STUDIA MATHEMATICA 232 (1) (2016)

Unconditionality for m-homogeneous polynomials on ℓ_∞^n

by

ANDREAS DEFANT (Oldenburg) and PABLO SEVILLA-PERIS (Valencia)

Dedicated to our good friends Pepe Bonet and Manolo Maestre,
on the happy occasion of their 60th birthdays

Abstract. Let $\chi(m,n)$ be the unconditional basis constant of the monomial basis z^α, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$, of the Banach space of all m-homogeneous polynomials in n complex variables, endowed with the supremum norm on the n-dimensional unit polydisc $\mathbb{D}^n$. We prove that the quotient of $\sup_m \sqrt[m]{\sup_m \chi(m,n)}$ and $\sqrt{n/\log n}$ tends to 1 as $n \to \infty$. This reflects a quite precise dependence of $\chi(m,n)$ on the degree m of the polynomials and their number n of variables. Moreover, we give an analogous formula for m-linear forms, a reformulation of our results in terms of tensor products, and as an application a solution for a problem on Bohr radii.

1. Introduction. Unconditional bases form one of the basic concepts in Banach space theory. A Schauder basis $(e_i)_{i\in I}$ of a (complex) Banach space X is said to be *unconditional* if all series representations $x = \sum_{i=1}^\infty \alpha_i e_i$ converge unconditionally. Equivalently, there is a constant $c > 0$ such that for every finite choice of complex numbers $x_1, \dots, x_n$ and of signs $\varepsilon_1, \dots, \varepsilon_n = \pm 1$ we have

$$\Big\| \sum_{i=1}^n \varepsilon_i x_i e_i \Big\| \le c \Big\| \sum_{i=1}^n x_i e_i \Big\|.$$

The best constant c in this inequality is called the *unconditional basis constant* of $(e_i)_{i\in I}$ and denoted by $\chi((e_i)_{i\in I}; X)$. A continuous function $P : X \to \mathbb{C}$ is an *m-homogeneous polynomial* if there exists an m-linear form $L : X \times \cdots \times X \to \mathbb{C}$ such that $P(x) = L(x, \dots, x)$ for every $x \in X$. We denote by $\mathcal{P}(^m X)$ and $\mathcal{L}(^m X)$ the spaces of m-homogeneous polynomials and m-linear forms on X, respectively, with the norms

2010 *Mathematics Subject Classification*: Primary 46G25, 46B04; Secondary 46B28.
Key words and phrases: unconditional basis constant, homogeneous polynomials.
Received 7 September 2015; revised 26 February 2016.
Published online 10 March 2016.

DOI: 10.4064/sm8386-2-2016

$$\|P\| = \sup_{\|x\|\le 1} |P(x)| \quad \text{and} \quad \|L\| = \sup_{\substack{\|x_i\|\le 1\\ i=1,\dots,m}} |L(x_1,\dots,x_m)|.$$

We focus on polynomials and multilinear forms on ℓ_∞^n (that is, $\mathbb{C}^n$ with the $\|\cdot\|_\infty$-norm).

Both Banach spaces $\mathcal{P}(^m\ell_\infty^n)$ and $\mathcal{L}(^m\ell_\infty^n)$ are finite-dimensional and have natural monomial bases. We write $\{e_1,\dots,e_n\}$ for the canonical basis of ℓ_∞^n and $\{e_1^*,\dots,e_n^*\}$ for its dual basis. For each index $\mathbf{i} = (i_1,\dots,i_m)$ with $1 \le i_1,\dots,i_m \le n$ (we denote the set of all such indices by $\mathcal{M}(m,n)$) we consider $e_{\mathbf{i}}^* \in \mathcal{L}(^m\ell_\infty^n)$ given by

$$e_{\mathbf{i}}^* : x = (x_1,\dots,x_m) \rightsquigarrow e_{i_1}^*(x_1)\cdots e_{i_m}^*(x_m).$$

Then $(e_{\mathbf{i}}^*)_{\mathbf{i}\in\mathcal{M}(m,n)}$ is obviously a basis of $\mathcal{L}(^m\ell_\infty^n)$. On the other hand, the monomials are the natural basis of $\mathcal{P}(^m\ell_\infty^n)$. These are defined as follows: Each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n = m$ (we denote by $\Lambda(m,n)$ the corresponding set) defines the monomial $z^\alpha \in \mathcal{P}(^m\ell_\infty^n)$,

$$z^\alpha : u = (u_1,\dots,u_n) \rightsquigarrow u_1^{\alpha_1}\cdots u_n^{\alpha_n}.$$

Schütt started the study of the unconditional basis constants of these two bases in [12], proving that

$$\frac{1}{8}\sqrt{n} \le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^2\ell_\infty^n)\big) \le (1+\sqrt{2})\sqrt{n}.$$

This study was continued in [5], where as a particular case of a more general result it is shown that for every m there is a constant $C(m) > 0$ such that for each n,

$$\frac{1}{C(m)}\sqrt{n}^{\,m-1} \le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big) \le C(m)\sqrt{n}^{\,m-1}$$

and

$$\frac{1}{C(m)}\sqrt{n}^{\,m-1} \le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \le C(m)\sqrt{n}^{\,m-1}.$$

These estimates are consequences of more general results that relate the unconditional basis constants of (full or symmetric) tensor products with classical concepts from Banach space theory, such as the Banach–Mazur distance to ℓ_1^n or the Gordon–Lewis property. The fact that we are specifically working with ℓ_∞^n played no rôle there. This was taken into account in [6], where the Bohnenblust–Hille inequality, a very particular property of ℓ_∞^n, was substantially improved and used to show that there exists a universal constant C such that

$$\begin{aligned} \frac{1}{C^m}\Big(\frac{n}{m}\Big)^{(m-1)/2} &\le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \le C^m\Big(\frac{n}{m}\Big)^{(m-1)/2} && \text{if } n \ge m,\\ 1 &\le \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \le C^m && \text{if } n < m. \end{aligned}$$

Our aim is to prove a refinement of these inequalities which in a very precise sense links the degree m of the polynomials with their number n of variables.

THEOREM 1.1. *We have*

$$\lim_{n\to\infty} \frac{\sup_m \sqrt[m]{\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)}}{\sqrt{n}} = 1 \tag{1.1}$$

and

$$\lim_{n\to\infty} \frac{\sup_m \sqrt[m]{\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)}}{\sqrt{n/\log n}} = 1. \tag{1.2}$$

Note that in the polynomial case there is a log term in n which distinguishes it from the multilinear case. Before we proceed to the proof, let us note that a simple calculation characterizes $\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)$ to be the best constant $c > 0$ such that for every $L \in \mathcal{L}(^m\ell_\infty^n)$,

$$\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |L(e_{i_1}, \ldots, e_{i_m})| \le c\|L\|. \tag{1.3}$$

Analogously, $\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)$ is the best constant $c > 0$ such that for every m-homogeneous polynomial $P = \sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$ in n variables,

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha| \le c\|P\|.$$

Formulated in this way, it is plain that $\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)$ is the Sidon constant of the characters $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ acting on the compact abelian group $\mathbb{T}^n$.

2. Proof of the main result

Proof of (1.1). We prove that

$$\begin{aligned} 1 \le \liminf_{n\to\infty} \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}} \\ \le \limsup_{n\to\infty} \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}} \le 1. \end{aligned} \tag{2.1}$$

To do this, we need a classical result due to Bohnenblust and Hille [3, Section 2]: For each m there is a (best) constant $\mathrm{BH}_m^{\mathrm{mult}} \ge 1$ such that for every $L \in \mathcal{L}(^m\ell_\infty^n)$ we have

$$\Big(\sum_{i_1,\ldots,i_m=1}^n |L(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \mathrm{BH}_m^{\mathrm{mult}}\|L\|. \tag{2.2}$$

A recent result from [2] shows that there exists a constant $\kappa > 1$ such that for all m,

$$\mathrm{BH}_m^{\mathrm{mult}} \leq \kappa m^{(1-\gamma)/2},$$

where γ is the Euler–Mascheroni constant. With this at hand, we start the proof of the upper bound from (2.1). Take $L \in \mathcal{L}({}^m\ell_\infty^n)$; then by Hölder's inequality

$$\begin{aligned}\sum_{i_1,\ldots,i_m=1}^{n} &|L(e_{i_1},\ldots,e_{i_m})| \\ &\leq \Big(\sum_{i_1,\ldots,i_m=1}^{n} |L(e_{i_1},\ldots,e_{i_m})|^{\frac{m+1}{2m}}\Big)^{\frac{2m}{m+1}} |\mathcal{M}(m,n)|^{(m-1)/(2m)} \\ &\qquad\qquad\qquad\qquad \leq \kappa m^{(1-\gamma)/2} n^{(m-1)/2} \|L\|.\end{aligned}$$

This, in view of (1.3), implies that for every n and m we have

$$\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}({}^m\ell_\infty^n)\big) \leq \kappa m^{(1-\gamma)/2} n^{(m-1)/2},$$

and hence

$$\chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}({}^m\ell_\infty^n)\big)^{1/m} \leq \frac{\kappa^{1/m} m^{(1-\gamma)/(2m)}}{n^{1/(2m)}} n^{1/2}.$$

Fix now some $\varepsilon > 0$ and choose m_0 such that

$$\sup_{m \geq m_0} \frac{\kappa^{1/m} m^{(1-\gamma)/(2m)}}{n^{1/(2m)}} \leq \sup_{m \geq m_0} \kappa^{1/m} m^{(1-\gamma)/(2m)} \leq 1 + \varepsilon.$$

But for each fixed m the sequence $(\kappa^{1/m} m^{(1-\gamma)/(2m)} n^{-1/(2m)})_n$ tends to 0, hence we can find n_0 such that for all $n \geq n_0$,

$$\sup_{m < m_0} \frac{\kappa^{1/m} m^{(1-\gamma)/(2m)}}{n^{1/(2m)}} \leq 1.$$

Then for all $n \geq n_0$ we have

$$\begin{aligned}&\frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}({}^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}} \\ &\quad \leq \max\bigg\{\frac{\sup_{m<m_0} \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}({}^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}}, \frac{\sup_{m \geq m_0} \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}({}^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}}\bigg\} \\ &\qquad\qquad\qquad\qquad \leq 1 + \varepsilon.\end{aligned}$$

This shows the right estimate in (2.1).

To prove the left estimate, by the Chevet type inequality from [1, Theorem 4.4] there is an absolute constant $C > 0$ such that for each choice of

m, n there are signs $\varepsilon_{\mathbf{i}} = \pm 1$ with $\mathbf{i} \in \mathcal{M}(m,n)$ for which

$$
\begin{aligned}
(2.3)\qquad n^m &= \Big(\sup_{z\in B_{\ell_\infty^n}} \sum_{i=1}^n |z_i|\Big)^m = \Big\| \sum_{\mathbf{i}\in\mathcal{M}(m,n)} e_{\mathbf{i}}^* \Big\|_{\mathcal{L}(^m\ell_\infty^n)} \\
&= \Big\| \sum_{\mathbf{i}\in\mathcal{M}(m,n)} \varepsilon_{\mathbf{i}}\varepsilon_{\mathbf{i}} e_{\mathbf{i}}^* \Big\|_{\mathcal{L}(^m\ell_\infty^n)} \\
&\le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big) \Big\| \sum_{\mathbf{i}\in\mathcal{M}(m,n)} \varepsilon_{\mathbf{i}} e_{\mathbf{i}}^* \Big\|_{\mathcal{L}(^m\ell_\infty^n)} \\
&\le \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big) Cm(\log n)^{3/2} n^{(m+1)/2}.
\end{aligned}
$$

As a consequence we have

$$
\sup_m \frac{1}{C^{1/m} m^{1/m} n^{1/(2m)} (\log n)^{3/(2m)}} \le \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}},
$$

and hence for $m = n$,

$$
\Big(\frac{1}{Cn^{1/2}(\log n)^{3/2}}\Big)^{1/n} \le \frac{\sup_m \chi\big((e_{\mathbf{i}}^*)_{\mathbf{i}}; \mathcal{L}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n}}.
$$

But if n tends to ∞, then we obtain the remaining left estimate in (2.1). ∎

Proof of (1.2). The basic idea to estimate $\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)$ is essentially the same, but technically more demanding. Again we split the proof in two steps, and check the following upper and lower bounds.

$$
\begin{aligned}
(2.4)\qquad 1 \le \liminf_{n\to\infty} &\frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n/\log n}} \\
&\le \limsup_{n\to\infty} \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n/\log n}} \le 1.
\end{aligned}
$$

To begin, we fix some $\varepsilon > 0$, and want to show that

$$
(2.5)\qquad \limsup_{n\to\infty} \frac{\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n/\log n}} \le 1+\varepsilon.
$$

One of the key tools is going to be again the Bohnenblust–Hille inequality, this time in its polynomial form. From (2.2) it is easy to prove, using the polarization formula (this is done in [3, Section 3]), that for each m there exists a (best) constant $\mathrm{BH}_m^{\mathrm{pol}} \ge 1$ such that for every m-homogeneous polynomial $P = \sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$ we have

$$
\Big(\sum_{\alpha\in\Lambda(m,n)} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \mathrm{BH}_m^{\mathrm{pol}} \|P\|.
$$

Again, a good control of the growth of $\mathrm{BH}_m^{\mathrm{pol}}$ is going to be crucial. In [6] it was shown that for each $\delta > 0$ there is a constant $c(\delta) \geq 1$ such that $\mathrm{BH}_m^{\mathrm{pol}} \leq c(\delta)(\sqrt{2}+\delta)^m$ for every m, and in [2] that $\sqrt{2}$ can even be replaced by 1. Using Hölder's inequality we find that for every n, m and every polynomial $P = \sum_{\alpha\in\Lambda(m,n)} c_\alpha z^\alpha$,

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha| \leq \mathrm{BH}_m^{\mathrm{pol}} |\Lambda(m,n)|^{(m-1)/(2m)} \sup_{z\in\mathbb{D}^n} \Big| \sum_{\alpha\in\Lambda(m,n)} c_\alpha(f) z^\alpha \Big|.$$

It is well known that

$$|\Lambda(m,n)| = \binom{n+m-1}{m} \leq e^m \Big(1+\frac{n}{m}\Big)^m,$$

hence for every n,

$$\sup_m \chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big) \leq \sup_m \bigg[\mathrm{BH}_m^{\mathrm{pol}} e^{(m-1)/2}\Big(1+\frac{n}{m}\Big)^{(m-1)/2}\bigg].$$

With this, in order to prove (2.5) it is enough to show that there is some n_0 such that for all $n \geq n_0$ and every m,

$$\bigg[\mathrm{BH}_m^{\mathrm{pol}} e^{(m-1)/2}\Big(1+\frac{n}{m}\Big)^{(m-1)/2}\Big(\frac{\log n}{n}\Big)^{m/2}\bigg] \leq (1+\varepsilon)^m. \tag{2.6}$$

Our strategy is going to be the following. First we find a proper n_0 and then consider three cases for m, showing that in each case, (2.6) holds. Let us note first that, since $\lim_m((1+\varepsilon)^m/m)^{1/(m-1)} = 1+\varepsilon$, we can choose $m_1 = m_1(\varepsilon)$ so that for every $m \geq m_1$,

$$1+\frac{\varepsilon}{2} \leq \Big(\frac{(1+\varepsilon)^m}{m}\Big)^{1/(m-1)}. \tag{2.7}$$

As we already mentioned, for each $\delta > 0$ there exists a constant $c(\delta) \geq 1$ such that for every m

$$\mathrm{BH}_m^{\mathrm{pol}} \leq c(\delta)(1+\delta)^m.$$

Choose $\delta > 0$ so that $1+\delta < (1+\varepsilon)^{1/4}$ and $m_2 = m_2(\varepsilon)$ such that $c(\delta)^{1/m} < (1+\varepsilon)^{1/4}$ for all $m \geq m_2$. Then

$$\sup_{m_2\leq m} \mathrm{BH}_m^{\mathrm{pol}} \leq (1+\varepsilon)^{m/2}. \tag{2.8}$$

We fix $m_0 = \max\{m_1, m_2\}$. Let us now take $n_1 = n_1(\varepsilon)$ such that for all $n \geq n_1$,

$$1+\frac{1}{\sqrt{n}} < 1+\frac{\varepsilon}{2}, \tag{2.9}$$

and $n_2 = n_2(\varepsilon)$ such that for all $n \geq n_2$,

$$\frac{\sqrt{2e\log n}}{n^{1/4}} \leq \sqrt{1+\varepsilon}. \tag{2.10}$$

Since for each fixed m the sequence $\big((1+\frac{n}{m})^{(m-1)/2}(\frac{\log n}{n})^{m/2}\big)_n$ obviously tends to zero, there clearly is some $n_3 = n_3(\varepsilon)$ such that for all $n \geq n_3$,

$$\sup_{m \leq m_0} \left[\mathrm{BH}_m^{\mathrm{pol}} e^{(m-1)/2} \left(1+\frac{n}{m}\right)^{(m-1)/2} \left(\frac{\log n}{n}\right)^{m/2} \right] \leq 1. \tag{2.11}$$

We now set $n_0 = \max\{n_1, n_2, n_3, (m_0+1)^2\}$. Observe that, although one may think that n_0 depends on ε and m_0, in fact m_0 only depends on ε; hence actually $n_0 = n_0(\varepsilon)$.

Now in order to prove (2.6) we fix $n \geq n_0$ and choose $m \geq 2$. We consider three different cases for m: $m \leq m_0$, and $m_0 \leq m < \sqrt{n}$, or $\sqrt{n} \leq m$. Observe that (2.11) already shows that (2.6) holds for $m \leq m_0$. For the remaining two cases let us note first that by (2.7) and (2.9) we have, for every $m_0 \leq m$,

$$1 + \frac{1}{\sqrt{n}} \leq \left(\frac{(1+\varepsilon)^m}{m}\right)^{1/(m-1)}.$$

Then, if $m_0 \leq m < \sqrt{n}$, a straightforward calculation gives

$$\begin{aligned} \left(1+\frac{n}{m}\right)^{(m-1)/2} &\leq \left(\frac{\sqrt{n}+n}{m}\right)^{(m-1)/2} \\ &\leq \frac{(1+\varepsilon)^{m/2} n^{(m-1)/2}}{\sqrt{m}} \frac{1}{m^{(m-1)/2}} \\ &= (1+\varepsilon)^{m/2} \frac{n^{m/2}}{n^{1/2} m^{m/2}}, \end{aligned} \tag{2.12}$$

and with (2.8) this implies that

$$\begin{aligned} &\mathrm{BH}_m^{\mathrm{pol}} e^{(m-1)/2} \left(1+\frac{n}{m}\right)^{(m-1)/2} \left(\frac{\log n}{n}\right)^{m/2} \\ &\leq (1+\varepsilon)^m e^{(m-1)/2} \frac{n^{m/2}}{n^{1/2} m^{m/2}} \left(\frac{\log n}{n}\right)^{m/2} \leq (1+\varepsilon)^m \left(\frac{e \log n}{n^{1/m} m}\right)^{m/2}. \end{aligned}$$

Now a simple calculation shows that the function $x \in (0,\infty) \mapsto x n^{1/x}$ has a global minimum $e \log n$ at $x = \log n$. This proves (2.6) in the second case: $m_0 \leq m < \sqrt{n}$ (remember that $n \geq n_0$ was fixed). Finally, for the third case $\sqrt{n} \leq m$, we trivially have

$$\left(1+\frac{n}{m}\right)^{(m-1)/2} \leq (2\sqrt{n})^{m/2},$$

and hence in this last case we get (2.6) using (2.8) and (2.10). This completes the proof of (2.5).

Finally, it remains to show the left inequality in (2.4). The main tool is again of probabilistic nature, and we are going to use the Kahane–Salem–

Zygmund inequality [11, Chapter 6, Theorem 4]: There is a universal constant $\mathrm{C}_{\mathrm{KSZ}} > 0$ such that for every scalar family $c_\alpha, \alpha \in \Lambda(m,n)$, there exists a choice of signs $\varepsilon_\alpha \in \{1,-1\}$ for $\alpha \in \Lambda(m,n)$ such that we have

$$\sup_{z\in\mathbb{D}^n}\Big|\sum_{\alpha\in\Lambda(m,n)}\varepsilon_\alpha c_\alpha z^\alpha\Big| \le \mathrm{C}_{\mathrm{KSZ}}\sqrt{n\log m\sum_{\alpha\in\Lambda(m,n)}|c_\alpha|^2}.$$

We consider $c_\alpha = m!/\alpha!, \alpha \in \Lambda(m,n)$. Then there are $\varepsilon_\alpha \in \{1,-1\}$ such that

$$\begin{aligned}\sum_{\alpha\in\Lambda(m,n)}\frac{m!}{\alpha!} &\le \chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)\sup_{z\in\mathbb{D}^n}\Big|\sum_{\alpha\in\Lambda(m,n)}\varepsilon_\alpha\frac{m!}{\alpha!}z^\alpha\Big|\\ &\le \chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)\mathrm{C}_{\mathrm{KSZ}}\sqrt{n\log m\sum_{\alpha\in\Lambda(m,n)}\Big(\frac{m!}{\alpha!}\Big)^2}.\end{aligned}$$

By the multi-binomial formula we have

$$\sum_{\alpha\in\Lambda(m,n)}\frac{m!}{\alpha!} = n^m$$

and

$$\sum_{\alpha\in\Lambda(m,n)}\Big(\frac{m!}{\alpha!}\Big)^2 \le m!\sum_{\alpha\in\Lambda(m,n)}\frac{m!}{\alpha!} = m!n^m.$$

This gives, for all m,n,

$$\frac{n^{(m-1)/(2m)}}{\mathrm{C}_{\mathrm{KSZ}}^{1/m}(m!\log m)^{1/(2m)}} \le \chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)^{1/m}.$$

Recall Stirling's formula $m! \le 2\sqrt{2\pi m}(m/e)^m$; hence for all m,n,

$$\frac{n^{(m-1)/(2m)}}{\mathrm{C}_{\mathrm{KSZ}}^{1/m}(\log m)^{1/(2m)}2^{1/(2m)}(2\pi m)^{1/(4m)}(m/e)^{1/2}} \le \chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)^{1/m}.$$

Now we set $m = \lceil\log n\rceil$ and divide by $\sqrt{n/\log n}$ to obtain, for all n,

$$\begin{aligned}&\frac{\sqrt{(\log n)/n}\,n^{\frac{\log n-1}{2\log n}}}{\mathrm{C}_{\mathrm{KSZ}}^{\frac{1}{\log n}}(\log(1+\log n))^{\frac{1}{2\log n}}2^{\frac{1}{2\log n}}(2\pi\log n)^{\frac{1}{4\log n}}\big(\frac{1+\log n}{e}\big)^{1/2}}\\ &\qquad\le \frac{\sup_m\chi\big((z^\alpha)_\alpha;\mathcal{P}(^m\ell_\infty^n)\big)^{1/m}}{\sqrt{n/\log n}}.\end{aligned}$$

But the left side of this inequality tends to 1 as $n\to\infty$. This clearly gives the left inequality in (2.4), and completes the proof of Theorem 1.1. ■

3. Two consequences

3.1. A tensor product formulation. It is well known that the Banach space $\mathcal{L}({}^m\ell_\infty^n)$ of m-linear forms can be represented as an m-fold tensor product, and the Banach space $\mathcal{P}({}^m\ell_\infty^n)$ of m-homogeneous polynomials as an m-fold symmetric tensor product. More precisely, we have

$$\mathcal{L}({}^m\ell_\infty^n) = \bigotimes\nolimits_{\varepsilon}^{m} \ell_1^n \quad \text{and} \quad \mathcal{P}({}^m\ell_\infty^n) = \bigotimes\nolimits_{\varepsilon_s}^{s,m} \ell_1^n \tag{3.1}$$

isometrically as Banach spaces (see e.g. [9, Chapter 1] or [10]); here ε stands for the injective tensor norm on the tensor product $\bigotimes^m \ell_1^n$, and ε_s for the symmetric injective tensor norm on the symmetric tensor product $\bigotimes^{s,m} \ell_1^n$. Under the identification from (3.1) the basis $(e_{\mathbf{i}}^*)_{\mathbf{i}\in\mathcal{M}(m,n)}$ of $\mathcal{L}({}^m\ell_\infty^n)$ transfers into the basis of $\bigotimes_\varepsilon^m \ell_1^n$ given by $e_{\mathbf{i}} = (e_{i_1}\otimes\cdots\otimes e_{i_m})$ for $\mathbf{i}\in\mathcal{M}(m,n)$. Analogously, the image of the monomial basis $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ in $\mathcal{P}({}^m\ell_\infty^n)$ is the basis

$$S(e_{\mathbf{j}}) = \frac{1}{m!}\sum_{\sigma\in\Pi_m}(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(m)}}), \qquad \mathbf{j}=(j_1,\ldots,j_m)\in\mathcal{J}(m,n),$$

of $\bigotimes_{\varepsilon_s}^{s,m}\ell_1^n$, where Π_m stands for all permutations of $\{1,\ldots,m\}$ and

$$\mathcal{J}(m,n) = \{\mathbf{j}\in\mathcal{M}(m,n) \colon 1\le j_1\le\cdots\le j_m\le n\}.$$

With this notation, Theorem 1.1 has the following immediate translation in terms of tensor products.

Corollary 3.1. *We have*

$$\lim_{n\to\infty}\frac{\sup_m \chi\big((e_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)};\ \bigotimes_\varepsilon^m \ell_1^n\big)^{1/m}}{\sqrt{n}} = 1$$

and

$$\lim_{n\to\infty}\frac{\sup_m \chi\big((Se_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)};\ \bigotimes_{\varepsilon_s}^{s,m} \ell_1^n\big)^{1/m}}{\sqrt{n/\log n}} = 1.$$

3.2. Bohr radius. The nth *Bohr radius* K_n is defined to be the supremum of $0\le r\le 1$ such that for all holomorphic functions $f:\mathbb{D}^n\to\mathbb{C}$ we have

$$\sup_{z\in r\mathbb{D}^n}\sum_{\alpha\in\mathbb{N}_0^n}\left|\frac{\partial^\alpha f(0)}{\alpha!}z^\alpha\right| \le \sup_{z\in\mathbb{D}^n}\left|\sum_{\alpha\in\mathbb{N}_0^n}\frac{\partial^\alpha f(0)}{\alpha!}z^\alpha\right|,$$

and K_n^m, the mth *homogeneous Bohr radius*, is defined analogously, taking only m-homogeneous polynomials $\sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=m} c_\alpha z^\alpha$. These objects have been extensively studied over the last years. By [7, Corollary 2.3] we have

$$\frac{1}{3}\inf_m K_n^m \le K_n \le \min\left\{\frac{1}{3},\inf_m K_n^m\right\}. \tag{3.2}$$

In the special case $n = 1$ obviously $K_1^m = 1$, hence

$$K_1 = \frac{1}{3}.$$

This is Bohr's famous power series theorem from [4], and it shows that the factor $1/3$ in (3.2) is indispensable at least for small n. But how important is this factor for large n ? Or, to put it in technical terms: Does the equality

$$\lim_{n\to\infty} \frac{K_n}{\inf_m K_n^m} = \frac{1}{3}$$

hold? This question appears explicitly in [8, Problem 4.4]. Let us see with Theorem 1.1 that this is not the case: First of all, from [2] we know that

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{(\log n)/n}} = 1$$

(improving the inequality $1 \le \liminf_n K_n \sqrt{n/\log n} \le \limsup_n K_n \sqrt{n/\log n} \le \sqrt{2}$ from [6]). On the other hand, straightforward arguments show (see [7, Lemma 2.1]) that for all n, m,

$$K_n^m = \frac{1}{\sqrt[m]{\chi\big((z^\alpha)_\alpha; \mathcal{P}(^m\ell_\infty^n)\big)}}.$$

These two facts, together with Theorem 1.1, readily give our last result which shows that, when n grows, the factor $1/3$ looses influence in (3.2).

Corollary 3.2. *We have*

$$\lim_{n\to\infty} \frac{K_n}{\inf_m K_n^m} = 1.$$

Acknowledgements. This research was supported by MCINN project MTM2014-57838-C2-2-P.

References

[1] F. Bayart, *Maximum modulus of random polynomials*, Quart. J. Math. 63 (2012), 21–39.

[2] F. Bayart, D. Pellegrino, and J. B. Seoane-Sepúlveda, *The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$*, Adv. Math. 264 (2014), 726–746.

[3] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) 32 (1931), 600–622.

[4] H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. (2) 13 (1914), 1–5.

[5] A. Defant, J. C. Díaz, D. García, and M. Maestre, *Unconditional basis and Gordon–Lewis constants for spaces of polynomials*, J. Funct. Anal. 181 (2001), 119–145.

[6] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) 174 (2011), 485–497.

[7] A. Defant, D. García, and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. Reine Angew. Math. 557 (2003), 173–197.

[8] A. Defant and C. Prengel, *Harald Bohr meets Stefan Banach*, in: Methods in Banach Space Theory, London Math. Soc. Lecture Note Ser. 337, Cambridge Univ. Press, Cambridge, 2006, 317–339.

[9] S. Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer Monogr. Math., Springer London, London, 1999.

[10] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. 17 (1997), 153–188 (1999).

[11] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge Stud. Adv. Math. 5, Cambridge Univ. Press, Cambridge, 1985.

[12] C. Schütt, *Unconditionality in tensor products*, Israel J. Math. 31 (1978), 209–216.

Andreas Defant
Institut für Mathematik
Universität Oldenburg
D-26111 Oldenburg, Germany
E-mail: defant@mathematik.uni-oldenburg.de

Pablo Sevilla-Peris
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
46022 Valencia, Spain
E-mail: psevilla@mat.upv.es