$, a contradiction. If $D = 3^3 \cdot 11, 3^2 \cdot 5 \cdot 11$ or $3 \cdot 11^2$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 31^{\alpha_4}) = 19 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} 11^{\alpha_3} 31^{\alpha_4}$, $3^{\alpha_1 - 1} 5^{\alpha_2 - 1} 11^{\alpha_3} 31^{\alpha_4 + 1}$, $23 \cdot 3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3 - 1} 31^{\alpha_4}$, respectively. We have $3 \mid \sigma(31^{\alpha_4})$, thus $\alpha_4 + 1 \equiv 3 \pmod 6$, hence $331 \mid \sigma(31^{\alpha_4})$, a contradiction.

Case 1.7: $p_4 = 43$. If $43 \nmid \sigma(11^{\alpha_3})$, then $\beta_4 = 0$, and so $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.9474\ldots$, which contradicts (2.13). If $43 \mid \sigma(11^{\alpha_3})$, then $\alpha_3 + 1 \equiv 7 \pmod{14}$, thus $\alpha_3 \ge 6$. Noting that

$$\begin{aligned} f_3(2, 2, 6, 2) \le f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \left(\frac{32}{33} + \frac{2^4}{D}\right) \cdot \frac{42}{43} < 1 - \frac{1}{3^3}, \end{aligned}$$

we obtain $988 < D < 1926$. Since $D = 3^{\alpha_1 - \beta_1 + 1} 5^{\alpha_2 - \beta_2} 11^{\alpha_3 - \beta_3 + 1} p_4^{\alpha_4 - \beta_4}$, we have $D = 3 \cdot 11 \cdot 43$ or $3^3 \cdot 5 \cdot 11$, thus $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 43^{\alpha_4}) = 29 \cdot 3^{\alpha_1 + 1} 5^{\alpha_2} 11^{\alpha_3} 43^{\alpha_4 - 1}$, $3^{\alpha_1 - 2} 5^{\alpha_2 - 1} 11^{\alpha_3} 43^{\alpha_4 - 1} \cdot 7 \cdot 13$, respectively. We have $5 \mid \sigma(11^{\alpha_3})$, thus $\alpha_3 + 1 \equiv 5 \pmod{10}$, hence $3221 \mid \sigma(11^{\alpha_3})$, a contradiction.

Case 1.8: $p_4 = 79$. Then $\alpha_2 \ge 4$. Noting that

$$\begin{aligned} f_3(2, 4, 2, 2) \le f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \left(\frac{32}{33} + \frac{2^4}{D}\right) \cdot \frac{78}{79} < 1 - \frac{1}{3^3}, \end{aligned}$$

we obtain $2851 < D < 3504$. Since $D = 3^{\alpha_1 - \beta_1 + 1} 5^{\alpha_2 - \beta_2} 11^{\alpha_3 - \beta_3 + 1} p_4^{\alpha_4 - \beta_4}$, we have $D = 3^3 \cdot 11^2$, thus $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 79^{\alpha_4}) = 199 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} 11^{\alpha_3 - 1} 79^{\alpha_4}$. We have $5 \mid \sigma(11^{\alpha_3})$, thus $\alpha_3 + 1 \equiv 5 \pmod{10}$, hence $3221 \mid \sigma(11^{\alpha_3})$, a contradiction.

Case 1.9: $p_4 = 109$. Then $\alpha_2 \ge 4$. Noting that

$$\begin{aligned} f_3(2, 4, 2, 2) \le f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \left(\frac{32}{33} + \frac{2^4}{D}\right) \cdot \frac{108}{109} < 1 - \frac{1}{3^3}, \end{aligned}$$

we obtain $7331 < D < 14027$. Since $D = 3^{\alpha_1 - \beta_1 + 1} 5^{\alpha_2 - \beta_2} 11^{\alpha_3 - \beta_3 + 1} p_4^{\alpha_4 - \beta_4}$ and $\alpha_1 = 2$, we have $D = 3^2 \cdot 5^3 \cdot 11, 3 \cdot 5^2 \cdot 11^2$ or $3^2 \cdot 11^3$, thus $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 109^{\alpha_4}) = 751 \cdot 3^{\alpha_1 - 1} 5^{\alpha_2 - 3} 11^{\alpha_3} 109^{\alpha_4}, 19 \cdot 29 \cdot 3^{\alpha_1} 5^{\alpha_2 - 2} 11^{\alpha_3 - 1} 109^{\alpha_4}, 727 \cdot 3^{\alpha_1 - 1} 5^{\alpha_2} 11^{\alpha_3 - 2}$.

109^{α_4}, respectively. We have $3 \mid \sigma(109^{\alpha_4})$, so $\alpha_4 + 1 \equiv 0 \pmod 3$, thus $571 \mid \sigma(109^{\alpha_4})$, which is a contradiction.

CASE 1.10: $p_4 = 127$. Then $\alpha_2 \geq 4$ and $(\alpha_1, \alpha_2, \alpha_3) \neq (2, 4, 2)$. If $\alpha_2 = 4$ then $\beta_3 = 1$, thus $D \geq 3 \cdot 11^4$. We have

$$\begin{aligned} 0.96264\ldots = f_3(2,4,4,2) \leq f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\ &\leq 0.96242\ldots, \end{aligned}$$

a contradiction. If $\alpha_2 \geq 6$ and $5 \nmid \sigma(11^{\alpha_3})$, then $\beta_2 = 0$, thus $D \geq 3 \cdot 5^6 \cdot 11$. We have

$$\begin{aligned} 0.96222\ldots = f_3(2,6,2,2) \leq f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\ &\leq 0.96209\ldots, \end{aligned}$$

a contradiction. If $\alpha_2 \geq 6$ and $5 \mid \sigma(11^{\alpha_3})$, then $\alpha_3 + 1 \equiv 5 \pmod{10}$, $\alpha_3 \geq 4$. As $f_3(2,6,4,2) \leq g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < 1 - 1/3^3$, we have $17610 \leq D \leq 17985$. Since $D = 3^{\alpha_1-\beta_1+1} 5^{\alpha_2-\beta_2} 11^{\alpha_3-\beta_3+1} p_4^{\alpha_4-\beta_4}$, no such D exists.

CASE 1.11: $p_4 = 139$. Then $\alpha_2 \geq 6$ and $\alpha_3 \geq 4$. Noting that

$$\begin{aligned} f_3(2,6,4,2) \leq f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\ &= \left(\frac{32}{33} + \frac{2^4}{D}\right) \cdot \frac{138}{139} < 1 - \frac{1}{3^3}, \end{aligned}$$

we have $65577 < D < 71052$. Since $D = 3^{\alpha_1-\beta_1+1} 5^{\alpha_2-\beta_2} 11^{\alpha_3-\beta_3+1} p_4^{\alpha_4-\beta_4}$, we have $D = 3^2 \cdot 5 \cdot 11 \cdot 139$. Thus $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 139^{\alpha_4}) = 3^{\alpha_1-1} 5^{\alpha_2-1} 11^{\alpha_3} 139^{\alpha_4-1} \cdot 43 \cdot 97$, but $97 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} 139^{\alpha_4})$, a contradiction.

CASE 2: $\alpha_1 \geq 4$. Then by (2.11) we have

$$f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{13^3}\right) = 0.98672\ldots. \tag{2.18}$$

If $D \geq 29 \cdot 3 \cdot 11$, then by (2.12) we have

$$g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < \frac{32}{33} + \frac{2^4}{33 \cdot 29} = 0.98641\ldots,$$

a contradiction.

If $D = 3^4 \cdot 11$ or $5^2 \cdot 3 \cdot 11$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} p_4^{\alpha_4}) = 3^{\alpha_1-3} 5^{\alpha_2+1} 11^{\alpha_3+1} p_4^{\alpha_4}$ or $17 \cdot 3^{\alpha_1+1} 5^{\alpha_2-2} 11^{\alpha_3} p_4^{\alpha_4}$. If $\alpha_2 = 2$, then $\sigma(5^2) = 31$, we get $p_4 = 31$. Since $31 \nmid \sigma(3^{\alpha_1} 11^{\alpha_3} 31^{\alpha_4})$, thus $\alpha_4 = 1$, which is impossible. If $\alpha_2 \geq 4$, then by (2.11) and (2.12) we have

$$f_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{13^3}\right) = 0.99436\ldots,$$

$$g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < \frac{32}{33} + \frac{2^4}{25 \cdot 33} = 0.98909\ldots,$$

a contradiction.

If $D = p_4 \cdot 3 \cdot 11$, $p_4 = 13, 17, 19$ or 23, then $\alpha_i = \beta_i$, $i = 1, 2, 3$, $\alpha_4 = \beta_4 + 1$. By (2.10) we have $\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1+3}5^{\alpha_2}11^{\alpha_3}13^{\alpha_4-1}$, $7 \cdot 3^{\alpha_1}5^{\alpha_2+1}11^{\alpha_3}17^{\alpha_4-1}$, $3^{\alpha_1+1}5^{\alpha_2}11^{\alpha_3}19^{\alpha_4-1} \cdot 13$ or $3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}23^{\alpha_4-1} \cdot 47$. We observe that $5 \mid \sigma(11^{\alpha_4})$, thus $\alpha_3 + 1 \equiv 5 \pmod{10}$, hence $3221 \mid \sigma(11^{\alpha_3})$, a contradiction.

If $D = 3^2 \cdot 5 \cdot 11$, then by (2.10) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}) = 31 \cdot 3^{\alpha_1-1}5^{\alpha_2-1}11^{\alpha_3}p_4^{\alpha_4}. \tag{2.19}$$

If $\alpha_3 = 2$, then $7 \mid \sigma(11^2)$, which is impossible. Thus $\alpha_3 \geq 4$. By (2.11) and (2.12) we have

$$f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{11^5}\right)\left(1 - \frac{1}{13^3}\right) = 0.98746\ldots,$$

and $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < 1$, hence $71 \leq p_4 \leq 493$. By (2.19) we have $3 \mid \sigma(p_4^{\alpha_4})$, thus $p_4 \equiv 1 \pmod 3$ and $\alpha_4 + 1 \equiv 3 \pmod 6$. If $5 \mid \sigma(11^{\alpha_3})$, then $\alpha_3 + 1 \equiv 5 \pmod{10}$; we have $3221 \mid \sigma(11^{\alpha_3})$, thus $p_4 = 3221$, but $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}3221^{\alpha_4})$, a contradiction. If $5 \nmid \sigma(11^{\alpha_3})$, then $5 \mid \sigma(p_4^{\alpha_4})$, $p_4 \equiv 1 \pmod{15}$. Therefore, $p_4 = 151, 181, 211, 241, 271, 331$ or 421. Note that we have $7 \mid \sigma(151^{\alpha_4}), \sigma(331^{\alpha_4})$; $79 \mid \sigma(181^{\alpha_4})$; $37 \mid \sigma(211^{\alpha_4})$; $19441 \mid \sigma(241^{\alpha_4})$; $24571 \mid \sigma(271^2)$; $59221 \mid \sigma(181^{\alpha_4})$. Thus (2.19) cannot hold.

If $D = 3 \cdot 11^2$, then $\alpha_3 = \beta_3 + 1$, $\alpha_i = \beta_i$, $i = 1, 2, 4$. By (2.10) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}) = 23 \cdot 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3-1}p_4^{\alpha_4}. \tag{2.20}$$

If $\alpha_2 = 2$, then $p_4 = 31$. Since $31 \nmid \sigma(3^{\alpha_1}11^{\alpha_3}31^{\alpha_4})$, we get $\alpha_4 = 1$, which is impossible. Thus $\alpha_2 \geq 4$. If $\alpha_3 = 2$, then $7 \mid \sigma(11^2)$, which is impossible. Thus $\alpha_3 \geq 4$. By (2.11) and (2.12) we have

$$f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{11^5}\right)\left(1 - \frac{1}{13^3}\right) = 0.9951\ldots,$$

and $g_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = f_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < 1$, so $59 \leq p_4 \leq 73$. By (2.20) we have $3 \mid \sigma(p_4^{\alpha_4})$, thus $p_4 \equiv 1 \pmod 3$, hence $p_4 = 61, 73$. Since $\mathrm{ord}_3(p_4)$, $\mathrm{ord}_5(p_4)$ and $\mathrm{ord}_{11}(p_4)$ are even, we have $p_4 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4})$, thus (2.20) cannot hold.

If $D = 3^3 \cdot 11$, then

$$\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}p_4^{\alpha_4}) = 19 \cdot 3^{\alpha_1}5^{\alpha_2}11^{\alpha_3-1}p_4^{\alpha_4}. \tag{2.21}$$

Similarly to the above, we know that $\alpha_2 \geq 4$ and $\alpha_3 \geq 4$, $37 \leq p_4 < 43$. By (2.21) we have $3 \mid \sigma(p_4^{\alpha_4})$, thus $p_4 \equiv 1 \pmod 3$, hence $p_4 = 37$, but $37 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}37^{\alpha_4})$, thus (2.21) cannot hold.

If $D = 5 \cdot 3 \cdot 11$, then for $p_4 = 13$ we have $\sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_3}13^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2-1}11^{\alpha_3+1}13^{\alpha_4}$. Thus $5 \mid \sigma(11^{\alpha_3})$, $\alpha_3 + 1 \equiv 5 \pmod{10}$, and consequently $3221 \mid \sigma(11^{\alpha_3})$, a contradiction. For $p_4 \geq 17$, we have

$$g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\frac{32}{33} + \frac{2^4}{5\cdot 33}\right)\left(1-\frac{1}{p_4}\right) \geq \left(\frac{32}{33} + \frac{2^4}{5\cdot 33}\right)\cdot\frac{16}{17} > 1,$$

a contradiction. If $D \leq 3^2\cdot 11$, then $g_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(\frac{32}{33}+\frac{2^4}{3^2\cdot 11}\right)\cdot\frac{12}{13} > 1$, a contradiction. ■

3. Propositions

Proposition 3.1. *There is no odd near-perfect number n of the form $n = 3^{\alpha_1}5^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}$.*

Proof. Assume that $n = 3^{\alpha_1}5^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}$ is an odd near-perfect number with redundant divisor $d = 3^{\beta_1}5^{\beta_2}p_3^{\beta_3}p_4^{\beta_4}$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}) = 2\cdot 3^{\alpha_1}5^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4} + 3^{\beta_1}5^{\beta_2}p_3^{\beta_3}p_4^{\beta_4}, \tag{3.1}$$

where $\beta_1+\beta_2+\beta_3+\beta_4 < \alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\beta_i \leq \alpha_i$, $i = 1,2,3,4$. Since $\sigma(n) \equiv 1 \pmod 2$, we see that α_i's are even. Let

$$f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(1-\frac{1}{3^{\alpha_1+1}}\right)\left(1-\frac{1}{5^{\alpha_2+1}}\right)\left(1-\frac{1}{p_3^{\alpha_3+1}}\right)\left(1-\frac{1}{p_4^{\alpha_4+1}}\right), \tag{3.2}$$

$$\begin{aligned} g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = {} & \frac{2^4(p_3-1)(p_4-1)}{3\cdot 5\cdot p_3\cdot p_4} \\ & + \frac{2^3\cdot(p_3-1)(p_4-1)}{3^{\alpha_1-\beta_1+1}5^{\alpha_2-\beta_2+1}p_3^{\alpha_3-\beta_3+1}p_4^{\alpha_4-\beta_4+1}}. \end{aligned} \tag{3.3}$$

By (3.1)–(3.3), we have $f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < 1$.

If $p_3 \geq 31$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{31}{30}\cdot\frac{37}{36} < 2,$$

which is absurd. Thus $p_3 = 7, 11, 13, 17, 19, 23$ or 29. By Lemmas 2.2, 2.4, 2.5, it is sufficient to consider $p_3 \in \{7, 13, 23, 29\}$.

Case 1: $p_3 = 7$. By (3.2) and (3.3) we have

$$\begin{aligned} f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\geq \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right) \\ &= 0.95175\ldots, \end{aligned} \tag{3.4}$$

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\frac{32}{35}+\frac{2^4}{D_1}\right)\left(1-\frac{1}{p_4}\right), \tag{3.5}$$

where $D_1 = 3^{\alpha_1-\beta_1}5^{\alpha_2-\beta_2+1}7^{\alpha_3-\beta_3+1}p_4^{\alpha_4-\beta_4}$.

Case 1.1: $D_1 \geq 13 \cdot 5 \cdot 7$. Then by (3.5) we have $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.94945\ldots$, which contradicts (3.4).

Case 1.2: $D_1 = 11 \cdot 5 \cdot 7$. Then $\alpha_i = \beta_i$, $i = 1, 2, 3$, $\beta_4 = \alpha_4 - 1$ and $p_4 = 11$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}11^{\alpha_4}) = 23 \cdot 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}11^{\alpha_4-1}. \tag{3.6}$$

Since $\mathrm{ord}_7(3) = \mathrm{ord}_7(5) = 6$, we have $7 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3})$, thus $7 \mid \sigma(11^{\alpha_4})$. Since $\mathrm{ord}_7(11) = 3$, we have $\alpha_4 + 1 \equiv 3 \pmod 6$; as $19 \mid \sigma(11^2)$, we have $19 \mid \sigma(11^{\alpha_4})$. So (3.6) cannot hold.

Case 1.3: $D_1 = 3^2 \cdot 5 \cdot 7$ or $5 \cdot 7^2$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p_4^{\alpha_4}) = 19 \cdot 3^{\alpha_1-2}5^{\alpha_2}7^{\alpha_3}p_4^{\alpha_4} \text{ or } 3^{\alpha_1+1}5^{\alpha_2+1}7^{\alpha_3-1}p_4^{\alpha_4}.$$

If $\alpha_1 = 2$, then $p_4 = 13$. Since $13 \nmid \sigma(5^{\alpha_2}7^{\alpha_3}13^{\alpha_4})$, we have $\alpha_4 = 1$, which is impossible. Thus $\alpha_1 \geq 4$. By (3.2) and (3.5) we have

$$\begin{aligned} f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &\geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right) \\ &= 0.98429\ldots, \\ g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &< \frac{32}{35} + \frac{2^4}{5 \cdot 7^2} = 0.97959\ldots, \end{aligned} \tag{3.7}$$

a contradiction.

Case 1.4: $D_1 = 5^2 \cdot 7$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p_4^{\alpha_4}) = 11 \cdot 3^{\alpha_1}5^{\alpha_2-1}7^{\alpha_3}p_4^{\alpha_4}. \tag{3.8}$$

Since $\mathrm{ord}_5(3) = \mathrm{ord}_5(7) = 4, \mathrm{ord}_7(3) = \mathrm{ord}_7(5) = 6$, and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$, we have $5 \cdot 7 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3})$, thus $5^{\alpha_2-1}7^{\alpha_3} \mid \sigma(p_4^{\alpha_4})$. If $3 \mid \sigma(7^{\alpha_3})$, then $\alpha_3 + 1 \equiv 3 \pmod 6$. Since $19 \mid \sigma(7^{\alpha_3})$, we have $p_4 = 19$, $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}19^{\alpha_4})$, thus $3 \nmid \sigma(7^{\alpha_3})$. If $11 \mid \sigma(3^{\alpha_1})$, then by $\mathrm{ord}_{11}(3) = 5$, we have $\alpha_1 + 1 \equiv 5 \pmod{10}$, $11^2 \mid \sigma(3^{\alpha_1})$, hence $p_4 = 11$. Since $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}11^{\alpha_4})$, we get $3 \mid \sigma(7^{\alpha_3})$, which is impossible. If $11 \mid \sigma(5^{\alpha_2})$, then as $\mathrm{ord}_{11}(5) = 5$, we have $\alpha_2 + 1 \equiv 5 \pmod{10}$, $11 \cdot 71 \mid \sigma(3^{\alpha_1})$, hence $p_4 = 71$. Since $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}71^{\alpha_4})$, we have $3 \mid \sigma(7^{\alpha_3})$, which is also impossible. By (3.8) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}) = p_4^{\alpha_4}, \qquad \sigma(p_4^{\alpha_4}) = 11 \cdot 3^{\alpha_1}5^{\alpha_2-1}7^{\alpha_3}.$$

Then $p_4(\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}) - 11 \cdot 3^{\alpha_1}5^{\alpha_2-1}7^{\alpha_3}) = -11 \cdot 3^{\alpha_1}5^{\alpha_2-1}7^{\alpha_3} + 1$, thus

$$-11 \cdot 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3} + 1 < \left(\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} - 11\right)3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p_4 = -\frac{141}{16} \cdot 3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p_4,$$

which is impossible.

Case 1.5: $D_1 = 3 \cdot 5 \cdot 7$. Then $\sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}7^{\alpha_3+1}p_4^{\alpha_4}$. By (3.4) and (3.5) we have $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{32}{35} + \frac{2^4}{3 \cdot 5 \cdot 7}\right)\left(1 - \frac{1}{p_4}\right) < 1$, thus $p_4 < 16$. For $p_4 = 11$, we have $3 \mid \sigma(7^{\alpha_3})$, $\alpha_3 + 1 \equiv 3 \pmod 6$. Since

$\mathrm{ord}_{19}(7) = 3$, we have $19 \mid \sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}11^{\alpha_4})$, which is impossible. For $p_4 = 13$, we have $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}7^{\alpha_3}13^{\alpha_4})$, which is also impossible.

CASE 1.6: $D_1 = 5 \cdot 7$. Then by (3.5) we have

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\frac{32}{35} + \frac{2^4}{5\cdot 7}\right)\left(1 - \frac{1}{p_4}\right) \geq \frac{48}{35}\cdot\frac{10}{11} > 1,$$

which is impossible.

CASE 2: $p_3 = 13$. By (3.3) we have

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \left(\frac{64}{65} + \frac{32}{D_2}\right)\left(1 - \frac{1}{p_4}\right), \tag{3.9}$$

where $D_2 = 3^{\alpha_1-\beta_1}5^{\alpha_2-\beta_2+1}13^{\alpha_3-\beta_3+1}p_4^{\alpha_4-\beta_4}$.

CASE 2.1: $\alpha_1 = 2$. If $p_4 \geq 47$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2}\cdot\frac{5}{4}\cdot\frac{13}{12}\cdot\frac{47}{46} < 2,$$

which is absurd. Thus $p_4 \in \{17, 19, 23, 29, 31, 37, 41, 43\}$. Note that if $p_4 \geq 37$ and $\alpha_1 = 2$, then $\alpha_2 \geq 4$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2\cdot 5^2)}{3^2\cdot 5^2}\cdot\frac{13}{12}\cdot\frac{37}{36} < 2,$$

a contradiction. Thus if $17 \leq p_4 \leq 31$, then

$$\begin{aligned} f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\geq \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^3}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{17^3}\right) \\ &= 0.95463\ldots. \end{aligned} \tag{3.10}$$

If $p_4 \geq 37$, then

$$\begin{aligned} f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &\geq \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{5^5}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{37^3}\right) \\ &= 0.96219\ldots. \end{aligned} \tag{3.11}$$

CASE 2.1.1: $p_4 \in \{17, 19, 23, 43\}$. By (3.1) we have $\beta_2 = 0$ and $\beta_3 = 1$, thus $D_2 \geq 5^3 \cdot 13^2$. If $p_4 \in \{17, 19, 23\}$, then by (3.9) we have

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \left(\frac{64}{65} + \frac{32}{5^3\cdot 13^2}\right)\cdot\frac{22}{23} = 0.94325\ldots,$$

which contradicts (3.10). If $p_4 = 43$, then from $\alpha_2 \geq 4$ we have $D_2 \geq 5^5 \cdot 13^2$. Thus

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \left(\frac{64}{65} + \frac{32}{5^5\cdot 13^2}\right)\cdot\frac{42}{43} = 0.96177\ldots,$$

which contradicts (3.11).

CASE 2.1.2: $p_4 \in \{29, 37\}$. By (3.1) we have $\beta_2 = \beta_4 = 0$, thus $D_2 \geq 5^3 \cdot 13 \cdot p_4^2$.

If $p_4 = 29$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.95068\ldots$, contrary to (3.10).

If $p_4 = 37$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.95801\ldots$, contrary to (3.11).

CASE 2.1.3: $p_4 = 31$. By (3.1) we have $\beta_3 = 1$, thus $D_2 = 3^{\alpha_1-\beta_1}5^{\alpha_2-\beta_2+1}13^{\alpha_3}31^{\alpha_4-\beta_4} \ge 5\cdot 13^2$. If $D_2 = 5\cdot 13^2$, $3\cdot 5\cdot 13^2$, $5^2\cdot 13^2$, $3^2\cdot 5\cdot 13^2$ or $3\cdot 5^2\cdot 13^2$, then $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4}) = 3^5\cdot 5^{\alpha_2}\cdot 13\cdot 31^{\alpha_4}$, $3\cdot 5^{\alpha_2}\cdot 13\cdot 31^{\alpha_4}\cdot 79$, $3^2\cdot 5^{\alpha_2-1}\cdot 13\cdot 31^{\alpha_4}\cdot 131$, $5^{\alpha_2+1}\cdot 13\cdot 31^{\alpha_4}\cdot 47$, $3\cdot 5^{\alpha_2-1}\cdot 13\cdot 31^{\alpha_4}\cdot 17\cdot 23$, respectively. We have $5 \mid \sigma(31^{\alpha_4})$, thus $\alpha_4+1 \equiv 5 \pmod{10}$, hence $11 \mid \sigma(31^{\alpha_4})$, a contradiction. If $D_2 \ge 5^3\cdot 13^2$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.95431\ldots$, which contradicts (3.10).

CASE 2.1.4: $p_4 = 41$. By (3.1) we have $\beta_4 = 0$ and $\beta_3 = 1$, thus $D_2 \ge 5\cdot 13^2\cdot 41^2$. By (3.9) we have $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 0.96062\ldots$, which contradicts (3.11).

CASE 2.2: $\alpha_1 \ge 4$. Then

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.98726\ldots. \tag{3.12}$$

If $\alpha_2 \ge 4$, then

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.99491\ldots. \tag{3.13}$$

If $\alpha_1 \ge 6, \alpha_2 \ge 4$, then

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.99856\ldots. \tag{3.14}$$

CASE 2.2.1: $D_2 \ge 5\cdot 13\cdot 191$. Then

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{64}{65} + \frac{32}{5\cdot 13\cdot 191} = 0.98719\ldots,$$

which contradicts (3.12).

CASE 2.2.2: $D_2 = 3^t\cdot 5\cdot 13$, $2 \le t \le 4$. Then $\alpha_1 = \beta_1 + t$, $\alpha_i = \beta_i$, $i = 2, 3, 4$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-t}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}(2\cdot 3^t + 1), \qquad t = 2, 3, 4.$$

If $\alpha_2 = 2$, then $\sigma(5^2) = 31$, thus $p_4 = 31$. Since $31 \nmid \sigma(3^{\alpha_1}13^{\alpha_3}31^{\alpha_4})$, we have $\alpha_4 = 1$, which is impossible. Thus $\alpha_2 \ge 4$.

If $t = 4$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) < \frac{64}{65} + \frac{32}{3^4\cdot 65} = 0.99069\ldots$, which contradicts (3.13).

If $t = 3$, then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-t}5^{\alpha_2+1}13^{\alpha_3}p_4^{\alpha_4}\cdot 11. \tag{3.15}$$

By (3.13) and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<1$, we have $127\le p_4<352$. Moreover $3\,|\,\sigma(p_4^{\alpha_4})$ and $5\,|\,\sigma(p_4^{\alpha_4})$, thus $p_4\equiv 1 \pmod{15}$. Therefore, $p_4\in\{151,181,211,241,271,331\}$. We have $7\,|\,\sigma(151^2),\sigma(331^2)$; $79\,|\,\sigma(181^2)$; $37\,|\,\sigma(211^2)$; $19441\,|\,\sigma(241^2)$; $24571\,|\,\sigma(271^2)$. Thus (3.15) cannot hold.

If $t=2$, then by (3.13) and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<1$, we have $23.4\le p_4<26$, which is impossible.

Case 2.2.3: $D_2=5\cdot 13^t$, $t=2,3$. Then $\alpha_3=\beta_3+t-1$, $\alpha_i=\beta_i$, $i=1,2,4$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})=3^{\alpha_1}5^{\alpha_2}13^{\alpha_3-t+1}p_4^{\alpha_4}(2\cdot 13^{t-1}+1),\qquad t=2,3. \tag{3.16}$$

Similarly, we have $\alpha_2\ge 4$. If $t=3$, then $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<\frac{64}{65}+\frac{32}{5\cdot 13^3}=0.98752\ldots$, which contradicts (3.13). If $t=2$, then by (3.13) and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<1$, we have $37<p_4\le 43\ldots$, thus $p_4=41,43$. We have $3\nmid\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}41^{\alpha_4})$, $5\nmid\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}43^{\alpha_4})$, thus (3.16) cannot hold.

Case 2.2.4: $D_2=3\cdot 5^t\cdot 13$, $t=2,3$. Then $\alpha_1=\beta_1+1$, $\alpha_2=\beta_2+t-1$, $\alpha_i=\beta_i$, $i=3,4$. By (3.1) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})=3^{\alpha_1-1}5^{\alpha_2-t+1}13^{\alpha_3}p_4^{\alpha_4}(2\cdot 3\cdot 5^{t-1}+1),\qquad t=2,3. \tag{3.17}$$

If $t=3$, then similarly to Case 2.2.2, we have $\alpha_2\ge 4$ and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<\frac{64}{65}+\frac{32}{3\cdot 5^3\cdot 13}=0.99117\ldots$, which contradicts (3.13). If $t=2$ and $\alpha_2\ge 4$, then by (3.13) and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<1$, we have $47\le p_4\le 53$. Thus $p_4=47,53$. We have $3\nmid\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, thus (3.17) cannot hold. If $t=2$ and $\alpha_2=2$, then we deduce that $37\le p_4\le 53$ and $3\nmid\sigma(3^{\alpha_1}5^2 13^{\alpha_3}p_4^{\alpha_4})$, thus (3.17) cannot hold.

Case 2.2.5: $D_2=5^t\cdot 13$, $t=3,4$. If $t=3$, then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})=3^{\alpha_1+1}5^{\alpha_2-2}13^{\alpha_3}p_4^{\alpha_4}\cdot 17.$$

Similarly to Case 2.2.2, we have $\alpha_2\ge 4$. By (3.13) and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<1$, we have $107\le p_4<233$. We see that $5\,|\,\sigma(p_4^{\alpha_4})$, thus $p_4\equiv 1\pmod 5$. Therefore, $p_4\in\{131,151,181,211\}$. We have $17\nmid\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, thus the above equality cannot hold.

If $t=4$, then $\alpha_2\ge 4$ and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<\frac{64}{65}+\frac{32}{5^4\cdot 13}=0.9885\ldots$, which contradicts (3.13).

Case 2.2.6: $D_2=5^2\cdot 13^2$, $3^2\cdot 5\cdot 13^2$ or $3^3\cdot 5^2\cdot 13$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})=3^{\alpha_1}5^{\alpha_2-1}13^{\alpha_3-1}p_4^{\alpha_4}\cdot 131,\,3^{\alpha_1-2}5^{\alpha_2+1}13^{\alpha_3-1}p_4^{\alpha_4}\cdot 47,\\ 3^{\alpha_1-3}5^{\alpha_2-1}13^{\alpha_3}p_4^{\alpha_4}\cdot 271.$$

Similarly to Case 2.2.2, we have $\alpha_2\ge 4$ and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4)<\frac{64}{65}+\frac{32}{5^2\cdot 13^2}=0.99218\ldots$, which contradicts (3.13).

CASE 2.2.7: $D_2 = 3\cdot 5\cdot 13^2$. Then $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3-1}p_4^{\alpha_4}\cdot 79$. Similarly to Case 2.2.2, we have $\alpha_2 \geq 4$. If $\alpha_1 = 4$, then $p_4 = 11$, which is impossible. Thus $\alpha_1 \geq 6$ and $g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < \frac{64}{65} + \frac{32}{3\cdot 5\cdot 13^2} = 0.99723\ldots$, which contradicts (3.14).

CASE 2.2.8: $D_2 = 3^2\cdot 5\cdot 13\cdot p_4$, $p_4 = 17, 19$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4-1}(2\cdot 3\cdot p_4 + 1). \tag{3.18}$$

Since $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, (3.18) cannot hold.

CASE 2.2.9: $D_2 = 5^2\cdot 13\cdot p_4$, $17 \leq p_4 \leq 37$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2-1}13^{\alpha_3}p_4^{\alpha_4-1}(2\cdot 5\cdot p_4 + 1). \tag{3.19}$$

If $p_4 \equiv 2, 3, 4 \pmod 5$, then $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, and (3.19) cannot hold.

If $p_4 = 31$, then by (3.19) we have

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2-1}13^{\alpha_3}31^{\alpha_4-1}\cdot 311.$$

Now, $5 \mid \sigma(31^{\alpha_4})$, thus $\alpha_4 + 1 \equiv 5 \pmod{10}$, hence $11 \mid \sigma(31^{\alpha_4})$, a contradiction.

CASE 2.2.10: $D_2 = 3\cdot 5\cdot 13\cdot p_4$, $17 \leq p_4 \leq 61$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4-1}(2\cdot 3\cdot p_4 + 1). \tag{3.20}$$

If $p_4 \equiv 2, 3, 4 \pmod 5$, then $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, and (3.20) cannot hold.

If $p_4 \in \{31, 41\}$, then by (3.20) we have $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4-1}\cdot 11\cdot 17$ or $3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3+1}31^{\alpha_4-1}\cdot 19$. But $17 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4})$ and $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}41^{\alpha_4})$, so (3.20) cannot hold. If $p_4 = 61$, then by (3.20) we have $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}61^{\alpha_4}) = 3^{\alpha_1-1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4-1}\cdot 367$, thus $5 \mid \sigma(61^{\alpha_4})$, $\alpha_4 + 1 \equiv 5 \pmod{10}$, $131 \mid \sigma(61^{\alpha_4})$, which is impossible.

CASE 2.2.11: $D_2 = 5\cdot 13\cdot p_4$, $17 \leq p_4 \leq 181$. Then

$$\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4-1}(2p_4 + 1). \tag{3.21}$$

If $p_4 \equiv 2, 3, 4 \pmod 5$, then $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, and (3.21) cannot hold.

If $p_4 \in \{101, 151, 181\}$, then $13 \nmid \sigma(5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4})$, thus $13 \mid \sigma(3^{\alpha_1})$, $\alpha_1 + 1 \equiv 3 \pmod 6$, hence $\alpha_1 \geq 8$. Therefore

$$g_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{64}{65} + \frac{32}{5\cdot 13\cdot 101} = 0.98948\ldots,$$

$$f_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1 - \frac{1}{3^9}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{13^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.99129\ldots,$$

a contradiction.

If $p_4 \in \{31, 41, 61\}$, then $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}p_4^{\alpha_4}) = 3^{\alpha_1+2}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4-1} \cdot 7$, $3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}41^{\alpha_4-1} \cdot 83$, $3^{\alpha_1+1}5^{\alpha_2}13^{\alpha_3}61^{\alpha_4-1} \cdot 41$. But $7 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}31^{\alpha_4})$, $41 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}41^{\alpha_4})$, $41 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}61^{\alpha_4})$, a contradiction.

If $p_4 = 71$, then $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}71^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2}13^{\alpha_3+1}71^{\alpha_4-1} \cdot 11$, and we have $5 \mid \sigma(71^{\alpha_4})$, $\alpha_4 + 1 \equiv 5 \pmod{10}$, $211 \mid \sigma(71^{\alpha_4})$, a contradiction.

If $p_4 = 131$, then $\sigma(3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}131^{\alpha_4}) = 3^{\alpha_1}5^{\alpha_2}13^{\alpha_3}131^{\alpha_4-1} \cdot 263$, and we have $5 \mid \sigma(131^{\alpha_4})$, $\alpha_4 + 1 \equiv 5 \pmod{10}$, $61 \mid \sigma(131^{\alpha_4})$, a contradiction.

Case 2.2.12: $D_2 \le 5^2 \cdot 13$. Then

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(\frac{64}{65} + \frac{32}{5^2 \cdot 13}\right) \cdot \frac{16}{17} > 1,$$

a contradiction.

Case 3: $p_3 = 23$. If $p_4 \ge 53$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{23}{22} \cdot \frac{53}{52} < 2,$$

which is absurd. Thus $p_4 \in \{29, 31, 37, 41, 43, 47\}$.

Note that $\alpha_1 \ge 4$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2} \cdot \frac{5}{4} \cdot \frac{23}{22} \cdot \frac{29}{28} < 2,$$

a contradiction. Similarly, if $p_4 \ge 37$, then $\alpha_2 \ge 4$; if $p_4 \ge 43$, then $\alpha_1 \ge 6$.

Case 3.1: $p_4 = 29$. By (3.1) we know that $\beta_1 = \beta_2 = 0$. Then

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{29^3}\right) = 0.98779\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le \frac{2^7 \cdot 7 \cdot 11}{3 \cdot 5 \cdot 23 \cdot 29} + \frac{2^6 \cdot 7 \cdot 11}{3^5 \cdot 5^3 \cdot 23 \cdot 29} = 0.98535\ldots,$$

a contradiction.

Case 3.2: $p_4 = 31$. By (3.2) we have

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.98780\ldots. \tag{3.22}$$

By (3.3),

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2^6 \cdot 11}{23 \cdot 31} + \frac{2^5 \cdot 11}{D_3}, \tag{3.23}$$

where $D_3 = 3^{\alpha_1-\beta_1}5^{\alpha_2-\beta_2}23^{\alpha_3-\beta_3+1}31^{\alpha_4-\beta_4+1}$. If $D_3 \le 23 \cdot 31^2$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge 1.003\ldots > 1$, a contradiction.

Now we consider $D_3 \ge 3^2 \cdot 5 \cdot 23 \cdot 31$.

Case 3.2.1: $\alpha_2 \geq 4$. Then by (3.2) we have

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.99545\ldots. \tag{3.24}$$

If $D_3 \geq 3 \cdot 23^2 \cdot 31$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99453\ldots$, contrary to (3.24).

If $D_3 = 3^2 \cdot 5 \cdot 23 \cdot 31$, then $\sigma(3^{\alpha_1} 5^{\alpha_2} 23^{\alpha_3} 31^{\alpha_4}) = 3^{\alpha_1 - 2} 5^{\alpha_2 - 1} 23^{\alpha_3} 31^{\alpha_4} \cdot 13 \cdot 7$. Thus $7 \mid \sigma(23^{\alpha_3})$, $\alpha_3 + 1 \equiv 3 \pmod 6$, hence $79 \mid \sigma(23^{\alpha_3})$, which is impossible.

Case 3.2.2: $\alpha_2 = 2$. Then $\beta_4 = 1$. If $5 \nmid \sigma(31^{\alpha_4})$, then $\beta_2 = 0$. Thus $D_3 \geq 5^2 \cdot 23 \cdot 31^2$. If $D_3 \geq 3 \cdot 5^2 \cdot 23 \cdot 31^2$, then $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.9875\ldots$, which contradicts (3.22). If $D_3 = 5^2 \cdot 23 \cdot 31^2$, then $\alpha_2 = \alpha_4 = 2$ and $\alpha_i = \beta_i$, $i = 1, 3$. Thus $\sigma(3^{\alpha_1} 5^2 23^{\alpha_3} 31^2) = 3^{\alpha_1 + 1} 23^{\alpha_3} \cdot 11 \cdot 31 \cdot 47$; but $331 \mid \sigma(31^2)$, a contradiction.

If $5 \mid \sigma(31^{\alpha_4})$, then $\alpha_4 + 1 \equiv 5 \pmod{10}$, $\alpha_4 \geq 4$. Thus $D_3 \geq 23 \cdot 31^4$, hence $g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.9873\ldots$, which contradicts (3.22).

Case 3.3: $p_4 = 37$. By (3.1) we know $\beta_2 = \beta_4 = 0$. By (3.2) and (3.3),

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{37^3}\right) = 0.99546\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^7 \cdot 3 \cdot 11}{5 \cdot 23 \cdot 37} + \frac{2^6 \cdot 3 \cdot 11}{5^5 \cdot 23 \cdot 37^3} = 0.99271\ldots,$$

a contradiction.

Case 3.4: $p_4 = 41$. By (3.1) we know $\beta_1 = \beta_4 = 0$. By (3.2) and (3.3),

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{41^3}\right) = 0.99546\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^8 \cdot 11}{3 \cdot 23 \cdot 41} + \frac{2^7 \cdot 11}{3^5 \cdot 23 \cdot 41^3} = 0.99540\ldots,$$

a contradiction.

Case 3.5: $p_4 = 43$. By (3.1) we know that $\beta_2 = 0$. By (3.2) and (3.3),

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{43^3}\right) = 0.99912\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^6 \cdot 7 \cdot 11}{5 \cdot 23 \cdot 43} + \frac{2^5 \cdot 7 \cdot 11}{5^5 \cdot 23 \cdot 43} = 0.99735\ldots,$$

a contradiction.

CASE 3.6: $p_4 = 47$. By (3.1) we know $\beta_1 = \beta_2 = 0$. By (3.2) and (3.3),

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^5}\right)\left(1 - \frac{1}{23^3}\right)\left(1 - \frac{1}{47^3}\right) = 0.99913\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^6 \cdot 11}{3 \cdot 5 \cdot 47} + \frac{2^5 \cdot 11}{3^7 \cdot 5^5 \cdot 47} = 0.99858\ldots,$$

a contradiction.

CASE 4: $p_3 = 29$. If $p_4 \geq 37$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{29}{28} \cdot \frac{37}{36} < 2,$$

which is absurd. Thus $p_4 = 31$. We have $29 \nmid \sigma(3^{\alpha_1}5^{\alpha_2}29^{\alpha_3}31^{\alpha_4})$, thus $\beta_3 = 0$. Moreover $\alpha_1 \geq 6$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^4)}{3^4} \cdot \frac{5}{4} \cdot \frac{29}{28} \cdot \frac{31}{30} < 2,$$

a contradiction. Then by (3.2) and (3.3),

$$f_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^7}\right)\left(1 - \frac{1}{5^3}\right)\left(1 - \frac{1}{29^3}\right)\left(1 - \frac{1}{31^3}\right) = 0.99914\ldots,$$

$$g_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^4 \cdot 28 \cdot 30}{3 \cdot 5 \cdot 29 \cdot 31} + \frac{2^3 \cdot 28 \cdot 30}{3 \cdot 5 \cdot 29^3 \cdot 31} = 0.99725\ldots,$$

a contradiction.

This completes the proof of Proposition 3.1. ∎

PROPOSITION 3.2. *If $n = 3^{\alpha_1}7^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}$ is an odd near-perfect number, then $n = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$ with redundant divisor $d = 3^2 \cdot 7 \cdot 11^2 \cdot 19^2$.*

Proof. Assume that $n = 3^{\alpha_1}7^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}$ is an odd near-perfect number with redundant divisor $d = 3^{\beta_1}7^{\beta_2}p_3^{\beta_3}p_4^{\beta_4}$. Then

$$\sigma(3^{\alpha_1}7^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4}) = 2 \cdot 3^{\alpha_1}7^{\alpha_2}p_3^{\alpha_3}p_4^{\alpha_4} + 3^{\beta_1}7^{\beta_2}p_3^{\beta_3}p_4^{\beta_4}, \tag{3.25}$$

where $\beta_1 + \beta_2 + \beta_3 + \beta_4 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\beta_i \leq \alpha_i$, $i = 1, 2, 3, 4$, and α_i's are even. Let

$$\begin{aligned} f_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ = \left(1 - \frac{1}{3^{\alpha_1+1}}\right)\left(1 - \frac{1}{7^{\alpha_2+1}}\right)\left(1 - \frac{1}{p_3^{\alpha_3+1}}\right)\left(1 - \frac{1}{p_4^{\alpha_4+1}}\right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2^3(p_3 - 1)(p_4 - 1)}{7 \cdot p_3 \cdot p_4} \\ + \frac{2^2 \cdot (p_3 - 1)(p_4 - 1)}{3^{\alpha_1-\beta_1}7^{\alpha_2-\beta_2+1}p_3^{\alpha_3-\beta_3+1}p_4^{\alpha_4-\beta_4+1}}. \end{aligned} \tag{3.27}$$

By (3.25)–(3.27), we have $f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) < 1$. If $p_3 \geq 17$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2}\cdot\frac{7}{6}\cdot\frac{17}{16}\cdot\frac{19}{18} < 2,$$

which is contradictory. Thus $p_3 = 11$ or 13.

CASE 1: $p_3 = 11$. If $p_4 \geq 29$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{29}{28} < 2,$$

which is absurd. Thus $p_4 \in \{13, 17, 19, 23\}$. Note that if $p_4 \geq 17$, then $\alpha_1 \geq 4$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{17}{16} < 2,$$

a contradiction. Similarly, we have $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \neq (2,2,2,2)$.

CASE 1.1: $p_4 = 13$. If $\alpha_1 = 2$, then $\beta_3 = 0$, $\beta_4 = 1$. Thus

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^6\cdot 5\cdot 3}{7\cdot 11\cdot 13} + \frac{2^5\cdot 5\cdot 3}{7\cdot 11^3\cdot 13^2} = 0.95934\ldots. \tag{3.28}$$

If $\alpha_2 \geq 4$; or $\alpha_2 = 2$, $\alpha_3 \geq 4$; or $\alpha_2 = \alpha_3 = 2$, $\alpha_4 \geq 4$, then

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{13^5}\right) = 0.95943\cdots,$$

which contradicts (3.28).

If $\alpha_1 \geq 4$, then

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{13^3}\right) = 0.99178\ldots. \tag{3.29}$$

By (3.27) we have

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \frac{2^6\cdot 5\cdot 3}{7\cdot 11\cdot 13} + \frac{2^5\cdot 5\cdot 3}{D}, \tag{3.30}$$

where $D = 3^{\alpha_1-\beta_1}7^{\alpha_2-\beta_2+1}11^{\alpha_3-\beta_3+1}13^{\alpha_4-\beta_4+1}$. Then $D > 7\cdot 11\cdot 13$.

If $D \leq 7\cdot 11^2\cdot 13$, then $g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq 1.00263\ldots$, contrary to (3.29).

If $D = 7\cdot 11\cdot 13^2$, then $\alpha_i = \beta_i$, $i = 1,2,3$, $\alpha_4 = \beta_4 + 1$. By (3.25) we have $\sigma(3^{\alpha_1}7^{\alpha_2}11^{\alpha_3}13^{\alpha_4}) = 3^{\alpha_1+3}7^{\alpha_2}11^{\alpha_3}13^{\alpha_4-1}$. We have $7 \mid \sigma(11^{\alpha_3})$, thus $\alpha_3+1 \equiv 3 \pmod 6$. Since $19 \mid \sigma(11^2)$, we have $19 \mid \sigma(11^{\alpha_3})$, which is impossible.

If $D \geq 3\cdot 7^2\cdot 11\cdot 13$, then $g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.98187\ldots$, which contradicts (3.29).

CASE 1.2: $p_4 = 17$. By (3.25), we have $\beta_4 = 0$. Thus

$$f_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{17^3}\right) = 0.99203\ldots,$$

$$g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq \frac{2^8 \cdot 5}{7 \cdot 11 \cdot 17} + \frac{2^7 \cdot 5}{7 \cdot 11 \cdot 17^3} = 0.97953\ldots,$$

a contradiction.

CASE 1.3: $p_4 = 19$. Then

$$f_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq \left(1 - \frac{1}{3^5}\right)\left(1 - \frac{1}{7^3}\right)\left(1 - \frac{1}{11^3}\right)\left(1 - \frac{1}{19^3}\right) = 0.99209\ldots. \tag{3.31}$$

By (3.27) we have

$$g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{2^5 \cdot 3^2 \cdot 5}{7 \cdot 11 \cdot 19} + \frac{2^4 \cdot 3^2 \cdot 5}{D},$$

where $D = 3^{\alpha_1 - \beta_1} 7^{\alpha_2 - \beta_2 + 1} 11^{\alpha_3 - \beta_3 + 1} 19^{\alpha_4 - \beta_4 + 1}$. Then $D > 7 \cdot 11 \cdot 19$.

If $D \leq 3^3 \cdot 7 \cdot 11 \cdot 19$, then $g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq 1.00251\ldots > 1$, a contradiction.

If $D \geq 7^2 \cdot 11^2 \cdot 19$, then $g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 0.99067\ldots$, contrary to (3.31).

If $D = 7^3 \cdot 11 \cdot 19$, then $\sigma(3^{\alpha_1} 7^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4}) = 3^{\alpha_1+2} 7^{\alpha_2-2} 11^{\alpha_3+1} 19^{\alpha_4}$. If $\alpha_1 = 4$, then $\alpha_3 = 1$, which is impossible. If $\alpha_1 \geq 6$, then $f_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq 0.99573\ldots$ and $g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0.99432\ldots$, a contradiction.

If $D = 3^2 \cdot 7^2 \cdot 11 \cdot 19$, then we have the following fact: if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (4, 2, 2, 2)$, then

$$f_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > f_5(4, 2, 2, 2) = \frac{10160}{10241} = g_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

a contradiction; if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (4, 2, 2, 2)$, then $\beta_2 = 1, \beta_i = 2$ for $i = 1, 3, 4$. That is, $\sigma(3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2) = 2 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 + 3^2 \cdot 7 \cdot 11^2 \cdot 19^2$.

If $D = 3 \cdot 7 \cdot 11 \cdot 19^2$, then $\sigma(3^{\alpha_1} 7^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4}) = 3^{\alpha_1-1} 7^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4-1} \cdot 5 \cdot 23$. We have $5 \mid \sigma(11^{\alpha_3})$, hence $\alpha_3 + 1 \equiv 5 \pmod{10}$ and $3221 \mid \sigma(11^{\alpha_3})$, which is impossible.

If $D = 3 \cdot 7 \cdot 11^2 \cdot 19$, then $\sigma(3^{\alpha_1} 7^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4}) = 3^{\alpha_1-1} 7^{\alpha_2} 11^{\alpha_3-1} 19^{\alpha_4} \cdot 67$. We get $67 \mid \sigma(19^{\alpha_4})$. From $\mathrm{ord}_{67}(19) = 33$, we have $\alpha_4 + 1 \equiv 33 \pmod{66}$ and $127 \mid \sigma(19^{\alpha_4})$, which is impossible.

CASE 1.4: $p_4 = 23$. Note that $(\alpha_1, \alpha_2) \neq (4, 2)$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^4 7^2)}{3^4 7^2} \cdot \frac{11}{10} \cdot \frac{23}{22} = 1.99837\ldots < 2,$$

a contradiction. As $\mathrm{ord}_{23}(7) = \mathrm{ord}_{23}(11) = 22$, we have $23 \nmid \sigma(7^{\alpha_2} 11^{\alpha_3} 23^{\alpha_4})$.

If $23 \nmid \sigma(3^{\alpha_1})$, then $\beta_4 = 0$. If $\alpha_1 \geq 6$, or $\alpha_1 = 4$, $\alpha_2 \geq 4$, then

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^5}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{23^3}\right) = 0.99499\ldots,$$

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^5\cdot 5}{7\cdot 23} + \frac{2^4\cdot 5}{7\cdot 23^3} = 0.99472\ldots,$$

a contradiction.

If $23 \mid \sigma(3^{\alpha_1})$, then $\alpha_1 + 1 \equiv 11 \pmod{22}$, thus $\alpha_1 \geq 10$. We have

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^{11}}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{11^3}\right)\left(1-\frac{1}{23^3}\right) = 0.99624\ldots.$$

By (3.27) we also have

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \frac{2^5\cdot 5}{7\cdot 23} + \frac{2^4\cdot 5}{D},$$

where $D = 3^{\alpha_1-\beta_1}7^{\alpha_2-\beta_2+1}11^{\alpha_3-\beta_3}23^{\alpha_4-\beta_4+1}$. So $D > 7\cdot 23$.

If $D \geq 3^2\cdot 7\cdot 23^2$, then $g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.99618\ldots$, which is impossible.

If $D \leq 7^2\cdot 11\cdot 23$, then $g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq 1.00024\cdots > 1$, which is impossible.

If $D = 3^2\cdot 7\cdot 11\cdot 23$, $3\cdot 7^3\cdot 23$, $3^3\cdot 7^2\cdot 23$, $3^4\cdot 7\cdot 23$ or $7\cdot 11^2\cdot 23$, then $\sigma(3^{\alpha_1}7^{\alpha_2}11^{\alpha_3}23^{\alpha_4}) = 3^{\alpha_1-2}7^{\alpha_2}11^{\alpha_3-1}23^{\alpha_4}\cdot 197$, $3^{\alpha_1-1}7^{\alpha_2-2}11^{\alpha_3}23^{\alpha_4}\cdot 5\cdot 59$, $3^{\alpha_1-3}7^{\alpha_2-1}11^{\alpha_3}23^{\alpha_4}\cdot 379$, $3^{\alpha_1-4}7^{\alpha_2}11^{\alpha_3}23^{\alpha_4}\cdot 163$, $3^{\alpha_1+5}7^{\alpha_2}11^{\alpha_3-2}23^{\alpha_4}$, respectively. We have $3 \mid \sigma(7^{\alpha_2})$. Thus $\alpha_2 + 1 \equiv 3 \pmod 6$ and $19 \mid \sigma(7^{\alpha_2})$, which is impossible.

If $D = 7^2\cdot 23^2$, then $\sigma(3^{\alpha_1}7^{\alpha_2}11^{\alpha_3}23^{\alpha_4}) = 3^{\alpha_1}7^{\alpha_2-1}11^{\alpha_3}23^{\alpha_4-1}\cdot 17\cdot 19$; but $17 \nmid \sigma(3^{\alpha_1}7^{\alpha_2}11^{\alpha_3}23^{\alpha_4})$, a contradiction.

CASE 2: $p_3 = 13$. If $p_4 \geq 23$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{23}{22} < 2,$$

which is absurd. Thus $p_4 \in \{17, 19\}$. Note that $\alpha_1 \geq 4$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^2)}{3^2}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{17}{16} < 2,$$

a contradiction.

CASE 2.1: $p_4 = 17$. Note that $(\alpha_1,\alpha_2,\alpha_4) \neq (4,2,2)$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^4 7^2 17^2)}{3^4 7^2 17^2}\cdot\frac{13}{12} = 1.99977\ldots < 2,$$

a contradiction. Similarly, $(\alpha_1,\alpha_2,\alpha_3) \neq (4,2,2)$. Thus $\alpha_1 \geq 6$; or $\alpha_1 = 4$,

$\alpha_2 \geq 4$; or $\alpha_1 = 4$, $\alpha_2 = 2$, $\alpha_3 \geq 4$, $\alpha_4 \geq 4$. Then

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^5}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{13^5}\right)\left(1-\frac{1}{17^5}\right) = 0.99297\ldots. \tag{3.32}$$

By (3.25) we find that $\beta_2 = \beta_4 = 0$. If $13 \mid \sigma(3^{\alpha_1})$, then $\alpha_1 \equiv 2 \pmod 6$, and so $\alpha_1 \geq 8$. Thus

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{7^3}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{17^3}\right) = 0.99637\ldots,$$

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 3}{7\cdot 13\cdot 17} + \frac{2^8\cdot 3}{7^3\cdot 13\cdot 17^3} = 0.99292\ldots,$$

a contradiction. If $13 \nmid \sigma(3^{\alpha_1})$, then $\beta_3 = 0$. We have

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq \frac{2^9\cdot 3}{7\cdot 13\cdot 17} + \frac{2^8\cdot 3}{7^3\cdot 13^3\cdot 17^3} = 0.99289\ldots,$$

which contradicts with (3.32).

Case 2.2: $p_4 = 19$. Note that $\alpha_1 \geq 6$, since otherwise

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{\sigma(3^4)}{3^4}\cdot\frac{7}{6}\cdot\frac{13}{12}\cdot\frac{19}{18} < 2,$$

a contradiction. Similarly, $\alpha_2 \geq 4$, $(\alpha_1,\alpha_3) \neq (6,2)$, $(\alpha_1,\alpha_4) \neq (6,2)$, $(\alpha_3,\alpha_4) \neq (2,2)$. Thus $\alpha_1 = 6$, $\alpha_3 \geq 4$, $\alpha_4 \geq 4$; or $\alpha_1 \geq 8$, $\alpha_4 \geq 4$; or $\alpha_1 \geq 8$, $\alpha_3 \geq 4$. Then

$$f_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq \left(1-\frac{1}{3^9}\right)\left(1-\frac{1}{7^5}\right)\left(1-\frac{1}{13^3}\right)\left(1-\frac{1}{19^5}\right) = 0.999434\ldots. \tag{3.33}$$

By (3.25) we have $\beta_2 = 0$. By (3.27) we get

$$g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \frac{2^6\cdot 3^3}{7\cdot 13\cdot 19} + \frac{2^5\cdot 3^3}{D}, \tag{3.34}$$

where $D = 3^{\alpha_1-\beta_1}7^{\alpha_2+1}13^{\alpha_3-\beta_3+1}19^{\alpha_4-\beta_4+1}$. Then $D \geq 7^5\cdot 13\cdot 19$.

If $D \geq 7^5\cdot 13\cdot 19^2$, then $g_5(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \leq 0.999432\ldots$, which contradicts (3.33).

If $D = 7^5\cdot 13\cdot 19$, $3\cdot 7^5\cdot 13\cdot 19$, $3^2\cdot 7^5\cdot 13\cdot 19$ or $7^5\cdot 13^2\cdot 19$, then we have $\alpha_2 = 4$ and $\sigma(3^{\alpha_1}7^4 13^{\alpha_3}19^{\alpha_4}) = 3^{\alpha_1+1}\cdot 13^{\alpha_3}\cdot 19^{\alpha_4}\cdot 1601$, $3^{\alpha_1-1}\cdot 13^{\alpha_3}\cdot 19^{\alpha_4}\cdot 14407$, $3^{\alpha_1-2}13^{\alpha_3}19^{\alpha_4}\cdot 11\cdot 3929$, $3^{\alpha_1+1}13^{\alpha_3-1}19^{\alpha_4}\cdot 20809$, respectively. But $\sigma(7^4) = 2801$, a contradiction.

This completes the proof of Proposition 3.2. ∎

4. Proof of Theorem 1.1. Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ is an odd near-perfect number. Then $\sigma(n) = 2n + d$, where $d \mid n$ and $d < n$. If $p_1 \geq 5$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} < 2,$$

which is absurd. Thus $p_1 = 3$. If $p_2 \geq 11$, then

$$2 = \frac{\sigma(n)}{n} - \frac{d}{n} < \frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} < 2,$$

a contradiction again. Thus $p_2 \in \{5, 7\}$. By Proposition 3.1, there is no odd near-perfect number of the form $3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$. By Proposition 3.2, there is only one near-perfect number of the form $3^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$, namely, $n = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.

This completes the proof of Theorem 1.1.

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