FUNDAMENTA
MATHEMATICAE
234 (2016)

Prolongational centers and their depths

by

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Abstract. In 1926 Birkhoff defined the center depth, one of the fundamental invariants that characterize the topological structure of a dynamical system. In this paper, we introduce the concepts of prolongational centers and their depths, which lead to a complete family of topological invariants. Some basic properties of the prolongational centers and their depths are established. Also, we construct a dynamical system in which the depth of a prolongational center is a prescribed countable ordinal.

1. Introduction. In a dynamical system, the closure of the set of points belonging to the Poisson stable trajectories is called the *center* and the trajectories in it are called *central trajectories*. As was indicated by Birkhoff [4], one of the fundamental invariants that characterize the topological structure of a dynamical system is the ordinal number of the central trajectories, that is, the center depth. In the literature, there are three different algorithms for finding center depths defined respectively by Birkhoff [4], Birkhoff and Smith [6], and Maier [7]. For a detailed description of these different processes, see Maier [8] and Neumann [10]. Those algorithms put the dynamical system into correspondence with different topological invariants—the ordinal numbers of the center depth: β_A (Birkhoff's algorithm), β_B (Birkhoff and Smith's algorithm), β_C (Maier's algorithm). In [8], it was shown that

$$\beta_A \geq \beta_B \geq \beta_C. \tag{1}$$

An important problem is the range of β_A, β_B and β_C. Much significant work in this direction has been done by Birkhoff, Maier, Neumann, Reed [14], Schwartz and Thomas [15], Sharkovskiĭ [16], Shil'nikov [17] and others.

To extend the notions of recurrent motion in a dynamical system, using continuous real valued functions on the phase space, Auslander [1]

2010 *Mathematics Subject Classification*: Primary 37B20; Secondary 54H20.
Key words and phrases: prolongational (Auslander) recurrence, prolongational center, center depth.
Received 5 February 2015; revised 22 September 2015.
Published online 17 March 2016.

introduced the notion of prolongational recurrence (or generalized recurrence). The set of prolongational recurrent points contains the periodic points, recurrent (or Poisson stable) points and non-wandering points. It is known that prolongational recurrence is an important concept in the theory of structural stability. As in the construction of central sequences by removing the set of wandering points, we establish prolongational central sequences from nested sets of prolongational recurrent points of subsystems. In this paper, we shall define prolongational centers and their depths, which leads to a complete family of topological invariants for dynamical systems. Some fundamental properties of the centers and their depths are presented. In general, a similar relation to (1) for different order centers is impossible, but if two different order centers are the same set, then an inequality between their depths holds. Also, it is shown that the depths of prolongational centers can be arbitrarily chosen countable ordinals.

2. Prolongational recurrence. Throughout the paper, X denotes a metric space with metric d. For a subset $A \subseteq X$, $\overline{A}$ denotes the closure of A. Let $B(x,\delta) = \{y \in X : d(x,y) < \delta\}$ be the open ball with center x and radius $\delta > 0$. Let $\mathbb{R}$ be the real line, $\mathbb{R}^+$ and $\mathbb{R}^-$ the subsets of $\mathbb{R}$ consisting of the non-negative and non-positive real numbers, respectively.

A *dynamical system* or (continuous) *flow* on X is a triple $(X, \pi, \mathbb{R})$, where π is a continuous mapping from $X \times \mathbb{R}$ onto X satisfying the following axioms:

(1) $\pi(x, 0) = x$ for each $x \in X$,
(2) $\pi(\pi(x,t), s) = \pi(x, t+s)$ for all $x \in X$ and $t, s \in \mathbb{R}$.

For brevity, we suppress the mapping π notationally and just write $x \cdot t$ in place of $\pi(x,t)$. Similarly, let $A \cdot J = \{x \cdot t : x \in A, t \in I\}$ for $A \subseteq X$ and $I \subseteq \mathbb{R}$. If either A or J is a singleton, i.e., $A = \{x\}$ or $I = \{t\}$, then we simply write $x \cdot J$ and $A \cdot t$ in place of $\{x\} \cdot J$ and $A \cdot \{t\}$, respectively. A set $S \subseteq X$ is *positively* [*negatively*] *invariant* if $S \cdot \mathbb{R}^+ = S$ [$S \cdot \mathbb{R}^- = S$], and is invariant if $S \cdot \mathbb{R} = S$. Recall that $y \in X$ is called an *ω-limit point* [*α-limit point*] of $x \in X$ if there is a sequence of real numbers $t_i \to \infty$ [$t_i \to -\infty$] such that $x \cdot t_i \to y$ as $i \to \infty$. The set of ω-limit points [α-limit points] of x is denoted by $\omega(x)$ [$\alpha(x)$]. A point $x \in X$ is *Poisson stable* if $x \in \omega(x) \cap \alpha(x)$. The *first prolongational set* and the *first prolongational limit set* are defined, respectively, by $D_1(x) = \{y \in X :$ there are sequences $\{x_n\} \subseteq X$, $\{t_n\} \subseteq \mathbb{R}^+$ such that $x_n \to x$ and $x_n \cdot t_n \to y\}$ and $J_1(x) = \{y \in X :$ there are sequences $\{x_n\} \subseteq X$, $\{t_n\} \subseteq \mathbb{R}^+$ such that $x_n \to x$, $t_n \to \infty$ and $x_n \cdot t_n \to y\}$. We say a point $x \in X$ is *non-wandering* if $x \in J_1(x)$. For each $x \in X$, we have $D_1(x) = x \cdot \mathbb{R}^+ \cup J_1(x)$. Note that $\omega(x)$, $\alpha(x)$ and $J_1(x)$ are closed invariant sets, and $D_1(x)$ is a closed and positively invariant set. For the elementary properties of dynamical systems, consult [3, 5, 9].

Let $\mathscr{X}$ be the collection of all subsets of X, and $\mathcal{M} = \{\Gamma : \Gamma \text{ is a map from } X \text{ to } \mathscr{X}\}$. For $\Gamma \in \mathcal{M}$ and $A \in \mathscr{X}$, we define $\Gamma(A) = \bigcup\{\Gamma(x) : x \in A\}$. If n is a positive integer, the map $\Gamma^n : X \to \mathscr{X}$ is defined inductively by $\Gamma^1 = \Gamma$ and $\Gamma^n = \Gamma \circ \Gamma^{n-1}$, i.e., $\Gamma^1(x) = \Gamma(x)$ and $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$ for $x \in X$. Now, we introduce two operators $\mathscr{D}$ and $\mathscr{S}$ on the collection $\mathcal{M}$. If $\Gamma \in \mathcal{M}$, we define

$$\mathscr{D}\Gamma(x) = \bigcap_{\delta>0} \overline{\Gamma(B(x,\delta))} \quad \text{and} \quad \mathscr{S}\Gamma(x) = \bigcup_{n=1}^{\infty} \Gamma^n(x) \qquad \text{for } x \in X.$$

It is easy to see that $y \in \mathscr{D}\Gamma(x)$ if and only if there are sequences $\{x_n\}$ and $\{y_n\}$ with $y_n \in \Gamma(x_n)$ such that $x_n \to x$ and $y_n \to y$. Also, $y \in \mathscr{S}\Gamma(x)$ if and only if there are points $x = x_0, x_1, \ldots, x_n = y$ with $x_i \in \Gamma(x_{i-1})$ $(i = 1, \ldots, n)$. Obviously, $\mathscr{D}$ and $\mathscr{S}$ may be regarded as a 'closure' operator and a 'transitizing' operator on $\mathcal{M}$ respectively.

In the study of a dynamical system, the set-valued maps ω, D_1 and J_1 are the most important ones in $\mathcal{M}$. Starting with D_1, we use the operators $\mathscr{D}$ and $\mathscr{S}$ to define higher prolongation maps $\{D_\alpha\}$ (α is an ordinal) as follows. By transfinite induction, if α is a successor ordinal, then having defined $D_{\alpha-1}$, we set $D_\alpha = \mathscr{D}\mathscr{S}D_{\alpha-1}$; if α is a limit ordinal, then having defined D_β for every $\beta < \alpha$, we set $D_\alpha = \mathscr{D}(\bigcup_{\beta<\alpha} \mathscr{S}D_\beta)$. In [2], it is shown that higher prolongations are important in the theory of stability. Similarly, we can consider the higher prolongational limit maps, which lead to the concept of prolongational recurrence (or Auslander recurrence). Let $\Gamma = J_1$, and define $J_2 = \mathscr{D}\mathscr{S}J_1$. Inductively, if α is a successor ordinal, having defined $J_{\alpha-1}$ we set $J_\alpha = \mathscr{D}\mathscr{S}J_{\alpha-1}$; if α is a limit ordinal, having defined J_β for each $\beta < \alpha$ we set $J_\alpha = \mathscr{D}(\bigcup_{\beta<\alpha} \mathscr{S}J_\beta)$. For each ordinal α and $x \in X$, $J_\alpha(x)$ is a closed invariant set and $D_\alpha(x) = \gamma^+(x) \cup J_\alpha(x)$. The reader is referred to [1] for details.

Definition 2.1. For each ordinal α, let $\mathcal{R}_\alpha = \{x \in X : x \in J_\alpha(x)\}$, called the *$\alpha$-order recurrent set*. Then define $\mathcal{R} = \bigcup \mathcal{R}_\alpha$, the *prolongational recurrent set*. An element of $\mathcal{R}$ is called a *prolongational recurrent point*.

The notion of prolongational recurrence was first introduced by Auslander, and in the literature it is also called *Auslander recurrence* or *generalized recurrence*. Some fundamental properties of prolongational recurrence were established by Auslander [1]. The prolongational recurrent set $\mathcal{R}$ is an invariant closed set, and it is closely related to the structural stability of a dynamical system (see [11]). In particular, Nitecki [12] proved that a dynamical system is C^0-explosive if and only if $\mathcal{R} \neq \emptyset$, and Peixoto [13] perturbed a vector field with a non-periodic prolongational recurrent point to get a periodic orbit (a closing lemma).

LEMMA 2.2. *For each ordinal α, $\mathcal{R}_\alpha$ is closed and invariant.*

Proof. In fact, $\mathcal{R}_1 = \{x \in X : x \in J_1(x)\}$ is a non-wandering set, which is closed and invariant. We proceed by transfinite induction. Assume that $\mathcal{R}_\lambda$ is closed for each ordinal $\lambda < \alpha$. Let $\{x_n\}$ be a sequence in $\mathcal{R}_\alpha$ such that $x_n \to y$. Since $x_n \in J_\alpha(x_n)$, for each n there exist sequences $\{p_k^n\}, \{q_k^n\}$ such that $p_k^n \to x_n$, $q_k^n \to x_n$ as $k \to \infty$ and $p_k^n \in J_{\lambda_k}^{l_k}(q_k^n)$ ($\lambda_k < \alpha$, l_k a positive integer). Thus, let subsequences $p_{k_n}^n$, $q_{k_n}^n$ be such that $p_{k_n}^n \to y$, $q_{k_n}^n \to y$ as $n \to \infty$, $p_{k_n}^n \in J_{\lambda_n}^{l_n}(q_{k_n}^n)$ ($\lambda_n < \alpha$, l_n a positive integer). It follows that $y \in J_\alpha(y)$, and hence $\mathcal{R}_\alpha$ is closed. Next, let $x \in \mathcal{R}_\alpha$ and $t \in \mathbb{R}$. By [1, p. 69, Lemma 3], it follows from $x \in J_\alpha(x)$ that $x \cdot t \in J_\alpha(x) \cdot t = J_\alpha(x \cdot t)$, i.e., $x \cdot t \in \mathcal{R}_\alpha$. Hence, $\mathcal{R}_\alpha$ is invariant. ∎

Let α and β be a pair of ordinals. If $\alpha < \beta$, by the definition of J_α we have $J_\alpha(x) \subseteq J_\beta(x)$ for $x \in X$, i.e., the map J_α is monotone. Thus, for $x \in X$ and $\alpha < \beta$, if $x \in J_\alpha(x)$ then $x \in J_\beta(x)$, which means that $\mathcal{R}_\alpha \subseteq \mathcal{R}_\beta$. In [2], it was shown that $D_\alpha = D_{\omega_1}$ for $\alpha > \omega_1$, where ω_1 is the first uncountable ordinal. Since $D_\alpha(x) = J_\alpha(x)$ for $x \in \mathcal{R}_\alpha$, we also have $J_\alpha(x) = J_{\omega_1}(x)$ if $\alpha > \omega_1$. Hence, we have the following result.

LEMMA 2.3. *There exists an ordinal $\varsigma \le \omega_1$ such that $\mathcal{R} = \mathcal{R}_\varsigma$.*

In the next section, it is proved that if X is a second countable topological space, then ς is countable. Note that if X is compact, then $\mathcal{R}_\alpha \neq \emptyset$ for each ordinal α.

DEFINITION 2.4. Let ς be the least ordinal such that $\mathcal{R} = \mathcal{R}_\varsigma$ $(\neq \emptyset)$; then we call ς the *recurrence order* of (X, π), and let $\varsigma = \mathfrak{D}(\pi)$.

A function $\rho : X \times \mathbb{R} \to \mathbb{R}$ is called a *reparametrization* if (i) $\rho(\,,x) : \mathbb{R} \to \mathbb{R}$ is surjective, and (ii) $\rho(\,,x) : \mathbb{R} \to \mathbb{R}$ is strictly increasing for each $x \in X$. Let π and ψ be dynamical systems defined in metric spaces X and Y, respectively. If there exist a homeomorphism $h : X \to Y$ and a reparametrization $\rho : X \times \mathbb{R} \to \mathbb{R}$ such that $h(\pi(x, \rho(x,t))) = \psi(h(x), t)$ for all $x \in X$ and $t \in \mathbb{R}$, then π and ψ are said to be *topologically equivalent* or *isomorphic*. The pair (h, ρ) is a *topological equivalence* from π to ψ. For the special case $\rho(x,t) = t$ for all $x \in X$ and $t \in \mathbb{R}$, π and ψ are said to be *topologically conjugate*. Obviously, these two notions show the topological similarity between dynamical systems, and many of their trajectory properties are preserved by topological equivalence. In particular, if $x \in X$, it is easy to see that $h(J_1(x)) = J_1(h(x))$. By transfinite induction, $h(J_\alpha(x)) = J_\alpha(h(x))$ for each ordinal $\alpha \ge 1$. So, we obtain the following result.

PROPERTY 2.5. *Let (h, ρ) be a topological equivalence from π to ψ. For every ordinal α, denote by $\mathcal{R}_\alpha^\pi$ and $\mathcal{R}_\alpha^\psi$ the respective α-order recurrent sets.*

Then $h(\mathcal{R}_\alpha^\pi) = \mathcal{R}_\alpha^\psi$. Also, $h(\mathcal{R}^\pi) = \mathcal{R}^\psi$, where $\mathcal{R}^\pi$ and $\mathcal{R}^\psi$ are the prolongational recurrent sets of π and ψ. In particular, $\mathfrak{D}(\pi) = \mathfrak{D}(\psi)$.

3. Prolongational center. Let X be a second countable topological space, i.e., X has a countable base. Every separable metric space is second countable. In the following, we always assume that X is a separable metric space, and let π be a dynamical system on X such that $\mathcal{R} \neq \emptyset$. We recall the Baire Theorem (see [9, p. 312]):

THEOREM 3.1 (Baire). *Let X be a second countable topological space. If*

$$\{K_1, \dots, K_n, \dots, K_{\omega_0}, K_{\omega_0+1}, \dots, K_\lambda, \dots\}$$

is a collection of closed subsets of X such that

$$K_1 \supseteq \dots \supseteq K_n \supseteq \dots \supseteq K_{\omega_0} \supseteq K_{\omega_0+1} \supseteq \dots \supseteq K_\lambda \cdots,$$

where ω_0 is the first infinite ordinal, then there exists a countable ordinal ς such that $K_\varsigma = K_\lambda$ for all ordinals $\lambda \geq \varsigma$.

Let $\Omega(\pi)$ be the set of all non-wandering points in the flow (X, π); it is a closed invariant set. A point $x \in \Omega(\pi)$ is not necessarily non-wandering in the subflow $\pi|_\Omega$ on $\Omega(\pi)$, i.e., $\Omega(\pi|_\Omega)$ may be a proper subset of $\Omega(\pi)$. Now, let $\Omega_1 = \Omega(\pi)$, and for each positive integer n define $\Omega_{n+1} = \Omega(\pi|_{\Omega_n})$. We let $\Omega_{\omega_0} = \bigcap_{n=1}^\infty \Omega_n$. The set Ω_{ω_0} is again a closed invariant set, thus define $\Omega_{\omega_0+1} = \Omega(\pi|_{\Omega_{\omega_0}})$. This process can be continued to all ordinals by transfinite induction. If $\lambda = \kappa + 1$ for some ordinal κ, then define $\Omega_\lambda = \Omega(\pi|_{\Omega_\kappa})$. If λ is a limit ordinal, then define $\Omega_\lambda = \bigcap_{\kappa<\lambda} \Omega_\kappa$. Thus, we obtain a transfinite sequence of closed sets

$$\Omega_1 \supseteq \dots \supseteq \Omega_n \supseteq \dots \supseteq \Omega_{\omega_0} \supseteq \Omega_{\omega_0+1} \supseteq \dots \supseteq \Omega_\lambda \supseteq \cdots.$$

By the Baire Theorem, there exists a countable ordinal σ_1 such that $\Omega_\lambda = \Omega_{\sigma_1}$ for all $\lambda > \sigma_1$. The set $\mathcal{C}_1 = \Omega_{\sigma_1}$ is the *center* of π. An element of the center is a *central point* of π. For a detailed discussion, see Birkhoff [5] and Nemytskiĭ and Stepanov [9]. By Lemma 2.2, the above procedure can be applied to each $\mathcal{R}_\alpha$ for an ordinal $\alpha > 1$ and the prolongational recurrence set $\mathcal{R}$.

DEFINITION 3.2. For an ordinal α, let $\mathcal{R}_\alpha^1 = \mathcal{R}_\alpha$. Define $\mathcal{R}_\alpha^\lambda$ inductively as follows. If λ is a successor ordinal, let $\mathcal{R}_\alpha^\lambda = \mathcal{R}_\alpha(\pi|_{\mathcal{R}_\alpha^{\lambda-1}})$ be the α-order recurrent set of the subflow $\pi|_{\mathcal{R}_\alpha^{\lambda-1}}$. If λ is a limit ordinal, let $\mathcal{R}_\alpha^\lambda = \bigcap_{\eta<\lambda} \mathcal{R}_\alpha^\eta$. Then there exists a countable ordinal σ_α such that $\mathcal{R}_\alpha^\lambda = \mathcal{R}_\alpha^{\sigma_\alpha}$ for all $\lambda > \sigma_\alpha$. The set $\mathcal{C}_\alpha = \mathcal{R}_\alpha^{\sigma_\alpha}$ is said to be the α-*order center*. In particular, $\mathcal{C}_1$ is called the *Birkhoff center*.

DEFINITION 3.3. Let $\mathcal{R}^1 = \mathcal{R}$. Define $\mathcal{R}^\lambda$ inductively as follows. If λ is a successor ordinal, let $\mathcal{R}^\lambda = \mathcal{R}(\pi|_{\mathcal{R}^{\lambda-1}})$ be the generalized recurrent set of the subflow $\pi|_{\mathcal{R}^{\lambda-1}}$. If λ is a limit ordinal, let $\mathcal{R}^\lambda = \bigcap_{\beta<\lambda} \mathcal{R}^\beta$. Then there exists a countable ordinal σ such that $\mathcal{R}^\lambda = \mathcal{R}^\sigma$ for all $\lambda > \sigma$. The set $\mathcal{C} = \mathcal{R}^\sigma$ is said to be the *prolongational center*.

LEMMA 3.4. *For two ordinals α and β, if $\alpha < \beta$, then $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$. In particular, $\mathcal{C}_\alpha \subseteq \mathcal{C}$ for each α.*

Proof. For $\alpha < \beta$, we have $\mathcal{R}_\alpha \subseteq \mathcal{R}_\beta$. Let $\mathcal{R}_\alpha^1 = \mathcal{R}_\alpha$ and $\mathcal{R}_\beta^1 = \mathcal{R}_\beta$. Assume that $\mathcal{R}_\alpha^\eta \subseteq \mathcal{R}_\beta^\eta$ for all $\eta < \lambda$. Thus, if λ is a successor ordinal, we have $\mathcal{R}_\alpha^\lambda = \mathcal{R}_\alpha(\pi|_{\mathcal{R}_\alpha^{\lambda-1}}) \subseteq \mathcal{R}_\alpha(\pi|_{\mathcal{R}_\beta^{\lambda-1}}) \subseteq \mathcal{R}_\beta(\pi|_{\mathcal{R}_\beta^{\lambda-1}}) = \mathcal{R}_\beta^\lambda$. If λ is a limit ordinal, then $\mathcal{R}_\alpha^\lambda = \bigcap_{\eta<\lambda} \mathcal{R}_\alpha^\eta \subseteq \bigcap_{\eta<\lambda} \mathcal{R}_\beta^\eta = \mathcal{R}_\beta^\lambda$. It follows that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$ for $\alpha < \beta$. ∎

By Lemma 2.3, we have the prolongational center $\mathcal{C} = \mathcal{C}_\varsigma$, where $\varsigma = \mathfrak{D}(\pi)$ is a countable ordinal. Thus, for a flow π on a separable metric space X there exists a collection of centers satisfying $\mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_\varsigma$. In the following example, we see that $\mathcal{C}_\alpha$ may be a proper subset of $\mathcal{C}_\beta$ for $\alpha < \beta$.

EXAMPLE 3.5. Consider a flow in $\mathbb{D} = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1\} \subset \mathbb{R}^2$, defined in polar coordinates as follows:

$$\frac{dr}{dt} = r(1-r)f(r,\theta) \quad \text{and} \quad \frac{d\theta}{dt} = f(r,\theta) \qquad (0 \le r \le 1), \tag{2}$$

where $f(r,\theta)$ is a smooth function such that $f(1,0) = 0$, $f(1,\pi/(2n)) = 0$ $(n = 1,2,\ldots)$ and $f(r,\theta) > 0$ elsewhere. It is easy to see that the Birkhoff center consists of all the rest points, i.e., $\mathcal{C}_1 = \{(1,\pi/(2n)) : n = 1,2,\ldots\} \cup \{(0,0),(1,0)\}$. Let $p_1 = (1,0)$ and $p_2 = (1,\pi)$. Note that for the subflow on the unit circle $\mathbb{S} = \partial\mathbb{D}$, we have $J_2(p_2) = \{p_1\}$ and $J_2(p_1) = \mathbb{S}$. Hence, $\mathcal{C}_2 = \mathcal{C}_1$. Finally, it is easy to see that $\mathcal{C}_3 = \mathbb{S} \cup \{(0,0)\}$, which is also the prolongational center $\mathcal{C}$.

By Property 2.5, we have the following conclusion.

PROPERTY 3.6. *The family $\{\mathcal{C}_1, \ldots, \mathcal{C}_{\mathfrak{D}(\pi)}\}$ of prolongational centers are invariants of topological equivalence.*

4. Depth of center. In the process of obtaining the Birkhoff center $\mathcal{C}_1$, there exists a least countable ordinal σ_1 such that $\Omega_\lambda = \Omega_{\sigma_1}$ for all $\lambda > \sigma_1$; this ordinal is said to be the *depth* of $\mathcal{C}_1$. Similarly, we can define the depth of an α-order center for each $\alpha > 1$.

DEFINITION 4.1. Let α be an ordinal. The *depth* of the α-order center $\mathcal{C}_\alpha$, denoted d_α, is the least ordinal σ_α such that $\mathcal{R}_\alpha^\lambda = \mathcal{R}_\alpha^{\sigma_\alpha} = \mathcal{C}_\alpha$ for all $\lambda > \sigma_\alpha$. In particular, the depth of the prolongational center $\mathcal{C}_{\mathfrak{D}(\pi)}$ is the least ordinal $d_{\mathfrak{D}(\pi)}$ such that $\mathcal{R}_{\mathfrak{D}(\pi)}^\lambda = \mathcal{R}_{\mathfrak{D}(\pi)}^{d_{\mathfrak{D}(\pi)}} = \mathcal{C}_{\mathfrak{D}(\pi)}$ for all $\lambda > \sigma_{\mathfrak{D}(\pi)}$.

Now, we construct two systems so that one satisfies $d_1 > d_2$ and the other $d_1 < d_2$.

EXAMPLE 4.2. Consider the flow π defined by (2) with $f(1,0)=0$ and $f(r,\theta)>0$ for $(r,\theta)\neq(1,0)$. It is easy to see that the Birkhoff center $\mathcal{C}_1$ is $\{(0,0),(1,0)\}$ and $d_1=2$. For the subflow $(\mathbb{S},\pi)$, let $p\in\mathbb{S}$, $J_2(p)=\{y : x_n\to p$ and $y_n\to y$ with $x_n\in\mathbb{S}$, $y_n\in J_1^{k_n}(x_n)$ (k_n a positive integer)$\}=\mathbb{S}$. Thus, $\mathcal{C}_2=\{(0,0)\}\cup\mathbb{S}$ and $d_2=1$.

EXAMPLE 4.3. First, we consider a simple system in $\mathbb{D}=\{(x_1,x_2) : x_1^2+x_2^2\leq 1\}$, defined in polar coordinate as follows:

$$\frac{dr}{dt}=r(1-r)\quad\text{and}\quad\frac{d\theta}{dt}=1\qquad(0\leq r\leq 1). \tag{3}$$

Let $p=(1/2,0)$ and $X=p\cdot\mathbb{R}\cup\mathbb{S}\cup\{(0,0)\}$, where X is a closed invariant set of the system (3). Then we define a dynamical system π on X by the following ordinary differential equations:

$$\frac{dr}{dt}=r(1-r)f(r,\theta)\quad\text{and}\quad\frac{d\theta}{dt}=f(r,\theta)\qquad(0\leq r\leq 1), \tag{4}$$

where $f(r,\theta)$ is a smooth function such that $f(r,\theta)=0$ if $\theta=0$ or $(r,\theta)=(1,\pi/(2n))$ $(n=1,2,\ldots)$, otherwise $f(r,\theta)>0$. Let F be the set of intersection points of the positive x_1-axis and the trajectory $p\cdot\mathbb{R}$ of (3). Clearly, each point in F is a rest point of (4). It is easy to see that the Birkhoff center $\mathcal{C}_1$ is

$$F\cup\{(0,0),(1,0)\}\cup\{(1,\pi/(2n)) : n=1,2,\ldots\}$$

and $d_1=1$. For each $p\in\mathbb{S}$, there exist sequences $x_n\in p\cdot\mathbb{R}$ and $y_n\in\mathscr{S}J_1(x_n)$ such that $x_n\to p$ and $y_n\to p$, hence $\mathcal{R}_2=F\cup\mathbb{S}\cup\{(0,0)\}$. Observe that

$$\mathcal{R}_2(\pi|_{\mathbb{S}})=\{(1,0)\}\cup\{(1,\pi/(2n)) : n=1,2,\ldots\}.$$

Thus, $\mathcal{C}_2=\mathcal{C}_1$ and $d_2=2$.

In the above system with $d_1>d_2$, we have $\mathcal{C}_1\neq\mathcal{C}_2$. In general, for the case $\mathcal{C}_\alpha=\mathcal{C}_\beta$ $(\alpha>\beta)$, since $\mathcal{R}_\beta\subseteq\mathcal{R}_\alpha$ always holds, it is easy to deduce the following result.

THEOREM 4.4. *For ordinals $\alpha>\beta\geq 1$, if $\mathcal{C}_\alpha=\mathcal{C}_\beta$, then $d_\alpha\geq d_\beta$.*

Now, we state our main result:

THEOREM 4.5. *For any countable ordinals α and β there exists a dynamical system with $d_\beta=\alpha$.*

To prove this theorem, we just construct a concrete system with $d_\beta=\alpha$. First, let $Y=(-1,1)$ and consider a flow on Y defined by $dx/dt=g(x)$, where $g(x)$ is a smooth function. Let $p=-1/2$ and $q=2/3$. If $g(0)=0$ and $g(x)>0$ for $x\neq 0$, it is easy to see that $q\notin J_1(p)=\{0\}$ and $q\in J_2(p)$. If $g(x)=0$ for $x=0$, $x=1/n$ $(n=2,3,\ldots)$ and $g(x)>0$ elsewhere, then we have $q\notin J_2(p)$ and $q\in J_3(p)$. Similarly, if $g(x)=0$ for $x=0$, $x=\frac{1}{n+1/m}$

($n = 2, 3, \dots$ and $m = 1, 2, \dots$) and $g(x) > 0$ elsewhere, then we obtain $q \notin J_3(p) = \{0\}$ and $q \in J_4(p)$.

Now, if n is any positive integer, it is clear that we may define a function g_n, similar to g above, so that in the dynamical system π_n determined by the equation $dx/dt = g_n(x)$, we have $q \notin J_{n-1}(p)$ and $q \in J_n(p)$.

By appropriately combining the π_n, we may define a dynamical system π_{ω_0} such that $q \notin J_n(p)$ for any positive integer n and $q \in J_{\omega_0}(p)$, where ω_0 is the first infinite ordinal. In fact, we may suppose that $g_{\omega_0}(x) = 0$ for $x = 0$ and $x = 1/n$ ($n = 2, 3, \dots$). Then, in each interval

$$Y_n = \left[\frac{1}{n+1}, \frac{1}{n}\right) \qquad (n \geq 2)$$

we define $g_{\omega_0}(x)$ as $g_n(x)$ in $[0, 2/3)$ using a linear contraction, i.e.,

$$g_{\omega_0}(x) = g_n\left(\frac{2}{3}n(n+1)\left(x - \frac{1}{n+1}\right)\right) \qquad \text{for } x \in Y_n.$$

If we let π_{ω_0} be the dynamical system determined by $dx/dt = g_{\omega_0}(x)$, then π_{ω_0} behaves on Y_n like π_n on $[0, 2/3)$. So, $q \notin J_n(p)$ for any positive integer n, but $q \in J_{\omega_0}(p)$.

Next, we define π_{ω_0+1} in a similar way so that the dynamical behavior of π_{ω_0+1} on each Y_n is the same as the behavior of π_{ω_0} on $[0, 2/3)$. Thus, for π_{ω_0+1}, we have $q \notin J_{\omega_0}(p)$ and $q \in J_{\omega_0+1}(p)$. Clearly, by transfinite induction, we can obtain the following result.

Property 4.6. *For each countable ordinal β, there exists a dynamical system π_β on Y such that if β is a successor ordinal then $q \notin J_{\beta-1}(p)$ and $q \in J_\beta(p)$, while if β is a limit ordinal then $q \notin J_\gamma(p)$ for all $\gamma < \beta$ and $q \in J_\beta(p)$.*

In order to construct a system such that for a given countable ordinal β, d_β is a given countable ordinal α, we apply Neumann's construction [10, Sec. 3, pp. 5–8]. In Neumann's notation, let E_α be a 2-manifold on which there exists a dynamical system (E_α, ϕ_α) such that the depth of the Birkhoff center $\mathcal{C}_1$ is α. We may adjust ϕ_α so that the points of some given subset in E_α become rest points. According to Property 4.6, let F be the set of rest points of π_β on Y. Define $F_\beta = \{(x, y) \in \mathbb{R}^2 : x \in F \text{ and } y \in (-1, 0]\}$, which is a subset of the open unit square $U = \{(x, y) \in \mathbb{R}^2 : |x| < 1 \text{ and } |y| < 1\}$. According to Neumann's construction, let U_α be the image of U in E_α and W be the corresponding image of F_β in U_α. Then, we adjust ϕ_α to a new flow ψ_α on E_α such that the points of W become rest points of ψ_α. Thus, we clearly get a dynamical system (E_α, ψ_α) with the following property.

Theorem 4.7. *For each countable ordinal $\beta \geq 1$, there exists a dynamical system such that d_β is a given countable ordinal α.*

Clearly, by Properties 2.5 and 3.6, the depths of centers are also invariants of topological equivalence. Thus, we can introduce the following concept, which is also an invariant of topological equivalence.

DEFINITION 4.8. Let (X,π) be a dynamical system on a separable metric space. The family of pairs $\{(\mathcal{C}_1, d_1), \ldots, (\mathcal{C}_{\mathfrak{D}(\pi)}, d_{\mathfrak{D}(\pi)})\}$ is said to be the *recurrent index* of (X,π).

Finally, we suggest a subject of further study: To what extent does the recurrent index determine the dynamical behavior of a dynamical system?

Acknowledgements. The authors sincerely thank the referee for many valuable suggestions which considerably improved the quality of this paper. Also, the authors greatly appreciate the editor's excellent work.

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