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Fragments of strong compactness, families of partitions and ideal extensions

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Abstract. We investigate some natural combinatorial principles related to the notion of mild ineffability, and use them to obtain new characterizations of mild ineffable and weakly compact cardinals. We also show that one of these principles may be satisfied by a successor cardinal. Finally, we establish a version for $\mathcal{P}_\kappa(\lambda)$ of the canonical Ramsey theorem for pairs.

1. Introduction. Abramson, Harrington, Kleinberg and Zwicker [1] showed that a number of large cardinals properties could be reformulated as flipping properties. In this general framework, an infinite cardinal κ has a certain large cardinal property P if and only if, for any family of a certain size of partitions of a set X into two pieces, there is a function that selects one piece of each partition in such a way that the collection of chosen pieces satisfies some property Q. For example, κ is a strong limit cardinal if, for any family of less than κ many partitions of κ into two pieces, there is a piece selection function such that the intersection of all chosen pieces has size at least two. This paper is concerned with generalized flipping properties. That is, we modify the basic setting in two ways. First, we allow partitions into more than two pieces, and second, our piece selection function needs not be total, as long as it selects a "large" number of pieces.

The properties that we will characterize in this way are fragments of strong compactness. Strong compactness is a global property. That is, a regular infinite cardinal κ is *strongly compact* if, for any cardinal $\lambda \geq \kappa$, a certain property $P(\kappa, \lambda)$ holds. There are several possibilities for $P(\kappa, \lambda)$. One of them affirms the existence of a prime κ-complete fine ideal on $\mathcal{P}_\kappa(\lambda)$.

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This of course can be seen as an ideal extension property: we are asserting that the non-cofinal ideal $\mathcal{I}_{\kappa,\lambda}$ can be extended to a prime κ-complete ideal on $\mathcal{P}_\kappa(\lambda)$. It can also be recast as a flipping property. Let $\langle Q_\zeta : \zeta < 2^{\lambda^{<\kappa}}\rangle$ be an enumeration of all partitions of $\mathcal{P}_\kappa(\lambda)$ into two pieces. Then our property asserts the existence of a function $f \in \prod_{\zeta<2^{\lambda^{<\kappa}}} Q_\zeta$ such that any intersection of less than κ many of the selected pieces is cofinal in $\mathcal{P}_\kappa(\lambda)$. Another possible choice for $P(\kappa,\lambda)$ is the stronger assertion that any κ-complete ideal on a set P of size λ can be extended to a κ-complete prime ideal on P. On the weaker side, there is the mild λ-ineffability of κ, a property introduced by Di Prisco and Zwicker [3]. Yet another possibility for $P(\kappa,\lambda)$ is the conjunction of "κ is inaccessible" and "$\mathrm{TP}(\kappa,\lambda)$ holds", this latter property being due to Weiß (see [17] and [18]). We investigate the interplay between various such assertions, trying in particular to determine which implies which.

The paper is organized as follows. In Sections 2 and 3 we give several characterizations of the mild $\lambda^{<\kappa}$-ineffability of κ in terms of ideal extensions and piece selections. Since κ is mildly $\kappa^{<\kappa}$-ineffable just in case it is weakly compact, making $\lambda = \kappa$ gives us a few assertions that can be added to the already existing long list of equivalent formulations of weak compactness. This is done in Section 4. In Section 3 it is also shown that κ is mildly $\lambda^{<\kappa}$-ineffable if and only if it is inaccessible and the piece selection principle $\mathrm{PS}(\kappa,\lambda)$ holds. In Section 5 we prove that it is consistent relative to large cardinals that $\mathrm{PS}(\kappa,\lambda)$ implies $\mathrm{TP}(\kappa,\lambda)$, and that it is consistent relative to a supercompact that "$\mathrm{PS}(\kappa,\lambda')$ holds for every cardinal $\lambda' \geq \kappa$, but κ is successor". In Section 6 we show that a principle that looks fairly weak does have some strength. Finally in Section 7 we establish a version for $\mathcal{P}_\kappa(\lambda)$ of the canonical Ramsey theorem for pairs.

2. Ideal extensions. In this section we compare several assertions dealing with extensions of ideals. We start with some definitions.

Throughout the paper, κ will denote a regular infinite cardinal, and λ a cardinal greater than or equal to κ. Given a set X and a cardinal ν, we denote by $\mathcal{P}_\nu(X)$ the set of all subsets of X of size less than ν, while $[X]^\nu$ denotes the set of all subsets of X of size ν. An *ideal* on X is a collection of subsets of X such that:

- $A \cup B \in J$ whenever $A, B \in J$,
- $P(A) \subseteq J$ for all $A \in J$,
- $X \notin J$,
- $\{x\} \in J$ for all $x \in X$.

Given an ideal J on a set X, we denote by J^+ the set $\{A \subseteq X : A \notin J\}$, while J^* denotes the set $\{A \subseteq X : X \setminus A \in J\}$. For any $A \in J^+$, we let $J|A := \{B \subseteq X : B \cap A \in J\}$. We say that J is *κ-complete* if for any collection Z of less than κ many sets in J one has $\bigcup Z \in J$. The *cofinality* of J, denoted $\mathrm{cof}(J)$, is the least cardinality of a subcollection I of J such that $J = \bigcup_{A\in I} \mathcal{P}(A)$. For a cardinal τ, we say that J is *τ-saturated* if there is no subset $Q \subseteq J^+$ of size τ with the property that $A \cap B \in J$ for any two distinct elements A, B of Q. Furthermore, J is *nowhere τ-saturated* if there is no $A \in J$ such that $J|A$ is τ-saturated. We say that J is *prime* if it is 2-saturated. An ideal K on X *extends* J if $J \subseteq K$. We denote by $\mathcal{I}_{\kappa,\lambda}$ the set of all $A \subseteq \mathcal{P}_\kappa(\lambda)$ for which there exists $a \in \mathcal{P}_\kappa(\lambda)$ such that $A \cap \{b \in \mathcal{P}_\kappa(\lambda) : a \subseteq b\} = \emptyset$. An ideal J on $\mathcal{P}_\kappa(\lambda)$ is *fine* if it extends $\mathcal{I}_{\kappa,\lambda}$.

We say that κ is *mildly λ-ineffable* if, for $s_a \subseteq a$ for $a \in \mathcal{P}_\kappa(\lambda)$, there exists $S \subseteq \lambda$ with the property that, for any $b \in \mathcal{P}_\kappa(\lambda)$, there is $a \in \mathcal{P}_\kappa(\lambda)$ such that $b \subseteq a$ and $S \cap b = s_a \cap b$.

FACT 2.1.

(1) (Carr [2]) *If κ is mildly λ-ineffable, then it is mildly μ-ineffable for any cardinal μ with $\kappa \le \mu \le \lambda$.*
(2) (Carr [2]) *κ is mildly λ-ineffable if and only if it is weakly compact.*
(3) (Usuba [16]) *Suppose κ is mildly λ-ineffable and* $\mathrm{cof}(\lambda) \ge \kappa$. *Then* $\lambda^{<\kappa} = \lambda$.

Thus:

(a) If $\mathrm{cof}(\lambda) \ge \kappa$, then κ is mildly $\lambda^{<\kappa}$-ineffable if and only if it is mildly λ-ineffable.
(b) If $\mathrm{cof}(\lambda) < \kappa$, then κ is mildly $\lambda^{<\kappa}$-ineffable if and only if it is mildly λ^+-ineffable.

For a cardinal $\sigma \ge \kappa$, κ is *σ-compact* if there exists a prime, κ-complete, fine ideal on $\mathcal{P}_\kappa(\sigma)$ extending $\mathcal{I}_{\kappa,\sigma}$. Next, κ is *strongly compact* if it is σ-compact for every cardinal $\sigma \ge \kappa$.

DEFINITION 2.2. Given an infinite set P and a κ-complete ideal J on P, we write $\mathrm{IE}^1_\kappa(J)$ when there exists a prime, κ-complete ideal on P that extends J.

Thus, κ is σ-compact if and only if $\mathrm{IE}^1_\kappa(\mathcal{I}_{\kappa,\sigma})$ holds.

PROPOSITION 2.3. *Suppose that κ is mildly 2^λ-ineffable. Let P be a set of size λ, and let J be a κ-complete ideal on P. Then* $\mathrm{IE}^1_\kappa(J)$ *holds.*

Proof. Let $\langle X^0_\beta : \beta < 2^\lambda\rangle$ be an enumeration of all subsets of P. For $\beta < 2^\lambda$, let $X^1_\beta := P \setminus X^0_\beta$. For every $a \in \mathcal{P}_\kappa(2^\lambda)$, pick $t_a : a \to 2$ such that $\bigcap_{\beta\in a} X_\beta^{t_a(\beta)} \in J^+$. There must be $g : 2^\lambda \to 2$ with the property that, for every $b \in \mathcal{P}_\kappa(2^\lambda)$, there is $a \in \mathcal{P}_\kappa(2^\lambda)$ such that $a \supseteq b$ and $g\restriction b = t_a\restriction b$. It is easy to check that $K := \{X_\beta^{1-g(\beta)} : \beta < 2^\mu\}$ is a prime κ-complete ideal on P extending J. ■

Kunen [8] established that if κ is measurable and uncountable, and either $2^\kappa > \kappa^+$ or $\mathrm{IE}^1_\kappa(J)$ holds for every κ-complete ideal J on κ, then for any ordinal θ, there exists an inner model where θ is a measurable cardinal. By Proposition 2.3 it follows that in case $\kappa > \omega$ the mild 2^κ-ineffability of κ is a much stronger property than its measurability.

Let X be a set. By a *partition* of X we mean a subset Q of $\mathcal{P}(X) \setminus \{\emptyset\}$ such that:

- $A \cap B = \emptyset$ for any two distinct members A, B of Q,
- $\bigcup Q = X$.

Definition 2.4. Given a set P and a κ-complete ideal J on P, we denote by $\mathrm{IE}^2_\kappa(J)$ the following statement: Suppose that for each $p \in P$ there is a partition Q_p of P with $|Q_p| < \kappa$; then there is $h \in \prod_{p\in P} Q_p$ and a κ-complete ideal K on P extending J such that $\mathrm{ran}(h) \subseteq K^*$.

Clearly, $\mathrm{IE}^1_\kappa(J)$ implies $\mathrm{IE}^2_\kappa(J)$.

Remark 2.5. Suppose $\kappa = \lambda = \omega$ and let P be a set of size $\aleph_0$ and J an ideal on P. Then it is easy to show that $\mathrm{IE}^2_\kappa(J)$ holds. Indeed, if $\langle p_n : n < \omega\rangle$ is an enumeration of P, and for each n, Q_n is a finite partition of P, then one can define by induction sets $A_n \in Q_n$ for $n < \omega$ such that $\bigcap_{i\le n} A_i \in J^+$. The statement is verified if we let $h(p_n) := A_n$.

Proposition 2.6. *Suppose that κ is mildly λ-ineffable. Then for any set P of size λ and any κ-complete ideal J on P, we have* $\mathrm{IE}^2_\kappa(J)$.

Proof. Suppose that for each $p \in P$, Q_p is a partition of P of size less than κ, say σ_p. Let $\langle A^i_p : i < \sigma_p\rangle$ be a one-to-one enumeration of Q_p. For $\sigma_p \le i < \kappa$, set $A^i_p := \emptyset$. Fix a bijection $\varphi : P \times \kappa \to \lambda$ and, for $p \in P$ and $i < \kappa$, set $B^0_{\varphi(p,i)} := A^i_p$ and $B^1_{\varphi(p,i)} := P \setminus B^0_{\varphi(p,i)}$. For each $d \in \mathcal{P}_\kappa(\lambda) \setminus \emptyset$, pick $t_d : d \to 2$ such that $\bigcap_{\delta\in d} B^{t_d(\delta)}_\delta \in J^+$. Then we may find $g : \lambda \to 2$ with the property that for any $c \in \mathcal{P}_\kappa(\lambda)$ there is $d \in \mathcal{P}_\kappa(\lambda)$ such that $c \subseteq d$ and $t_d{\restriction}c = g{\restriction}c$.

Claim 2.7. *Let $p \in P$. Then $g(\varphi(p,i)) = 0$ for some $i < \sigma_p$.*

Proof. Suppose otherwise. Let $c := \{\varphi(p,i) : i < \sigma_p\}$. There must be $d \in \mathcal{P}_\kappa(\lambda)$ such that $c \subseteq d$ and $t_d{\restriction}c = g{\restriction}c$. Then

$$\bigcap_{\delta\in d} B^{t_d(\delta)}_\delta \subseteq \bigcap_{\delta\in c} B^{t_d(\delta)}_\delta = \bigcap_{i<\sigma_p} B^1_{\varphi(p,i)} = \bigcap_{i<\sigma_p} (P \setminus A^i_p) = \emptyset,$$

a contradiction. ∎

Using Claim 2.7, pick $i_p < \sigma_p$ for each $p \in P$ so that $g(\varphi(p, i_p)) = 0$.

Claim 2.8. *Let $x \in \mathcal{P}_\kappa(P) \setminus \{\emptyset\}$. Then $\bigcap_{p\in x} A^{i_p}_p \in J^+$.*

Proof. Define $c := \{\varphi(p, i_p) : p \in x\}$. Then we may find $d \in \mathcal{P}_\kappa(\lambda)$ such that $c \subseteq d$ and $t_d \restriction c = g \restriction c$. We have

$$\bigcap_{\delta \in d} B_\delta^{t_d(\delta)} \subseteq \bigcap_{\delta \in c} B_\delta^{t_d(\delta)} = \bigcap_{p \in x} B^0_{\varphi(p,i_p)} = \bigcap_{p \in x} A_p^{i_p}.$$

Hence $\bigcap_{p \in x} A_p^{i_p} \in J^+$. ∎

Now, let K be the collection of all $Y \subseteq P$ such that $(Y \setminus A) \cap \bigcap_{p\in x} A_p^{i_p} = \emptyset$ for some $A \in J$ and some $x \in \mathcal{P}_\kappa(P) \setminus \{\emptyset\}$. It is easy to check that K is a κ-complete ideal on P extending J. Moreover, $\{A_p^{i_p} : p \in P\} \subseteq K^*$. That completes the proof of Proposition 2.6. ∎

The following can be found in [4, p. 48].

Fact 2.9. *Let σ be an infinite cardinal. Then $\{\sigma^\rho : 2^\rho < \sigma\}$ is finite.*

Lemma 2.10. *Suppose $2^{<\kappa} \leq \lambda$. Then $(\lambda^{<\kappa})^{<\kappa} = \lambda^{<\kappa}$.*

Proof. If $\lambda = 2^\rho$ for some cardinal $\rho < \kappa$, then clearly $\lambda^{<\kappa} = \lambda$. Otherwise, we can infer from Fact 2.9 that there is a cardinal $\nu < \kappa$ such that $\lambda^{<\kappa} = \lambda^\nu$. Then of course $(\lambda^{<\kappa})^{<\kappa} = \lambda^\nu = \lambda^{<\kappa}$. ∎

Fact 2.11 (Carr [2], Di Prisco and Zwicker [3]). *Suppose that $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$ holds. Then κ is a strong limit cardinal.*

Proof. Suppose toward a contradiction that there is a cardinal $\nu < \kappa$ such that $2^\nu \geq \kappa$. Let $\langle f_\zeta : \zeta < \kappa\rangle$ be a sequence of pairwise distinct functions from ν to 2. For $\eta < \nu$ and $l < 2$, write $W_\eta^l := \{a \in \mathcal{P}_\kappa(\lambda) : f_{\sup(a\cap\kappa)}(\eta) = l\}$. Then we may find $u : \nu \to 2$ such that $\bigcap_{\eta<\nu} W_\eta^{u(\nu)} \in \mathcal{I}^+_{\kappa,\lambda}$. Now clearly $\bigcap_{\eta<\nu} W_\eta^{u(\nu)} := \{a \in \mathcal{P}_\kappa(\lambda) : f_{\sup(a\cap\kappa)} = u\}$, which yields the desired contradiction. ∎

Lemma 2.12. *Suppose that $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$ holds. Then κ is mildly $\lambda^{<\kappa}$-ineffable.*

Proof. By Lemma 2.10 and Fact 2.11, we have $(\lambda^{<\kappa})^{<\kappa} = \lambda^{<\kappa}$. Select two bijections $\varphi : \mathcal{P}_\kappa(\lambda) \to \mathcal{P}_\kappa(\lambda^{<\kappa})$ and $f : \mathcal{P}_\kappa(\lambda) \to \lambda^{<\kappa}$. Now let $t(x) : x \to 2$ be given for $x \in \mathcal{P}_\kappa(\lambda^{<\kappa})$. For $d \in \mathcal{P}_\kappa(\lambda)$, set

$$A_d^j := \Big\{e \in \mathcal{P}_\kappa(\lambda) : f(d) \in \bigcup_{a\subseteq e} \varphi(a) \text{ and } t\Big(\bigcup_{a\subseteq e} \varphi(a)(f(d))\Big) = j\Big\}$$

for $j = 0, 1$ and $A_d^3 := \mathcal{P}_\kappa(\lambda) \setminus (A_d^0 \cup A_d^1)$. Note that $A_d^3 \in \mathcal{I}_{\kappa,\lambda}$. We may find $h : \mathcal{P}_\kappa(\lambda) \to 3$ with the property that for any $z \in \mathcal{P}_\kappa(\lambda)$, $\bigcap_{d\subseteq z} A_d^{h(d)} \in \mathcal{I}^+_{\kappa,\lambda}$. Define $T : \lambda^{<\kappa} \to 2$ by letting $T(f(d)) = h(d)$ for every $d \in \mathcal{P}_\kappa(\lambda)$. Now fix $w \in \mathcal{P}_\kappa(\lambda^{<\kappa})$. Select $z \in \mathcal{P}_\kappa(\lambda)$ so that $w \subseteq \{f(d) : d \subseteq z\}$ and pick $e \in \bigcap_{d \subseteq z} A_d^{h(d)}$. Let $x = \bigcup_{a \subseteq e} \varphi(a)$. Then clearly $w \subseteq x$. Moreover, $t(x) \restriction w = T \restriction w$. ∎

THEOREM 2.13. *The following are equivalent:*

(i) *κ is mildly $\lambda^{<\kappa}$-ineffable.*
(ii) *For any set P of size $\lambda^{<\kappa}$, for any κ-complete ideal J on P,* $\mathrm{IE}^2_\kappa(J)$ *holds.*
(iii) $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$ *holds.*
(iv) $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda^{<\kappa}})$ *holds.*

Proof. (i)→(ii) follows from Proposition 2.6; (ii)→(iii) is trivial; (iii)→(i) follows from Lemma 2.12; (ii)→(iv) follows from Lemma 2.10 and Fact 2.11; (iv)→(i) follows from Lemmas 2.10, 2.12 and Fact 2.11. ∎

Theorem 2.13 strengthens a result of Carr [2, Theorem 3.2]. We also remark the following. By a result of Usuba [16], it is consistent relative to some large cardinal to have two infinite cardinals $\kappa' \leq \lambda'$ such that κ' is mildly λ'-ineffable (in fact, completely λ'-ineffable) but not $(\lambda')^{<\kappa'}$-ineffable. Combining this with Theorem 2.13, we see that $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$ does not necessarily follow in case κ is mildly λ-ineffable.

The *splitting number* $\mathfrak{s}(\kappa)$ is the least cardinality of any $F \subseteq [\kappa]^\kappa$ with the property that for any $A \in [\kappa]^\kappa$ there is $B \in F$ such that $|A \cap B| = |A \setminus B| = \kappa$.

COROLLARY 2.14. *The following are equivalent:*

(i) *κ is weakly compact.*
(ii) *For any set P of size κ and any κ-complete ideal J on P,* $\mathrm{IE}^2_\kappa(J)$ *holds.*
(iii) $\mathfrak{s}(\kappa) > \kappa$.

The equivalence between (i) and (iii) in Corollary 2.14 is due to Suzuki [15].

3. Piece selection, part 1. Now we devote three sections to the following two principles.

DEFINITION 3.1. We let $\mathrm{PS}^+(\kappa,\lambda)$ (respectively $\mathrm{PS}(\kappa,\lambda)$) assert the following: For $b \in \mathcal{P}_\kappa(\lambda)$, let Q_b be a partition of $\mathcal{P}_\kappa(\lambda)$ with $|Q_b| < \kappa$. Then there is $B \in \mathcal{I}^+_{\kappa,\lambda}$ and $h \in \prod_{b \in \mathcal{P}_\kappa(\lambda)} Q_b$ such that for any $a, b \in B$ the set $\{c \in h(a) \cap h(b) : a \cup b \subseteq c\}$ is non-empty (respectively, there is $x \in \mathcal{P}_\kappa(\lambda)$ such that $a \cup b \subseteq x$ and the two sets $\{c \in h(a) \cap h(x) : x \subseteq c\}$ and $\{d \in h(b) \cap h(x) : x \subseteq d\}$ are non-empty).

LEMMA 3.2.

(i) $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$ *implies* $\mathrm{PS}^+(\kappa,\lambda)$.
(ii) $\mathrm{PS}^+(\kappa,\lambda)$ *implies* $\mathrm{PS}(\kappa,\lambda)$.

Proof. (i) Trivial.

(ii) Suppose that for each $b \in \mathcal{P}_\kappa(\lambda)$, Q_b is a partition of $\mathcal{P}_\kappa(\lambda)$ with $|Q_b| < \kappa$. Then we may find $B \in \mathcal{I}^+_{\kappa,\lambda}$ and $h \in \prod_{b\in\mathcal{P}_\kappa(\lambda)} Q_b$ such that for any $r, s \in B$ we have $\{e \in h(r) \cap h(s) : r \cup s \subseteq e\} \neq \emptyset$. Now fix $a, b \in B$. Pick $x \in B$ with $a \cup b \subseteq x$. Then $\{c \in h(a) \cap h(x) : x \subseteq c\} \neq \emptyset$, and moreover $\{d \in h(b) \cap h(x) : x \subseteq d\} \neq \emptyset$. ∎

Lemma 3.3. *For κ inaccessible,* $\mathrm{PS}(\kappa, \lambda)$ *implies* $\mathrm{IE}^2_\kappa(\mathcal{I}_{\kappa,\lambda})$.

Proof. Suppose that for each $z \in \mathcal{P}_\kappa(\lambda)$, R_z is a partition of $\mathcal{P}_\kappa(\lambda)$ with $|R_z| < \kappa$. For $b \in \mathcal{P}_\kappa(\lambda)$, set $Q_b := \{\bigcap_{z\subseteq b} t(z) : t \in \prod_{z\subseteq b} R_z\} \setminus \{\emptyset\}$. Then we may find $B \in \mathcal{I}^+_{\kappa,\lambda}$ and $T \in \prod_{b\in\mathcal{P}_\kappa(\lambda)}(\prod_{z\subseteq b} R_z)$ such that, for any $a, b \in B$, there is $x_{ab} \in B$ such that $a \cup b \subseteq x_{ab}$ and

$$\Big\{c \in \Big(\bigcap_{w\subseteq a} T(a)(w)\Big) \cap \Big(\bigcap_{v\subseteq x} T(x_{ab})(v) : x_{ab} \subseteq c\Big)\Big\} \neq \emptyset,$$

$$\Big\{d \in \Big(\bigcap_{z\subseteq b} T(b)(z)\Big) \cap \Big(\bigcap_{v\subseteq x} T(x_{ab})(v) : x_{ab} \subseteq d\Big)\Big\} \neq \emptyset.$$

Claim 3.4. *Let $a, b \in B$. Then $T(a)\restriction\mathcal{P}(a \cap b) = T(b)\restriction\mathcal{P}(a \cap b)$.*

Proof. Fix $v \subseteq a \cap b$. Then $T(a)(v) \cap T(x_{ab})(v) \neq \emptyset$ and $T(b)(v) \cap T(x_{ab})(v) \neq \emptyset$, so $T(a)(v) = T(x_{ab})(v) = T(b)(v)$. ∎

Set $h := \bigcup_{b\in B} T(b)$. By the claim, $h \in \prod_{z\in\mathcal{P}_\kappa(\lambda)} R_z$. Now fix y in $\mathcal{P}_\kappa(\mathcal{P}_\kappa(\lambda)) \setminus \{\emptyset\}$. We will show that $\bigcap_{z\in y} h(z) \in \mathcal{I}^+_{\kappa,\lambda}$. Given $e \in \mathcal{P}_\kappa(\lambda)$, pick $a \in B$ such that $e \cup \bigcup y \subseteq a$. Then we may find $c \in \bigcap_{w\subseteq a} T(a)(w)$ with $a \subseteq c$. Clearly, $e \subseteq c$. Moreover, $c \in \bigcap_{z\in y} h(z)$. ∎

Theorem 3.5. *The following are equivalent:*

(i) *κ is mildly $\lambda^{<\kappa}$-ineffable.*
(ii) *κ is inaccessible and* $\mathrm{PS}^+(\kappa, \lambda)$ *holds.*
(iii) *κ is inaccessible and* $\mathrm{PS}(\kappa, \lambda)$ *holds.*
(iv) *κ is inaccessible and* $\mathrm{PS}^+(\kappa, \lambda^{<\kappa})$ *holds.*
(v) *κ is inaccessible and* $\mathrm{PS}(\kappa, \lambda^{<\kappa})$ *holds.*

Proof. (i)→(ii) and (i)→(iv) follow from Theorem 2.13 and Lemma 3.2. (i)→(iii) and (iv)→(v) follow from Lemma 3.2. (iii)→(i) and (v)→(i) follow from Theorem 2.13 and Lemma 3.3. ∎

4. Piece selection, part 2. In this section we concentrate on the case $\lambda = \kappa$.

Definition 4.1. Given an infinite cardinal ρ, we let $\mathrm{PS}^+(\rho)$ (respectively $\mathrm{PS}(\rho)$) assert the following: For $\beta \in \rho$, let Q_β be a partition of ρ with $|Q_\beta| < \rho$. Then there is a cofinal subset B of ρ, and there is $h \in \prod_{\beta\in\rho} Q_\beta$ such that for any $\alpha, \beta \in B$ we have $\{\gamma \in h(\alpha) \cap h(\beta) : \max(\alpha, \beta) \leq \gamma\} \neq \emptyset$

(respectively, there is $\zeta \in \rho$ such that $\max(\alpha, \beta) \leq \zeta$ and we have $\{\gamma \in h(\alpha) \cap h(\zeta) : \zeta \leq \gamma\} \neq \emptyset$ and $\{\delta \in h(\beta) \cap h(\zeta) : \zeta \leq \delta\} \neq \emptyset$).

Lemma 4.2. $\mathrm{PS}^+(\rho)$ *implies* $\mathrm{PS}(\rho)$.

Proof. Just as the proof of Lemma 3.2(ii). ∎

Lemma 4.3. *Suppose that* $\mathrm{PS}(\rho)$ *holds. Then* ρ *is regular.*

Proof. Suppose otherwise, and let $\rho := \lim_{i<\mathrm{cf}(\rho)} \rho_i$ where $\langle \rho_i : 0 < i < \mathrm{cf}(\rho)\rangle$ is an increasing and continuous sequence of infinite cardinals. Write $\rho_0 := 0$. Define $\varphi : \rho \to \mathrm{cf}(\rho)$ by letting $\varphi(\eta)$ be the unique $i < \mathrm{cf}(\rho)$ such that $\rho_i \leq \eta < \rho_{i+1}$. Then we may find a cofinal subset B of ρ and $k : \rho \to \mathrm{cf}(\rho)$ such that for any $\alpha, \beta \in B$ there is $\zeta_{\alpha,\beta} \in \kappa$ such that $\max(\alpha, \beta) \leq \zeta_{\alpha,\beta}$ and we have

$$\{\gamma \in \varphi^{-1}(\{k(\alpha)\}) \cap \varphi^{-1}(\{k(\zeta_{\alpha,\beta})\}) : \zeta_{\alpha,\beta} \leq \gamma\} \neq \emptyset,$$
$$\{\delta \in \varphi^{-1}(\{k(\beta)\}) \cap \varphi^{-1}(\{k(\zeta_{\alpha,\beta})\}) : \zeta_{\alpha,\beta} \leq \delta\} \neq \emptyset.$$

Pick $\alpha, \beta \in B$ with $\varphi(\beta) > k(\alpha)$, and $\delta \in \varphi^{-1}(\{k(\beta)\}) \cap \varphi^{-1}(\{k(\zeta_{\alpha,\beta})\})$ with $\delta \geq \beta$. Then

$$k(\alpha) < \varphi(\beta) \leq \varphi(\delta) = k(\beta) = k(\zeta_{\alpha,\beta}) = k(\alpha),$$

a contradiction. ∎

Lemma 4.4. *The following are equivalent:*

(i) $\mathrm{PS}^+(\rho)$.
(ii) $\mathrm{PS}^+(\rho, \rho)$.

Proof. To prove (i)→(ii), use the fact that $[\rho]^\rho \subseteq \mathcal{I}^+_{\rho,\rho}$.

(ii)→(i). Suppose that $\mathrm{PS}^+(\rho, \rho)$ holds. For $\beta \in \rho$, let Q_β be a partition of ρ with $|Q_\beta| < \rho$. For $b \in \mathcal{P}_\rho(\rho)$, let

$$R_b := \{\{d \in \mathcal{P}_\rho(\rho) : \sup(d) \in H\} : H \in Q_{\sup(b)}\}.$$

Then we may find $D \in \mathcal{I}^+_{\rho,\rho}$ and $k \in \prod_{b\in D} R_b$ such that for all $a, b \in D$ we have $\{c \in k(a) \cap k(b) : a \cup b \subseteq c\} \neq \emptyset$. Set $B := \{\sup(b) : b \in D\}$ and pick a one-to-one function $f : B \to D$ such that $\sup(f(\beta)) = \beta$ for all $\beta \in B$. Now define $h \in \prod_{\beta\in B}(Q_\beta)$ by letting $h(\beta)$ be the set $\{\sup(d) : d \in k(f(\beta))\}$. For $\alpha, \beta \in D$, there must be $c \in k(f(d)) \cap k(f(\beta))$ such that $f(\alpha) \cup f(\beta) \subseteq c$. Set $\gamma := \sup(c)$. Then clearly $\max(\alpha, \beta) \leq \gamma$, and moreover $\gamma \in h(\alpha) \cap h(\beta)$. ∎

Theorem 4.5. *For an infinite cardinal* ρ, *the following are equivalent:*

(i) ρ *is weakly compact.*
(ii) ρ *is a strong limit cardinal and* $\mathrm{PS}^+(\rho)$ *holds.*
(iii) ρ *is a strong limit cardinal and* $\mathrm{PS}(\rho)$ *holds.*

Proof. (i)→(ii) follows from Fact 2.1, Theorem 3.5 and Lemma 4.4; and (ii)→(iii) follows from Lemma 4.2. To prove (iii)→(i), observe that $\mathrm{PS}(\rho)$ implies $\mathrm{PS}(\rho,\rho)$, and use Fact 2.1, Lemma 4.3 and Theorem 3.5. ∎

PROPOSITION 4.6. *Suppose that* $\mathrm{PS}^+(\rho)$ *holds. Then* ρ *is a limit cardinal.*

Proof. Suppose otherwise and let $\rho = \tau^+$. For $\gamma \in \rho \setminus \tau$ pick a bijection $g_\gamma : \gamma \to \tau$. Then there must be $D \in [\rho \setminus \tau]^\rho$ and $f : D \to \tau$ such that for any $\delta, \eta \in D$ we have

$$\{\gamma \in \rho : \gamma \geq \max(\delta,\eta),\ g_\gamma(\delta) = f(\delta) \text{ and } g_\gamma(\eta) = f(\eta)\} \neq \emptyset.$$

Clearly f is one-to-one, which yields the desired contradiction. ∎

It remains an open problem whether $\mathrm{PS}^+(\rho)$ implies that ρ is a strong limit cardinal.

5. Piece selection, part 3. This section is devoted to $\mathrm{PS}(\kappa,\lambda)$. Recall that a subset C of $\mathcal{P}_\kappa(\lambda)$ is *strongly closed* if $\bigcup X \in C$ for all $X \subseteq C$ with $0 < |X| < \kappa$.

NOTATION 5.1. $\mathrm{SNS}_{\kappa,\lambda}$ denotes the collection of all $B \subseteq \mathcal{P}_\kappa(\lambda)$ such that $B \cap C \neq \emptyset$ for some strongly closed, cofinal subset of $\mathcal{P}_\kappa(\lambda)$.

It is known [14, 12] that:

- $\mathrm{SNS}_{\kappa,\lambda}$ is a κ-complete ideal on $\mathcal{P}_\kappa(\lambda)$ extending $\mathcal{I}_{\kappa,\lambda}$.
- $\mathrm{SNS}_{\kappa,\lambda}$ is the collection of all $B \subseteq \mathcal{P}_\kappa(\lambda)$ such that $\{a \in \mathcal{P}_\kappa(\lambda) : f''a \subseteq \mathcal{P}(a)\} \cap B = \emptyset$ for some $f : \lambda \to \mathcal{P}_\kappa(\lambda)$.
- Let j be a one-to-one function from $\lambda \times \lambda$ to λ, and D be the set of all $a \in \mathcal{P}_\kappa(\lambda)$ such that $j(\alpha,\beta) \in a$ whenever $\alpha < \beta$ are in a. Then $\mathrm{SNS}_{\kappa,\lambda} | D$ is the non-stationary ideal on $\mathcal{P}_\kappa(\lambda)$.

DEFINITION 5.2. $\mathrm{TP}(\kappa,\lambda)$ (respectively $\mathrm{TP}^-(\kappa,\lambda)$) asserts the following: Let $s_a \subseteq a$ for $a \in \mathcal{P}_\kappa(\lambda)$ be such that for some $C \in \mathrm{NS}^*_{\kappa,\lambda}$ (respectively $\mathrm{SNS}^*_{\kappa,\lambda}$) we have $|\{s_a \restriction c : c \subseteq a\}| < \kappa$ for all $c \in C$; then there is $S \subseteq \lambda$ with the property that for every $b \in \mathcal{P}_\kappa(\lambda)$ there is $a \in \mathcal{P}_\kappa(\lambda)$ such that $b \subseteq a$ and $S \cap b = s_a \cap b$.

REMARK 5.3. The following are clearly equivalent:

(1) κ is mildly λ-ineffable.
(2) κ is inaccessible and $\mathrm{TP}^-(\kappa,\lambda)$ holds.
(3) κ is inaccessible and $\mathrm{TP}(\kappa,\lambda)$ holds.

PROPOSITION 5.4. $\mathrm{PS}(\kappa,\lambda)$ *implies* $\mathrm{TP}^-(\kappa,\lambda)$.

Proof. Suppose that $\mathrm{PS}(\kappa,\lambda)$ holds. Let C be a strongly closed cofinal subset of $\mathcal{P}_\kappa(\lambda)$, and let $s_a \subseteq a$ for $a \in \mathcal{P}_\kappa(\lambda)$ be such that $|\{s_e \cap c : c \subseteq e\}| < \kappa$ for all $c \in C$. Define $f : \mathcal{P}_\kappa(\lambda) \to C \cup \{\emptyset\}$ by $f(b) = \bigcup(C \cup \mathcal{P}(b))$. For

$b \in \mathcal{P}_\kappa(\lambda)$, set $X_b = \{e \in \mathcal{P}_\kappa(\lambda) : b \setminus e \neq \emptyset\}$, $Z_b^t := \{e \in \mathcal{P}_\kappa(\lambda) : b \subseteq e$ and $s_e \cap f(b) = t\}$ for all $t \in \mathcal{P}(f(b))$, and $Q_b = (\{Z_b^t : t \in \mathcal{P}(f(b))\} \setminus \{\emptyset\}) \cup \{X_b\}$. Note that $|Q_b| < \kappa$. Select $B \in \mathcal{I}^+_{\kappa,\lambda}$ and $h \in \prod_{b \in \mathcal{P}_\kappa(\lambda)} Q_b$ such that for any $a, b \in B$ there is $x_{ab} \in \mathcal{P}_\kappa(\lambda)$ such that $a \cup b \subseteq x_{ab}$, $\{c \in h(a) \cap h(x_{ab}) : x_{ab} \subseteq c\} \neq \emptyset$ and $\{d \in h(b) \cap h(x_{ab}) : x_{ab} \subseteq d\} \neq \emptyset$.

Claim 5.5. *Let $b \in B$. Then $h(b) \neq X_b$.*

Proof. There is $d \in h(b) \cap h(x_{bb})$ such that $b \subseteq d$. Clearly $d \neq X_b$. ∎

Claim 5.6. *Let $a, b \in B$. Then $h(x_{ab}) \neq X_{x_{ab}}$.*

Proof. There is $c \in h(a) \cap h(x_{ab})$ such that $x_{ab} \subseteq c$. Clearly $c \notin X_{x_{ab}}$. ∎

By Claims 5.5 and 5.6, there is $k \in \prod_{b \in \mathcal{P}_\kappa(\lambda)} \mathcal{P}(f(b))$ such that:

- $h(b) = Z_b^{k(b)}$ for all $b \in B$,
- $h(x_{ab}) = Z_{x_{ab}}^{k(x_{ab})}$ for all $a, b \in B$.

Claim 5.7. *Let $a, b \in B$. Then $k(a) \cap f(a) \cap b = k(b) \cap f(a) \cap b$.*

Proof. We may find $c \in h(a) \cap h(x_{ab})$ with $x_{ab} \subseteq c$, and $d \in h(b) \cap h(x_{ab})$ with $x_{ab} \subseteq d$. Then

$$\begin{aligned} k(a) \cap f(b) &= s_c \cap f(a) \cap f(b) = s_c \cap f(x_{ab}) \cap f(a) \cap f(b) \\ &= k(x_{ab}) \cap f(a) \cap f(b) = s_d \cap f(x_{ab}) \cap f(a) \cap f(b) \\ &= s_d \cap f(b) \cap f(a) = k(b) \cap f(a). \ \blacksquare \end{aligned}$$

Let $S = \bigcup_{b \in B} (k(b) \cap b)$. Given $q \in \mathcal{P}_\kappa(\lambda)$, we may find $p \in C$ and $b \in B$ such that $q \subseteq p \subseteq b$. Since $Z_b^{k(b)} \neq \emptyset$, there must be $e \in \mathcal{P}_\kappa(\lambda)$ such that $b \subseteq e$ and $s_e \cap f(b) = k(b)$.

Claim 5.8. *$s_e \cap q = S \cap q$.*

Proof. First we show $s_e \cap q \subseteq S \cap q$. Fix $\alpha \in s_e \cap q$. Then $\alpha \in s_e \cap f(b) = k(b) \subseteq S$, and hence $\alpha \in S \cap q$. Now we show $s_e \cap q \supseteq S \cap q$. Fix $\beta \in s \cap q$. There must be $a \in B$ such that $\beta \in k(a)$. Clearly $q \subseteq f(b)$, so $\beta \in k(a) \cap f(b)$. Then by Claim 5.7, β lies in $k(b) \cap f(a)$, and hence in $s_e \cap q$. ∎

That completes the proof of Proposition 5.4. ∎

Remark 5.9. Let $\mathrm{PS}^*(\kappa, \lambda)$ be the strengthening of $\mathrm{PS}(\kappa, \lambda)$ obtained by replacing "$x \in \mathcal{P}_\kappa(\lambda)$" in the statement of $\mathrm{PS}(\kappa, \lambda)$ with "$x \in B$". It is not difficult to see that

$$\mathrm{PS}^+(\kappa, \lambda) \to \mathrm{PS}^*(\kappa, \lambda) \to \mathrm{TP}(\kappa, \lambda).$$

The following questions remain open:

- Is $\mathrm{TP}^-(\kappa, \lambda)$ equivalent to $\mathrm{TP}(\kappa, \lambda)$?
- Is $\mathrm{PS}(\kappa, \lambda)$ equivalent to $\mathrm{PS}^*(\kappa, \lambda)$?
- Does $\mathrm{PS}(\kappa, \lambda)$ imply $\mathrm{TP}(\kappa, \lambda)$?

Suppose that κ is the successor of a singular limit of strongly compact cardinals. Then by a result of Magidor and Shelah [9], κ has the tree property, and in fact, as shown in [9], $\mathrm{TP}(\kappa, \lambda')$ holds for every cardinal $\lambda' \geq \kappa$. We modify the proof so as to obtain the following.

THEOREM 5.10. *Suppose that $\kappa = \nu^+$, where ν is a singular limit of λ-compact cardinals. Then* $\mathrm{PS}(\kappa, \lambda)$ *holds.*

Proof. Set $\sigma = \mathrm{cof}(\nu)$, and select an increasing sequence $\langle \nu_i : i < \sigma \rangle$ of λ-supercompact cardinals with $\sigma < \nu_0$ and $\sup\{\nu_i : i < \sigma\} = \nu$. Now suppose that for each $a \in \mathcal{P}_\kappa(\lambda)$, $\langle Q_a^j : j < \sigma \rangle$ is a sequence of subsets of $\mathcal{P}_\kappa(\lambda)$ with $\bigcup_{j<\sigma} Q_a^j = \mathcal{P}_\kappa(\lambda)$.

Pick a ν_0-complete ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ extending the dual of the non-cofinal ideal $\mathcal{I}_{\kappa,\lambda}$. For $a \in \mathcal{P}_\kappa(\lambda)$, define $f_a : \mathcal{P}_\kappa(\lambda) \to \sigma$ by letting $f_a(e)$ be the least $k < \sigma$ with $e \in \bigcup_{j<\sigma_k} Q_a^j$, and pick $X_a \in U \cap \mathcal{P}(\{v \in \mathcal{P}_\kappa(\lambda) : a \subseteq v\})$ and $k_a < \sigma$ such that f_a takes the constant value k_a on X_a. There must be $S \in \mathcal{I}_{\kappa,\lambda}^+$ and $k < \sigma$ such that $k_a = k$ for all $a \in S$. Define $g : S \times S \to \mathcal{I}_{\kappa,\lambda}$, $h_1 : S \times S \to \nu_k$ and $h_2 : S \times S \to \nu_k$ so that $g(a, x) \in X_a \cap X_x \cap Q_a^{h(a,x)} \cap Q_x^{h_2(a,x)}$.

Pick a ν_{k+1}-complete ultrafilter W on $\mathcal{P}_\kappa(\lambda)$ extending $\mathcal{I}_{\kappa,\lambda}|S$. For $a \in S$, select $Y_a \in W \cap \mathcal{P}_\kappa(\lambda)$ and $r_a, s_a \in \nu_k$ so that h_1 (respectively h_2) takes the constant value r_a (respectively s_a) on $\{a\} \times Y_a$. We may find $B \in \mathcal{I}_{\kappa,\lambda}^+ \cap \mathcal{P}(S)$ and $(r, s) \in \nu_k \times \nu_k$ such that $(r_a, s_a) = (r, s)$ for all $a \in B$. Now fix $a, b \in B$. Pick $x \in Y_a \cap Y_b$ with $a \cup b \subseteq x$. Then:

- $g(a, x) \in Q_a^r \cap Q_x^s$,
- $a \cup x \subseteq g(a, x)$,
- $g(b, x) \in Q_b^r \cap Q_x^s$,
- $b \cup x \subseteq g(b, x)$. ■

To conclude this section we establish that it is possible to have "κ is weakly inaccessible but not strongly inaccessible and $\mathrm{PS}^*(\kappa, \lambda)$ holds".

THEOREM 5.11. *Suppose that κ is supercompact, and let $\mathbb{A}$ be the forcing notion to add κ Cohen reals. Then in $V^{\mathbb{A}}$ the principle* $\mathrm{PS}^*(\kappa, \lambda')$ *holds for any cardinal $\lambda' \geq \kappa$.*

Proof. Let G be generic for $\mathbb{A}$ over V. Then in $V[G]$, κ is not strong limit. We show that

$$V[G] \models \forall \lambda' \geq \kappa \ \ \mathrm{PS}^*(\kappa, \lambda').$$

Fix $\lambda' \geq \kappa$. Assume that in $V[G]$, for every $x \in \mathcal{P}_\kappa(\lambda')$ there is a partition $\{Q_x^i\}_{i<\gamma_x}$ of $\mathcal{P}_\kappa(\lambda')$ where $\gamma_x < \kappa$. Fix in V a λ'-supercompact embedding $j : V \to M$ with critical point κ. Force over $V[G]$ to get a generic object $H \subseteq j(\mathbb{A})$ such that $j[G] \subseteq H$. Since $\mathbb{A}$ is κ-c.c. (it is even $\aleph_1$-c.c.) we see that $j \restriction \mathbb{A}$ is a complete embedding from $\mathbb{A}$ to $j(\mathbb{A})$, so we may lift j to an elementary embedding $j^* : V[G] \to M[H]$ that we rename j.

We work in $V[H]$. Consider $a^* := j[\lambda']$. By the closure of M we have $a^* \in M[H]$. By elementarity, for every $x \in \mathcal{P}_j(\kappa)(j(\lambda'))$ there is a partition $j(Q)_x$ of $\mathcal{P}_\kappa(\lambda)$ into less than $j(\kappa)$ many pieces. Moreover, κ is the critical point of j, so for every $x \in \mathcal{P}_\kappa(\lambda')$, $j(Q)_{j[x]}$ is a partition into γ_x many pieces; let $\{j(Q)^i_{j[x]}\}_{i<\gamma_x}$ enumerate it. For every $x \in \mathcal{P}_\kappa(\lambda')$, we let $\eta_x < \kappa$ be such that $a^* \in j(Q)^{\eta_x}_{j[x]}$. Then for every $x, y \in \mathcal{P}_\kappa(\lambda')$, we have

$$j(Q)^{\eta_x}_{j[x]} \cap j(Q)^{\eta_y}_{j[y]} \cap \{z \in \mathcal{P}_{j(\kappa)}(j(\lambda')) : j[x] \cup j[y] \subseteq z\} \neq \emptyset$$

(because a^* is in this intersection); it follows by elementarity that $Q^{\eta_x}_x \cap Q^{\eta_y}_y \cap \{z \in \mathcal{P}_\kappa(\lambda') : x \cup y \subseteq z\} \neq \emptyset$. So, if we let $h^* \in \prod_{x \in \mathcal{P}_\kappa(\lambda')} Q_x$ be given by $h^*(x) := Q^{\eta_x}_x$, then the function so defined would satisfy the conclusion of $\mathrm{PS}^*(\kappa, \lambda')$ (with $B = \mathcal{P}_\kappa(\lambda')$). However, h is defined in a $j(\mathbb{A})/\mathbb{A}$ generic extension of $V[G]$, so we have instead to prove that such a sequence exists in $V[G]$. We fix names $\langle \dot\eta_x : x \in \mathcal{P}_\kappa(\lambda')\rangle$ for the sequence $\langle \eta_x : x \in \mathcal{P}_\kappa(\lambda')\rangle$.

Now we work in $V[G]$. For every $x \in \mathcal{P}_\kappa(\lambda')$, fix $p_x \in j(\mathbb{A})/\mathbb{A}$ and $\zeta_x < \kappa$ such that $p_x \Vdash \dot\eta_x = \zeta_x$. Fix θ large enough so that all the relevant objects are in H_θ, then let C be the set of all countable elementary substructures $N \prec H_\theta$ containing all the relevant objects for the following argument to work; observe that C is a club. For every $N \in C$ consider $p_{N\cap\lambda'} \cap N$; this is a finite subset of N, hence it actually belongs to N. By Fodor's theorem we can find a condition q and a stationary subset S of C such that for every $N \in S$, $p_{N\cap\lambda'} \cap N = q$. Note that, for every $N \in S$, if $N' \in S$ contains N and $p_{N\cap\lambda'}$ as subsets, then $p_{N\cap\lambda'} \cap p_{N'\cap\lambda'} = p_{N\cap\lambda'} \cap p_{N'\cap\lambda'} \cap N' = p_{N\cap\lambda'} \cap q = q$, hence $p_{N\cap\lambda'}$ and $p_{N'\cap\lambda'}$ are compatible.

Now, let $B := \{N \cap \lambda' : N \in S\}$ and define h as follows: $h(x) := Q^{\zeta_x}_x$ if $x \in B$, and $h(x) := Q^0_x$ otherwise. We claim that B and h satisfy the conclusion of $\mathrm{PS}^*(\kappa, \lambda')$. For $a, b \in B$, let $N, N' \in S$ be such that $a = N \cap \lambda'$ and $b := N' \cap \lambda'$. Pick $O \in S$ such that O contains $N, N', p_{N\cap\lambda'}, p_{N'\cap\lambda'}$ as subsets and let $x := O \cap \lambda'$. Then by the remark above, we have $p_{N\cap\lambda'} \| p_{O\cap\lambda'}$ and $p_{N'\cap\lambda'} \| p_{O\cap\lambda'}$. Let $r \le p_{M\cap\lambda'}, p_{O\cap\lambda'}$ and $s \le p_{N\cap\lambda'}, p_{O\cap\lambda'}$. Then r forces $\zeta_{M\cap\lambda'} = \dot\eta_{O\cap\lambda'}$, and s forces $\zeta_{N\cap\lambda'} = \dot\eta_{O\cap\lambda'}$; this means that

- $Q^{\zeta_a}_a \cap Q^{\zeta_x}_x \cap \{z \in \mathcal{P}_\kappa(\lambda') : x \subseteq z\} \neq \emptyset$,
- $Q^{\zeta_b}_b \cap Q^{\zeta_x}_x \cap \{z \in \mathcal{P}_\kappa(\lambda') : x \subseteq z\} \neq \emptyset$.

Therefore,

- $h(a) \cap h(x) \cap \{z \in \mathcal{P}_\kappa(\lambda') : x \subseteq z\} \neq \emptyset$,
- $h(b) \cap h(x) \cap \{z \in \mathcal{P}_\kappa(\lambda') : x \subseteq z\} \neq \emptyset$,

as required by the property. ∎

Weiß [18] established the consistency relative to a supercompact cardinal of the statement "$\mathrm{TP}(\omega_2, \lambda')$ holds for every cardinal $\lambda' \geq \omega_2$" (and in fact the stronger $\mathrm{ISP}(\omega_2, \lambda')$ holds in his model for every $\lambda' \geq \omega_2$). It remains open whether it is consistent relative to a large cardinal that "$\mathrm{PS}(\omega_2, \lambda')$ holds for every $\lambda' \geq \omega_2$".

6. More on ideal extensions. Let μ be a cardinal with $\kappa \leq \mu \leq \lambda$. We define $p_\mu : \mathcal{P}_\kappa(\lambda) \to \mathcal{P}_\kappa(\mu)$ by $p_\mu(a) = a \cap \mu$. Given a complete fine ideal H on $\mathcal{P}_\kappa(\lambda)$, we let $p_\mu(H) = \{B \subseteq \mathcal{P}_\kappa(\mu) : p_\mu^{-1}(B) \in H\}$. It is easy to see that $p_\mu(H)$ is a κ-complete, fine ideal on $\mathcal{P}_\kappa(\mu)$.

Definition 6.1. For each cardinal $\pi \geq 2$, the principle $\mathrm{IE}(\kappa, \mu, \lambda, \pi)$ asserts the following. Suppose that for each $x \in \mathcal{P}_\kappa(\lambda)$, $\langle Q_x^i : i < \pi\rangle$ is a sequence of pairwise disjoint sets (possibly empty) such that $\bigcup_{i<\pi} Q_x^i = \mathcal{P}_\kappa(\mu)$. Let H be a κ-complete fine ideal on $\mathcal{P}_\kappa(\lambda)$. Then there is $i < \pi$, a κ-complete ideal K on $\mathcal{P}_\kappa(\lambda)$ extending H, and $E \in K^*$ such that $\{Q_x^i : x \in E\} \subseteq p_\mu(K)$.

This section is devoted to the study of $\mathrm{IE}(\kappa, \mu, \lambda, \pi)$. We start with the following, which is readily checked.

Remark 6.2. (1) For any cardinal ν with $\kappa \leq \nu \leq \mu$, and any cardinal $\chi \geq \pi$, $\mathrm{IE}(\kappa, \mu, \lambda, \pi)$ implies $\mathrm{IE}(\kappa, \nu, \lambda, \chi)$.

(2) Suppose that $\mathrm{IE}_\kappa^2(H)$ holds for every κ-complete fine ideal H on $\mathcal{P}_\kappa(\lambda)$. Then $\mathrm{IE}(\kappa, \lambda, \lambda, 2)$ holds.

(3) $\mathrm{IE}(\kappa, \kappa, \kappa, 2)$ holds if and only if κ is weakly compact

Lemma 6.3. *Let J be a κ-complete fine ideal on $\mathcal{P}_\kappa(\mu)$. Then there exists a κ-complete fine ideal H on $\mathcal{P}_\kappa(\lambda)$ such that $p_\mu(H) = J$.*

Proof. Write $K := \{X \subseteq \mathcal{P}_\kappa(\lambda) : p_\mu'' X \in J\}$. Note that $\emptyset \in K$. Let H be the set of all $Z \subseteq \mathcal{P}_\kappa(\lambda)$ such that $Z \subseteq X \cup Y$ for some $X \in K$ and some $Y \in \mathcal{I}_{\kappa,\lambda}$. It is easy to check that H is a κ-complete fine ideal on $\mathcal{P}_\kappa(\lambda)$. For any $B \in J$ we have $p_\mu''(p_\mu^{-1}(B)) \subseteq B$, and therefore $p_\mu^{-1}(B) \in K$. It follows that $J \subseteq p_\mu(H)$. For the reverse inclusion, fix $B \in p_\mu(H)$. Suppose by contraposition that $B \in J^+$. Pick $X \in K$ and $t \in \mathcal{P}_\kappa(\lambda)$ such that $p_\mu^{-1}(B) \subseteq X \cup \{x \in \mathcal{P}_\kappa(\lambda) : t \setminus x \neq \emptyset\}$. We may find $b \in B \setminus p_\mu'' X$ such that $t \cap \mu \subseteq b$. Now let $x = b \cup (t \setminus x)$. Then clearly $p_\mu(x) = b$. Since $x \in p_\mu^{-1}(B)$ and $t \subseteq x$, we must have $x \in X$. It follows that $b \in p_\mu'' X$, which yields the desired contradiction. ∎

Notation 6.4. Let $\mathcal{J}_{\kappa,\mu}$ denote the collection of all κ-complete fine nowhere κ-saturated ideals on $\mathcal{P}_\kappa(\mu)$.

Fact 6.5 (Matet [10]). *Let $X \subseteq \mathcal{J}_{\kappa,\mu}$ with $0 < |X| < \kappa$. Then there is a partition Q of $\mathcal{P}_\kappa(\mu)$ such that $|Q| = \kappa$ and $Q \subseteq \bigcap_{I \in X} I^+$.*

For a cardinal $\pi \geq \lambda$, $\mathcal{A}_{\pi,\lambda}(\kappa,\pi)$ asserts the existence of $X \subseteq \mathcal{P}_\kappa(\lambda)$ of size π such that $|X \cap \mathcal{P}(x)| < \kappa$ for all $x \in \mathcal{P}_\kappa(\lambda)$.

The following is immediate.

Remark 6.6. (1) $\mathcal{A}_{\kappa,\lambda}(\kappa,\lambda)$ holds.

(2) If κ is inaccessible, then $\mathcal{A}_{\kappa,\lambda}(\kappa,\lambda^{<\kappa})$ holds.

Theorem 6.7. *Suppose that there is a cardinal $\pi \geq \min(\lambda, |\mathcal{J}_{\kappa,\mu}|)$ such that $\mathrm{IE}(\kappa,\mu,\lambda,\pi)$ and $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ both hold. Then any κ-complete fine ideal on $\mathcal{P}_\kappa(\mu)$ extends to a κ-saturated κ-complete ideal on $\mathcal{P}_\kappa(\mu)$.*

Proof. Let J be a κ-complete fine ideal on $\mathcal{P}_\kappa(\mu)$. By Lemma 6.3, there is a κ-complete fine ideal H on $\mathcal{P}_\kappa(\lambda)$ such that $p_\mu(H) = J$. Select $Y \subseteq \mathcal{P}_\kappa(\lambda)$ of size π so that $|Y \cap \mathcal{P}(x)| < \kappa$ for all $x \in \mathcal{P}_\kappa(\lambda)$. Pick $Z \subseteq Y$ with $|Z| = |\mathcal{J}_{\kappa,\lambda}|$, and fix a bijection $u : \mathcal{J}_{\kappa,\mu} \to Z$. Define a partial function $G : \mathcal{P}_\kappa(\lambda) \times \mathcal{P}_\kappa(\mu) \to Y$ as follows. Let $x \in \mathcal{P}_\kappa(\lambda)$ with $|Z \cap \mathcal{P}(x)| \neq \emptyset$. By Fact 6.5 we may find C^x_d for $d \in Y \cap \mathcal{P}(x)$ such that:

- $\{C^x_d : d \in Y \cap \mathcal{P}(x)\} \subseteq \bigcap_{I \in u^{-1}(Z \cap \mathcal{P}(x))} I^+$,
- $\bigcup\{C^x_d : d \in Y \cap \mathcal{P}(x)\} = \mathcal{P}_\kappa(\mu)$,
- $C^x_d \cap C^x_e = \emptyset$ for any two distinct elements x, e of $Y \cap \mathcal{P}(x)$.

Now let $G(x,a) = d$ whenever $d \in Y \cap \mathcal{P}(x)$ and $a \in C^x_d$. There must be $d \in Y$, a κ-complete ideal K on $\mathcal{P}_\kappa(\lambda)$ extending H, and $E \in K^*$ such that $\{y \in E : F(x, y \cap \mu) = d\} \in K$ for every $x \in E$.

Claim 6.8. $p_\mu(K) \notin \mathcal{J}_{\kappa,\mu}$.

Proof. Suppose otherwise. Pick $x \in E$ with $d \cup u(p_\mu(K)) \subseteq x$. Then $C^x_d \notin p_\mu(K)$ and therefore $E \cap p_\mu^{-1}(C^x_d) \notin K$. This is a contradiction since

$$E \cap p_\mu^{-1}(C^x_d) \subseteq \{y \in E : F(x, y \cap \mu) = d\}. \ \blacksquare$$

By the claim, we may find $A \in (p_\mu(K))^+$ such that $p_\mu(K)|A$ is κ-saturated. It remains to observe that $J = p_\mu(H) \subseteq p_\mu(K) \subseteq p_\mu(K)|A$. $\blacksquare$

The following is due to Levy and Silver (see [6, Proposition 16.4(b)]).

Fact 6.9. *Suppose that κ is weakly compact, and let K be a κ-saturated, κ-complete fine ideal on $\mathcal{P}_\kappa(\mu)$. Then $K|A$ is prime for some $A \in K^+$.*

Corollary 6.10. *Suppose that κ is weakly compact, $2^{2^{\mu^{<\kappa}}} \leq \lambda^{<\kappa}$ and $\mathrm{IE}(\kappa,\mu,\lambda,\lambda^{<\kappa})$ holds. Then $\mathrm{IE}^1_\kappa(J)$ holds for any κ-complete fine ideal J on $\mathcal{P}_\kappa(\mu)$.*

Proof. Use Remark 6.6, Theorem 6.7 and Fact 6.9. $\blacksquare$

7. A two-cardinal version of the canonical Ramsey theorem for pairs. This section is concerned with partition properties on $\mathcal{P}_\kappa(\lambda)$. To prove our first result we will work, as in the preceding sections, with families of partitions of $\mathcal{P}_\kappa(\lambda)$ into less than κ many pieces. The result can be seen as a $\mathcal{P}_\kappa(\lambda)$ version of the regressive function theorem of Kanamori and McAloon:

FACT 7.1 ([7]). *Suppose that* $0 < n < \omega$, $A \in [\omega]^\omega$, *and* $F : [A]^{n+1} \to \omega$ *is such that* $F(x) < \min(x)$ *for all* $x \in [A]^{n+1}$. *Then there is* $C \in [A]^\omega$ *such that* $F(x) = F(y)$ *whenever* $x, y \in [C]^{n+1}$ *and* $\min(x) = \min(y)$.

We first recall some definitions. Given $A \subseteq \mathcal{P}_\kappa(\lambda)$ and $0 < n < \omega$, we let $[A]^n_\subset = \{(a_1, \ldots, a_n) \in A \times \cdots \times A : a_1 \subset \cdots \subset a_n\}$. We will occasionally abuse notation and write $\{a_1, \ldots, a_n\}$ for $(a_1, \ldots, a_n)$.

Let J be an ideal on $\mathcal{P}_\kappa(\lambda)$. We call J *weakly selective* if, given $A \in J^+$ and $B_a \in J$ for $a \in A$, we can find $C \in J^+ \cap \mathcal{P}(A)$ such that $b \notin B_a$ whenever $a, b \in C$ and $a \subset b$. We let $\mathcal{Z}_J$ denote the collection of all $\mathcal{Z} \subseteq J^+$ such that:

- $\bigcap t \in J^+$ for any $t \in \mathcal{P}_\kappa(Z) \setminus \{\emptyset\}$,
- for any $c \in J^+$, there is $A \in Z$ with $C \setminus A \in J^+$.

We let $\mathfrak{p}_J$ be the least cardinality of any Z in $\mathcal{Z}_J$ if $\mathcal{Z}_J \neq \emptyset$, and $\mathfrak{p}_J = (2^{\lambda^{<\kappa}})^+$ otherwise.

Let ρ be a cardinal greater than or equal to κ. We endow ${}^\rho 2$ with the topology obtained by taking as basic open sets $\emptyset$ and O^ρ_s for $s \in \bigcup\{{}^x 2 : x \in \mathcal{P}_\kappa(\rho)\}$, where $O^\rho_s = \{f \in {}^\rho 2 : s \subset f\}$. We denote by $\mathrm{cov}(M_{\kappa,\rho})$ the least cardinality of any non-empty family of dense open subsets of ${}^\rho 2$ with empty intersection.

FACT 7.2 ([11, 13]). *Let J be a κ-complete fine ideal on $P_\kappa(\lambda)$ such that* $\mathrm{cof}(J) < \mathrm{cov}(M_k, \lambda^{<\kappa})$. *Then J is weakly selective, and moreover* $\mathfrak{p}_J > \lambda^{<\kappa}$.

We will improve the following result of [11]:

FACT 7.3. *Suppose that κ is mildly $\lambda^{<\kappa}$-ineffable and J is a weakly selective, κ-complete, fine ideal on $\mathcal{P}_\kappa(\lambda)$ such that $\mathfrak{p}_J > \lambda^{<\kappa}$. Let $F : [P_\kappa(\lambda)]^{n+1}_\subset \to 2$, where $0 < n < \omega$. Then F is constant on $[D]^n_\subset$ for some $D \in J^+$.*

LEMMA 7.4. *Suppose that κ is mildly $\lambda^{<\kappa}$-ineffable and J is a weakly selective, κ-complete, fine ideal on $\mathcal{P}_\kappa(\lambda)$ such that $\mathfrak{p}_J > \lambda^{<\kappa}$, and σ is a cardinal. Let $0 < n < \omega$, $F : [\mathcal{P}_\kappa(\lambda)]^{n+1}_\subset \to 2$ and $A \in J^+$ be given such that $|\{F(x \cup \{c\}) : \bigcup x \subset c \in A\}| < \kappa$ for all $x \in [\mathcal{P}_\kappa(\lambda)]^n$. Then there are $D \in J^+ \cap \mathcal{P}(A)$ and $f : [\mathcal{P}_\kappa(\lambda)]^n \to \sigma$ such that $F(x \cup \{c\}) = f(x)$ whenever $x \in [\mathcal{P}_\kappa(\lambda)]^n$, $c \in D$ and there is $b \in D$ with $\bigcup x \subseteq b \subset c$.*

Proof. For $x \in [\mathcal{P}_\kappa(\lambda)]^n$ and $i \in \sigma$, set $W^i_x = \{c \in A : \bigcup x \subseteq c$ and $F(x \cup \{c\}) = i\}$. By Proposition 2.6 there is $f : [\mathcal{P}_\kappa(\lambda)]^n \to \sigma$ such that $\bigcap_{x \in X} W^{f(x)}_x \in J^+$ for every $X \in \mathcal{P}_\kappa([\mathcal{P}_\kappa(\lambda)]^n) \setminus \{\emptyset\}$. Select $B \in J^+ \cap \mathcal{P}(A)$

such that $B \setminus W^f_x(x) \in J$ for every $x \in [\mathcal{P}_\kappa(\lambda)]^n$. Finally, pick $D \in J^+ \cap \mathcal{P}(B)$ such that $c \notin \bigcup\{B \setminus W^{f(x)}_x : x \in [\mathcal{P}(b)]^n\}$ whenever $(b,c) \in [D]^2_c$. ■

THEOREM 7.5. *Suppose that κ is mildly $\lambda^{<\kappa}$-ineffable, J is a weakly selective, κ-complete, fine ideal on $\mathcal{P}_\kappa(\lambda)$ such that $\mathfrak{p}_J > \lambda^{<\kappa}$, and σ is a cardinal. Let $0 < n < \omega$, $F : [\mathcal{P}_\kappa(\lambda)]^{n+1}_\subset \to \sigma$ and $A \in J^+$ be given such that for any $b \in \mathcal{P}_\kappa(\lambda)$, $|\{F(b, d_1, \ldots, d_n) : (d_1, \ldots, d_n) \in [A]^n_\subset$ and $b \subset d_1\}| < \kappa$. Then there are $C \in J^+ \cap \mathcal{P}(A)$ and $g : \mathcal{P}_\kappa(\lambda) \to \sigma$ such that:*

(1) *$F(a_1, \ldots, a_{n+1}) = g(a_1)$ whenever $a_1 \in \mathcal{P}_\kappa(\lambda)$, $(a_2, \ldots, a_{n+1}) \in [C]^n_\subset$ and there is $b \in C$ such that $a_1 \subseteq b \subset a_2$;*
(2) *either g is constant on C, or $g(a) < g(b)$ whenever $(a, b) \in [C]^2_\subset$.*

Proof. Using repeatedly Lemma 7.4 define $A_l \in J^+$ and $F_l : [\mathcal{P}_\kappa(\lambda)]^{(n+1)-l} \to \sigma$ for $l \leq n$ so that:

- $A_0 = A$, and $F_0(a_1, \ldots, a_{n+1})$ equals $F(a_1, \ldots, a_{n+1})$ if $(a_2, \ldots, a_{n+1})$ is in A, and 0 otherwise;
- for $l < n$, $A_{l+1} \subseteq A_l$ and $F_l(a_1, \ldots, a_{n-l}, a_{n_l+1}) = F_{l+1}(a_1, \ldots, a_{n-l})$ whenever $a_{n-l+1} \in A_{l+1}$ and there is $b \in A_{l+1}$ such that $a_{n-l} \subseteq b \subset a_{n-l+1}$.

Then clearly, given $a_1 \in \mathcal{P}_\kappa(\lambda)$ and $(a_2, \ldots, a_{n+1}) \in [A_n]^n$ such that $a_1 \subseteq b \subset a_2$ for some $b \in A_n$, we get

$$F(a_1, \ldots, a_{n+1}) = F_0(a_1, \ldots, a_{n+1}) = F_1(a_1, \ldots, a_n) = \cdots = F_n(a_1).$$

Define $f : [A_n]^2_\subset \to 3$ by letting $f(a, b)$ equal 0 if and only if $F_n(a) = F_n(b)$, and 1 if and only if $F_n(a) < F_n(b)$. By Fact 7.3 we may find $C \in J^+ \cap \mathcal{P}(A_n)$ and $s < 3$ such that f takes the constant value s on $[C]^2_\subset$. It is easy to see that $s \neq 2$. ■

In the remainder of this section we restrict our attention to the case $n = 1$. We will prove a $\mathcal{P}_\kappa(\lambda)$ version of the canonical Ramsey theorem for pairs due to Erdős and Rado.

FACT 7.6 ([5]). *For any $F : [\omega]^2 \to \omega$, there is $C \in [\omega]^\omega$ such that one of the following holds:*

(1) *F is constant on $[C]^2$.*
(2) *For $x, y \in [C]^2$, $F(x) = F(y)$ if and only if $\min(x) = \min(y)$.*
(3) *For $x, y \in [C]^2$, $F(x) = F(y)$ if and only if $\max(x) = \max(y)$.*
(4) *F is one-to-one on $[C]^2$.*

Given $Z \subseteq \mathcal{P}_\kappa(\lambda)$ and a function F defined on $[\mathcal{P}_\kappa(\lambda)]^2$, we let $\phi(Z, F)$ assert that $F(e, d) \neq F(a, b)$ whenever $(a, b) \in [\mathcal{P}_\kappa(\lambda)]^2_\subset$, $e, d \in Z$ and there is $c \in Z$ such that $b \cup e \subseteq c \subset d$. Furthermore, we let $\psi(Z, F)$ assert that $F(a, d) \neq F(c, d)$ whenever $(c, d) \in [Z]^2_\subset$ and there is $b \in Z$ such that $a \subseteq b \subset c$.

THEOREM 7.7. *Suppose that κ is mildly $\lambda^{<\kappa}$-ineffable, J is a weakly selective, κ-complete, fine ideal on $\mathcal{P}_\kappa(\lambda)$ and σ is a cardinal. Let $F : [\mathcal{P}_\kappa(\lambda)]^2_\subset \to \sigma$. Then one of the following holds:*

(a) *There is $Q \subseteq J^+$ such that:*
- *$S \cap S' \in J$ for any two distinct elements S, S' of Q,*
- *for any $T \in J^+$, there is $S \in Q$ such that $S \cap T \in J^+$,*
- *for each $S \in Q$, either F is constant on $[S]^2_\subset$, or there is $g : S \to \sigma$ such that for any $(a,b) \in [S]^2_\subset$, $F(a,b) = g(a) < g(b)$.*

(b) *There is $Z \in J^+$ such that:*
- *$\phi(Z,F)$ holds,*
- *$F(a,b) = F(a',b)$ whenever $(a,b),(a',b') \in [Z]^2_\subset$.*

(c) *There is $Z \in J^+$ such that:*
- *$\phi(Z,F)$ and $\psi(Z,F)$ both hold,*
- *$F(a,c) < F(b,c)$ whenever $(a,b,c) \in [Z]^3_\subset$.*

(d) *There is $Z \in J^+$ such that:*
- *$\phi(Z,F)$ and $\psi(Z,F)$ both hold,*
- *$F(a,c) > F(b,c)$ whenever $(a,b,c) \in [Z]^3_\subset$.*

Proof. CASE 1: for any $B \in J^+$, there is $A \in J^+ \cap \mathcal{P}(B)$ such that for any $b \in A$, we have $|\{F(b,d) : b \subset d \in A\}| < \kappa$.

CLAIM 7.8. *Let $B \in J^+$. Then there is $S \in J^+ \cap \mathcal{P}(B)$ such that either F is constant on $[S]^2_\subset$, or there is $g : S \to \sigma$ such that $F(a,b) = g(a) < g(b)$ for every $(a,b) \in [S]^2_\subset$.*

Proof. Select $A \in J^+$ such that $|\{F(b,d) : b \subset d \in A\}| < \kappa$ for all $b \in A$. Define $F' : [\mathcal{P}_\kappa(\lambda)]^2_\subset \to \sigma$ by letting $F'(b,d)$ equal $F(b,d)$ if b and d are both in A, and 0 otherwise. Then by Theorem 7.5, we may find $S \in J^+ \cap \mathcal{P}(A)$ and $g : S \to \sigma$ such that:

- for any $(a,b) \in [S]^2_\subset$, we have $g(a) = F'(a,b) = F(a,b)$,
- either g is constant on S, or $g(a) < g(b)$ whenever $(a,b) \in [S]^2_\subset$.

This proves the claim. ∎

The existence of Q as in (a) follows immediately from the claim.

CASE 2: there exists $B \in J^+$ with the property that for any $A \in J^+ \cap \mathcal{P}(B)$, there is $b \in A$ such that $|\{F(b,d) : b \subset d \in A\}| \geq \kappa$. Set $W := \mathcal{P}_\kappa(\lambda) \times \sigma$. For $(a,i) \in W$, let

$$X^0_{(a,i)} := \{b \in \mathcal{P}_\kappa(\lambda) : a \subset b \text{ and } F(a,b) \neq i\}$$

and $X^1_{(a,i)} = \mathcal{P}_\kappa(\lambda) \setminus X^0_{(a,i)}$. By Theorem 2.13, there is $h : W \to 2$ such that $\bigcap_{(a,i)\in t} X^{h(a,i)}_{(a,i)} \in (J|B)^+$ for all $t \in \mathcal{P}_\kappa(W) \setminus \{\emptyset\}$. We my find $C \in J^+ \cap \mathcal{P}(B)$ such that $C \setminus X^{h(a,i)}_{(a,i)} \in J$ for every $(a,i) \in W$.

Claim 7.9. *The set* $E := \{a \in C : \exists i < \sigma\ (h(a,i) = 1)\}$ *lies in* J.

Proof. Suppose otherwise. Define $f : E \to \sigma$ so that $h(a, f(a)) = 1$ for all $a \in E$. We may find $E' \in J^+ \cap \mathcal{P}(E)$ such that $b \in X^1_{(a,f(a))}$ whenever $(a,b) \in [E']^2_{\subset}$. But then for every $a \in E'$, $|\{F(a,b) : a \subset b \in E'\}| = 1 < \kappa$. This contradiction completes the proof of the claim. ∎

Let $D = C \setminus E$. For $c \in D$, set $x_c := \{F(a,b) : (a,b) \in [\mathcal{P}(c)]^2_{\subset}\}$. Now define $G : [D]^2_{\subset} \to 2$ by letting $G(c,d) = 0$ if and only if there is e in $D \cap \mathcal{P}(c)$ such that $F(e,d) \in x_c$. By Fact 7.3 there must be $D' \in J^+ \cap \mathcal{P}(D)$ and $u < 2$ such that G takes the constant value u on $[D']^2_{\subset}$.

Claim 7.10. $u = 1$.

Proof. Suppose otherwise. Pick $c \in D'$, and let $D'' := \{d \in D' : c \subset d\}$. Since $\mathcal{P}(c)$ and x_c have both size less than κ, we may find $D''' \in J^+ \cap \mathcal{P}(D'')$, $e \in D \cap \mathcal{P}(c)$ and $i \in x_c$ such that $F(e,d) = i$ for all $d \in D'''$. But then $C \setminus X^0_{(e,i)} \in J^+$. This contradiction completes the proof of the claim. ∎

It follows from this claim that $\phi(Z,F)$ holds for every $Z \in J^+ \cap \mathcal{P}(D')$. For $y \in \mathcal{P}_\kappa(\lambda)$, define $q_y : [D']^2_{\subset} \to 2$ by letting $q_y(a,b) = 0$ if and only if $y \subset b$ and $F(y,b) = F(a,b)$.

Subcase 1: there is $y \in \mathcal{P}_\kappa(\lambda)$ and $Y \in J^+ \cap \mathcal{P}(D')$ such that q_y takes the constant value 0 on $[Y]^2_{\subset}$. Then $F(a,b) = F(y,b) = F(a',b)$ whenever $(a,b), (a',b) \in [Y]^2_{\subset}$.

Subcase 2: for any $z \in \mathcal{P}_\kappa(\lambda)$ and any $L \in J^+ \cap \mathcal{P}(D')$, q_z is not identically 0 on $[L]^2_{\subset}$. Define $p : [D']^3_{\subset} \to 2$ by $p(b,c,d) = 0$ if and only if $F(a,d) = F(c,d)$ for some $a \subseteq b$. By Fact 7.3 we may find $H \in J^+ \cap \mathcal{P}(D')$ and $v < 2$ such that p is identically v on $[H]^2_c$.

Claim 7.11. $v = 1$.

Proof. Suppose otherwise. Pick $b \in H$ and set $H' := \{c \in H : b \subset c\}$. Define $p' : [H']^2_{\subset} \to \mathcal{P}(b)$ so that $F(p'(c,d),d) = F(c,d)$. By Fact 7.3 there must be $H'' \in J^+ \cap \mathcal{P}(H')$ and $a \subseteq b$ such that p' is constant with value a on $[H'']^2_{\subset}$. But then q_a is identically 0 on $[H'']^2_{\subset}$. This contradiction completes the proof of the claim. ∎

Note that by this claim, $\psi(Z,F)$ holds for every $Z \in J^+ \cap \mathcal{P}(H)$. Finally, define $K : [H]^3_{\subset} \to 2$ by letting $K(a,b,c) = 0$ if and only if $F(a,c) < F(b,c)$. By Fact 7.3 there are $Z \in J^+ \cap \mathcal{P}(H)$ and $j < 2$ such that $K(a,b,c) = j$ for every $(a,b,c) \in [Z]^3_{\subset}$. It remains to observe that if $j = 1$, then $F(a,c) > F(b,c)$ for each $(a,b,c) \in [Z]^3_{\subset}$. ∎

Remark 7.12. Note that (d) can only occur in case $\kappa = \omega$. Concerning (a), let us observe the following. Suppose $\lambda > \kappa$ and $F : [\mathcal{P}_\kappa(\lambda)]^2_{\subset} \to \kappa$ is defined by $F(a,b) = |a|$. Then clearly it is not possible to find $S \in J^+$

and a <u>one-to-one</u> function $g : S \to \kappa$ such that $F(a, b) = g(a)$ for every $(a, b) \in [S]^2_{\subset}$. Similarly, in (c) and (d), it may be impossible to find $Z \in J^+$ such that $F(a, b) \neq F(b, c)$ whenever $(a, c), (b, c) \in [Z]^2_{\subset}$. To see this, suppose that $\kappa = \omega < \lambda$ and define $F : [\mathcal{P}_\kappa(\lambda)]^2_{\subset} \to \kappa$ by $F(x, y) = |x| + |y|$ and $F'(x, y) = |y| - |x|$. Now given $Z \in J^+$, we may find $a \neq b$ in Z such that $|a| = |b|$. Pick $c \in Z$ with $a \cup b \subset c$. Then clearly $F(a, c) = F(b, c)$ and $F'(a, c) = F'(b, c)$.

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References

[1] F. G. Abramson, L. A. Harrington, E. M. Kleinberg and W. S. Zwicker, *Flipping properties: a unifying thread in the theory of large cardinals*, Ann. Math. Logic 12 (1977), 25–58.

[2] D. M. Carr, *$\mathcal{P}_\kappa\lambda$-generalizations of weak compactness*, Z. Math. Logik Grundlag. Math. 31 (1985), 393–401.

[3] C. A. Di Prisco and W. S. Zwicker, *Flipping properties and supercompact cardinals*, Fund. Math. 109 (1980), 31–36.

[4] P. Erdős, A. Hajnal, A. Máté and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Stud. Logic Found. Math. 106, North-Holland, Amsterdam, 1984.

[5] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. 25 (1950), 249–255.

[6] A. Kanamori, *The Higher Infinite*, Perspect. Math. Logic, Springer, Berlin, 1994.

[7] A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, Ann. Pure Appl. Logic 33 (1987), 23–41.

[8] K. Kunen, *On the GCH at measurable cardinals*, in: Logic Colloquium ‘69, R. O. Gandy and C. E. M. Yates (eds.), North-Holland, Amsterdam, 1971, 107–110.

[9] M. Magidor and S. Shelah, *The tree property at successors of singular cardinals*, Arch. Math. Logic 35 (1996), 385–404.

[10] P. Matet, *Strong compactness and a partition property*, Proc. Amer. Math. Soc. 134 (2006), 2147–2152.

[11] P. Matet, *Covering for category and combinatorics on $\mathcal{P}_\kappa(\lambda)$*, J. Math. Soc. Japan 58 (2006), 153–181.

[12] P. Matet, *Concerning stationary subsets of* $[\lambda]^{<\kappa}$, in: Set Theory and Its Applications, J. Steprāns and S. Watson (eds.), Lecture Notes in Math. 1401, Springer, Berlin, 1989, 119–127.

[13] P. Matet, C. Péan and S. Todorcevic, *Prime ideals on $\mathcal{P}_\omega(\lambda)$ with the partition property*, Arch. Math. Logic 41 (2002), 743–764.

[14] T. K. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic 7 (1974), 327–359.

[15] T. Suzuki, *About splitting numbers*, Proc. Japan Acad. Ser. A Math. Sci. 74 (1998), 33–35.

[16] T. Usuba, *Ineffability of $\mathcal{P}_\kappa\lambda$ for λ with small cofinality*, J. Math. Soc. Japan 60 (2008), 935–954.
[17] C. Weiß, *Subtle and ineffable tree properties*, Ph.D. thesis, Ludwig-Maximilians-Universität München, 2010.
[18] C. Weiß, *The combinatorial essence of supercompactness*, Ann. Pure Appl. Logic 163 (2012), 1710–1717.

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