

Measures on Suslinean spaces

by

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Abstract. We study the existence of nonseparable compact spaces that support a measure and are small from the topological point of view. In particular, we show that under Martin's Axiom there is a nonseparable compact space supporting a measure which has countable π -character and which cannot be mapped continuously onto $[0, 1]^{\omega_1}$. On the other hand, we prove that in the random model there is no nonseparable compact space having countable π -character and supporting a measure.

1. Introduction. The well known Suslin Hypothesis can be expressed in the following form:

Every linearly ordered compact space satisfying the countable chain condition is separable.

We shall call an arbitrary nonseparable ccc compact space a *Suslinean space*. Clearly, Suslinean spaces do exist in ZFC. Perhaps the most obvious example is $[0, 1]^\kappa$ with the standard product topology, where $\kappa > \mathfrak{c}$. Such a space, however, is far from being linearly ordered. Indeed, linearly ordered spaces cannot be mapped continuously onto $[0, 1]^{\omega_1}$.

The Suslin Hypothesis was shown to hold under MA_{ω_1} . Later it turned out that linearity of the space in question can be relaxed to some kind of topological *smallness*. For instance, Hajnal and Juhász [Juh71] proved that under MA_{ω_1} every ccc compact space with a π -base of size less than \mathfrak{c} is separable. By purely topological methods one can deduce from this result that under MA_{ω_1} there are no Suslinean first-countable spaces or Suslinean spaces of countable tightness (see Tall [Tal74]).

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Let us recall that no space which is first-countable or has countable tightness can be continuously mapped onto $[0, 1]^{\omega_1}$. In the light of the results mentioned above it is natural to ask if the following statement is consistent:

Every Suslinean space can be mapped continuously onto $[0, 1]^{\omega_1}$.

In [Tod00] Todorčević called such a conjecture “the ultimate version of Suslin hypothesis”. Another question which suggests itself here (cf. [Juh77]) is:

Is it consistent that there is no Suslinean space of countable π -character?

Both the questions have been answered negatively. First, Bell [Bel96] proved that MA_{ω_1} is consistent with the existence of a Suslinean space of countable π -character which does not map continuously onto $[0, 1]^{\omega_1}$. Then Moore [Moo99] gave an example of such a space whose existence is implied by MA. Finally, Todorčević [Tod00] constructed in ZFC a compact space with the same properties.

The aim of this article is to consider similar questions for the class of compacta that support a measure. In particular, we discuss the following two questions.

PROBLEM 1.1. *Is there a Suslinean space supporting a measure which cannot be mapped continuously onto $[0, 1]^{\omega_1}$?*

PROBLEM 1.2. *Is there a Suslinean space of countable π -character supporting a measure?*

We say that a space K *supports a measure* if there is a measure μ such that $\mu(U) > 0$ for every nonempty open $U \subseteq K$ (in other words, μ is *strictly positive* on K). Clearly, if a compact space supports a measure then it is ccc.

Assuming the continuum hypothesis Kunen [Kun81] constructed a first-countable Suslinean space supporting a measure so the answer to 1.1 and 1.2 is positive under CH. Later Kunen and van Mill [KvM95] proved the following (cf. Plebanek [Ple97, Theorem 5.1]).

THEOREM 1.3. *The following are equivalent:*

- (i) *there is a Suslinean first-countable Corson compact space supporting a measure,*
- (ii) $\text{cov}(\mathcal{N}_{\omega_1}) = \omega_1$ ⁽¹⁾.

For the definition of a Corson compact space see e.g. [KvM95]; it is worth recalling that every separable subspace of a Corson compactum is metrizable.

⁽¹⁾ See Section 2 for the notation used here and Section 4 for further discussion.

We give a consistent negative solution to Problem 1.2: we show that under a certain axiom (satisfied e.g. in the standard random model) there is no Suslinean space of countable π -character and supporting a measure.

On the other hand, in Section 3 assuming MA we construct a space K giving a positive solution to both Problems 1.1 and 1.2, i.e. our space K is a nonseparable compact space supporting a measure and of countable π -character. The space K is moreover linearly fibered, i.e. it can be mapped continuously onto 2^ω in such a way that all fibers are linearly ordered. Hence, it is in a sense a direct generalization of the Suslin line. In particular, K cannot be mapped continuously onto $[0, 1]^{\omega_1}$. Moreover, using slightly different methods, we show that under a weaker version of Martin's Axiom there is a small compact space which supports a measure but not a countably determined one (so, in a sense, it is close to being nonseparable).

We do not know if Problem 1.1 can be resolved in the usual set theory. However, Theorems 1.3 and 3.3 indicate that *usually* the answer to 1.1 is positive: if we seek for a consistent negative answer, we have to assume Martin's Axiom for the measure algebras (that is, $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$) and the negation of MA_{ω_1} , so the standard model obtained by adding ω_2 random reals might be a natural candidate.

The above results seem to reveal an interesting phenomenon. Usually Martin's Axiom eliminates pathological objects (in particular various kinds of Suslinean spaces) and it has been used for this purpose from the very beginning of its creation. However, the space from Theorem 3.3 is a pathology tolerated by Martin's Axiom but not by the classical random model. Perhaps this is an indication that the random model is the universe in which measures on compact spaces behave in a more orderly way than in other models.

It seems that under additional axioms the methods used in [Tod00] can provide yet another example of a small nonseparable space supporting a measure. This is a subject under investigation (see [BNI16]).

2. Preliminaries. We use the standard terminology and notation concerning topology, so $w(K)$ is the topological weight of a space K , while $\pi(K)$ is the π -weight, the smallest cardinality of a π -base of K .

Let K be a compact space and $x \in K$. Recall that a family of nonempty open sets \mathcal{P} is a π -base at x if for every open $V \ni x$ there is $U \in \mathcal{P}$ such that $U \subseteq V$. By $\pi(x, K)$ we denote the smallest cardinality of a π -base at x . The π -character of K is defined as $\pi\chi(K) = \sup\{\pi(x, K) : x \in K\}$.

Given a Boolean algebra \mathfrak{A} , a family $\mathcal{P} \subseteq \mathfrak{A}^+$ is its π -base if for every $A \in \mathfrak{A}^+$ there is $B \in \mathcal{P}$ with $B \leq A$. Clearly, every π -base of a Boolean algebra defines a π -base of its Stone space.

If \mathcal{I} is an ideal on a set K , then

$$\begin{aligned} \text{add}(\mathcal{I}) &= \min\left\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\right\}, \\ \text{non}(\mathcal{I}) &= \min\left\{|X|: X \subseteq K, X \notin \mathcal{I}\right\}, \\ \text{cov}(\mathcal{I}) &= \min\left\{|\mathcal{A}|: \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = K\right\}. \end{aligned}$$

By a *measure* on a topological space K we mean a finite Borel measure which is Radon, i.e. inner-regular with respect to compact sets. Typically, we consider Radon measures on compacta.

If μ is a *measure* on K and \mathcal{N}_μ is the ideal of μ -null sets, then we write $\text{non}(\mu) = \text{non}(\mathcal{N}_\mu)$ and $\text{cov}(\mu) = \text{cov}(\mathcal{N}_\mu)$.

By a measure μ on a Boolean algebra \mathfrak{A} we mean a finite and *finitely additive* function $\mu: \mathfrak{A} \rightarrow \mathbb{R}$ (note that usually we consider Boolean algebras which are not σ -complete).

Let \mathfrak{A} be a Boolean algebra and let $K = \text{ult}(\mathfrak{A})$ be its Stone space (of all ultrafilters). If $A \in \mathfrak{A}$, then we denote by $\widehat{A} = \{x \in K: A \in x\}$ the corresponding clopen subset of K . Recall that a measure μ on \mathfrak{A} can be transferred to a measure $\widehat{\mu}$ on the algebra of clopen subsets of K via the formula $\widehat{\mu}(\widehat{A}) = \mu(A)$. In turn, $\widehat{\mu}$ can be uniquely extended to a (σ -additive and Radon) measure on K .

If μ is a measure on K , then we denote by $\mathfrak{A}(\mu)$ the *measure algebra* of μ , i.e. $\mathfrak{A}(\mu) = \text{Bor}(K)/\{A: \mu(A) = 0\}$.

Let κ be a cardinal number. Denote by λ_κ the standard product measure on 2^κ (we also write λ for λ_ω). Recall that for $\kappa > \omega$, although the product σ -algebra Σ of 2^κ is much smaller than the family of Borel sets, by the classical Kakutani theorem, $\text{Bor}(2^\kappa)$ lies in the completion of Σ with respect to λ_κ . Therefore, λ_κ is in fact defined for all Borel sets. Let \mathcal{N}_κ be the ideal of λ_κ -null subsets of 2^κ . Denote by \mathfrak{A}_κ the measure algebra of λ_κ .

Let μ be the Radon measure on a space K . Let us recall that the *Maharam type* of μ is the least cardinal number κ such that there exists a family \mathcal{E} of Borel sets of size κ approximating μ with respect to symmetric difference, that is,

$$\inf\{\mu(E \triangle B): E \in \mathcal{E}\} = 0$$

for every Borel set B .

By the Maharam structure theorem, if a Radon measure μ is of Maharam type $\leq \kappa$ then the algebra $\mathfrak{A}(\mu)$ embeds into \mathfrak{A}_κ , the measure algebra of λ_κ . Moreover, if \mathfrak{A} is *homogeneous*, i.e. the measure $\mu|_A$ has the same Maharam type for each nonzero $A \in \mathfrak{A}$, then $\mathfrak{A}(\mu)$ and \mathfrak{A}_κ are isomorphic.

3. Suslinean spaces supporting a measure under MA. The main aim of this section is to give a partial positive answer to Problems 1.1 and 1.2.

The space K we construct under MA has the following property: there is a continuous mapping $f: K \rightarrow 2^\omega$ such that its fibers are homeomorphic to ordinal numbers (in particular, they are linearly ordered and scattered). Recall that no scattered compact space can be mapped onto $[0, 1]$ by a continuous function. Consequently, by the proposition given below, our space K does not admit a continuous surjection onto $[0, 1]^{\omega_1}$.

PROPOSITION 3.1 ([Tka91], see also [BN16]). *Assume K is a compact space, M is compact and metric and $f: K \rightarrow M$ is a continuous mapping. If K can be mapped continuously onto $[0, 1]^{\omega_1}$, then there is $t \in M$ such that $f^{-1}[t]$ can be mapped continuously onto $[0, 1]^{\omega_1}$.*

We shall use the following consequence of Martin’s Axiom.

LEMMA 3.2 ([Fre84, 33D]). *Assume MA. Let $\alpha < \mathfrak{c}$ and let $\{E_n^\xi: \xi < \alpha, n \in \omega\}$ be a family of closed subsets of 2^ω such that for each $\xi < \alpha$ the sequence $(E_n^\xi)_n$ is increasing and $\lim_n \lambda(E_n^\xi) = 1$. Then for each $\varepsilon > 0$ there is a closed set $E \subseteq 2^\omega$ such that*

- (i) $\lambda(E) > 1 - \varepsilon$,
- (ii) the set $\{i: E \not\subseteq E_i^\xi\}$ is finite for every $\xi < \alpha$.

We shall construct the required space K as a Stone space $\text{ult}(\mathfrak{A})$ of a certain Boolean subalgebra \mathfrak{A} of Borel subsets of 2^ω .

THEOREM 3.3. *Assume MA. There is a compact space K and a continuous mapping $f: K \rightarrow 2^\omega$ such that*

- K is nonseparable,
- K has countable π -character,
- K supports a measure,
- each fiber of f is homeomorphic to an ordinal number (and consequently K does not map continuously onto $[0, 1]^{\omega_1}$).

Proof. If F, G are closed subsets of the Cantor set 2^ω , then we write $F \subseteq^* G$ to denote that for every $x \in G$ there is an open neighbourhood $U \ni x$ such that $F \cap U \subseteq G$ (in other words, this means that $F \setminus G$ is closed). Notice that although the relation \subseteq^* is not transitive, if $F_0 \subseteq F_1 \subseteq^* F_2$, then $F_0 \subseteq^* F_2$.

Fix an enumeration $2^\omega = \{s_\alpha: \alpha < \mathfrak{c}\}$. Let \mathcal{U} be the family of clopen subsets of 2^ω . Let λ be the Lebesgue measure on 2^ω . We are going to construct inductively a sequence $(F_\alpha)_{\alpha < \mathfrak{c}}$ of closed subsets of 2^ω such that for every $\alpha < \mathfrak{c}$:

- (1) $\lambda(F_\alpha) > 0$,
- (2) $F_\alpha \cap \{s_\xi: \xi \leq \alpha\} = \emptyset$,
- (3) if $\beta < \alpha$ then $F_\alpha \subseteq^* F_\beta$,
- (4) $\lambda(\bigcup_n F_{\alpha+n}) = 1$.

Let \mathfrak{A} be the Boolean algebra of subsets of 2^ω generated by \mathcal{U} and by the family $\{F_\alpha: \alpha < \mathfrak{c}\}$. Let $K' = \text{ult}(\mathfrak{A})$ be its Stone space. The measure λ restricted to \mathfrak{A} defines a Radon measure μ on K' (see Section 2). Let $K \subseteq K'$ be the support of μ , i.e. $K = K' \setminus \bigcup\{\widehat{A}: \lambda(A) = 0\}$.

We shall first prove that the resulting space K has the required properties. Let $f: K \rightarrow 2^\omega = \text{ult}(\mathfrak{A})$ be the standard continuous mapping induced by the restriction map:

$$f(x) = \{C: C \in \mathfrak{A}, C \in x\} \in \text{ult}(\mathfrak{A}).$$

CLAIM. *The space K supports a measure and is not separable.*

Obviously, K supports the measure μ . Conditions (1) and (2) imply that the space K is nonseparable. Indeed, given a family $\{x_n: n < \omega\} \subseteq \text{ult}(\mathfrak{A})$ set $f(x_n) = t_n$ for every n . Then $\{t_n: n < \omega\} \cap F_\alpha = \emptyset$ for α large enough, so $\widehat{F_\alpha} \cap K \neq \emptyset$ contains no x_n .

CLAIM. *Each fiber of f is homeomorphic to an ordinal number $< \mathfrak{c}$.*

Indeed, fix $t \in 2^\omega$ and take $\beta < \alpha$ such that $t \in F_\alpha \cap F_\beta$. Then by property (3) there is $U \in \mathcal{U}$ such that $t \in U$ and $F_\alpha \cap U \subseteq F_\beta$, which means that $f^{-1}[t] \cap \widehat{F_\alpha} \subseteq f^{-1}[t] \cap \widehat{F_\beta}$. It follows that the algebra of clopen subsets of $f^{-1}[t]$ is generated by a well-ordered chain of length $< \mathfrak{c}$, so $f^{-1}[t]$ is homeomorphic to some ordinal number $< \mathfrak{c}$.

CLAIM. $\pi\chi(K) = \omega$.

Consider $x \in K$ and set $\gamma = \sup\{\alpha: x \in \widehat{F_\alpha}\}$. Then $\gamma < \mathfrak{c}$. Otherwise, $f(x) \in 2^\omega$ would belong to cofinally many F_α 's, contradicting property (2). We claim that the family

$$\mathcal{P}_x = \{\widehat{F_{\gamma+n}} \cap \widehat{U} \cap K: n \in \omega, U \in \mathcal{U}\} \setminus \{\emptyset\}$$

forms a (countable) local π -base at x .

The family of sets of the form $\widehat{B} \cap K$, where

$$B = C \cap (F_{\xi_1} \cap \dots \cap F_{\xi_k}) \cap (F_{\xi_{k+1}}^c \cap \dots \cap F_{\xi_l}^c),$$

$C \in \mathcal{U}$, $\xi_i \leq \gamma$ for every $i \leq k$ and $\lambda(B) > 0$, is a local base at x .

We will show that every nonempty $\widehat{B} \cap K$, where B is as above, has a subset from \mathcal{P}_x . By (4) there is n such that $\lambda(F_{\gamma+n} \cap B) > 0$. Let $t \in F_{\gamma+n} \cap B$ be such that whenever $t \in V$, $V \in \mathcal{U}$, we have $\lambda(V \cap F_{\gamma+n} \cap B) > 0$. There is $U' \in \mathcal{U}$ such that $t \in U'$ and $U' \subseteq C \cap (F_{\xi_{k+1}}^c \cap \dots \cap F_{\xi_l}^c)$ (because the latter set is open). For every $i \leq k$ we have $F_{\gamma+n} \subseteq^* F_{\xi_i}$ and so there is $U_i \in \mathcal{U}$ such that $t \in U_i$ and $F_{\gamma+n} \cap U_i \subseteq F_{\xi_i}$. Let $U = U' \cap U_0 \cap \dots \cap U_k$. Then $t \in U \cap F_{\gamma+n} \subseteq B$. By the choice of t we have $\lambda(U \cap F_{\gamma+n}) > 0$ and so $\widehat{F_{\gamma+n}} \cap \widehat{U} \cap K \neq \emptyset$. Finally, $\widehat{F_{\gamma+n}} \cap \widehat{U} \subseteq \widehat{B}$ and $\widehat{F_{\gamma+n}} \cap \widehat{U} \cap K \in \mathcal{P}_x$, as required.

To complete the proof, we shall carry out an inductive construction. For F_0 take any closed set such that $\lambda(F_0) > 0$ and $s_0 \notin F_0$. Assume that we have constructed $\{F_\xi: \xi < \alpha\}$.

Notice that for every $\xi < \alpha$ there is an increasing sequence $(E_n^\xi)_n$ of closed sets in 2^ω such that

- (A) $\lim_n \lambda(E_n^\xi) = 1$,
- (B) $E_n^\xi \subseteq^* F_\xi$.

Indeed, for every n there is a (possibly empty) closed set $K_n \subseteq 2^\omega \setminus F_\xi$ such that $\lambda(K_n) > \lambda(K \setminus F_\xi) - 1/n$. We can assume that the sequence $(K_n)_n$ is increasing. Then $E_n^\xi = F_\xi \cup K_n$ is as desired, since $E_n^\xi \setminus F_\xi = K_n$ is closed.

Now the family $\{E_n^\xi: \xi < \alpha, n \in \omega\}$ fulfils the assumptions of Lemma 3.2. Let n be such that $\alpha = \gamma + n$ where γ is a limit ordinal. Let E be the set given by Lemma 3.2 for $\varepsilon = 1/(n + 1)$. As $\text{non}(\mathcal{N}) = \mathfrak{c}$ by Martin's Axiom, the set $\{s_\xi: \xi \leq \alpha\}$ is of measure zero, so we can find a closed set F_α such that $F_\alpha \subseteq E \setminus \{s_\xi: \xi \leq \alpha\}$ and $\lambda(F_\alpha) > 1 - 1/(n + 1)$. It is then enough to check condition (3). Let $\beta < \alpha$ and $x \in F_\beta$. As $F_\alpha \subseteq E$, by Theorem 3.2(ii), there is $k \in \omega$ such that $F_\alpha \subseteq E_k^\beta$. Hence, by (B), $F_\alpha \subseteq^* F_\beta$, and we are done. ■

Using a weaker version of Martin's Axiom we can construct an example of a space with slightly weaker properties than those of the above theorem. Recall that a measure μ on a space K is *countably determined* if there is a countable family \mathcal{F} of closed subsets of K such that

$$\mu(U) = \sup\{\mu(F): F \in \mathcal{F}, F \subseteq U\}$$

for every open $U \subseteq K$. Clearly, if K supports a countably determined measure, then it is separable. Also, separable spaces support countably determined measures (since purely atomic measures are countably determined). On the other hand, $2^{\mathfrak{c}}$ is separable but the product measure on $2^{\mathfrak{c}}$ is not countably determined.

Recall that the assertion $\mathfrak{p} = \mathfrak{c}$, which is equivalent to $\text{MA}(\sigma\text{-centered})$, that is Martin's Axiom for σ -centered posets, says that every family \mathcal{A} of subsets of ω of size less than \mathfrak{c} such that every finite intersection of elements of \mathcal{A} is infinite has an infinite pseudo-intersection P (i.e. $P \setminus A$ is finite for each $A \in \mathcal{A}$).

THEOREM 3.4. *Assume $\mathfrak{p} = \mathfrak{c}$. There is a compact space K supporting a measure which is not countably determined and a continuous mapping $f: K \rightarrow 2^\omega$ such that each fiber of f is homeomorphic to an ordinal number (so in particular K does not map continuously onto $[0, 1]^{\omega_1}$).*

Proof. Let λ be the usual measure on 2^ω . Fix an enumeration $(H_\alpha)_{\alpha < \mathfrak{c}}$ of all closed subsets of 2^ω with $\lambda(H_\alpha) > 0$.

Consider $\kappa \leq \mathfrak{c}$ and a sequence $(F_\alpha)_{\alpha < \kappa}$ of closed subsets of 2^ω such that for every $\alpha < \kappa$:

- (1) $\lambda(F_\alpha) > 0$,
- (2) if $\beta < \alpha$ then $F_\alpha \subseteq^* F_\beta$,
- (3) $F_\alpha \subseteq H_\alpha$.

Here we follow the notation used in the proof of Theorem 3.3.

We again let \mathfrak{A} be the Boolean algebra generated by $\{F_\alpha : \alpha < \kappa\}$ and the family \mathcal{U} of clopen subsets of 2^ω . Let μ be the measure on K uniquely determined by $\mu(K \cap \widehat{A}) = \lambda(A)$ for $A \in \mathfrak{A}$.

Assume that μ is countably determined and let \mathcal{D} be the countable family of closed subsets of K determining the measure μ . We can assume that \mathcal{D} is closed under finite unions and intersections. Let $\mathcal{E} = \{f[D] : D \in \mathcal{D}\}$, where $f : K \rightarrow 2^\omega$ is defined by the restriction map, as in the proof of Theorem 3.3.

CLAIM A. $\kappa < \mathfrak{c}$.

Indeed, otherwise \mathcal{E} is a countable family of closed subsets of 2^ω and every closed $H \subseteq 2^\omega$ with $\lambda(H) > 0$ contains some $E \in \mathcal{E}$, but this is plainly impossible.

CLAIM B. *There is a closed set $F_\kappa \subseteq H_\kappa$ such that $\lambda(F_\kappa) > 0$ and $F_\kappa \subseteq^* F_\alpha$ for every $\alpha < \kappa$.*

To verify Claim B we first prove the following.

CLAIM C. *For every $\xi < \kappa$ there is a sequence $(E_n^\xi)_n$ of elements of \mathcal{E} such that*

- (i) $\lim_n \lambda(E_n^\xi) = 1$,
- (ii) $E_n^\xi \subseteq^* F_\xi$.

Indeed, fix $\xi < \kappa$. Notice that both $\widehat{F_\xi} \cap K$ and $K \setminus f^{-1}[F_\xi]$ are open subsets of K . Moreover

$$\mu(\widehat{F_\xi} \cup (K \setminus f^{-1}[F_\xi])) = 1.$$

Therefore for any $\varepsilon > 0$ we can find $D_1, D_2 \in \mathcal{D}$ such that $D_1 \subseteq \widehat{F_\xi}$, $D_2 \subseteq K \setminus f^{-1}[F_\xi]$ and $\mu(D_1 \cup D_2) > 1 - \varepsilon$. Then $E = f[D_1 \cup D_2] \in \mathcal{E}$, $\lambda(E) > 1 - \varepsilon$ and $E \subseteq^* F_\xi$.

Using this observation we can define E_n^ξ satisfying (i) and (ii) of Claim C.

Now we can prove Claim B. Let us fix an enumeration $\mathcal{E} = \{E_n : n \in \omega\}$. For $\xi < \alpha$, $k > 0$ and $\{E_k^\xi : \xi < \kappa, n \in \omega\}$ as in Claim C let

$$T_k = \{n \in \omega : \exists k E_n \subseteq E_k^\xi\}, \quad N_k = \{n \in \omega : \lambda(E_n) > 1 - 1/k\}.$$

Notice that if $I \subseteq \kappa$ is finite, then for every k ,

$$\left| \bigcap_{\xi \in I} T_\xi \cap N_k \right| = \omega,$$

since \mathcal{D} is closed under finite intersections and, by (i),

$$\lim_n \lambda \left(\bigcap_{\xi \in I} E_n^\xi \right) = 1.$$

Now, since $\mathfrak{p} = \mathfrak{c}$, there is an infinite T such that $T \setminus T_\xi$ is finite for every $\xi < \alpha$ and $T \setminus N_k$ is finite for every k . In particular, $\lambda(E_n) \rightarrow 1$ for $n \in T$ so we can pick an infinite $T' \subseteq T$ with the property that

$$\lambda(F) > 1 - \lambda(H_\kappa), \quad \text{where } F = \bigcap_{n \in T'} E_n.$$

Now, for every $\xi < \kappa$ there is $n \in T' \cap T_\xi$. Then $F \subseteq E_n$ and so, by the definition of T_ξ , there is k such that $F \subseteq E_k^\xi$. Since $E_k^\xi \subseteq^* F_\xi$ we get $F \subseteq^* F_\xi$. We set $F_\kappa = F \cap H_\kappa$. Then $\lambda(F_\kappa) > 0$ and $F_\kappa \subseteq F \subseteq^* F_\xi$ for every $\xi < \kappa$, as required.

Claims A and B imply that there must be $\kappa < \mathfrak{c}$ such that μ is not countably determined, and the proof is complete. ■

It is not clear for us if the space K from Theorem 3.4 can be constructed without additional set-theoretic assumptions. There is, however, a ZFC example of a compact space K which cannot be mapped continuously onto $[0, 1]^{\omega_1}$ and which supports a measure which is not strongly countably determined. Recall that a measure is *strongly countably determined* if there is a countable family \mathcal{F} of closed G_δ sets such that

$$\mu(U) = \sup\{\mu(F) : F \in \mathcal{F}, F \subseteq U\}.$$

Indeed, the space constructed in [Bel96] is separable (hence, it supports a measure), it does not map continuously onto $[0, 1]^{\omega_1}$ and it does not have a countable π -base (and thus it cannot support a strongly countably determined measure).

4. Measures on spaces of small π -weight. As we have mentioned in the introduction, under MA_{ω_1} every space with π -weight not exceeding ω_1 is separable. In this short section we prove that for spaces supporting a measure we can relax the set-theoretic assumption.

Recall that countably determined measures were defined in the previous section. The following fact is an immediate corollary of [Fre84, Corollary 32H].

PROPOSITION 4.1. *Under MA_{ω_1} , if K is a compact space with $\pi(K) \leq \omega_1$, and K supports a measure of countable Maharam type, then each measure supported by K is countably determined (and consequently K is separable).*

In particular, under MA_{ω_1} , if K does not map continuously onto $[0, 1]^{\omega_1}$ and $\pi(K) \leq \omega_1$ then every measure supported by K is countably determined. The assumption on π -weight is essential here, as demonstrated in Theorem 3.3.

We shall prove a result similar to Proposition 4.1 assuming Martin’s Axiom for measure algebras. It will be convenient to recall several formulations of such an axiom. The following fact is a combination of results due to Fremlin and Cichoń (see [Fre08, 525J] and [DP04, Section 4] for details).

THEOREM 4.2. *The following are equivalent:*

- (i) $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$,
- (ii) ω_1 is a precaliber of measure algebras,
- (iii) ω_1 is a caliber of Radon measures.

Here we say that ω_1 is a *caliber of Radon measures* if for any such measure μ on a compact space K and any family $\{B_\alpha : \alpha < \omega\}$ of Borel sets of positive measure there is $x \in K$ such that $x \in B_\alpha$ for uncountably many α .

COROLLARY 4.3. *If $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$ then $\text{cov}(\mu) > \omega_1$ for every Radon measure μ defined on some compact space K .*

Proof. Otherwise, $K = \bigcup_{\alpha < \omega_1} N_\alpha$ where $\mu(N_\alpha) = 0$. Then there are $F_\alpha \subseteq K \setminus \bigcup_{\beta < \alpha} N_\beta$ with $\mu(F_\alpha) > 0$ for $\alpha < \omega_1$, and the family $\{F_\alpha : \alpha < \omega_1\}$ witnesses that ω_1 is not a caliber of μ . ■

The next theorem is a slight generalization of [Kam89, Lemma 3.6], where the result was proved only for Boolean spaces under a stronger assumption on weight rather than π -weight. The proof given below is a modification of the proof of [Juh71, Theorem 1.3].

THEOREM 4.4. *Assume $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$. If K is a compact space supporting a measure and $\pi(K) \leq \omega_1$, then K is separable.*

Proof. Let μ be a strictly positive probability measure on K . We work in the product space K^ω , and denote by ν the product measure $\prod_{n \in \omega} \mu$, so that

$$\nu(B_0 \times \cdots \times B_n \times K \times K \times \cdots) = \mu(B_0) \cdots \mu(B_n)$$

for all Borel rectangles.

Let \mathcal{P} be a π -base of size at most ω_1 . For every $P \in \mathcal{P}$ define

$$\mathcal{G}_P = \{\pi_n^{-1}[K \setminus P] : n \in \omega\} \quad \text{and} \quad D_P = \bigcap \mathcal{G}_P.$$

Notice that $\nu(D_P) = 0$ since $\mu(K \setminus P) < 1$.

Our assumption and Corollary 4.3 imply that there is $x \in K^\omega$ such that $x \notin \bigcup_{P \in \mathcal{P}} D_P$. The set $S = \{\pi_n(x) : n \in \omega\}$ is then dense in K . Indeed, for every open $U \subseteq K$ there is $P \in \mathcal{P}$ such that $P \subseteq U$. As $x \in K^\omega \setminus D_P$ there is $n \in \omega$ such that $\pi_n(x) \in P$. ■

Let us also state a corollary to a result due to Todorčević [Tod90] (where the notion of free sequences of pairs and its connection with tightness is explained).

COROLLARY 4.5. *Assuming $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$, every countably tight compact space supporting a measure has a countable π -base.*

Proof. [Tod90, Lemma 1] states that in every compact space K there is a family $\{(F_t, G_t) : t \in T\}$ such that

- (a) $F_t \subseteq G_t \subseteq K$, where F_t is closed and G_t is open for every t ,
- (b) the interiors of F_t are nonempty and form a π -base of K ,
- (c) for every $T_0 \subseteq T$ the subfamily $\{(F_t, G_t) : t \in T_0\}$ is free if and only if $\bigcap_{t \in T_0} F_t \neq \emptyset$.

It is enough to note that in our setting T must be countable. Indeed, otherwise take a measure μ which is strictly positive on K . Then $\mu(F_t) > 0$, so by Theorem 4.2 and (4) we get an uncountable free family, which implies that the tightness of K is uncountable. ■

5. Measures on spaces of small π -character. In this section we give a relatively consistent negative answer to Problem 1.2: there may be no Suslinean space of countable π -character supporting a measure.

We denote by (\mathfrak{R}) the following statement:

$$\mathfrak{c} = \omega_2 \ \& \ \text{non}(\mathcal{N}) = \omega_1 \ \& \ \text{cov}(\mathcal{N}_{\omega_2}) = \mathfrak{c}.$$

(\mathfrak{R}) holds e.g. in the model obtained by forcing with \mathfrak{A}_{ω_2} . Since $\text{cov}(\mathcal{N}_{\omega_1}) \geq \text{cov}(\mathcal{N}_{\omega_2})$, axiom (\mathfrak{R}) is stronger than the assumption $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$ used in the previous section.

Axiom (\mathfrak{R}) has an interesting impact on several properties of measures on topological spaces. Let us mention the following two results proved in [Ple97] and [Ple00], respectively.

THEOREM 5.1. *Assume (\mathfrak{R}) .*

- (a) *A compact space K carries a measure of Maharam type \mathfrak{c} if and only if there is a continuous surjection from K onto $[0, 1]^{\mathfrak{c}}$.*
- (b) *Every Radon measure on a first-countable compactum is strongly countably determined.*

We shall need a theorem due to Fremlin explaining that non and cov are cardinal coefficients of measure algebras rather than of concrete measure spaces.

THEOREM 5.2 ([Fre89, Theorem 6.13(c)&(d)]). *Suppose ν_1, ν_2 are Radon measures. If ν_1 and ν_2 have isomorphic measure algebras, then $\gamma(\nu_1) = \gamma(\nu_2)$, where $\gamma \in \{\text{non}, \text{cov}\}$.*

This observation allows us to prove the following.

LEMMA 5.3. *Let μ be a measure on a compact space K . Assume that K cannot be mapped continuously onto $[0, 1]^{\mathfrak{c}^+}$. If $\text{non}(\mathcal{N}) = \omega_1$, then $\text{non}(\mu) = \omega_1$, and if $\text{cov}(\mathcal{N}_{\omega_2}) = \omega_2$, then $\text{cov}(\mu) = \omega_2$.*

Proof. By Kraszewski’s theorem (see [Kra01, Corollary 3.11])

$$\text{non}(\mathcal{N}) = \text{non}(\mathcal{N}_{\omega_1}) = \text{non}(\mathcal{N}_{\omega_2}).$$

It is not difficult to see that $\text{cov}(\mathcal{N}_\kappa) \geq \text{cov}(\mathcal{N}_\lambda)$ if $\kappa \leq \lambda$ (see [Kra01, Fact 4.1]). Hence

$$\text{cov}(\mathcal{N}) = \text{cov}(\mathcal{N}_{\omega_1}) = \text{cov}(\mathcal{N}_{\omega_2}) = \omega_2.$$

Now it is enough to notice that the Maharam type of μ is at most \mathfrak{c} . Indeed, by a theorem due to Haydon, if K carries a Radon measure of Maharam type \mathfrak{c}^+ , then K can be mapped continuously onto $[0, 1]^{\mathfrak{c}^+}$ (see [Hay77, Theorem 2.4]). But, if K carries a measure of type $\lambda > \mathfrak{c}$, then it carries a measure of type \mathfrak{c}^+ (see Fremlin [Fre08, Section 531]).

By the Maharam theorem and Theorem 5.2, we are done. ■

PROPOSITION 5.4. *Assume (\mathfrak{R}) . If K is a compact space supporting a measure and K cannot be mapped continuously onto $[0, 1]^{\mathfrak{c}^+}$, then $\text{d}(K) \leq \omega_1$.*

Proof. We will show that there is $X \subseteq K$ such that $|X| \leq \omega_1$ and $\mu^*(X) = 1$. As μ is strictly positive, we will be done. A priori, Lemma 5.3 and Theorem 5.2 only imply that there is $X' \subseteq K$ of size ω_1 such that $\mu^*(X') > 0$. But then we can consider the measurable hull of X' and repeat the same argument for $\mu|_{K \setminus X'}$, which is still a Radon measure. Proceeding in this manner, at some countable step we obtain a (countable) family of sets of size ω_1 whose union has outer measure 1 in K . Define X as this union. ■

We shall now prove the main results of the section.

THEOREM 5.5. *Assume (\mathfrak{R}) . If K is a compact space supporting a measure and $\pi\chi(K) \leq \omega_1$, then K is separable.*

Proof. Notice that if K is a compact space supporting a measure and $\pi\chi(K) \leq \omega_1$, then K cannot be mapped continuously onto $[0, 1]^{\mathfrak{c}^+}$. Indeed, it is known that $w(X) \leq \pi\chi(X)^{\mathfrak{c}(X)}$ for all T_3 spaces (see [Šap74]), where $\mathfrak{c}(X)$ is the *cellularity* of X , i.e. the supremum of the sizes of families of pairwise disjoint open subsets of K . Hence $w(K) \leq \omega_1^\omega = \mathfrak{c}$. Moreover, a continuous mapping cannot increase the weight of compact spaces.

Now it follows from Proposition 5.4 that $\text{d}(K) \leq \omega_1$. Fix a dense set $X \subseteq K$ of size ω_1 and then for each $x \in X$ fix a π -base \mathcal{F}_x at x of size ω_1 . Then $\mathcal{P} = \bigcup_{x \in X} \mathcal{F}_x$ is a π -base of size at most ω_1 . Indeed, if V is a nonempty open subset of K , then it contains $x \in X$. So \mathcal{F}_x contains a nonempty open $U \subseteq V$.

We conclude that $\pi(K) \leq \omega_1$, and Theorem 4.4 gives the separability of K . ■

THEOREM 5.6. *Assume (\mathfrak{R}) . Let K be a space of countable π -character, supporting a measure. Then K has a countable π -base.*

Proof. We already know (see the proof of Theorem 5.5) that there is a π -base \mathcal{P} of K of size at most ω_1 . Enumerate $\mathcal{P} = \{P_\xi : \xi < \omega_1\}$. Denote $\mathcal{P}_\alpha = \{P_\xi : \xi < \alpha\}$ and

$$Z_\alpha = \{x \in X : \mathcal{P}_\alpha \text{ contains a local } \pi\text{-base of } x\}.$$

Notice that $K = \bigcup_\alpha Z_\alpha$ and that Z_α is closed for every α .

As $\text{cov}(\mu) > \omega_1$, there is $\alpha < \omega_1$ such that $\mu(Z_\alpha) = 1$. So, $Z_\alpha = K$. One can easily check, as above, that \mathcal{P}_α is a (countable) π -base of K . ■

Let us recall that by Theorem 1.3 if $\text{cov}(\mathcal{N}_{\omega_1}) = \omega_1$, then there is a first-countable nonseparable compact space supporting a measure. Of course, first-countable spaces have countable π -character, so we need to assume at least $\text{cov}(\mathcal{N}_{\omega_1}) > \omega_1$ to prove Theorem 5.6.

We conclude by some remarks on the so called normal measures. Recall that a Radon measure on a topological space μ is *normal* if each nowhere dense set is null with respect to μ . There are several results concerning the existence of normal measures on certain kind of spaces, e.g. Fishel and Papert [FP64] proved that there is no normal measure on locally connected spaces. On the other hand, Plebanek [Ple14] constructed an example of a normal measure on a connected space (which can be made first-countable under CH). Zindulka [Zin00] proved that consistently there is no normal measure on a first-countable locally compact space. Since the support of a normal measure cannot be separable, Theorem 5.6 implies the following.

COROLLARY 5.7. *Assume (\mathfrak{R}) . There is no normal strictly positive measure on a compact space of a countable π -character.*

In fact, using [Zin00, Lemma 3.2, Theorem 3.3 and Lemma 3.4] and Theorem 5.5 one can show that consistently there is no locally compact space with a countable π -character carrying a nontrivial residual measure.

6. Open problems. Problem 1.1 remains unsolved in ZFC. In particular, we still do not know the answer to the following question.

PROBLEM 6.1. *Does (\mathfrak{R}) imply that there is no compact nonseparable space K supporting a measure and such that K cannot be mapped continuously onto $[0, 1]^{\omega_1}$?*

Perhaps the construction from Section 3 can be improved in such a way that it works in ZFC.

PROBLEM 6.2. *Is there a compact nonseparable space supporting a measure and a continuous mapping $f: K \rightarrow 2^\omega$ such that $f^{-1}[t]$ is scattered for each $t \in 2^\omega$?*

If the answer to the above is “yes”, then we would have to use different methods than those from Section 3. In the proof of Theorem 3.3 we added a condition (4) to ensure that the constructed space has countable π -character. Even without adding (4), the space in question would have the π -character at most ω_1 (if $\mathfrak{c} = \omega_2$). So, by Theorem 5.5, it would have to be separable under axiom (\mathfrak{R}) . There is a slightly more general reason why under axiom (\mathfrak{R}) we would have to devise more subtle methods to construct a similar example.

PROPOSITION 6.3. *Assume (\mathfrak{R}) . Suppose that K is a compact space supporting a measure μ , M is a compact metric space and $f: K \rightarrow M$ is a continuous mapping such that for each $t \in M$ the set $f^{-1}(t)$ contains a dense set of isolated points of size at most ω_1 . Then K is separable.*

Proof. Fix a countable base \mathcal{U} of M . Let $X \subseteq K$ be a set of size at most ω_1 such that $\mu^*(X) = 1$ (see Proposition 5.4). Let $Y = f[X]$. For each $t \in Y$ let Y_t be the set of isolated points of $f^{-1}(t)$. For each $t \in Y$ and $y \in Y_t$ let P_y be an open set such that $P_y \cap f^{-1}(t) = \{y\}$. We claim that

$$\mathcal{P} = \{P_y \cap f^{-1}[U] : y \in Y_t, t \in Y, U \in \mathcal{U}\}$$

is a π -base of K and thus, according to Theorem 5.5, K is separable.

Let $V \subseteq K$ be an open set. Then there is $x \in V \cap X$. As the set of isolated points is dense in $f^{-1}(f(x))$, there is $y \in Y_{f(x)}$ such that $y \in V$. We claim that there is $U \in \mathcal{U}$ such that $P_y \cap f^{-1}[U] \subseteq V$ and $f(x) \in U$ (and so $P_y \cap f^{-1}[U]$ is nonempty). Suppose to the contrary that

$$(P_y \cap f^{-1}[U]) \setminus V \neq \emptyset$$

for each $U \in \mathcal{U}$ such that $f(x) \in U$. Then also

$$\overline{P_y} \cap \overline{f^{-1}[U]} \setminus V \neq \emptyset$$

and so by compactness

$$\overline{P_y} \cap f^{-1}(f(x)) \setminus V \neq \emptyset.$$

But $\overline{P_y} \cap f^{-1}(f(x)) = \{y\}$ and $y \in V$, a contradiction. ■

Finally, we do not know if one can prove a theorem similar to Corollary 5.7 for spaces which cannot be mapped onto $[0, 1]^{\omega_1}$.

PROBLEM 6.4. *Is it consistent that there is no compact space which cannot be mapped continuously onto $[0, 1]^{\omega_1}$ and which supports a normal measure?*

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