COLLOQUIUM MATHEMATICUM

VOL. 144 2016 NO. 1

GENERAL REILLY-TYPE INEQUALITIES FOR SUBMANIFOLDS OF WEIGHTED EUCLIDEAN SPACES

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Abstract. We prove new upper bounds for the first positive eigenvalue of a family of second order operators, including the Bakry–Émery Laplacian, for submanifolds of weighted Euclidean spaces.

1. Introduction. A *weighted manifold* $(\bar{M}, \bar{g}, \bar{\mu}_f)$ is a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with a weighted volume form $\bar{\mu}_f = e^{-f} dv_{\bar{g}}$, where f is a real-valued smooth function on $\bar{M}$ and $dv_{\bar{g}}$ is the Riemannian volume form associated with the metric $\bar{g}$. In the present note, we will focus on the case where $(\bar{M}, \bar{g})$ is the Euclidean space $(\mathbb{R}^N, \mathrm{can})$ with its canonical flat metric, and we will consider isometric immersions of Riemannian manifolds (M^n, g) into $(\mathbb{R}^N, \mathrm{can})$. For such an immersion, we define the *weighted mean curvature vector* $\mathbf{H}_f = \mathbf{H} - (\overline{\nabla} f)^\perp$, where $\mathbf{H}$ is the mean curvature vector of the immersion and $(\overline{\nabla} f)^\perp$ is the projection of $\overline{\nabla} f$ on the normal bundle $T^\perp M$.

We can define on M a divergence and a Laplace operator associated with the volume form $\mu_f = e^{-f} dv_g$ by

$$\mathrm{div}_f(Y) = \mathrm{div}(Y) - \langle \nabla f, Y \rangle \quad \text{and} \quad \Delta_f u = -\mathrm{div}_f(\nabla u) = \Delta u + \langle \nabla f, \nabla u \rangle,$$

where ∇ is the gradient on M, that is, the projection of $\overline{\nabla}$ on TM. We call them the *f-divergence* and the *f-Laplacian*; the latter is often called the Bakry–Émery Laplacian, Witten Laplacian or drifting Laplacian in the literature. It is a classical fact that Δ_f has a discrete spectrum composed of an infinite sequence of non-negative real numbers

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

The eigenvalue $\lambda_0 = 0$ has multiplicity one and corresponds to constant functions.

2010 *Mathematics Subject Classification*: 53C42, 53C21.
Key words and phrases: weighted manifolds, drifting Laplacian.
Received 9 April 2015; revised 7 December 2015.
Published online 4 March 2016.

In [4], Batista, Cavalcante and Pyo proved the following upper bound for the first positive eigenvalue of Δ_f:

$$\lambda_1(\Delta_f) \leq \frac{\int_M \|\mathbf{H}_f - \overline{\nabla} f\|^2 \mu_f}{n \operatorname{vol}_f(M)} = \frac{\int_M (\|\mathbf{H}\|^2 + \|\nabla f\|^2)\mu_f}{n \operatorname{vol}_f(M)}, \tag{1.1}$$

where $\operatorname{vol}_f(M) = \int_M \mu_f$ is the *f-volume* of M. (1.1) is a weighted version of the classical Reilly inequality (see [9])

$$\lambda_1(\Delta) \leq \frac{1}{n \operatorname{vol}(M)} \int_M \|\mathbf{H}\|^2 \, dv_g.$$

Very recently, Domingo-Juan and Miquel [6] obtained (1.1) with a more complete characterization of the equality case by the use of mean curvature flow.

The aim of this note is to give a general inequality, which contains the above one, for a larger class of f-divergence-type operators. More precisely, for a positive symmetric divergence-free $(1,1)$-tensor T, we define the operator $L_{T,f}$ by

$$L_{T,f} u = -\operatorname{div}_f(T \nabla u)$$

for any $\mathcal{C}^2$ function u on M. We prove the following theorem.

THEOREM 1.1. *weighted Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into the weighted Euclidean space $\mathbb{R}^N$ with density e^{-f}. Let S and T be two symmetric divergence-free $(1,1)$-tensors over M. Assume moreover that T is positive. Then the first positive eigenvalue of the operator $L_{T,f}$ satisfies the inequality*

$$\lambda_1(L_{T,f}) \Big(\int_M \operatorname{tr}(S) \mu_f \Big)^2 \leq \Big(\int_M \operatorname{tr}(T) \mu_f \Big) \int_M (\|H_S\|^2 + \|S \nabla f\|^2) \mu_f. \tag{1.2}$$

Moreover, if equality holds in the case $S = \operatorname{Id}$ then M is a self-shrinker for the mean curvature flow and $f_{|M} = a - \frac{1}{2} c r_p^2$, where r_p is the Euclidean distance to p, the center of mass of M. In particular, if $n = N - 1$ and $H > 0$, or $n = 2$, $N = 3$ and M is embedded and has genus 0, then M is a geodesic hypersphere.

As a corollary, we obtain a similar inequality for submanifolds of the sphere $\mathbb{S}^N$ which generalizes the corresponding inequality of [4] and [6] for the operator $L_{T,f}$ (see Corollary 4.4). We also prove a general non-weighted Reilly-type inequality (Theorem 5.1).

2. Preliminaries. Let (M^n, g) be a connected and oriented closed Riemannian manifold isometrically immersed into $\mathbb{R}^N$. We denote by X its position vector, B its second fundamental form and $\mathbf{H} = \operatorname{tr}(B)$ its mean curvature vector. For the case of hypersurfaces, we will also consider the real-valued mean curvature $H = \langle \mathbf{H}, \nu \rangle$, where ν is a unit normal vector field (H is defined up to sign, depending of the choice of ν). We denote by $\{\partial_1, \dots, \partial_N\}$ the canonical frame of $\mathbb{R}^N$ and for $k \in \{1, \dots, N\}$ by $X^k = \langle X, \partial_k \rangle$ the coordinate functions. We begin by giving the following elementary lemma.

LEMMA 2.1. *If A is a field of endomorphisms on M, then*

$$\sum_{k=1}^{N} \langle A(\nabla X^k), \nabla X^k \rangle = \operatorname{tr}(A).$$

Proof. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame of TM. It is a classical fact that $\nabla X^k = \partial_k^\top = \sum_{i=1}^n \langle \partial_k, e_i \rangle e_i$. Hence,

$$\begin{aligned} \sum_{k=1}^{N} \langle A(\nabla X^k), \nabla X^k \rangle &= \sum_{k=1}^{N} \sum_{i,j=1}^{n} \langle \partial_k, e_i \rangle \langle \partial_k, e_j \rangle \langle A e_i, e_j \rangle \\ &= \sum_{i,j=1}^{n} \Big(\sum_{k=1}^{N} \langle \partial_k, e_i \rangle \langle \partial_k, e_j \rangle \Big) \langle A e_i, e_j \rangle \\ &= \sum_{i,j=1}^{n} \langle e_i, e_j \rangle \langle A e_i, e_j \rangle = \operatorname{tr}(A). \ \blacksquare \end{aligned}$$

Note that, in particular, for $A = \mathrm{Id}$, we recover the well-known identity $\sum_{k=1}^{N} \|\nabla X^k\|^2 = n$.

Now, we recall briefly some basic facts about the f-divergence. We first have the weighted version of the divergence theorem:

$$\int_M \operatorname{div}_f(Y) \mu_f = 0 \tag{2.1}$$

for any vector field Y on M. From this, we easily deduce the integration by parts formula

$$\int_M u \operatorname{div}_f(Y) \mu_f = -\int_M \langle \nabla u, Y \rangle \mu_f \tag{2.2}$$

for any smooth function u and any vector field Y on M.

Now, let T be a divergence-free symmetric $(1,1)$-tensor. We associate with T the second order differential operator L_T defined by $L_T u := -\operatorname{div}(T \nabla u)$ for any $\mathcal{C}^2$ function u on M. We also associate with T the normal vector field

$$H_T = \sum_{i,j=1}^{n} \langle T e_i, e_j \rangle B(e_i, e_j), \tag{2.3}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of TM. We defined in Section 1 a corresponding weighted operator by $L_{T,f} u = -\operatorname{div}_f(T \nabla u)$ for any $\mathcal{C}^2$ function u. We have the following weighted Hsiung–Minkowski formula.

LEMMA 2.2. *We have*

$$\int_M (\langle X, H_T - T\nabla f\rangle + \mathrm{tr}(T))\mu_f = 0.$$

Proof. First, it is well known that $L_T X = -H_T$. The proof is standard and completely analogous to the case $T = \mathrm{Id}$, that is, $\Delta X = -n\mathbf{H}$, and uses the fact that $\mathrm{div}(T) = 0$. From this, we deduce

$$\begin{aligned} L_T\|X\|^2 &= \sum_{k=1}^{N} L_T((X^k)^2) = -2\sum_{k=1}^{N}\mathrm{div}(X^k T(\nabla X^k)) \\ &= 2\sum_{k=1}^{N}\big(X^k L_T X^k - \langle \nabla X^k, T(\nabla X^k)\rangle\big) \\ &= -2\langle X, H_T\rangle - 2\mathrm{tr}(T), \end{aligned} \tag{2.4}$$

where we have also used Lemma 2.1 for the last line. Therefore,

$$\begin{aligned} \tfrac{1}{2}L_{T,f}\|X\|^2 &= \tfrac{1}{2}L_T\|X\|^2 + \tfrac{1}{2}\langle T(\nabla\|X\|^2), \nabla f\rangle \\ &= -\langle X, H_T\rangle - \mathrm{tr}(T) + \tfrac{1}{2}\langle \nabla\|X\|^2, T\nabla f\rangle \\ &= -\langle H_T - T\nabla f, X\rangle - \mathrm{tr}(T), \end{aligned}$$

where we have used (2.4), the symmetry of T and the fact that $\nabla\|X\|^2 = 2X^\top$. We conclude by integrating over M for the measure μ_f and using the fact that $\int_M L_{T,f}\|X\|^2\mu_f = 0$ by (2.1). ∎

3. Proof of Theorem 1.1. Now, we have all the ingredients to prove the main theorem of this note. First, since we assume that the tensor T is positive, the operator $L_{T,f}$ has a discrete non-negative spectrum. The first eigenvalue $\lambda_0 = 0$ is of multiplicity one and the associated eigenfunctions are constants. We denote by $\lambda_1(L_{T,f})$ the first positive eigenvalue. From the definition of $L_{T,f}$ and (2.2) we have the following variational characterization of $\lambda_1(L_{T,f})$:

$$\lambda_1(L_{T,f}) = \inf\left\{ \frac{\int_M \langle T\nabla u, \nabla u\rangle\mu_f}{\int_M u^2\mu_f} \;\middle|\; u \in \mathcal{C}^\infty(M),\ \int_M u\mu_f = 0 \right\}.$$

Up to a translation if needed, we may assume that the μ_f-center of mass of M is the origin, that is, $\int_M X\mu_f = \vec{0}$. Hence, the coordinates can be used as test functions in the Rayleigh quotient and we have

$$\lambda_1(L_{T,f})\int_M \|X\|^2\mu_f \le \int_M \sum_{k=1}^{N}\langle T\nabla X^k, \nabla X^k\rangle\mu_f,$$

which gives, by Lemma 2.1,

$$\lambda_1(L_{T,f}) \int_M \|X\|^2 \mu_f \leq \int_M \mathrm{tr}(T)\mu_f. \tag{3.1}$$

Now,

$$\begin{aligned}
\lambda_1(L_{T,f})\Big(\int_M \mathrm{tr}(S)\mu_f\Big)^2 &= \lambda_1(L_{T,f})\Big(\int_M (\langle X, H_S - S\nabla f\rangle)\mu_f\Big)^2 \\
&\leq \lambda_1(L_{T,f})\Big(\int_M \|X\|^2\mu_f\Big)\Big(\int_M \|H_S - S\nabla f\|^2\mu_f\Big) \\
&\leq \Big(\int_M \mathrm{tr}(T)\mu_f\Big)\Big(\int_M \|H_S - S\nabla f\|^2\mu_f\Big),
\end{aligned}$$

where we have used successively the weighted Hsiung–Minkowski formula, the Cauchy–Schwarz inequality and (3.1). Since H_S is normal and $S\nabla f$ is tangent to M, we get the required upper bound (1.2).

Equality case. Now, we assume that $S = \mathrm{Id}$. Then (1.2) becomes

$$\lambda_1(L_{T,f}) \leq \frac{\int_M \mathrm{tr}(T)\mu_f}{n^2 \mathrm{Vol}_f(M)^2} \int_M (\|\mathbf{H}\|^2 + \|\nabla f\|^2)\mu_f.$$

If equality occurs, then all the above inequalities are equalities. In particular, equality occurs in the Cauchy–Schwarz inequality and we have $\mathbf{H} - \nabla f = cX$ for some constant c. Identifying tangential and normal parts, we get $\nabla f = -cX^\top$ and $\mathbf{H} = cX^\perp$.

The normal equation $\mathbf{H} = cX^\perp$ is exactly the definition of a self-similar solution of the mean curvature flow. Since M is a compact submanifold of $\mathbb{R}^N$, c cannot be zero. The case $c > 0$ is no more possible. Indeed, if $c > 0$, then M is a self-expander, but it is well known that there exists no compact self-expander. Hence, the only possibility is $c < 0$, that is, M is a self-shrinker.

In addition, as $X^\top = \frac{1}{2}\nabla\|X\|^2$, the tangential equation becomes $\nabla(f + \frac{1}{2}c\|X\|^2) = 0$. Since M is connected, there exists a constant a with $f_{|M} = a - \frac{1}{2}c\|X\|^2$.

In the particular cases $N = n + 1$ and $H > 0$, or $n = 2$, $N = 3$ and M embedded and of genus 0, we know from [8] and [5] respectively that M has to be a geodesic hypersphere. This finishes the proof of the equality case.

4. Some corollaries. In this section, we state some corollaries of Theorem 1.1. The first one is just a particular case of the theorem involving higher order mean curvatures. Before stating it, we recall briefly the definition of higher order mean curvatures and their associated tensors. For

$r \in \{1, \dots, n\}$, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i,i_1,\dots,i_r \\ j,j_1,\dots,j_r}} \epsilon \begin{pmatrix} i, i_1, \dots, i_r \\ j, j_1, \dots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle e_i^* \otimes e_j^*$$

if r is even, and

$$T_r = \frac{1}{r!} \sum_{\substack{i,i_1,\dots,i_r \\ j,j_1,\dots,j_r}} \epsilon \begin{pmatrix} i, i_1, \dots, i_r \\ j, j_1, \dots, j_r \end{pmatrix} \langle B_{i_1 j_1} B_{i_2 j_2} \rangle \cdots \langle B_{i_{r-1} j_{r-1}} B_{i_r j_r} \rangle B_{i_r, j_r} \otimes e_i^* \otimes e_j^*$$

if r is odd, where the B_{ij}'s are the coefficients of the second fundamental form B in a local orthonormal frame $\{e_1, \dots, e_n\}$ and ϵ is the standard signature for permutations. Here, $\{e_1^*, \dots, e_n^*\}$ is the dual coframe of $\{e_1, \dots, e_n\}$. By definition, the rth mean curvature is

$$H_r = \frac{1}{c(r)} \mathrm{tr}(T_r), \quad \text{where} \quad c(r) = (n-r) \binom{r}{n}.$$

Note that H_r is a real function if r is even, and a normal vector field if r is odd; in the latter case, we will denote it by $\mathbf{H}_r$. By convention, we set $H_0 = 1$. Moreover, if r is even, we easily show that $H_{T_r} = c(r)\mathbf{H}_{r+1}$, where H_{T_r} is given by (2.3).

In the case of hypersurfaces, we can consider the higher order mean curvatures as scalar functions also for odd indices by taking for B the real-valued second fundamental form.

By the symmetry of B, these tensors are clearly symmetric. Moreover, we have the following well-known lemma (as can be found in [7] for instance).

Lemma 4.1.

(1) *If* $n = N - 1$, *then* $\mathrm{div}(T_r) = 0$ *for any* $r \in \{0, \dots, n-1\}$.
(2) *If* $n \leq N - 2$, *then* $\mathrm{div}(T_r) = 0$ *for any even* $r \in \{0, \dots, n-1\}$.

The tensor T_r is the linearized operator associated with the rth mean curvature and plays a crucial role in the study of the r-stability of hypersurfaces with constant rth mean curvature (see [1] for instance).

The following corollary of Theorem 1.1 is immediate, since the tensors T_r are divergence-free. Note that this is a weighted version of an inequality of Alias and Malacarne [2].

Corollary 4.2. *Let* (M^n, g) *be a connected and oriented closed Riemannian manifold isometrically immersed into the weighted Euclidean space* $\mathbb{R}^N$ *with density* e^{-f}. *Let* $r, s \in \{1, \dots, n-1\}$. *Assume that* r *and* s *are even if* $N > n+1$ *and assume moreover that* T_r *is positive. Then the* *first positive eigenvalue of the operator* $L_{r,f} = L_{T_r,f}$ *satisfies the inequality*

$$\lambda_1(L_{r,f})\Big(\int_M H_s\mu_f\Big)^2 \leq \frac{c(r)}{c(s)^2}\Big(\int_M H_r\mu_f\Big)\int_M (c(s)^2\|\mathbf{H}_{s+1}\|^2 + \|T_s\nabla f\|^2)\,\mu_f.$$

REMARK 4.3. In the case of hypersurfaces, it is sufficient to have $H_{r+1} > 0$ to ensure that T_r is positive (see [3] for instance).

Now, using the embedding of the sphere $\mathbb{S}^N$ into the Euclidean space $\mathbb{R}^{N+1}$, we can prove another corollary for submanifolds of the sphere $\mathbb{S}^N$. More precisely:

COROLLARY 4.4. *Let* (M^n, g) *be a connected and oriented closed Riemannian manifold isometrically immersed into the sphere* $\mathbb{S}^N$ *endowed with a density* e^{-f}. *Let* S *and* T *be two symmetric divergence-free* $(1,1)$*-tensors over* M. *Assume moreover that* T *is positive. Then the first positive eigenvalue of the operator* $L_{T,f}$ *satisfies the inequality*

$$\lambda_1(L_{T,f})\Big(\int_M \operatorname{tr}(S)\mu_f\Big)^2 \leq \Big(\int_M \operatorname{tr}(T)\mu_f\Big)\int_M (\|H_S\|^2 + \operatorname{tr}(S)^2 + \|S\nabla f\|^2)\mu_f.$$

Proof. The proof comes easily from Theorem 1.1. We denote by ϕ the immersion of M into $\mathbb{S}^N$, we consider the canonical immersion i of $\mathbb{S}^N$ into $\mathbb{R}^{N+1}$, and we extend the weight f defined on $\mathbb{S}^N$ to a weight $\widetilde{f}$ on $\mathbb{R}^{N+1}$, for instance by taking $\widetilde{f}(x) = |x|f(x/|x|)$ for any $x \in \mathbb{S}^N$ and $\widetilde{f}(0) = 0$. From Theorem 1.1 we have

$$\lambda_1(L_{T,f})\Big(\int_M \operatorname{tr}(S)\mu_f\Big)^2 \leq \Big(\int_M \operatorname{tr}(T)\mu_f\Big)\int_M (|H'_S|^2 + |S\nabla\widetilde{f}|^2)\mu_f, \tag{4.1}$$

where H'_S is defined by $H_S = \sum_{i,j=1}^n S(e_i,e_j)B'(e_i,e_j)$ with B' the second fundamental form of the immersion of M into $\mathbb{R}^{N+1}$. Obviously, the second fundamental forms B of ϕ and B' of $i\circ\phi$ are linked by $B' = B - g\phi$. Hence, we immediately get $H'_S = H_S - \operatorname{tr}(S)\phi$. Therefore, $\|H'_S\|^2 = \|H_S\|^2 + \operatorname{tr}(S)^2$ since H_S and ϕ are orthogonal, and $\|\phi\| = 1$ since M is contained in the sphere $\mathbb{S}^N$. Inserting this in (4.1), since f coincides with $\widetilde{f}$ on M, we have $\nabla\widetilde{f} = \nabla f$ and so

$$\lambda_1(L_{T,f})\Big(\int_M \operatorname{tr}(S)\mu_f\Big)^2 \leq \Big(\int_M \operatorname{tr}(T)\mu_f\Big)\int_M (\|H_S\|^2 + \operatorname{tr}(S)^2 + \|S\nabla f\|^2)\mu_f. \ ■$$

For submanifolds of spheres, we immediately have the following corollary involving higher order mean curvatures.

COROLLARY 4.5. *Let (M^n, g) be a connected, oriented closed Riemannian manifold isometrically immersed into the sphere $\mathbb{S}^N$ endowed with a density e^{-f}. Let $r, s \in \{1, \dots, n-1\}$. Assume that r and s are even if $N > n+1$ and assume moreover that T_r is positive. Then the first eigenvalue of the operator $L_{r,f}$ satisfies*

$$\lambda_1(L_{r,f})\Big(\int_M H_s \mu_f\Big)^2 \le \frac{c(r)}{c(s)^2}\Big(\int_M H_r \mu_f\Big)\int_M \big(c(s)^2\|\mathbf{H}_{s+1}\|^2 + c(s)^2 H_s^2 + \|T_s \nabla f\|^2\big)\mu_f.$$

5. A general non-weighted inequality. In the classical case, that is, without density, the equality case can be characterized in a more rigid way. Namely, we have the following result.

THEOREM 5.1. *Let (M^n, g) be a connected, oriented closed Riemannian manifold isometrically immersed into $\mathbb{R}^N$. Assume that M is endowed with two symmetric and divergence-free $(1,1)$-tensors S and T. Assume in addition that T is positive. Then the first positive eigenvalue of the operator L_T satisfies*

$$\lambda_1(L_T)\Big(\int_M \operatorname{tr}(S)\, dv_g\Big)^2 \le \Big(\int_M \operatorname{tr}(T)\, dv_g\Big)\Big(\int_M \|H_S\|^2\, dv_g\Big). \tag{5.1}$$

Moreover, if $N > n+1$ and H_S does not vanish identically and equality occurs, then $\operatorname{tr}(S)$ and $\|H_S\|$ are non-zero constants and M is S-minimally immersed into a geodesic hypersphere of $\mathbb{R}^N$ of radius $|\operatorname{tr}(S)|/\|H_S\|$.

In particular, if $n = N-1$ and H_S does not vanish identically, then if equality holds, then $\operatorname{tr}(S)$ and H_S are non-zero constants and M is a geodesic hypersphere of radius $|\operatorname{tr}(S)|/|H_S|$.

REMARKS 5.2. (1) Note that for this theorem, in contrast to Theorem 1.1, we do not need to assume that M is embedded to characterize the equality case, the embedding being obtained as a consequence.

(2) For $T = \mathrm{Id}$, we have

$$\lambda(\Delta)\Big(\int_M \operatorname{tr}(S)\, dv_g\Big)^2 \le n \operatorname{vol}(M) \int_M \|H_S\|^2\, dv_g,$$

which was proved by Grosjean [7].

Proof of Theorem 5.1. Inequality (5.1) is immediate from Theorem 1.1 with f identically zero. If equality occurs, then all the inequalities in the proof of Theorem 1.1 become equalities. In particular, we have $H_S = cX$ from the equality case of the Cauchy–Schwarz inequality, where c is a non-zero constant. This means that the position vector X is everywhere normal to M. But, on the other hand, since $\nabla\|X\|^2 = 2X^\top$, we get $\nabla\|X\|^2 = 0$.

Hence, M being connected, $\|X\| = r$ is constant and M lies in a geodesic hypersphere of radius r. Moreover, $H_S = cX$ shows that $\|H_S\|$ is also constant, and from (2.3) we conclude that $\mathrm{tr}(S) = -\langle X, H_S\rangle = c^{-1}\|H_S\|^2$. Thus, $\mathrm{tr}(S)$ is also constant. Note that, since we assume that H_S does not vanish identically, $\mathrm{tr}(S)$ and $\|H_S\|$ are non-zero constants and we have $r = |\mathrm{tr}(S)|/\|H_S\|$.

Now, we will show that the immersion of M in the hypersphere $\mathbb{S}^{N-1}(r)$ is S-minimal, that is, $\widetilde{H}_S = 0$, where $\widetilde{H}_S$ is defined by

$$\widetilde{H}_S = \sum_{i,j=1}^{n} S(e_i, e_j)\widetilde{B}(e_i, e_j),$$

with $\widetilde{B}$ the second fundamental form of M in $\mathbb{S}^{N-1}(r)$. Clearly, $B = \widetilde{B} + \overline{B}$ where $\overline{B}$ is the second fundamental form of $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ and is given by $\overline{B}_{ij} = -r^{-2}\delta_{ij}X$. From this fact and the definition of H_S and $\widetilde{H}_S$, we get

$$\begin{aligned} H_S &= \widetilde{H}_S - \frac{1}{r^2}\sum_{i,j}^{n} S(e_i, e_j)\delta_{ij}X = \widetilde{H}_S - \frac{1}{r^2}\mathrm{tr}(S)X \\ &= \widetilde{H}_S - \frac{|H_S|^2}{\mathrm{tr}(S)}X = \widetilde{H}_S + cX = \widetilde{H}_S + H_S. \end{aligned}$$

We deduce that $\widetilde{H}_S = 0$, that is, M is S-minimally immersed into $\mathbb{S}^{N-1}(r)$.

If $n = N - 1$, then if equality occurs, by the above discussion and since M has no boundary, we deduce that M is $\mathbb{S}^{N-1}(r)$. ■

REFERENCES

[1] H. Alencar, M. do Carmo and H. Rosenberg, *On the first eigenvalue of the linearized operator of the r-th mean curvature of a hypersurface*, Ann. Global Anal. Geom. 11 (1993), 387–395.

[2] L. J. Alias and J. M. Malacarne, *On the first eigenvalue of the linearized operator of the higher order mean curvature for closed hypersurfaces in space forms*, Illinois J. Math. 48 (2004), 219–240.

[3] J. L. M. Barbosa and A. G. Colares, *Stability of hypersurfaces with constant r-mean curvature*, Ann. Global Anal. Geom. 15 (1997), 277–297.

[4] M. Batista, M. P. Cavalcante and J. Pyo, *Some isomperimetric inequalities and eigenvalue estimates in weighted manifolds*, J. Math. Anal. Appl. 419 (2014), 617–626.

[5] S. Brendle, *Embedded self-similar shrinkers of genus 0*, arXiv:1411.4640 (2014).

[6] M. C. Domingo-Juan and V. Miquel, *Reilly's type inequality for the Laplacian associated to a density related with shrinkers for MCF*, arXiv:1503.01332 (2015).

[7] J.-F. Grosjean, *Upper bounds for the first eigenvalue of the Laplacian on compact manifolds*, Pacific J. Math. 206 (2002), 93–112.

[8] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. 31 (1990), 285–299.

[9] R. C. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. 52 (1977), 525–533.

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