COLLOQUIUM MATHEMATICUM

VOL. 144 2016 NO. 1

MARKOV–KREIN TRANSFORM

BY

JACQUES FARAUT (Paris) and FAIZA FOURATI (Tunis)

Abstract. The Markov–Krein transform maps a positive measure on the real line to a probability measure. It is implicitly defined through an identity linking two holomorphic functions. In this paper an explicit formula is given. Its proof is obtained by considering boundary values of holomorhic functions. This transform appears in several classical questions in analysis and probability theory: Markov moment problem, Dirichlet distributions and processes, orbital measures. An asymptotic property for this transform involves Thorin–Bondesson distributions.

1. Introduction. A probability measure μ and a bounded positive measure ν on $\mathbb{R}$ are said to be linked by the *Markov–Krein relation* if

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa}}\,\mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\,\nu(du)\Big),$$

where $\kappa = \nu(\mathbb{R})$. The study of this relation is motivated by the following observation by Okounkov (see [O, Proposition 8.2, p. 172]). Consider the action of the orthogonal group $O(n)$ on the space $\mathrm{Sym}(n,\mathbb{R})$ of $n\times n$ real symmetric matrices, or the action of the unitary group $U(n)$ on the space $\mathrm{Herm}(n,\mathbb{C})$ of $n\times n$ Hermitian matrices. An orbit $\mathcal{O}$ for this action is determined by the eigenvalues $a_1,\dots,a_n$ of a matrix in $\mathcal{O}$. The projection of the associated orbital measure on the straight line generated by a rank one matrix is a probability measure μ on $\mathbb{R}$ which satisfies the relation

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{nd/2}}\,\mu(dt) = \prod_{i=1}^{n} \frac{1}{(z-a_i)^{d/2}},$$

where $d=1$ in the case of $\mathrm{Sym}(n,\mathbb{R})$, and $d=2$ in the case of $\mathrm{Herm}(n,\mathbb{C})$. This formula is a special case of the Markov–Krein relation with

$$\nu = \sum_{i=1}^{n} \frac{d}{2}\delta_{a_i}.$$

2010 *Mathematics Subject Classification*: Primary 44A60; Secondary 60B10, 60E10, 65D07.
Key words and phrases: Markov–Krein transform, orbital measure, Dirichlet distribution, spline distribution, Thorin–Bondesson distribution, Pólya distribution.
Received 14 April 2014; revised 21 September 2015.
Published online 9 March 2016.

DOI: 10.4064/cm6235-10-2015

The probability measures appearing in this geometric setting are generalized spline distributions we will study in Section 2. In Section 3 it will be proven that given a positive measure ν with compact support, there is a unique probability measure μ with compact support satisfying the Markov–Krein relation. Hence we get a map: to the positive measure ν the Markov–Krein transform associates the probability measure μ. We will see in Section 2 that this transform is related to the Dirichlet distributions in case ν is a discrete measure. An explicit formula for the transform is given in Section 4 by using boundary values of holomorphic functions. This formula is essentially a special case of the one obtained in [Ci]. In Section 6 we consider a sequence (ν_n) of positive measures and the sequence (μ_n) of the Markov–Krein transforms. We study the asymptotic of μ_n as $\nu_n(\mathbb{R})$ goes to infinity. The result we will establish involves Thorin–Bondesson distributions (or extended generalized gamma convolutions, EGGC), a class of probability measures introduced by Thorin [T1], [T2] (see also [B]).

The Markov–Krein transform shows up in several questions of classical analysis. We have mentionned its relation to orbital measures. It appears in the solution of the Markov moment problem by Krein and Nudel'man [Kr]. It plays a central role in the theory of Dirichlet processes. See [Ci], [J]. A large part of the book by Kerov [Ke] is devoted to the Markov–Krein correspondence in the framework of the asymptotic analysis for representations of the symmetric group. It has been a source of inspiration for our work.

2. The generalized spline distributions $M_n(a;\tau)$. We recall definitions and results from [F1]. For $\tau=(\tau_1,\dots,\tau_n)\in(\mathbb{R}_+^*)^n$ $(n\geq 2)$, the *Dirichlet distribution* $D_n^{(\tau)}$ is the probability measure on the simplex

$$\varDelta_{n-1}=\{u=(u_1,\dots,u_n)\in\mathbb{R}^n\mid u_i\geq 0,\, u_1+\dots+u_n=1\}$$

defined by

$$\int_{\varDelta_{n-1}} f(u)\,D_n^{(\tau)}(du)=\frac{1}{C_n(\tau)}\int_{\varDelta_{n-1}} f(u)u_1^{\tau_1-1}\dots u_n^{\tau_n-1}\,\alpha(du),$$

where α is the uniform probability measure on $\varDelta_{n-1}$, i.e. the normalized restriction to $\varDelta_{n-1}$ of the Lebesgue measure on the hyperplane $u_1+\dots+u_n=1$, and

$$C_n(\tau)=\int_{\varDelta_{n-1}} u_1^{\tau_1-1}\dots u_n^{\tau_n-1}\,\alpha(du).$$

The evaluation of the constant $C_n(\tau)$ gives

$$C_n(\tau)=(n-1)!\frac{\varGamma(\tau_1)\dots\varGamma(\tau_n)}{\varGamma(|\tau|)},$$

where $|\tau|=\tau_1+\dots+\tau_n$.

For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, with $a_1 \le \dots \le a_n$, the probability measure $M_n(a;\tau)$ on $\mathbb{R}$ is the image of the Dirichlet distribution $D_n^{(\tau)}$ by the map

$$\Delta_{n-1} \to \mathbb{R}, \qquad u \mapsto a_1u_1 + \dots + a_nu_n,$$

i.e., for a continuous function F on $\mathbb{R}$,

$$\int_{\mathbb{R}} F(t)\, M_n(a;\tau;dt) = \int_{\Delta_{n-1}} F(a_1u_1 + \dots + a_nu_n)\, D_n^{(\tau)}(du).$$

The support of $M_n(a;\tau)$ is compact, $\operatorname{supp}(M_n(\tau;a)) \subset [a_1, a_n]$. If $\tau_1 = \dots = \tau_n = 1$, then $M_n(a;\tau)$ is a *spline distribution* (see [Cu]). For $\tau_i > 0$, we will say that $M_n(a;\tau)$ is a *generalized spline distribution.*

For instance, for $n = 2$,

$$\int_{\mathbb{R}} F(t)\, M_2(a;\tau;dt) = \frac{\Gamma(\tau_1+\tau_2)}{\Gamma(\tau_1)\Gamma(\tau_2)} \int_0^1 F\big(a_1(1-u) + a_2u\big)(1-u)^{\tau_1-1}u^{\tau_2-1}\, du.$$

By the change of variable $t = a_1(1-u) + a_2u$ we get

$$\int_{\mathbb{R}} F(t)\, M_2(a;\tau;dt) = \frac{(a_2-a_1)^{-(\tau_1+\tau_2-1)}}{B(\tau_1,\tau_2)} \int_{a_1}^{a_2} F(t)(t-a_1)^{\tau_2-1}(a_2-t)^{\tau_1-1}\, dt.$$

We define the function $\log z$ on $\mathbb{C} \setminus]-\infty, 0]$ and, for $\alpha \in \mathbb{C}$, the function z^α as follows: if $z = re^{i\theta}$, with $r > 0$, $-\pi < \theta < \pi$, then $\log z = \log r + i\theta$ and $z^\alpha = e^{\alpha \log z} = r^\alpha e^{i\alpha\theta}$.

THEOREM 2.1. *The probability measure $M_n(a;\tau)$ satisfies the relation*

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{|\tau|}}\, M_n(a;\tau;dt) = \prod_{i=1}^n \left(\frac{1}{z-a_i}\right)^{\tau_i}$$

for $z \in \mathbb{C} \setminus]-\infty, a_n]$.

(See [F1, Theorem 3.1].) This is a special case of the Markov–Krein relation we will consider in the next section.

3. The Markov–Krein transform. Let ν be a nonzero positive measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \log(1+|u|)\, \nu(du) < \infty,$$

and μ a probability measure on $\mathbb{R}$. We say that the measures μ and ν are linked by the *Markov–Krein relation* if, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\, \mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\, \nu(du)\Big),$$

where $\kappa = \nu(\mathbb{R})$, the total measure of ν. By Theorem 2.1, the measures $\mu = M_n(\tau; a)$ and

$$\nu = \sum_{i=1}^{n} \tau_i \delta_{a_i}$$

are linked by the Markov–Krein relation. In fact, in this case, the relation becomes

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\, \mu(dt) = \prod_{i=1}^{n} \left(\frac{1}{z-a_i}\right)^{\tau_i}, \qquad \kappa = \tau_1 + \cdots + \tau_n.$$

Let us assume that the measures μ and ν are compactly supported, and denote by h_m and p_m their moments:

$$h_m = \int_{\mathbb{R}} t^m\, \mu(dt), \qquad p_m = \int_{\mathbb{R}} t^m\, \nu(dt).$$

(Observe that $\kappa = \nu(\mathbb{R}) = p_0$.) Being compactly supported, μ and ν are determined by the sequences of their moments. Hence, by expanding in power series both sides of the Markov–Krein relation, one obtains:

Proposition 3.1. *The measures μ and ν are linked by the Markov–Krein relation if and only if the moments h_m and p_m of μ and ν satisfy*

$$\sum_{m=0}^{\infty} \frac{(\kappa)_m}{m!} h_m w^m = \exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m} w^m\right)$$

for sufficiently small w. It follows that h_m can be written as a polynomial in $p_1, \dots, p_m$:

$$h_m = \frac{m!}{(\kappa)_m} \sum_{k=1}^{m} \frac{1}{k!} \sum_{\alpha_i \geq 1,\, \alpha_1 + \cdots + \alpha_k = m} \frac{p_{\alpha_1}}{\alpha_1} \cdots \frac{p_{\alpha_k}}{\alpha_k}.$$

(Recall the *Pochhammer symbol* $(\kappa)_m = \kappa(\kappa+1)\dots(\kappa+m-1)$.)

Theorem 3.2. *For a given nonzero positive measure ν on $\mathbb{R}$ with compact support, there is a unique probability measure μ with compact support such that ν and μ are linked by the Markov–Krein relation.*

By definition the *Markov–Krein transform* is the map which associates to the positive measure ν the probability measure μ. (Theorem 3.2 can also be obtained from an explicit inversion formula of the transform, a special case of [Ci, Theorem 1].)

Proof of Theorem 3.2. If the measure μ exists, it is unique, since, by Proposition 3.1, the moments of μ are determined by those of ν.

Assume $\operatorname{supp}(\nu) \subset [a, b]$. There is a sequence $\nu^{(n)}$ of measures with finite support in $[a, b]$,

$$\nu^{(n)} = \sum_{i=1}^{n} \tau_i^{(n)} \delta_{a_i^{(n)}},$$

which converges weakly to ν. By Theorem 2.1 the measures $\nu^{(n)}$ and $\mu^{(n)} = M_n(\tau^{(n)}; a^{(n)})$ are linked by the Markov–Krein relation. The moment $p_m^{(n)}$ of ν_n converges to the corresponding moment p_m of ν. Observe that $h_m^{(0)} = 1$, and, for $m \geq 1$, by Proposition 3.1, the moments $h_m^{(n)}$ have limits h_m. The numbers h_m are moments of a probability measure μ, and μ is the weak limit of $\mu^{(n)}$. Furthermore, μ and ν are linked by the Markov–Krein relation. ∎

4. An explicit formula for the Markov–Krein transform. We first recall the definition of hyperfunctions of one variable and some of their elementary properties (see for instance [M]). Let $U \subset \mathbb{R}$ be open and $W \subset \mathbb{C}$ a complex open neighborhood of U with $W \cap \mathbb{R} = U$. The space $\mathcal{B}(U)$ of *hyperfunctions* on U is defined as

$$\mathcal{B}(U) = \mathcal{O}(W \setminus U)/\mathcal{O}(W),$$

where, for $V \subset \mathbb{C}$ open, $\mathcal{O}(V)$ is the space of holomorphic functions on V. For $F \in \mathcal{O}(W \setminus U)$, the equivalence class of F is denoted by $[F]$. Define

$$F^+ = \begin{cases} F & \text{on } W^+, \\ 0 & \text{on } W^-, \end{cases} \qquad F^- = \begin{cases} 0 & \text{on } W^+, \\ -F & \text{on } W^-. \end{cases}$$

($W^{\pm} = \{z \in W \mid \pm \operatorname{Im} z > 0\}$.) The hyperfunctions $[F^+]$ and $[F^-]$ are denoted by $F(x + i0)$ and $F(x - i0)$, and called the *boundary values* of F. Hence

$$[F] = F(x + i0) - F(x - i0).$$

Intuitively $[F]$ is the jump of F along U. A hyperfunction $f \in \mathcal{B}(U)$ *vanishes* on an open set $U_0 \subset U$ if there is a representative F of f which is holomorphic on $(W \setminus U) \cup U_0$. The *support* $\operatorname{supp}(f)$ of the hyperfunction $f \in \mathcal{B}(U)$ is the smallest closed set $C \subset U$ such that f vanishes on $U \setminus C$. The space of hyperfunctions on U with support contained in C is denoted by $\mathcal{B}_C(U)$.

Recall the space $\mathfrak{A}(K)$ of holomorphic functions in a neighborhood of the compact set $K \subset \mathbb{R}$:

$$\mathfrak{A}(K) = \bigcup_{U \supset K} \mathcal{O}(U),$$

where U is a complex open neighborhood of K. The space $\mathfrak{A}(K)$ is endowed with the inductive limit topology. An *analytic functional* on K is a continuous linear form on $\mathfrak{A}(K)$, and the space of analytic functionals on K is denoted by $\mathfrak{A}'(K)$. The *Cauchy transform* G_T of $T \in \mathfrak{A}'(K)$, defined by

$$G_T(z) = \left\langle T_t, \frac{1}{z-t} \right\rangle,$$

is holomorphic on $\mathbb{C} \setminus K$, and defines a hyperfunction $[G_T]$. The map $\Phi : T \mapsto f = [G_T]$ is an isomorphism from $\mathfrak{A}'(K)$ onto $\mathcal{B}_K(\mathbb{R})$. It follows that the space $\mathcal{D}'_K$ of distributions supported in K can be seen as a subspace of $\mathcal{B}_K(\mathbb{R})$.

Let $U \subset \mathbb{R}$ be open, and $\varepsilon > 0$. A function F defined on

$$\{z = x + iy \mid x \in U,\, 0 < |y| < \varepsilon\}$$

is said to be *of moderate growth along* U if, for every $K \subset U$ compact, there is a constant $C > 0$ and an integer $N > 0$ such that

$$|F(x+iy)| \le \frac{C}{|y|^N} \qquad (x \in K,\, 0 < |y| < \varepsilon).$$

Let $T \in \mathfrak{A}'(K)$, let $f \in \mathcal{B}_K(\mathbb{R})$ be its image by the isomorphism Φ, and F a representative of f. Then T is a distribution if and only if F is of moderate growth along $\mathbb{R}$. In such a case, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0,\, \varepsilon > 0} \int_{\mathbb{R}} (F(t+i\varepsilon) - F(t-i\varepsilon))\varphi(t)\, dt.$$

Furthermore $\operatorname{supp}(T) = \operatorname{supp}(f)$. For a compactly supported distribution T, the classical Cauchy–Stieltjes formula can be written $[G_T] = -2i\pi T$.

The distribution Y_α is defined, for $\operatorname{Re}\alpha > 0$, by

$$\langle Y_\alpha, \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty \varphi(t) t^{\alpha-1}\, dt \qquad (\varphi \in \mathcal{D}(\mathbb{R})),$$

and admits an analytic continuation for $\alpha \in \mathbb{C}$. These distributions Y_α satisfy

$$Y_\alpha * Y_\beta = Y_{\alpha+\beta}, \qquad Y_0 = \delta, \qquad Y_{-m} = \delta^{(m)} \quad (m \in \mathbb{N}).$$

In particular $Y_\alpha * Y_{-\alpha} = \delta$.

Recall that, for $\alpha \in \mathbb{C}$, the holomorphic function z^α in $\mathbb{C} \setminus\,]-\infty, 0]$ is defined as follows: if $z = re^{i\theta}$ with $r > 0$, $-\pi < \theta < \pi$, then $z^\alpha = r^\alpha e^{i\alpha\theta}$. The function z^α is of moderate growth along $\mathbb{R}$, and

$$[z^\alpha] = -2i\pi \frac{1}{\Gamma(-\alpha)} \check{Y}_{\alpha+1}.$$

(For a distribution T on $\mathbb{R}$, $\check{T}$ is the image of T by the symmetry $t \mapsto -t$: $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$ with $\check{\varphi}(t) = \varphi(-t)$.) In particular, for $m \in \mathbb{N}$, $[z^m] = 0$, and for $m \ge 1$,

$$[z^{-m}] = -2i\pi \frac{1}{(m-1)!} \delta^{(m-1)}.$$

We will now give an explicit formula for the Markov–Krein transform. Let ν be a positive measure on $\mathbb{R}$ with compact support, $\kappa = \nu(\mathbb{R})$. Recall that the Markov–Krein transform μ of ν is the unique probability measure μ such that

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\,\mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\,\nu(du)\Big)$$

(Theorem 3.2). Furthermore, the support of μ is compact.

THEOREM 4.1. *Let q be the holomorphic function defined on $\mathbb{C}\setminus\mathbb{R}$ by*

$$q(z) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\,\nu(du)\Big).$$

Then q is of moderate growth, and

$$\mu = -\frac{1}{2i\pi}\Gamma(\kappa)\check{Y}_{\kappa-1} * [q].$$

Observe that, if $\kappa = 1$, then one obtains the classical Cauchy–Stieltjes formula $\mu = -\frac{1}{2i\pi}[q]$.

LEMMA 4.2. *Let f be a holomorphic function on $\mathbb{C}\setminus\mathbb{R}$, and μ a measure on $\mathbb{R}$ with compact support. Then the function F defined by*

$$F(z) = \int_{\mathbb{R}} f(z-t)\,\mu(dt)$$

is holomorphic on $\mathbb{C}\setminus\mathbb{R}$. If f is of moderate growth along $\mathbb{R}$, then F is of moderate growth as well, and the distributions $[f]$ and $[F]$ satisfy

$$[F] = [f] * \mu.$$

Proof. If f is of moderate growth along $\mathbb{R}$, since μ is compactly supported, an easy estimate shows that F is of moderate growth as well. Then, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0,\,\varepsilon>0} \int_{\mathbb{R}} \big(F(t+i\varepsilon) - F(t-i\varepsilon)\big)\varphi(t)\,dt \\
&\qquad = \lim_{\varepsilon\to 0,\,\varepsilon>0} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \big(f(t+i\varepsilon-s) - f(t-i\varepsilon-s)\big)\,\mu(ds)\Big)\varphi(t)\,dt \\
&\qquad = \int_{\mathbb{R}} \Big(\lim_{\varepsilon\to 0,\,\varepsilon>0} \int_{\mathbb{R}} \big(f(t+i\varepsilon) - f(t-i\varepsilon)\big)\varphi(t-s)\,dt\Big)\,\mu(ds).
\end{aligned}$$

This equality means that $[F] = [f] * \mu$. Let us explain why it is possible to interchange the limit and the integration. In fact, for $0 < \varepsilon < \varepsilon_0$ and $|s| \le A$, there is a constant C such that

$$\Big|\int_{\mathbb{R}} \big(f(t+i\varepsilon) - f(t-i\varepsilon)\big)\varphi(t-s)\,dt\Big| \le C,$$

since the distribution T_ε defined by

$$\langle T_\varepsilon, \varphi\rangle = \int_{\mathbb{R}} \big(f(t+i\varepsilon) - f(t-i\varepsilon)\big)\varphi(t)\,dt$$

converges as $\varepsilon \to 0$, and the set of functions $\varphi(\cdot - s)$ is bounded in $\mathcal{D}(\mathbb{R})$ for $|s| \le A$. ∎

Proof of Theorem 4.1. The Markov–Krein relation can be written

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\,\mu(dt) = q(z).$$

(This means that q is a generalized Stieltjes transform of μ.) By Lemma 4.2 the function q is of moderate growth along $\mathbb{R}$, and

$$[z^{-\kappa}] * \mu = [q].$$

We saw that

$$[z^{-\kappa}] = -2i\pi \frac{1}{\Gamma(\kappa)} \check{Y}_{1-\kappa}.$$

Therefore, since $\check{Y}_{\kappa-1} * \check{Y}_{1-\kappa} = \delta$,

$$\mu = -\frac{1}{2i\pi}\Gamma(\kappa)\check{Y}_{\kappa-1} * [q].$$

(Recall that, if distributions T_1, T_2, T_3 have supports bounded from above, the following associativity holds: $(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3)$.) ∎

The logarithmic potential of the measure ν is defined on $\mathbb{R}$ by

$$U^\nu(x) = \int_{\mathbb{R}} \log\frac{1}{|x-u|}\,\nu(du),$$

with values in $]-\infty, \infty]$.

THEOREM 4.3. *If* $\exp U^\nu$ *is locally integrable and* $\kappa = \nu(\mathbb{R}) \ge 1$, *then the probability measure* μ *has a density* h. *Define*

$$g(x) = \frac{1}{\pi}\sin\big(\pi\nu(]x,\infty[)\big)\exp U^\nu(x).$$

(i) *If* $\kappa = 1$, *then* $h(x) = g(x)$.
(ii) *If* $\kappa > 1$, *then*

$$h(x) = (\kappa - 1)\int_x^\infty (s-x)^{\kappa-2} g(s)\,ds.$$

This formula can be obtained from one in [Ci] (part (ii) of Theorem 1, with $\tau = -\infty$, $A(-\infty) = 0$, $\alpha^* = \kappa - 1$). The proof there is obtained by using results of Widder and Hirschman about generalized Stieltjes transforms.

Proof of Theorem 4.3. By Theorem 4.1 all we have to show is that the distribution $-\frac{1}{2i\pi}[q]$ is defined by the locally integrable function g. Let

$$H(z) = \int_{\mathbb{R}} \log \frac{1}{z-u}\,\nu(du).$$

The function $\log z$ can be written

$$\log z = \log|z| + i\,\mathrm{Arg}(z),$$

and

$$\lim_{\varepsilon\to 0,\,\varepsilon>0} \log(x \pm i\varepsilon) = \begin{cases} \log|x| & \text{if } x > 0, \\ \log|x| \pm i\pi & \text{if } x < 0. \end{cases}$$

It follows that

$$\lim_{\varepsilon\to 0,\,\varepsilon>0} H(x \pm i\varepsilon) = U^\nu(x) \mp i\pi\nu([x,\infty[)$$

and

$$\begin{aligned} -\frac{1}{2i\pi} & \lim_{\varepsilon\to 0,\,\varepsilon>0} \big(q(x+i\varepsilon) - q(x-i\varepsilon)\big) \\ &= -\frac{1}{2i\pi} \lim_{\varepsilon\to 0,\,\varepsilon>0} \big(\exp H(x+i\varepsilon) - \exp H(x-i\varepsilon)\big) \\ &= -\frac{1}{2i\pi} \exp U^\nu(x) \big(e^{-i\pi\nu([x,\infty[)} - e^{i\pi\nu([x,\infty[)}\big) \\ &= \frac{1}{\pi} \exp U^\nu(x) \sin\big(\pi\nu([x,\infty[)\big) = g(x). \end{aligned}$$

To prove that this limit holds in the distribution sense one observes that, for $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$\begin{aligned} |q(x \pm i\varepsilon)| &= \exp\Big(-\int_{\mathbb{R}} \log\sqrt{(x-u)^2+\varepsilon^2}\,\nu(du)\Big) \\ &\le \exp\Big(-\int_{\mathbb{R}} \log|x-u|\,\nu(du)\Big) = \exp U^\nu(x). \ \blacksquare \end{aligned}$$

Examples. Assume the measure ν is discrete,

$$\nu = \sum_{i=1}^n \tau_i \delta_{a_i} \qquad (a_1 < \cdots < a_n,\ n \ge 3).$$

Then its Markov–Krein transform is $M_n(a_1, \ldots, a_n; \tau_1, \ldots, \tau_n)$. In that case

$$q(z) = \prod_{i=1}^n \Big(\frac{1}{z-a_i}\Big)^{\tau_i}.$$

(a) Assume $\tau_1 = \cdots = \tau_n = 1$. Then q is a rational function which can be written

$$q(z) = \sum_{i=1}^{n} c_i \frac{1}{z - a_i} \qquad \text{with } c_i = \prod_{j \neq i} \frac{1}{a_j - a_i}.$$

Therefore

$$[q] = -2i\pi \sum_{j=1}^{n} c_i \delta_{a_i}.$$

Since

$$\check{Y}_{n-1} * \delta_a = \frac{1}{(n-2)!}(a - x)_+^{n-2},$$

the measure μ has a density h given by

$$h(x) = (n-1) \sum_{a_i > x} c_i (a_i - x)^{n-2}.$$

This density is a spline function with knots $a_1, \dots, a_n$: the function h is of class $\mathcal{C}^{n-3}$, and its restriction to each interval $[a_j, a_{j+1}]$ is a polynomial of degree $\leq n-2$. In this case $M_n(a; \tau)$ is a spline distribution.

(b) Assume $0 < \tau_i < 1$ $(1 \leq i \leq n)$, $\kappa = \tau_1 + \cdots + \tau_n \geq 1$. Then the function

$$\exp U^\nu(x) = \prod_{i=1}^{n} |x - a_i|^{-\tau_i}$$

is locally integrable and

$$g(x) = \frac{1}{\pi} \sin\Big(\pi \sum_{a_i > x} \tau_i\Big) \prod_{i=1}^{n} |x - a_i|^{-\tau_i}.$$

The map

$$\nu \mapsto (\mu, \kappa), \qquad \mathcal{M}_c(\mathbb{R}) \to \mathcal{M}_c^1(\mathbb{R}) \times \mathbb{R}_+,$$

where μ is the Markov–Krein transform of ν and $\kappa = \nu(\mathbb{R})$, is injective, but not surjective. It is an open problem to determine the image of this map. In case ν is a probability measure, Kerov has the following result. He defined a *continuous diagram supported by a compact interval* $[a, b]$ to be a real function ω defined on $\mathbb{R}$ such that

$$|\omega(u_1) - \omega(u_2)| \leq |u_1 - u_2| \quad (u_1, u_2 \in \mathbb{R}),$$

and there is $c \in \mathbb{R}$ such that, for $u \notin [a, b]$,

$$\omega = |u - c|.$$

To a continuous diagram $\omega \in \mathcal{D}[a, b]$ we associate the distribution $\nu_\omega = \frac{1}{2}\omega''$ (the second derivative is taken in the distribution sense). Then $\langle \nu_\omega, 1 \rangle = 1$ and ν_ω is a probability measure if and only if ω is convex. The map $\omega \mapsto \nu''_\omega$

is injective, and if ν_ω is a measure, then

$$\omega(u) = \int_{\mathbb{R}} |u-x|\,\nu_\omega(dx).$$

By [Ke, p. 152], the map which associates to a continuous diagram $\omega \in \mathcal{D}[a,b]$ the Markov transform μ of ν_ω, determined by the relation

$$\int_{[a,b]} \frac{1}{z-t}\,\mu(dt) = \exp\bigl(-\langle \nu_\omega, \log(z-u)\rangle\bigr),$$

is a homeomorphism from $\mathcal{D}[a,b]$ onto the set $\mathcal{M}^1[a,b]$ of probability measures on $[a,b]$.

5. Thorin–Bondesson distributions. For $\xi \in \mathbb{R}^*$, $\tau > 0$, let $\gamma(\xi,\tau)$ denote the gamma distribution on $\mathbb{R}$ with density

$$Y(\xi u)\frac{|\xi|^\tau}{\Gamma(\tau)} e^{-\xi u}|u|^{\tau-1}.$$

(Recall the Heaviside function: $Y(t) = 1$ for $t \geq 0$ and $Y(t) = 0$ for $t < 0$.) The Fourier–Laplace transform φ of $\gamma(\xi,\tau)$ is given by

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\gamma(\xi,\tau;dt) = \left(\frac{\xi}{\xi - z}\right)^\tau.$$

It is defined for $\operatorname{Re} z < \xi$ if $\xi > 0$, and for $\operatorname{Re} z > \xi$ if $\xi < 0$, and admits a holomorphic extension to $\mathbb{C} \setminus [\xi,\infty[$ if $\xi > 0$, and to $\mathbb{C} \setminus]-\infty,\xi]$ if $\xi < 0$.

A *Thorin–Bondesson distribution* (or *extended generalized gamma convolution*, EGGC) is a probability measure μ on $\mathbb{R}$ which is a limit for the tight topology of convolution products of gamma distributions:

$$\mu = \lim_{n\to\infty} \Bigl(\prod_{i=1}^{n}\Bigr)^{*} \gamma(\xi_i^{(n)}, \tau_i^{(n)})$$

(see [T1], [T2], [B]). The set $\mathcal{T}_e$ of Thorin–Bondesson distributions is closed in the tight topology and a semigroup for the convolution. Chapter 9 in [S] is devoted to the measures in the Bondesson class, denoted BO. These measures are sub-probabilities supported by $[0,\infty[$. The probability measures in the Bondesson class are precisely the Thorin–Bondesson distributions (in our terminology) which are supported by $[0,\infty[$.

The Fourier–Laplace transform φ of

$$\gamma(\xi_1,\ldots,\xi_n;\tau_1,\ldots,\tau_n) := \gamma(\xi_1,\tau_1) * \cdots * \gamma(\xi_n,\tau_n)$$

is given by

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\gamma(\xi_1,\ldots,\xi_n;\tau_1,\ldots,\tau_n;dt) = \prod_{i=1}^{n}\left(\frac{\xi_i}{\xi_i - z}\right)^{\tau_i}.$$

It is defined for $|\mathrm{Re}\, z| < \sigma$, with $\sigma = \inf |\xi_i|$, and admits a holomorphic continuation to $\mathbb{C} \setminus]-\infty, -\sigma] \cup [\sigma, \infty[$. Let us observe that the function φ can be written

$$\varphi(z) = \exp\left(\int_{\mathbb{R}} \log\left(\frac{\xi}{\xi - z}\right) \nu(d\xi)\right),$$

with

$$\nu = \sum_{i=1}^{n} \tau_i \delta_{\xi_i}.$$

The measure $\gamma(\xi_1, \ldots, \xi_n; \tau_1, \ldots, \tau_n)$ is infinitely divisible. In fact, for $t > 0$, the measures

$$\mu_t = \gamma(\xi_1, \ldots, \xi_n; t\tau_1, \ldots, t\tau_n)$$

form a continuous semigroup of probability measures. Since a limit of infinitely divisible probability measures is infinitely divisible, every measure μ in $\mathcal{T}_e$ is infinitely divisible. Its Fourier–Laplace transform has the form

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\, \mu(dt) = e^{\psi(z)},$$

where ψ is a continuous function on $i\mathbb{R}$. Let $\mathcal{B}_e$ denote the set of continuous functions $\psi(z)$ on $i\mathbb{R}$ such that $e^{\psi(z)}$ is the Fourier–Laplace transform of a measure μ in $\mathcal{T}_e$.

THEOREM 5.1. *Let ψ be a continuous function on $i\mathbb{R}$, with $\psi(0) = 0$. The following properties are equivalent:*

(i) *ψ belongs to $\mathcal{B}_e$: For every $t > 0$, the function $e^{t\psi}$ is the Fourier–Laplace transform of a probability measure in $\mathcal{T}_e$.*
(ii) *The restriction of ψ to $i\mathbb{R}^*$ admits a holomorphic extension to $\mathbb{C}\setminus\mathbb{R}$, the derivative of which is a Pick function.*
(iii) *ψ admits the representation*

$$\psi(z) = \beta z + \gamma \frac{z^2}{2} + \int_{\mathbb{R}^*} \left(\log \frac{\xi}{\xi - z} - \frac{\xi z}{1 + \xi^2}\right) \nu(d\xi),$$

where $\beta \in \mathbb{R}$, $\gamma \geq 0$, and ν is a positive measure on $\mathbb{R}^*$ such that

$$\int_{0<|\xi|\leq 1} \log \frac{1}{|\xi|}\, \nu(d\xi) < \infty, \qquad \int_{|\xi|\geq 1} \frac{1}{\xi^2}\, \nu(d\xi) < \infty,$$

or equivalently

$$\int_{\mathbb{R}^*} \log\left(1 + \frac{1}{\xi^2}\right) \nu(d\xi) < \infty.$$

Furthermore

$$\beta = \mathrm{Re}\, \psi'(i), \qquad \gamma = \lim_{y\to\infty} \frac{1}{y} \mathrm{Im}\, \psi'(iy), \qquad \nu = \frac{1}{2i\pi}[\psi'].$$

This is a reformulation of results in [B, Section 7]. By the change of variable $\xi \mapsto u = 1/\xi$, we get the representation

$$\psi(z) = \beta z + \gamma \frac{z^2}{2} - \int_{\mathbb{R}^*} \left(\log(1 - uz) + \frac{uz}{u^2+1} \right) \nu_0(du),$$

where the measure ν_0, the image of ν by this map, satisfies

$$\int_{\mathbb{R}^*} \log(1 + u^2)\, \nu_0(du) < \infty.$$

Observe that

$$\operatorname{Re} \psi(i) = -\frac{1}{2} \Big(\gamma + \int_{\mathbb{R}^*} \log(1 + u^2)\, \nu_0(du) \Big).$$

To the measure ν_0 on $\mathbb{R}^*$ we associate the bounded positive measure $\tilde{\nu}$ on $\mathbb{R}$ defined, for bounded continuous functions f on $\mathbb{R}$, by

$$\int_{\mathbb{R}} f(u)\, \tilde{\nu}(du) = \gamma f(0) + \int_{\mathbb{R}^*} f(u) \log(1 + u^2)\, \nu_0(du).$$

Noticing that

$$\lim_{u \to 0} \frac{1}{u^2} \left(\log(1 - uz) + \frac{uz}{u^2+1} \right) = -\frac{1}{2} z^2,$$

we obtain the representation

$$\psi(z) = \beta z - \int_{\mathbb{R}} \left(\log(1 - uz) + \frac{uz}{1+u^2} \right) \frac{\tilde{\nu}(du)}{\log(1 + u^2)}.$$

By slightly modifying the statement of [B, Theorem 7.1.1], one gets the following one. On the set $\mathcal{B}_e$ we consider the topology of uniform convergence on compact sets in $i\mathbb{R}$, and on the set $\mathcal{M}(\mathbb{R})$ of positive bounded measures, the tight topology.

THEOREM 5.2. *The map*

$$\mathcal{B}_e \to \mathbb{R} \times \mathcal{M}(\mathbb{R}), \qquad \psi \mapsto (\beta, \tilde{\nu}),$$

is a homeomorphism.

EXAMPLE (Symmetric stable laws). For $0 < \alpha \leq 2$, the function ψ defined on $i\mathbb{R}$ by $\psi(iy) = -|y|^\alpha$ belongs to $\mathcal{B}_e$. It extension to $\mathbb{C} \setminus \mathbb{R}$ is given by

$$\psi(z) = \begin{cases} -(-iz)^\alpha & \text{if } \operatorname{Im} z > 0, \\ -(iz)^\alpha & \text{if } \operatorname{Im} z < 0, \end{cases}$$

which is a Pick function. If $0 < \alpha < 2$, then ψ admits the representation

$$\psi(z) = \frac{\alpha}{\pi} \cos(\alpha - 1)\frac{\pi}{2} \int_{\mathbb{R}^*} \left(\log \frac{\xi}{\xi - z} - \frac{\xi z}{1 + \xi^2} \right) |\xi|^{\alpha - 1}\, d\xi.$$

If $\alpha = 2$, then $\psi(z) = z^2$. In that case $\beta = 0$, $\gamma = 2$, and $\nu = 0$.

6. An asymptotic property for the Markov–Krein transform. In this section we consider a sequence (ν_n) in $\mathcal{M}_c(\mathbb{R})$ and the sequence (μ_n) of the Markov–Krein transforms: for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} (1-zt)^{-\kappa_n}\, \mu_n(dt) = \exp\Big(\int_{\mathbb{R}} -\log(1-zu)\, \nu_n(du)\Big),$$

where $\kappa_n = \nu_n(\mathbb{R})$. We will study the convergence of (μ_n) assuming that $\kappa_n = \nu_n(\mathbb{R})$ goes to infinity.

First consider a simple example. Recall that $M_n(a_1,\ldots,a_n;\tau_1,\ldots,\tau_n)$ is the Markov–Krein transform of the discrete measure $\nu = \sum_{i=1}^n \tau_i \delta_{a_i}$.

PROPOSITION 6.1. *Fix $\xi \in \mathbb{R}^*$ and $\tau > 0$. For the tight topology we have*

$$\lim_{n\to\infty} M_2\Big(0, \frac{n}{\xi}; n, \tau\Big) = \gamma(\xi,\tau).$$

Proof. Assume $\xi > 0$. For a bounded continuous function f on $\mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} f(t)\, M_2\Big(0, \frac{n}{\xi}; n, \tau; dt\Big) &= \frac{(n/\xi)^{-(n+\tau-1)}}{B(n,\tau)} \int_0^{n/\xi} f(t)\Big(\frac{n}{\xi}-t\Big)^{n-1} t^{\tau-1}\, dt \\ &= \frac{\xi^\tau}{n^\tau \frac{\Gamma(n)\Gamma(\tau)}{\Gamma(n+\tau)}} \int_0^{n/\xi} f(t)\Big(1-\frac{t\xi}{n}\Big)^{n-1} t^{\tau-1}\, dt.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\, M_2\Big(0, \frac{n}{\xi}; n, \tau\Big)\, dt = \frac{\xi^\tau}{\Gamma(\tau)} \int_0^\infty f(t) e^{-\xi t} t^{\tau-1}\, dt. \qquad \blacksquare$$

More generally:

PROPOSITION 6.2. *Fix $\xi_1,\ldots,\xi_k \in \mathbb{R}^*$ and $\tau_1,\ldots,\tau_k > 0$. For the tight topology we have*

$$\lim_{n\to\infty} M_{k+1}\Big(0, \frac{n}{\xi_1}, \ldots, \frac{n}{\xi_k}; n, \tau_1, \ldots, \tau_k\Big) = \gamma(\xi_1,\ldots,\xi_k;\tau_1,\ldots,\tau_k).$$

Proof. Let

$$\nu_n = n\delta_0 + \sum_{i=1}^k \tau_i \delta_{(n/\xi_i)}, \qquad \mu_n = M_{k+1}\Big(0, \frac{n}{\xi_1}, \ldots, \frac{n}{\xi_k}; n, \tau_1, \ldots, \tau_n\Big).$$

By Theorem 2.1,

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa_n}}\, \mu_n(dt) = z^{-n} \prod_{i=1}^k \frac{1}{(z-n/\xi_i)^{\tau_i}}$$

with $\kappa_n = \tau_1 + \cdots + \tau_k + n$. This relation can also be written as

$$\int_{\mathbb{R}} \frac{1}{(1 - tz/n)^{\kappa_n}} \, \mu_n(dt) = \prod_{i=1}^{k} \left(\frac{\xi_i}{\xi_i - z} \right)^{\tau_i}.$$

The first two moments of ν_n are given by

$$p_1^{(n)} = \sum_{i=1}^{k} \tau_i \left(\frac{n}{\xi_i} \right) = n \sum_{i=1}^{n} \frac{\tau_i}{\xi_i}, \qquad p_2^{(n)} = \sum_{i=1}^{k} \tau_i^2 \left(\frac{n}{\xi} \right)^2 = n^2 \sum_{i=1}^{k} \left(\frac{\tau_i}{\xi_i} \right)^2.$$

Therefore the second moment of μ_n, given by

$$h_2^{(n)} = \frac{2}{\kappa_n(\kappa_n + 1)} \big((p_1^{(n)})^2 + p_2^{(n)} \big),$$

is bounded. It follows that the sequence (μ_n) is relatively compact.

LEMMA 6.3 (see [Cu, Lemma 3, p. 92]). *Let (μ_n) be a sequence in $\mathcal{M}(\mathbb{R})$ which converges for the tight topology to a measure μ, and let (κ_n) be a sequence of positive numbers going to infinity. Then, for $y \in \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left(1 - i \frac{yt}{\kappa_n} \right)^{-\kappa_n} \mu_n(dt) = \int_{\mathbb{R}} e^{iyt} \, \mu(dt)$$

uniformly on compact sets.

We continue the proof of Proposition 6.2. Let μ_0 be the limit of a converging subsequence (μ_{n_j}). Then, by Lemma 6.3, for $z \in i\mathbb{R}$,

$$\int_{\mathbb{R}} e^{zt} \, \mu_0(dt) = \prod_{i=1}^{k} \left(\frac{\xi_i}{\xi_i - z} \right)^{\tau_i}.$$

It follows that $\mu_0 = \gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k)$, and it is the only possible limit for a converging subsequence. This proves that (μ_n) converges with the limit $\gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k)$. ∎

PROPOSITION 6.4. *Assume that $\lim_{n\to\infty} \kappa_n = \infty$, and that (μ_n) converges to a probability measure μ in the tight topology. Then μ is a Thorin–Bondesson distribution. Moreover, every Thorin–Bondesson distribution is obtained in that way.*

Proof. Define

$$F_n(z) = \int_{\mathbb{R}} \left(1 - \frac{zt}{\kappa_n} \right)^{-\kappa_n} \mu_n(dt).$$

Then, by Lemma 6.3,

$$\lim_{n\to\infty} F_n(iy) = F(iy) := \int_{\mathbb{R}} e^{ity} \, \mu(dt)$$

uniformly on compact sets in $\mathbb{R}$. On the other hand,

$$F_n(z) = \exp\biggl(\int_{\mathbb{R}} -\log\Bigl(1-\frac{zu}{\kappa_n}\Bigr)\,\nu_n(dt)\biggr) = \exp\Bigl(\int_{\mathbb{R}} -\log(1-zu)\,\widetilde{\nu}_n(du)\Bigr),$$

where $\widetilde{\nu}_n$ is the image of ν_n by the dilation of ratio $1/\kappa_n$. By Theorem 5.1 there are Thorin–Bondesson distributions $\widetilde{\mu}_n$ such that, for $z \in i\mathbb{R}$,

$$F_n(z) = \int_{\mathbb{R}} e^{zt}\,\widetilde{\mu}_n(dt).$$

By the Lévy–Cramer Theorem,

$$\lim_{n\to\infty} \widetilde{\mu}_n = \mu$$

in the tight topology. Since the set $\mathcal{T}_e$ of Thorin–Bondesson distributions is closed in the tight topology, it follows that μ is a Thorin–Bondesson distribution.

The set of such limits is closed. On the other hand, by Proposition 6.2, this set contains the gamma convolutions $\gamma(\xi_1,\dots,\xi_k;\tau_1,\dots,\tau_k)$. Hence this set is dense in $\mathcal{T}_e$. Being closed and dense it is equal to $\mathcal{T}_e$. ∎

The following theorem describes a representation for the Fourier–Laplace transform of the Thorin–Bondesson distribution μ, the limit of (μ_n). Define

$$\beta_n = \int_{\mathbb{R}} u\widetilde{\nu}_n, \qquad \sigma_n(du) = u^2\,\widetilde{\nu}_n(du),$$

where $\widetilde{\nu}_n$ is, as before, the image of ν_n by the dilation of ratio $1/\kappa_n$.

THEOREM 6.5. *Assume that β_n and σ_n have limits,*

$$\lim_{n\to\infty} \beta_n = \beta, \qquad \lim_{n\to\infty} \sigma_n = \sigma$$

(*in the tight topology*). *Then μ_n has a limit μ whose Fourier–Laplace transform is given by*

$$\int_{\mathbb{R}} e^{zt}\,\mu(dt) = \exp\biggl(\beta z - \int_{\mathbb{R}} \frac{\log(1-zu)+zu}{u^2}\,\sigma(du)\biggr).$$

Observe that

$$\lim_{u\to 0} \frac{\log(1-zu)+zu}{u^2} = -\frac{z^2}{2}.$$

Therefore the function

$$u \mapsto \frac{\log(1-zu)+zu}{u^2}$$

has a continuous extension to $\mathbb{R}$, and the formula in the theorem can be written

$$\int_{\mathbb{R}} e^{zt}\,\mu(dt) = \exp\biggl(\beta z + \frac{1}{2}\gamma z^2 - \int_{\mathbb{R}^*} (\log(1-zu)+zu)\,\sigma_0(du)\biggr),$$

with $\gamma = \sigma(\{0\})$, and σ_0 is the measure on $\mathbb{R}^*$ given by $\sigma_0(du) = u^{-2}\sigma(du)$.

Proof of Theorem 6.5. Let us prove that the sequence (μ_n) is relatively compact. For this we will show that the second moments $h_2^{(n)}$ of the measures μ_n are bounded. We know that

$$h_2^{(n)} = \frac{2}{\kappa_n(\kappa_n+1)}\big((p_1^{(n)})^2 + p_2^{(n)}\big),$$

where $p_m^{(n)}$ are the moments of order m of the measures ν_n. Since

$$p_1^{(n)} = \kappa_n\beta_n, \qquad p_2^{(n)} = \kappa_n^2\sigma_n(\mathbb{R}),$$

we get

$$h_2^{(n)} = \frac{2\kappa_n}{\kappa_n+1}(\beta_n^2 + \sigma_n(\mathbb{R})).$$

The sequences $\big(\sigma_n(\mathbb{R})\big)$ and (β_n) are converging, and hence the sequence $(h_2^{(n)})$ is bounded. Therefore (μ_n) is relatively compact. Let μ_0 be the limit of a converging subsequence of (μ_n). We get

$$\int_{\mathbb{R}} e^{zt}\,\mu_0(dt) = \exp\left(\beta z - \int_{\mathbb{R}} \frac{\log(1-zu)+zu}{u^2}\,\sigma(du)\right).$$

This shows that there exists only one possible limit for a converging subsequence. Therefore the whole sequence (μ_n) converges. ∎

Let us consider the case where

$$\nu_n = \sum_{k=1}^{n} \tau_i^{(n)}\delta_{a_i^{(n)}},$$

where $a^{(n)} = (a_1^{(n)},\ldots,a_n^{(n)})$ and $\tau^{(n)} = (\tau_1^{(n)},\ldots,\tau_n^{(n)})$ are n-tuples of real numbers. Then $\mu_n = M_n(\tau^{(n)}; a^{(n)})$, and

$$\kappa_n = \sum_{i=1}^{n}\tau_i^{(n)}, \qquad \beta_n = \sum_{i=1}^{n}\tau_i^{(n)}\alpha_i^{(n)}, \qquad \sigma_n = \sum_{i=1}^{n}\tau_i^{(n)}(\alpha_i^{(n)})^2\delta_{\alpha_i^{(n)}},$$

with $\alpha_i^{(n)} = \kappa_n^{-1}a_i^{(n)}$.

THEOREM 6.6. *Assume that the numbers $\tau_i^{(n)}$ satisfy $\tau_i^{(n)} \geq \tau > 0$. Assume that the measures σ_n converge to a measure σ in the tight topology.*

(i) *Then σ has the form*

$$\sigma = \sum_{j=1}^{\infty}\tau_j\alpha_j^2\delta_{\alpha_j} + \gamma\delta_0,$$

where $\gamma \geq 0$, (α_j) is a sequence of real numbers, $\tau_j \geq \tau$, and

$$\sum_{j=1}^{\infty}\tau_j\alpha_j^2 < \infty.$$

(ii) *Assume moreover that* $\lim_{n\to\infty}\beta_n=\beta$. *Then the measure* $\mu_n = M_n(\tau^{(n)};a^{(n)})$ *converges to a Thorin–Bondesson distribution* μ *such that*

$$\int_{\mathbb{R}} e^{zt}\,\mu(dt)=e^{\frac{1}{2}\gamma z^2}e^{\beta z}\prod_{j=1}^{\infty}\left(\frac{e^{-\alpha_j z}}{1-z\alpha_j}\right)^{\tau_j}.$$

LEMMA 6.7. *Let* (μ_n) *be a sequence of discrete measures of the form*

$$\mu_n=\sum_{i=1}^{n}\tau_i^{(n)}\delta_{\alpha_i^{(n)}},$$

where $\alpha_i^{(n)}$ *and* $\tau_i^{(n)}$ *are real numbers. Assume that* $\tau_i^{(n)}\geq\tau>0$ *for all* n *and* i, *and* μ_n *converges to* μ *in the vague topology. Then* μ *is of the form*

$$\mu=\sum_{j=1}^{\infty}\tau_j\delta_{\alpha_j},$$

where (α_j) *is a sequence of real numbers, and* $\tau_j\geq\tau$.

Proof. Let $\mathcal{A}$ denote the set of atoms of the measure μ. Then $\mathcal{A}$ is countable. Let $a<b$ be real numbers not in $\mathcal{A}$. Then $\mu([a,b])=\mu(]a,b[)$, hence $\lim_{n\to\infty}\mu_n(]a,b[)=\mu([a,b])$, therefore either $\mu([a,b])=0$ or $\mu([a,b])\geq\tau$. For an atom a of μ, there are two sequences (a_n) and (b_n) such that for every n we have $a_n<a<b_n$, and $a_n,b_n\notin\mathcal{A}$, with limit a. Hence $\mu([a_n,b_n])\geq\tau$, and since $\{a\}=\bigcap_{n=0}^{\infty}[a_n,b_n]$,

$$\mu(\{a\})=\lim_{n\to\infty}\mu([a_,b_n])\geq\tau.$$

It follows that every bounded interval contains only a finite number of atoms. Hence $\mathcal{A}$ is discrete.

Let $a<b$ be two consecutive atoms of μ. Let a_0 and b_0 be such that $a<a_0<b_0<b$. There are a finite number of intervals $[a_i,b_i]$ such that

$$[a_0,b_0]\subset\bigcup[a_i,b_i]\quad\text{and}\quad\mu([a_i,b_i])<\tau.$$

Therefore $\mu([a_0,b_0])=0$ and $\mu(]a,b[)=0$. This shows that there is an increasing sequence (a_i), possibly finite, and real numbers $\tau_j\geq\tau$ such that

$$\mu=\sum_{j=1}^{\infty}\tau_j\delta_{\alpha_j}.\ \blacksquare$$

Proof of Theorem 6.6. For (i) consider the sequence of measures μ_n defined on $\mathbb{R}\setminus\{0\}$ by $\mu_n(du)=u^{-2}\sigma_n$,

$$\mu_n=\sum_{i,\,a_i^{(n)}\neq 0}\tau_i^{(n)}\delta_{\alpha_i^{(n)}}.$$

Then μ_n converges to $\mu = u^{-2}\sigma$ on $\mathbb{R}\setminus\{0\}$ in the vague topology. By Lemma 6.7 the measure μ has the form

$$\mu = \sum_{j=1}^{\infty} \tau_j \delta_{\alpha_j}.$$

Hence σ restricted to $\mathbb{R}\setminus\{0\}$ is equal to $u^2\mu$, therefore there exists $\gamma \geq 0$ such that $\sigma = u^2\mu + \gamma\delta_0$.

Part (ii) follows from Theorem 6.5. ∎

For $a_1 < \cdots < a_n$ and $\tau_1 = \cdots = \tau_n = 1$, the probability measure $M_n(a_1, \ldots, a_n; 1, \ldots, 1)$ is a spline distribution. In that special case one obtains the following theorem, originally established by Schoenberg and Curry:

THEOREM 6.8 ([Cu, Theorem 6, p. 93]). *Assume that the sequence $\mu_n = M_n(a_1^{(n)}, \ldots, a_n^{(n)}; 1, \ldots, 1)$ converges to a measure μ. Then μ is a Pólya distribution: its Fourier–Laplace transform is a Pólya function,*

$$\Phi(z) = \int e^{zt}\, \mu(dt) = e^{\frac{1}{2}\gamma z^2} e^{\beta z} \prod_{j=1}^{\infty} \frac{e^{-\alpha_j z}}{1 - z\alpha_j},$$

with

$$\gamma \geq 0, \qquad \beta \in \mathbb{R}, \qquad \alpha_j \in \mathbb{R}, \qquad \sum_{j=1}^{\infty} \alpha_j^2 < \infty.$$

Conversely, every Pólya distribution is the limit of such a sequence of spline distributions.

Acknowledgements. This paper originates in Chapter 2 of the *Thèse de Doctorat* of the second author [F2]. The work benefited from the support of the CMCU program 10G 1503 (Programme Hubert Curien France–Tunisie, Analyse Harmonique & Probabilités).

REFERENCES

[B] L. Bondesson, *Generalized Gamma Convolutions and Related Classes of Distributions and Densities*, Lecture Notes in Statist. 76, Springer, 1992.

[Ci] D. M. Cifarelli and E. Regazzini, *Distribution functions of means of a Dirichlet process*, Ann. Statist. 18 (1990), 429–442.

[Cu] H. B. Curry and I. J. Schoenberg, *On Pólya frequency functions IV. The fundamental spline functions and their limits*, J. Anal. Math. 17 (1976), 71–107.

[F1] F. Fourati, *Dirichlet distribution and orbital measures*, J. Lie Theory 21 (2011), 189–203.

[F2] F. Fourati, *Distributions de Dirichlet, mesures orbitales et transformation de Markov–Krein*, thèse, Univ. Paris VI & Univ. El Manar, Tunis, 2011.

[J] L. F. James, *Functionals of Dirichlet processes, the Cifarelli–Regazzini identity and Beta-Gamma processes*, Ann. Statist. 33 (2005), 647–660.

[Ke] S. V. Kerov, *Asymptotic Representation Theory of the Symmetric Group and Its Applications in Analysis*, Transl. Math. Monogr. 219, Amer. Math. Soc., 2003.

[Kr] M. G. Krein and A. A. Nudel'man, *The Markov Moment Problem and Extrema Problems*, Transl. Math. Monogr. 50, Amer. Math. Soc., 1977.

[M] M. Morimoto, *An Introduction to Sato's Hyperfunctions*, Transl. Math. Monogr. 129, Amer. Math. Soc., 1993.

[O] G. Olshanski and A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*, in: Contemporary Mathematical Physics, Amer. Math. Soc. Transl. (2) 175, Amer. Math. Soc., 1996, 137–175.

[S] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions. Theory and Applications*, 2nd ed., de Gruyter, 2012.

[T1] O. Thorin, *On the infinite divisibility of the lognormal distribution*, Scand. Actuar. J. 1977, 121–148.

[T2] O. Thorin, *An extension of the notion of a generalized Γ-convolution*, Scand. Actuar. J. 1978, 141–149.

Jacques Faraut
Institut de Mathématiques de Jussieu
Université Pierre et Marie Curie
4 place Jussieu, case 247
75252 Paris Cedex 05, France
E-mail: jacques.faraut@imj-prg.fr

Faiza Fourati
Institut Préparatoire
aux Études d'Ingénieurs de Tunis
Université de Tunis
1089 Montfleury, Tunis, Tunisie
E-mail: fayza.fourati@ipeit.rnu.tn