COLLOQUIUM MATHEMATICUM

VOL. 144 | 2016 | NO. 2

STRONGLY PARACOMPACT METRIZABLE SPACES

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Abstract. Strongly paracompact metrizable spaces are characterized in terms of special S-maps onto metrizable non-Archimedean spaces. A similar characterization of strongly metrizable spaces is obtained as well. The approach is based on a sieve-construction of "metric"-continuous pseudo-sections of lower semicontinuous mappings.

1. Introduction. All spaces in this paper are assumed to be Hausdorff topological spaces. A cover $\mathscr{U}$ of a space X is *star-finite* if the set $\{W \in \mathscr{U} : W \cap U \neq \emptyset\}$ is finite for every $U \in \mathscr{U}$. A space X is *strongly paracompact* (also called *hypocompact*) if every open cover of X has a star-finite open refinement. Every strongly paracompact space is paracompact, but the converse is not necessarily true (see for instance [4]).

A space Z is *non-Archimedean* if it has a base such that if B_1 and B_2 are members of this base with $B_1 \cap B_2 \neq \emptyset$, then either $B_1 \subset B_2$ or $B_2 \subset B_1$. Sometimes a base with this property is said to be *of rank* 1 (see [15]). A typical example of a metrizable non-Archimedean space is the countable power T^ω of a discrete space T. This space is often called the *Baire space of weight* $\tau = |T|$, and denoted by $B(\tau)$.

According to a result of Morita [13] (see also [18, Theorem 2.3]), every strongly paracompact metrizable space X is a subset of $[0,1]^\omega \times B(\tau)$ for $\tau = w(X)$ being the weight of X. A similar result was obtained by Smirnov [19, Theorem 5], namely that every strongly paracompact metrizable space can be mapped continuously onto a non-Archimedean metrizable space by an S-map. Here, a map $g : X \to Z$ is an *S-map* if each $g^{-1}(z)$, $z \in Z$, is second countable. Smirnov's paper [19] contains a list of examples of such S-images of metrizable spaces. Briefly, it was shown in [19] that there exists a metrizable space which cannot be mapped continuously onto a non-Archimedean metrizable space by an S-map; that there exists a metrizable space which

2010 *Mathematics Subject Classification*: 54C60, 54C65, 54D20, 54E40, 54F65.
Key words and phrases: strongly paracompact space, strongly metrizable space, set-valued mapping, set-valued selection, semicontinuity, subcontinuity.
Received 12 May 2015; revised 29 December 2015.
Published online 10 March 2016.

DOI: 10.4064/cm6630-1-2016

is not strongly paracompact, but can be mapped continuously onto a non-Archimedean metrizable space by an S-map; that there exists a metrizable strongly paracompact space which cannot be mapped continuously onto a non-Archimedean metrizable space by a closed S-map. To these examples, let us also add a result of Nagata [14] that the product $(0,1) \times B(\omega_1)$ is not strongly paracompact. Hence, not every subset of $[0,1]^\omega \times B(\tau)$ is strongly paracompact.

In this paper, we are interested in S-maps with an extra property of being inversely "subcontinuous" (see Section 4). Let (X,ρ) be a metric space. For $\varepsilon > 0$, the *open ε-ball* centred at $x \in X$ is $B^\rho_\varepsilon(x) = \{y \in X : \rho(y,x) < \varepsilon\}$. We say that $g : X \to Z$ is a *strongly S-map* if for every $\varepsilon > 0$, every $z \in Z$ has a neighbourhood $V \subset Z$ such that $g^{-1}(V)$ is covered by countably many open ε-balls. The following theorem will be proved.

THEOREM 1.1. *A metrizable space X is strongly paracompact if and only if for every compatible (with the topology) metric ρ on X, the metric space (X,ρ) can be mapped continuously onto a non-Archimedean metrizable space by a strongly S-map.*

A metrizable space is called *strongly metrizable* if it has a base which is the union of countably many star-finite open covers. Following the arguments in [18], it was shown in [17, Proposition 3.27] that each strongly metrizable space of weight τ is a subset of $[0,1]^\omega \times B(\tau)$. Accordingly, strongly metrizable spaces of weight $\leq \tau$ are precisely the subsets of $[0,1]^\omega \times B(\tau)$. Another interesting result is that X is strongly metrizable if and only if it admits a compatible metric d such that, for each $\varepsilon > 0$, the collection of all open ε-balls $B^d_\varepsilon(x)$, $x \in X$, is locally finite [1]. Let d_1 be a metric on $[0,1]^\omega$, d_2 be a metric on $B(\tau)$, and $d = \max\{d_1, d_2\}$ be the box-metric on $[0,1]^\omega \times B(\tau)$, all compatible with the corresponding topologies. Given a subset $X \subset [0,1]^\omega \times B(\tau)$, let ρ be the restriction of d on X, and $g = \pi\restriction X$ be the restriction of the projection $\pi : [0,1]^\omega \times B(\tau) \to B(\tau)$. Then $g : X \to Z = g(X) \subset B(\tau)$ is a continuous strongly S-map of X onto the non-Archimedean space Z. This shows that the requirement "for every compatible (with the topology) metric ρ on X" in Theorem 1.1 is essential. In fact, this gives the following characterization of strongly metrizable spaces illustrating the subtle difference from strongly paracompact metrizable ones.

THEOREM 1.2. *A metrizable space X is strongly metrizable if and only if X admits a compatible (with the topology) metric ρ so that (X,ρ) can be mapped continuously onto a non-Archimedean metrizable space by a strongly S-map.*

The proof of Theorem 1.1 is based on a new sieve-construction of set-valued mappings when the range is not necessarily completely metrizable;

this is done in the next section. In Section 3, this is applied to construct pseudo-sections of lower semicontinuous mappings defined on strongly paracompact spaces (see Theorem 3.3). Section 4 is devoted to strongly S-maps, relating them to separable ρ-subcontinuous set-valued mappings (see Theorem 4.2). Finally, Theorems 1.1 and 1.2 are proved in Sections 5 and 6, respectively.

2. A sieve-construction. A partially ordered set $(T, \preceq)$ is a *tree* if $\{s \in T : s \preceq t\}$ is well ordered for every $t \in T$. For a tree $(T, \preceq)$, we use $T(0)$ to denote the set of all minimal elements of T. Given an ordinal α, if $T(\beta)$ is defined for every $\beta < \alpha$, then $T(\alpha)$ denotes the minimal elements of $T \setminus \bigcup_{\beta<\alpha} T(\beta)$. The set $T(\alpha)$ is called the *αth level* of T, while the *height* of T is the least ordinal α such that $T = \bigcup_{\beta<\alpha} T(\beta)$. We say that $(T, \preceq)$ is an *α-levelled* tree if its height is α. A maximal linearly ordered subset of a tree $(T, \preceq)$ is called a *branch*, and $\mathscr{B}(T)$ is used to denote the set of all branches of T. A tree $(T, \preceq)$ is *pruned* if every element of T has a successor in T, equivalently if for every $s \in T$ there exists $t \in T$ with $s \prec t$. In these terms, an ω-levelled tree $(T, \preceq)$ is pruned if each branch $\beta \in \mathscr{B}(T)$ is infinite.

Non-Archimedean spaces are naturally related to trees. According to [16, Theorem 2.9], every non-Archimedean space has a base which is a tree with respect to reverse inclusion. The relation is not formal, and following Nyikos [16], for a tree $(T, \preceq)$ and $t \in T$, let

$$\mathscr{O}(t) = \{\beta \in \mathscr{B}(T) : t \in \beta\}. \tag{2.1}$$

If $(T, \preceq)$ is a pruned ω-levelled tree, then the family $\{\mathscr{O}(t) : t \in T\}$ is a base for a completely metrizable non-Archimedean topology on $\mathscr{B}(T)$. We will refer to this topology as the *branch topology*, and to the resulting topological space as the *branch space*. Throughout this paper, $\mathscr{B}(T)$ will be always endowed with the branch topology when it comes to consider it as a topological space.

For a set Y, let 2^Y be the collection of all subsets of Y. Given a pruned ω-levelled tree $(T, \preceq)$, a set-valued mapping $\mathscr{R} : T \to 2^Y$ is a *sieve* on Y if

$$X = \bigcup_{t \in \mathrm{node}(\emptyset)} \mathscr{R}(t) \quad \text{and} \quad \mathscr{R}(t) = \bigcup_{s \in \mathrm{node}(t)} \mathscr{R}(s) \quad \text{for every } t \in T.$$

Here, $\mathrm{node}(\emptyset) = T(0)$ and $\mathrm{node}(t) \subset T$ is the set of all immediate successors of t, called the *node* of t in T. To every sieve $\mathscr{R} : T \to 2^Y$ one can associate the mapping $\Omega_{\mathscr{R}} : \mathscr{B}(T) \to 2^Y$ defined by $\Omega_{\mathscr{R}}(\beta) = \bigcap_{t\in\beta} \mathscr{R}(t)$, $\beta \in \mathscr{B}(T)$. It is commonly called the *polar mapping* associated to $\mathscr{R}$. The *inverse polar mapping* $\Omega_{\mathscr{R}}^{-1} : Y \to 2^{\mathscr{B}(T)}$ is defined by $\Omega_{\mathscr{R}}^{-1}(y) = \{\beta \in \mathscr{B}(T) : y \in \Omega_{\mathscr{R}}(\beta)\}$, $y \in Y$, and also denoted by $\mho_{\mathscr{R}}$. The polar mappings $\Omega_{\mathscr{R}}$ and $\mho_{\mathscr{R}}$ were used in several constructions (see for instance [6–9]).

We now turn to another natural construction associating a set-valued mapping to a sieve. Let (Y,ρ) be a metric space. For a subset $A\subset Y$, we write

$$B^\rho_\varepsilon(A)=\{y\in Y:\rho(y,A)<\varepsilon\}=\bigcup_{a\in A}B^\rho_\varepsilon(a).$$

A mapping (usually nonempty-valued) $\varphi:X\to 2^Y$ is called *ρ-continuous* (sometimes also *Hausdorff continuous*) if for every $\varepsilon>0$, every $x\in X$ has a neighbourhood U with $\varphi(x)\subset B^\rho_\varepsilon(\varphi(z))$ and $\varphi(z)\subset B^\rho_\varepsilon(\varphi(x))$ for every $z\in U$. In the proposition below and the rest of the paper, we will use $\mathscr{S}(Y)$ for the collection of all nonempty, closed and second countable subsets of Y:

$$\mathscr{S}(Y)=\{S\subset Y: S \text{ is nonempty, closed and second countable}\}.$$

Proposition 2.1. *Let (Y,ρ) be a metric space, $(T,\preceq)$ be a pruned ω-levelled tree, and $\eta:T\to\mathscr{S}(Y)$ be a mapping such that $\eta(s)\subset B^\rho_{2^{-n}}(\eta(t))$ for every $t\in T(n)$ and $s\in\operatorname{node}(t)$. Define $\Sigma_\eta:\mathscr{B}(T)\to\mathscr{S}(Y)$ by $\Sigma_\eta(\beta)=\overline{\bigcup_{t\in\beta}\eta(t)}$, $\beta\in\mathscr{B}(T)$. Then Σ_η is ρ-continuous.*

Proof. It is evident that Σ_η is separable-valued. Take $\varepsilon>0$ and $n<\omega$ with $2^{-n+1}<\varepsilon$. Next, take $t\in T(n)$ and branches $\alpha,\beta\in\mathscr{O}(t)\subset\mathscr{B}(T)$. To show that Σ_η is ρ-continuous, it suffices to show that $\Sigma_\eta(\alpha)\subset B^\rho_\varepsilon(\Sigma_\eta(\beta))$. To this end, for convenience, let $\alpha=\{s_k:k<\omega\}$ and $\beta=\{t_k:k<\omega\}$, where $s_k,t_k\in T(k)$, $k<\omega$. Since $\eta(s_{k+1})\subset B^\rho_{2^{-k}}(\eta(s_k))$ for every $k<\omega$, it follows that

$$\eta(s_{n+k+1})\subset B^\rho_{2^{-n}+\dots+2^{-(n+k)}}(\eta(s_n))\subset B^\rho_{2^{-n+1}}(\eta(s_n)),\qquad k\ge 1.$$

By (2.1), $s_k=t_k$ for every $k\le n$. Therefore, $\eta(s_k)=\eta(t_k)\subset\Sigma_\eta(\beta)$ for $k\le n$, and $\eta(s_k)\subset B^\rho_{2^{-n+1}}(\eta(s_n))=B^\rho_{2^{-n+1}}(\eta(t_n))\subset B^\rho_{2^{-n+1}}(\Sigma_\eta(\beta))$ for every $k>n$. Since $2^{-n+1}<\varepsilon$, we finally have $\Sigma_\eta(\alpha)=\overline{\bigcup_{k<\omega}\eta(s_k)}\subset B^\rho_\varepsilon(\Sigma_\eta(\beta))$. ∎

For a set D, let $[D]^{\le\omega}=\{S\subset D:1\le|S|\le\omega\}$. Following [8], for a pruned ω-levelled tree $(T,\preceq)$, set $\mathbf{L}_T=\bigcup_{n<\omega}[T(n)]^{\le\omega}$. Next, define a relation $\preceq$ on $\mathbf{L}_T$ by letting $\sigma\prec\mu$ for $\sigma,\mu\in\mathbf{L}_T$ if

$$\mu\subset\bigcup_{s\in\sigma}\operatorname{node}(s)\quad\text{and}\quad\mu\cap\operatorname{node}(s)\neq\emptyset,\ s\in\sigma.\tag{2.2}$$

Finally, extend the relation to a partial order on $\mathbf{L}_T$ by making it transitive. Thus, we get a pruned ω-levelled tree $(\mathbf{L}_T,\preceq)$ because so is T. We now have the following consequence (cf. [8, Theorem 2.2]).

Corollary 2.2. *Let (Y,ρ) be a metric space, $(T,\preceq)$ be a pruned ω-levelled tree, and $h:T\to Y$ be a map such that $\rho(h(s),h(t))<2^{-(n+1)}$ for every $t\in T(n)$ and $s\in\operatorname{node}(t)$. Define $\eta:\mathbf{L}_T\to\mathscr{S}(Y)$ by $\eta(\sigma)=\overline{h(\sigma)}$*

for every $\sigma \in \mathbf{L}_T$. Then the mapping $\Sigma_\eta : \mathscr{B}(\mathbf{L}_T) \to \mathscr{S}(Y)$ defined by $\Sigma_\eta(\boldsymbol{\beta}) = \overline{\bigcup_{\sigma\in\boldsymbol{\beta}} \eta(\sigma)}$, $\boldsymbol{\beta} \in \mathscr{B}(\mathbf{L}_T)$, is ρ-continuous.

Proof. This follows from Proposition 2.1 because $\sigma \in \mathbf{L}_T(n)$ and $\mu \in \operatorname{node}(\sigma)$ implies $\eta(\mu) \subset B^{\rho}_{2^{-n}}(\eta(\sigma))$ (see (2.2)). ∎

3. Factorizing pseudo-sections. For a space Y, let $\mathscr{F}(Y)$ be the collection of all nonempty closed subsets of Y. A mapping $\Phi : X \to \mathscr{F}(Y)$ is *lower semicontinuous*, or l.s.c., if the set

$$\Phi^{-1}[U] = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$$

is open in X for every open $U \subset Y$. The following characterization of strongly paracompact spaces was obtained in [8, Theorem 1.3].

THEOREM 3.1. *A space X is strongly paracompact if and only if for every complete metric space (Y, ρ), every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has a ρ-continuous section $\varphi : X \to \mathscr{S}(Y)$.*

Here, φ is a *section* of Φ if $\varphi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$. In fact, Theorem 3.1 was obtained as a consequence of the following slightly more general result (see [8, Corollary 3.5]).

THEOREM 3.2. *A space X is strongly paracompact if and only if for every complete metric space (Y, ρ), every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ admits a ρ-continuous mapping $\varphi : X \to \mathscr{S}(Y)$ with*

$$\rho(\varphi(x), \Phi(x)) = \inf\{\rho(y, z) : y \in \varphi(x) \text{ and } z \in \Phi(x)\} = 0, \qquad x \in X.$$

In the setting of these theorems, just paracompactness is enough to construct a compact-valued l.s.c. mapping $\Psi : X \to \mathscr{F}(Y)$ with $\Psi(x) \subset \Phi(x)$, $x \in X$ [12, Theorem 1.1]. By Theorem 3.2 applied to this Ψ, there exists a ρ-continuous $\varphi : X \to \mathscr{S}(Y)$ with $\rho\big(\varphi(x), \Psi(x)\big) = 0$ for every $x \in X$. Accordingly, φ will be a section of Ψ (hence, of Φ as well) because Ψ is nonempty-compact-valued. Thus, Theorem 3.2 implies Theorem 3.1. The purpose of this section is to show that Theorem 3.2 remains valid without the requirement on (Y, ρ) to be complete.

THEOREM 3.3. *Let X be a strongly paracompact space, (Y, ρ) be a metric space, and $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping. Then there exists a non-Archimedean metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{S}(Y)$ such that $\rho\big(\psi(g(x)), \Phi(x)\big) = 0$ for every $x \in X$.*

Proof. It is a slight modification of the proof of [8, Theorem 3.3]. Namely, take a nonempty-open-valued locally finite sieve $\mathscr{M} : T \to 2^Y$ on Y such that $\operatorname{diam}_\rho(\mathscr{M}(t)) < 2^{-(n+1)}$ for every $t \in T(n)$ and $n < \omega$. This means that each $\mathscr{M}(t)$, $t \in T$, is open and each collection $\{\mathscr{M}(t) : t \in T(n)\}$, $n < \omega$, is locally finite. Next, consider the tree $(\mathbf{L}_T, \preceq)$ associated to T as in (2.2), and define $\mathscr{P} : \mathbf{L}_T \to 2^X$ by

$$\mathscr{P}(\sigma) = \bigcup_{t\in\sigma} \Phi^{-1}[\mathscr{M}(t)], \qquad \sigma \in \mathbf{L}_T\,. \tag{3.1}$$

Being strongly paracompact, by [8, Proposition 3.2], the space X has an open-valued sieve $\mathscr{L} : \mathbf{L}_T \to 2^X$ such that $\mathscr{L}(\sigma) \subset \mathscr{P}(\sigma)$, $\sigma \in \mathbf{L}_T$, and each collection $\{\mathscr{L}(\sigma) : \sigma \in \mathbf{L}_T(n)\}$, $n < \omega$, is discrete. As in the proof of [8, Theorem 3.3], the inverse polar mapping $\mho_{\mathscr{L}} = \Omega_{\mathscr{L}}^{-1} : X \to 2^{\mathscr{B}(\mathbf{L}_T)}$ is singleton-valued and continuous. So, we can take $Z = \mathscr{B}(\mathbf{L}_T)$ and $g = \mho_{\mathscr{L}}$.

As for $\psi : Z \to \mathscr{S}(Y)$, define $h : T \to Y$ by $h(t) \in \mathscr{M}(t)$, $t \in T$, so that $\rho(h(s), h(t)) < 2^{-(n+1)}$ for every $t \in T(n)$ and $s \in \operatorname{node}(t)$. Let $\eta : \mathbf{L}_T \to \mathscr{S}(Y)$ and $\Sigma_\eta : \mathscr{B}(\mathbf{L}_T) \to \mathscr{S}(Y)$ be defined as in Corollary 2.2, associated to this h. Then, by this corollary, Σ_η is ρ-continuous and we can take $\psi = \Sigma_\eta$. To show that these Z, g and ψ are as required, take a point $x \in X$ and the branch $\boldsymbol{\beta} \in \mathscr{B}(\mathbf{L}_T)$ such that $x \in \mathscr{L}(\sigma)$ for every $\sigma \in \boldsymbol{\beta}$. Write $\boldsymbol{\beta} = \{\sigma_n : n < \omega\}$, where $\sigma_n \in \mathbf{L}_T(n)$ for each $n < \omega$. Since $\mathscr{L}(\sigma) \subset \mathscr{P}(\sigma)$ for every $\sigma \in \boldsymbol{\beta}$, by (3.1) for every $n < \omega$ there exists $t_n \in \sigma_n$ with $\Phi(x) \cap \mathscr{M}(t_n) \neq \emptyset$. Since $h(t_n) \in \mathscr{M}(t_n)$ and $\eta(\sigma_n) = \overline{h(\sigma_n)}$, it follows that $\rho(\eta(\sigma_n), \Phi(x)) \le \rho(h(t_n), \Phi(x)) < 2^{-(n+1)}$. Therefore, $\rho(\psi(g(x)), \Phi(x)) = \rho(\Sigma_\eta(\mho_{\mathscr{L}}(x)), \Phi(x)) = 0$ because $\eta(\sigma_n) \subset \Sigma_\eta(\boldsymbol{\beta})$ for every $n < \omega$. ■

4. Separable subcontinuity. A map $f : X \to Y$ is a *selection* (or a *single-valued selection*) for $\Phi : X \to 2^Y$ if $f(x) \in \Phi(x)$ for every $x \in X$. A mapping $\psi : X \to 2^Y$ is a *set-valued selection* (also called a *multi-selection*, or a *subset-selection*) for $\Phi : X \to 2^Y$ if $\psi(x) \subset \Phi(x)$ for every $x \in X$. For convenience, we sometimes write $\psi \subset \Phi$ to express that ψ is a set-valued selection for Φ. A mapping $\psi : X \to \mathscr{F}(Y)$ is called *upper semicontinuous*, or u.s.c., if the set

$$\psi^\#[U] = X \setminus \psi^{-1}[Y \setminus U] = \{x \in X : \psi(x) \subset U\}$$

is open in X for every open $U \subset Y$. We often say that $\psi : X \to \mathscr{F}(Y)$ is *usco* if it is u.s.c. and compact-valued.

In this section, we are interested in those set-valued mappings which are set-valued selections for separable-valued "metric"-u.s.c. mappings. The problem seems naturally related to a property known as *subcontinuity*. A map $f : X \to Y$ is called *subcontinuous* if for every net $\{x_\alpha : \alpha \in D\} \subset X$ convergent in X, the net $\{f(x_\alpha) : \alpha \in D\} \subset Y$ has a convergent subnet [5]. It was shown in [5, Theorem 3.4] that, for a Hausdorff space Y, a map $f : X \to Y$ is continuous if and only if it is subcontinuous and has a closed graph. The result was naturally extended to set-valued mappings in [20], where a nonempty-valued mapping $\varphi : X \to 2^Y$ was called *subcontinuous* if for every net $\{x_\alpha : \alpha \in D\} \subset X$ convergent in X and $y_\alpha \in \varphi(x_\alpha)$, $\alpha \in D$, the net $\{y_\alpha : \alpha \in D\} \subset Y$ has a convergent subnet.

It was shown in [20, Theorem 3.1] that each subcontinuous mapping $\varphi : X \to 2^Y$ with a closed graph $\{(x, y) : y \in \varphi(x)\} \subset X \times Y$ is usco. A natural characterization of *closed-graph* mappings was given in [10, Theorem 3.3], where it was shown that a nonempty-valued $\varphi : X \to 2^Y$ has a closed graph if and only if $\varphi(x) = \bigcap\{\overline{\varphi[V]} : V \subset X \text{ is open and } x \in V\}$ for every $x \in X$. Here, $\varphi[V] = \bigcup_{x\in V} \varphi(x)$ is the image of V by φ. Another natural characterization of subcontinuity of set-valued mappings was stated in [3, Theorem 7.1] and credited to [11]. It asserts that a nonempty-valued $\varphi : X \to 2^Y$ is subcontinuous if and only if for every open cover $\mathscr{U}$ of Y, every $x \in X$ has a neighbourhood V such that $\varphi[V]$ is covered by finitely many members of $\mathscr{U}$. Gathering all these results, we have the following interesting characterization of subcontinuity.

Theorem 4.1. *For a nonempty-valued mapping $\varphi : X \to 2^Y$ into a regular space Y, the following are equivalent*:

(a) *φ is subcontinuous.*
(b) *For every open cover $\mathscr{U}$ of Y, every $x \in X$ has a neighbourhood V such that $\varphi[V]$ is covered by finitely many members of $\mathscr{U}$.*
(c) *φ is a set-valued selection for some usco mapping.*

In the present section, we are interested in the equivalence of (b) and (c), which allows us to extend subcontinuity to separable-valued mappings in metric spaces. Namely, given a metric space (Y, ρ), we shall say that a nonempty-valued mapping $\varphi : X \to 2^Y$ is *separable ρ-subcontinuous* if for every $\varepsilon > 0$, every $x \in X$ is contained in an open set $V \subset X$ such that $\varphi[V]$ is covered by countably many open ε-balls. The following theorem will be proved.

Theorem 4.2. *Let X be a paracompact space and (Y, ρ) be a metric space. Then a nonempty-valued mapping $\varphi : X \to 2^Y$ is separable ρ-subcontinuous if and only if it is a set-valued selection for some ρ-u.s.c. mapping $\psi : X \to \mathscr{S}(Y)$.*

Here, $\psi : X \to 2^Y$ is *ρ-u.s.c.* if for every $\varepsilon > 0$, every $x \in X$ has a neighbourhood $U \subset X$ with $\psi(z) \subset B^\rho_\varepsilon(\psi(x))$ for every $z \in U$.

The proof of Theorem 4.2 is based on the technique of sieves developed in the previous sections, and the following observations.

Proposition 4.3. *Let X be a space, (Y, ρ) be a metric space and $\varphi : X \to 2^Y$ be a nonempty-valued mapping. For a point $x \in X$, the following are equivalent*:

(a) *There exists* $A \in \mathscr{S}(Y)$ *such that* $\varphi^{\#}[B^{\rho}_{\varepsilon}(A)]$ *is a neighbourhood of* x *for every* $\varepsilon > 0$.

(b) *For every* $\varepsilon > 0$ *there exists a neighbourhood* $V \subset X$ *of* x *such that* $\varphi[V]$ *is covered by countably many open* ε*-balls.*

Proof. The implication (a)⇒(b) is obvious. Briefly, if $A \in \mathscr{S}(Y)$ is as in (a) and $\varepsilon > 0$, take an open set $V \subset X$ with $x \in V \subset \varphi^{\#}[B^{\rho}_{\varepsilon}(A)]$. Since A is separable, it has a countable dense subset $C \subset A$. Accordingly, $\varphi[V] \subset B^{\rho}_{\varepsilon}(A) = B^{\rho}_{\varepsilon}(C)$, which is (b). Conversely, suppose that (b) holds and, for every $n \in \mathbb{N}$, let V_n be as in (b) corresponding to $\varepsilon = 1/n$. Next, take a nonempty countable set $C_n \subset Y$ with $\varphi[V_n] \subset B^{\rho}_{1/n}(C_n)$. Finally, $A = \overline{\bigcup_{n\in\mathbb{N}} C_n}$ is a nonempty separable closed subset of Y such that $V_n \subset \varphi^{\#}[B^{\rho}_{1/n}(A)]$ for every $n \in \mathbb{N}$, which is (a). ∎

Proposition 4.4. *Let* (Y, ρ) *be a metric space and* $\varphi : X \to 2^Y$ *be separable* ρ*-subcontinuous. Whenever* $D \in \mathscr{S}(Y)$ *and* $\delta > 0$, *set* $X_D = \varphi^{\#}[B^{\rho}_{\delta}(D)]$ *and* $Y_D = \overline{B^{\rho}_{2\delta}(D)}$. *Then the restriction* $\varphi_D = \varphi\restriction X_D$ *is also separable* ρ*-subcontinuous as a set-valued mapping from* X_D *to* 2^{Y_D}.

Proof. Take a point $x \in X_D$. By Proposition 4.3, there exists $A \in \mathscr{S}(Y)$ such that $\varphi^{\#}[B^{\rho}_{\varepsilon}(A)]$ is a neighbourhood of x for every $\varepsilon > 0$. Whenever $0 < \varepsilon < \delta/2$, it follows that

$$B^{\rho}_{\varepsilon}(A) \cap B^{\rho}_{\delta}(D) \subset B^{\rho}_{\varepsilon}(A \cap B^{\rho}_{3\delta/2}(D)) \subset B^{\rho}_{2\delta}(D) \subset Y_D.$$

Indeed, for $y \in B^{\rho}_{\varepsilon}(A) \cap B^{\rho}_{\delta}(D)$, there are $s \in A$ and $t \in D$ such that $\rho(y, s) < \varepsilon$ and $\rho(y, t) < \delta$. Accordingly, $\rho(s, t) < \delta + \varepsilon$ and we have $s \in B^{\rho}_{\delta+\varepsilon}(D) \subset B^{\rho}_{3\delta/2}(D)$. We can now take $A_D = \overline{A \cap B^{\rho}_{3\delta/2}(D)} \in \mathscr{S}(Y_D)$, which has the property that $\varphi^{\#}[B^{\rho}_{\varepsilon}(A)] \cap X_D \subset \varphi^{\#}_D[B^{\rho}_{\varepsilon}(A_D)]$. By Proposition 4.3, φ_D is also separable ρ-subcontinuous. ∎

Proposition 4.5. *Let* X *be a paracompact space,* (Y, ρ) *be a metric space and* $\varphi : X \to 2^Y$ *be separable* ρ*-subcontinuous. Then for every* $\varepsilon > 0$ *there exists a locally finite open cover* $\{U_s : s \in S\}$ *of* X *and a map* $\eta : S \to \mathscr{S}(Y)$ *such that* $\overline{U_s} \subset \varphi^{\#}[B^{\rho}_{\varepsilon}(\eta(s))]$ *for every* $s \in S$.

Proof. By Proposition 4.3, for every $x \in X$ there exists an open set $V_x \subset X$ and $A_x \in \mathscr{S}(Y)$ such that $x \in V_x \subset \overline{V_x} \subset \varphi^{\#}[B^{\rho}_{\varepsilon}(A_x)]$. Since X is paracompact, the open cover $\{V_x : x \in X\}$ has an open locally finite refinement $\{U_s : s \in S\}$. For every $s \in S$ there exists $x(s) \in X$ with $U_s \subset V_{x(s)}$. This defines a map $\eta : S \to \mathscr{S}(Y)$ by letting $\eta(s) = A_{x(s)}$, $s \in S$. ∎

Lemma 4.6. *Let* X *be a paracompact space,* (Y, ρ) *be a metric space and* $\varphi : X \to 2^Y$ *be a separable* ρ*-subcontinuous mapping. Then there exists a nonempty-open-valued locally finite sieve* $\mathscr{U} : T \to 2^X$ *and* $\eta : T \to \mathscr{S}(Y)$

such that, for every $t \in T(n)$ and $s \in \mathrm{node}(t)$,

$$\overline{\mathscr{U}(t)} \subset \varphi^{\#}[B^{\rho}_{2^{-(n+1)}}(\eta(t))] \quad \textit{and} \quad \eta(s) \subset B^{\rho}_{2^{-n}}(\eta(t)).$$

Proof. It is enough to construct $\mathscr{U}$ and η on $\mathrm{node}(\emptyset) = T(0)$, and illustrate how to extend them to $\mathrm{node}(t)$ for a given $t \in \mathrm{node}(\emptyset)$. The existence of $\mathscr{U} : T(0) \to 2^X$ and $\eta : T(0) \to \mathscr{S}(Y)$ such that $\overline{\mathscr{U}(t)} \subset \varphi^{\#}[B^{\rho}_{2^{-1}}(\eta(t))]$, $t \in T(0)$, follows from Proposition 4.5. Take $t \in T(0)$, and consider the paracompact space $X_t = \overline{\mathscr{U}(t)}$. Since $\varphi(x) \subset B^{\rho}_{2^{-1}}(\eta(t)) \subset B^{\rho}_{2^0}(\eta(t))$, we can view $\varphi_t = \varphi\restriction X_t$ as a mapping from X_t to the subsets of $Y_t = \overline{B^{\rho}_{2^0}(\eta(t))}$. According to Proposition 4.4, φ_t remains separable ρ-subcontinuous. Hence, we can apply Proposition 4.5 to get a locally finite open cover $\{U_s : s \in S_t\}$ of X_t and a map $\eta : S_t \to \mathscr{S}(Y_t)$ such that $\overline{U_s} \subset \varphi^{\#}[B^{\rho}_{2^{-2}}(\eta(s))]$ for every $s \in S_t$. Finally, take $\mathrm{node}(t) = S_t$, and define $\mathscr{U}(s) = U_s \cap \mathscr{U}(t)$, $s \in \mathrm{node}(t)$. The proof can be extended by induction. ∎

Proof of Theorem 4.2. Let $\mathscr{U} : T \to 2^X$ and $\eta : T \to \mathscr{S}(Y)$ be as in Lemma 4.6. Set $\overline{\mathscr{U}}(t) = \overline{\mathscr{U}(t)}$, $t \in T$. According to [6, Lemma 5.3], the inverse polar mapping $\mho_{\overline{\mathscr{U}}} : X \to 2^{\mathscr{B}(T)}$ is usco, namely nonempty-compact-valued and u.s.c. Define $\Sigma_\eta : \mathscr{B}(T) \to \mathscr{S}(Y)$ by $\Sigma_\eta(\beta) = \overline{\bigcup_{t\in\beta} \eta(t)}$ for $\beta \in \mathscr{B}(T)$. By Proposition 2.1 and Lemma 4.6, Σ_η is ρ-continuous. We finalize the proof by showing that the composite mapping $\psi(x) = \Sigma_\eta[\mho_{\overline{\mathscr{U}}}(x)]$, $x \in X$, is as required. Since $\mho_{\overline{\mathscr{U}}}$ is usco, ψ is ρ-u.s.c.; moreover, each $\psi(x)$, $x \in X$, is separable because each $\mho_{\overline{\mathscr{U}}}(x)$ is compact. It is also easy to see that each $\psi(x)$, $x \in X$, is closed. Finally, take a point $x \in X$, a branch $\beta \in \mho_{\overline{\mathscr{U}}}(x)$ and $t \in \beta \cap T(n)$. By the properties of $\mathscr{U}$ (see Lemma 4.6), it follows that

$$\varphi(x) \subset B^{\rho}_{2^{-(n+1)}}(\eta(t)) \subset B^{\rho}_{2^{-n}}(\Sigma_\eta(\beta)) \subset B^{\rho}_{2^{-n}}(\Sigma_\eta[\mho_{\overline{\mathscr{U}}}(x)]) = B^{\rho}_{2^{-n}}(\psi(x)).$$

Accordingly, $\varphi(x) \subset \psi(x)$, and the proof is complete. ∎

For a metric space (Y, ρ), a mapping $\varphi : X \to \mathscr{F}(Y)$ is *ρ-l.s.c.* if for every $\varepsilon > 0$, every $x \in X$ has a neighbourhood U such that $\varphi(x) \subset B^{\rho}_{\varepsilon}(\varphi(z))$ for every $z \in U$. It is evident that φ is ρ-continuous if and only if it is both ρ-l.s.c. and ρ-u.s.c. In the proof of Theorem 4.2, the inverse polar mapping $\mho_{\mathscr{U}}$ is nonempty-compact-valued and l.s.c. [6, Propositions 5.1 and 5.2]. Hence, the composition $\phi(x) = \Sigma_\eta[\mho_{\mathscr{U}}(x)]$, $x \in X$, is a ρ-l.s.c. mapping with $\phi \subset \psi$. This gives the following consequence.

Corollary 4.7. *Let X be a paracompact space, (Y, ρ) be a metric space and $\varphi : X \to 2^Y$ be a separable ρ-subcontinuous mapping. Then there exists a (completely) metrizable non-Archimedean space Z, a pair $(\Phi, \Psi) : X \to \mathscr{F}(Z)$ of nonempty-compact-valued mappings and $\theta : Z \to \mathscr{S}(Y)$ such that:*

(a) *Φ is l.s.c., Ψ is u.s.c. and $\Phi \subset \Psi$,*

(b) *θ is ρ-continuous with $\varphi \subset \theta \circ \Phi \subset \theta \circ \Psi$.*

5. Proof of Theorem 1.1. Suppose that X is a strongly paracompact metrizable space and ρ is a metric on X compatible with the topology of X. We follow the idea of [8, Corollary 4.3]. Namely, define an l.s.c. mapping $\Phi : X \to \mathscr{F}(X)$ by $\Phi(x) = \{x\}$, $x \in X$. By Theorem 3.3, there exists a non-Archimedean (completely) metrizable space Z, a continuous map $g : X \to Z$ and a ρ-continuous mapping $\psi : Z \to \mathscr{S}(X)$ such that $\rho(\psi(g(x)), \Phi(x)) = 0$ for every $x \in X$. Since $\Phi(x) = \{x\}$, we have $x \in \psi(g(x))$, and hence g^{-1} is a set-valued selection for ψ. Accordingly, g is a strongly S-map because ψ is ρ-continuous (see Theorem 4.2).

Conversely, let $\mathscr{U}$ be a locally finite open cover of X, and $\{V_U : U \in \mathscr{U}\}$ be another open cover of X such that $\overline{V_U} \subset U$ for all $U \in \mathscr{U}$. For every $U \in \mathscr{U}$, take a continuous function $\xi_U : X \to [0,1]$ such that

$$\xi_U^{-1}(1) = \overline{V_U} \quad \text{and} \quad \xi_U^{-1}(0) = X \setminus U. \tag{5.1}$$

Take a compatible metric d on X, and next define another metric ρ by

$$\rho(x,y) = d(x,y) + \sum \{|\xi_U(x) - \xi_U(y)| : U \in \mathscr{U}\}, \qquad x, y \in X. \tag{5.2}$$

Because the functions ξ_U, $U \in \mathscr{U}$, are continuous and $d \leq \rho$, the metrics d and ρ are equivalent. We take a nonempty subset $A \subset X$, and check that

$$U \cap A = \emptyset \quad \text{implies} \quad \overline{V_U} \cap B_1^{\rho}(A) = \emptyset. \tag{5.3}$$

Indeed, if $y \in \overline{V_U} \cap B_1^{\rho}(A)$, then there exists $x \in A$ with $\rho(x,y) < 1$. According to (5.2), this implies that $|\xi_U(x) - \xi_U(y)| < 1$. On the other hand, by (5.1), we have $\xi_U(y) = 1$. Hence, $\xi_U(x) > 0$, and therefore $x \in U$. Thus, $U \cap A = \emptyset$ implies $\overline{V_U} \cap B_1^{\rho}(A) = \emptyset$.

We now complete the proof in the following way. Since ρ is a compatible metric on X, by the conditions of the theorem, there exists a non-Archimedean metrizable space Z and a continuous map $g : X \to Z$ such that the mapping $g^{-1} : Z \to \mathscr{F}(X)$ is a set-valued selection for some ρ-u.s.c. mapping $\psi : Z \to \mathscr{S}(X)$; the latter follows by Theorem 4.2. Take a point $z \in Z$. Since $\psi(z) \in \mathscr{S}(X)$ and $\mathscr{U}$ is locally finite, the set $\mathscr{U}(z) = \{U \in \mathscr{U} : U \cap \psi(z) \neq \emptyset\}$ is countable. So, by (5.3), $B_1^{\rho}(\psi(z))$ is covered by countably many elements of $\{V_U : U \in \mathscr{U}\}$, in fact $B_1^{\rho}(\psi(z)) \subset \bigcup \mathscr{U}(z)$. Since ψ is ρ-u.s.c. and $g^{-1} \subset \psi$, we deduce that $O(z) = \psi^{\#}[B_1^{\rho}(\psi(z))]$ is a neighbourhood of z such that

$$g^{-1}(O(z)) \subset \bigcup \mathscr{U}(z). \tag{5.4}$$

Since $\{O(z) : z \in Z\}$ is an open cover of the metrizable non-Archimedean space Z, it is refined by a discrete open cover $\mathscr{D}$ of Z. According to (5.4), $\mathscr{W} = g^{-1}(\mathscr{D})$ is a discrete open cover of X which refines the collection of all countable unions of elements of $\mathscr{U}$. Thus, X is strongly paracompact (see [2, Theorem 2.3]).

6. Proof of Theorem 1.2. As was shown in the Introduction, every strongly metrizable space has this property. Conversely, suppose that ρ is a metric on X compatible with the topology of X, and such that (X, ρ) can be mapped onto a non-Archimedean space Z by a continuous strongly S-map $g : X \to Z$. For any $n \in \mathbb{N}$ and $x \in X$, there exists an open set $V_n(x)$ containing $g(x)$ such that $g^{-1}(V_n(x))$ is covered by countably many open $(1/n)$-balls in (X, ρ). Since Z is non-Archimedean, the open cover $\{V_n(x) : x \in X\}$ is refined by a discrete cover $\mathscr{V}_n$. Thus, each $h^{-1}(V)$, $V \in \mathscr{V}_n$, is covered by a collection $\mathscr{B}_V$ of countably many open $(1/n)$-balls in (X, ρ). Since $h^{-1}(V)$ is countably paracompact (being metrizable), the countable open cover $\{B \cap h^{-1}(V) : B \in \mathscr{B}_V\}$ of $h^{-1}(V)$ is refined by a star-finite open refinement $\mathscr{U}_V$ in $h^{-1}(V)$ (see [4, Theorem 5.2.6]). Thus, $\mathscr{U}_n = \bigcup_{V \in \mathscr{V}_n} \mathscr{U}_V$ is an open start-finite cover of X with $\operatorname{diam}_\rho(U) < 2/n$ for every $U \in \mathscr{U}_n$. Accordingly, $\mathscr{U} = \bigcup_{n \in \mathbb{N}} \mathscr{U}_n$ is a base for the topology of X, and X is strongly metrizable.

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