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On the characteristic initial value problem for nonlinear symmetric hyperbolic systems, including Einstein equations

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Abstract

We consider a characteristic initial value problem for a class of symmetric hyperbolic systems with initial data given on two smooth null intersecting characteristic surfaces. We prove existence of solutions on a future neighborhood of the initial surfaces. The result is applied to general semilinear wave equations, as well as the Einstein equations with or without sources, and conformal variations thereof.

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1. Introduction

There are several reasons why a characteristic Cauchy problem is of interest in general relativity. First, the general relativistic constraint equations on characteristic surfaces are trivial to solve (see e.g. [7, 12, 38]), while they are not on spacelike ones. Thus, a good understanding of the characteristic Cauchy problem is likely to provide more flexibility in constructing space-times with interesting properties. Next, an observer can in principle measure the initial data on her past light cone, and use those to determine the physical fields throughout her past by solving the field equations backwards in time; on the other hand, initial data on a spacelike surface near the observer cannot be measured instantaneously. Finally, Friedrich's conformal field equations may be used to construct space-times using initial data prescribed on past null infinity [13, 23, 27] which, at least in some situations, is a null cone emerging from a single point representing past timelike infinity.

The characteristic initial value problem for the vacuum Einstein equations with initial data given on two smooth null intersecting hypersurfaces has been studied by several authors [5, 6, 9, 15, 16, 21, 22, 34, 37, 39]; compare, in different settings, [3, 4, 28]. The most satisfactory treatment of the local evolution problem, for a large class of quasi-linear wave equations and symmetric hyperbolic systems, has been given by Rendall [38], who proved existence of a solution in a neighborhood of the intersection of the initial data hypersurfaces. A similar result for a neighborhood of the tip of a light-cone has been established by Dossa [17]. The region of existence has been extended by Cabet [1, 2] for a class of nonlinear wave equations satisfying certain structure conditions. In these last papers existence of the solution in a whole neighborhood of the initial data hypersurfaces, rather than of their intersection, is established. We will refer to this kind of results as "the neighborhood theorem". Similar results have been established by Dossa and collaborators [18–20, 29, 30] for various families of semilinear wave equations. Finally, Luk [33] established the neighborhood theorem for the vacuum Einstein equations in four space-time dimensions, through an argument which makes use of the specific structure of the nonlinearities occurring in those equations.

The aim of this work is to show that no conditions on the nonlinearity are necessary for existence near an (optimal) maximal subset of the initial data hypersurfaces for the large class of nonlinear wave equations which can be written in a doubly-null form.

We further show that our result applies to Einstein equations in four space-time dimensions, as well as to a version, due to Paetz [35], of the conformal field equations of Friedrich.

As a result we deduce that vacuum general relativistic characteristic initial data with suitable asymptotic behavior (as analyzed in detail in [14, 36]) lead to space-times with a piece of smooth Scri, without any smallness conditions on the data ([1]). Moreover, a *global-to-the-future* Scri is obtained if the data are sufficiently close to Minkowskian ones.

Higher-dimensional Einstein equations can be handled by a variation of our techniques; this will be discussed elsewhere.

Our analysis is tailored to a setting where the initial data are given on two transversely intersecting smooth characteristic surfaces. The characteristic initial value problem with initial data on a light cone issued from a point is readily reduced to the one considered here, by first solving locally near the tip (see [10, 17] and references therein), and then using the results proved here to obtain a solution near the maximal domain, within the light-cone, of existence of solutions of the transport equations.

([1]) Once this work was completed we have been made aware of a similar result in [32].

2. The basic energy identity

Let Y be an $(n-1)$-dimensional compact manifold without boundary. We are interested in quasi-linear first order symmetric hyperbolic systems of the form

$$Lf = G, \tag{2.1}$$

on subsets of

$$\widetilde{\mathcal{M}} := \{u \in [0,\infty),\ v \in [0,\infty),\ y \in Y\}. \tag{2.2}$$

In (2.1), f is assumed to be a section of a real vector bundle over $\widetilde{\mathcal{M}}$, equipped with a scalar product; similarly for G. We will use the same symbol ∇, respectively $\langle\cdot,\cdot\rangle$, to denote connections, respectively scalar products, on all relevant vector bundles. Both the scalar product and the connection coefficients are allowed to depend upon f, and we assume that ∇ is compatible with $\langle\cdot,\cdot\rangle$. Similarly, $\widetilde{\mathcal{M}}$ will be assumed to be equipped with a measure $d\mu$, possibly dependent upon f. Furthermore, L is a first order operator of the form

$$L = A^\mu \nabla_\mu,$$

where the A^μ's are self-adjoint, and are smooth functions of f and of the space-time coordinates. The summation convention is used throughout.

Let q_r, $r = 1,\dots,m$, denote a collection of smooth vector fields on Y such that for each $y \in Y$ the vectors $q_r(y)$ span T_yY; clearly $m \ge \dim Y$. For $k \in \mathbb{N}$ let $\mathcal{P}^k$ denote the collection of differential operators of the form

$$\mathring{\nabla}_{q_{r_1}} \dots \mathring{\nabla}_{q_{r_\ell}}, \qquad 0 \le \ell \le k. \tag{2.3}$$

Here $\mathring{\nabla}$ is a fixed, arbitrarily chosen, smooth connection which is f-, u-, and v-independent. We number the operators (2.3) in an arbitrary way and call them P_r, thus

$$\mathcal{P}^k = \{P_r : r = 1,\dots,N(k)\}$$

for a certain $N(k)$, with $P_1 = 1$, the identity map. We will often write $\mathring{\nabla}_r$ for $\mathring{\nabla}_{q_r}$.

Let w_r be any smooth functions on $\widetilde{\mathcal{M}}$. We set

$$X^\mu(k) := \sum_{r=1}^{N(k)} w_r \langle P_r f, A^\mu P_r f\rangle, \tag{2.4}$$

so that

$$\nabla_\mu(X^\mu(k)) = \sum_r \Big\{ \underbrace{\langle P_r f, A^\mu P_r f\rangle \partial_\mu w_r}_{\mathrm{I}_r} + w_r \Big(\underbrace{\langle P_r f, (\nabla_\mu A^\mu) P_r f\rangle}_{\mathrm{II}_r} + \underbrace{2\langle P_r f, L P_r f\rangle}_{\mathrm{III}_r} \Big) \Big\}. \tag{2.5}$$

Let

$$\Omega_{a,b} = \underbrace{[0,a]}_{\ni u} \times \underbrace{[0,b]}_{\ni v} \times \underbrace{Y}_{\ni x^B},$$

and let $d\mu = du\, dv\, d\mu_Y$ be any measure, absolutely continuous with respect to the coordinate Lebesgue measure, on Ω_{ab}, with smooth density function. From Stokes' theorem we have

$$\int_{\partial\Omega_{a,b}} X^\alpha(k)\, dS_\alpha = \int_{\Omega_{a,b}} \nabla_\mu (X^\mu(k))\, d\mu,$$

so that

$$\int_{u=a} X^\alpha(k)\, dS_\alpha + \int_{v=b} X^\alpha(k)\, dS_\alpha = \int_{u=0} X^\alpha(k)\, dS_\alpha + \int_{v=0} X^\alpha(k)\, dS_\alpha + \int_{\Omega_{a,b}} \nabla_\mu (X^\mu(k))\, d\mu. \tag{2.6}$$

From now on we specialise to f's which are of the form

$$f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{2.7}$$

with A^v and A^u satisfying

$$A^u = \begin{pmatrix} A^u_{\varphi\varphi} & 0 \\ 0 & 0 \end{pmatrix}, \quad A^v = \begin{pmatrix} 0 & 0 \\ 0 & A^v_{\psi\psi} \end{pmatrix}, \quad \text{and} \quad A^v_{\psi\psi}, A^u_{\varphi\varphi} > 0. \tag{2.8}$$

It is further assumed that the connections ∇ and $\mathring{\nabla}$ preserve the splitting (2.7). We will write

$$G = \begin{pmatrix} G_\varphi \\ G_\psi \end{pmatrix}. \tag{2.9}$$

From (2.7)–(2.8) we obtain, for fields supported in a compact set K,

$$\begin{aligned} \int_{u=a} X^\alpha(k)\, dS_\alpha &\geq c(K) \sum_r \int_{u=a} w_r \langle P_r\varphi, P_r\varphi\rangle\, dv\, d\mu_Y, \\ &=: c(K)\, E_{k,\{w_r\}}[\varphi, a], \end{aligned} \tag{2.10}$$

$$\begin{aligned} \int_{v=b} X^\alpha(k)\, dS_\alpha &\geq c(K) \sum_r \int_{v=b} w_r \langle P_r\psi, P_r\psi\rangle\, du\, d\mu_Y \\ &=: c(K)\, \mathcal{E}_{k,\{w_r\}}[\psi, b], \end{aligned} \tag{2.11}$$

for some constant $c(K)$. Equations (2.5)–(2.6) thus give

$$E_{k,\{w_r\}}[\varphi, a] + \mathcal{E}_{k,\{w_r\}}[\psi, b] \leq C_1(K)\big\{E_{k,\{w_r\}}[\varphi, 0] + \mathcal{E}_{k,\{w_r\}}[\psi, 0] + \int_{\Omega_{a,b}} \sum_r (\mathrm{I}_r + w_r(\mathrm{II}_r + \mathrm{III}_r))\big\}, \tag{2.12}$$

for some constant $C_1(K)$.

Let $\lambda \geq 0$. We choose the weights to be independent of r:

$$w_r = e^{-\lambda(u+v)}, \tag{2.13}$$

and we will write $E_{k,\lambda}$ for $E_{k,\{w_r\}}$ with this choice of weights, and similarly for $\mathcal{E}_{k,\lambda}$. From (2.10) we find

$$E_{k,\lambda}[\varphi,a] = \sum_{0\le j\le k}\int_{[0,b]\times Y} |\mathring{\nabla}_{q_{r_1}}\dots\mathring{\nabla}_{q_{r_j}}\varphi(a,v,\cdot)|^2 e^{-\lambda(a+v)}\,dv\,d\mu_Y$$
$$=: \int_0^b e^{-\lambda(a+v)}\|\varphi(a,v)\|^2_{H^k(Y)}\,dv, \tag{2.14}$$

where one recognises the usual Sobolev norms $H^k(Y)$ on Y. One similarly has

$$\mathcal{E}_{k,\lambda}[\psi,b] = \sum_{0\le j\le k}\int_{[0,a]\times Y} |\mathring{\nabla}_{q_{r_1}}\dots\mathring{\nabla}_{q_{r_j}}\psi(u,b,\cdot)|^2 e^{-\lambda(u+b)}\,du\,d\mu_Y$$
$$=: \int_0^a e^{-\lambda(u+b)}\|\psi(u,b)\|^2_{H^k(Y)}\,du. \tag{2.15}$$

We recall some general inequalities, which will be used repeatedly. Recall that Y is a compact manifold without boundary (compare, however, Remark 3.10). First, we have the Moser product inequality

$$\|fg\|_{H^k(Y)} \le C_M(Y,k)\big(\|f\|_{L^\infty(Y)}\|g\|_{H^k(Y)} + \|f\|_{H^k(Y)}\|g\|_{L^\infty(Y)}\big). \tag{2.16}$$

Next, we have the Moser commutation inequality, for $0\le r\le k$

$$\|P_r(fg)-P_r(f)g\|_{L^2(Y)}$$
$$\le C_M(Y,k)\big(\|f\|_{L^\infty(Y)}\|g\|_{H^k(Y)} + \|f\|_{H^{k-1}(Y)}\|g\|_{W^{1,\infty}(Y)}\big). \tag{2.17}$$

We shall also need the Moser composition inequality:

$$\|F(f,\cdot)\|_{H^k(Y)} \le \hat{C}_M\big(Y,k,F,\|f\|_{L^\infty(Y)}\big)\big(\|F(f=0,\cdot)\|_{H^k(Y)} + \|f\|_{H^k(Y)}\big). \tag{2.18}$$

The constants C_M and $\hat{C}_M$ also depend upon the connection $\mathring{\nabla}$.

We return to the energy identity on a set $\mathcal{U}\times Y$, with $\mathcal{U}$ coordinatised by u and v. If $X(k)$ is given by (2.4), with $w_r = e^{-\lambda(u+v)}$, then writing LP_rf as $P_rLf + [L,P_r]f$, and assuming

$$\langle\varphi, A^u_{\varphi\varphi}\varphi\rangle \ge c|\varphi|^2, \qquad \langle\psi, A^v_{\psi\psi}\psi\rangle \ge c|\psi|^2, \tag{2.19}$$

with $c>0$, for $k>(n-1)/2$ one obtains

$$\int_{\mathcal{U}\times Y}\nabla_\alpha(X^\alpha(k))\,d\mu$$
$$\le \int_{\mathcal{U}} e^{-\lambda(u+v)}\Big\{\big(\|\nabla_\mu A^\mu\|_{L^\infty(Y)} - c\lambda\big)\|f\|^2_{H^k(Y)} + C(Y,k)\|f\|_{H^k(Y)}\|G\|_{H^k(Y)}$$
$$+ 2\int_{\mathcal{U}\times Y}\langle P_rf, [L,P_r]f\rangle e^{-\lambda(u+v)}\,d\mu\Big\}. \tag{2.20}$$

Some special cases are worth pointing out:

1. The case of ODE's in u with a parameter v, or vice versa, corresponds to Y being a single point, and $k=0$.
2. The usual energy inequality for symmetric hyperbolic systems is obtained when $\mathcal{U}=I$ is an interval in $\mathbb{R}$.

To control the commutators we will assume (2.8). We identify $(\varphi, 0)$ with φ, and similarly for $(0, \psi)$ and ψ, and write

$$\begin{aligned}[A^\mu \nabla_\mu, P_r]f &= [A^u \nabla_u, P_r]f + [A^v \nabla_v, P_r]f + [A^B \nabla_B, P_r]f \\ &= [A^u \nabla_u, P_r]\varphi + [A^v \nabla_v, P_r]\psi + [A^B \nabla_B, P_r]f. \end{aligned} \tag{2.21}$$

Thus, it suffices to estimate $[A^u_{\varphi\varphi} \nabla_u, P_r]\varphi$, $[A^v_{\psi\psi} \nabla_v, P_r]\psi$, and $[A^B \nabla_B, P_r]f$. We define the relative connection coefficients Γ_μ by the formula

$$\Gamma_\mu f := \nabla_\mu f - \mathring{\nabla}_\mu f. \tag{2.22}$$

By hypothesis the connections preserve the (φ, ψ) decomposition, so that Γ_μ can be written as

$$\Gamma_\mu = \begin{pmatrix} \Gamma_{\varphi\varphi,\mu} & 0 \\ 0 & \Gamma_{\psi\psi,\mu} \end{pmatrix}. \tag{2.23}$$

This leads to the following form of $[A^B \nabla_B, P_r]f$:

$$\|[A^B \nabla_B, P_r]f\|_{L^2(Y)} = \|[A^B \mathring{\nabla}_B, P_r]f + [A^B \Gamma_B, P_r]f\|_{L^2(Y)}.$$

By using (2.17)–(2.18), the first term is estimated as

$$C'_M \big(\|A\|_{W^{1,\infty}(Y)} \|f\|_{H^k(Y)} + \|A\|_{H^k(Y)} \|f\|_{W^{1,\infty}(Y)} \big),$$

and the second as

$$C''_M \big(\|A^B \Gamma_B\|_{W^{1,\infty}(Y)} \|f\|_{H^{k-1}(Y)} + \|A^B \Gamma_B\|_{H^k(Y)} \|f\|_{L^\infty(Y)} \big),$$

leading, by (2.16), to an overall estimation

$$\begin{aligned} \|[A^B \nabla_B, P_r]f\|_{L^2(Y)} \le C\big(Y, k, \|f\|_{W^{1,\infty}(Y)}, \|A\|_{W^{1,\infty}(Y)}, \|\Gamma\|_{W^{1,\infty}(Y)}\big) \\ \times \big(\|f\|_{H^k(Y)} + \|A\|_{H^k(Y)} + \|\Gamma\|_{H^k(Y)} \big). \end{aligned} \tag{2.24}$$

Here we have written

$$\|A\|_{H^k(Y)} = \sum_\mu \|A^\mu\|_{H^k(Y)}, \qquad \|\Gamma\|_{H^k(Y)} = \sum_\mu \|\Gamma_\mu\|_{H^k(Y)}. \tag{2.25}$$

Writing $\nabla_\mu \varphi$ as $\partial_\mu \varphi + \gamma_{\varphi\varphi,\mu} \varphi$ we have

$$\begin{aligned} \underbrace{[A^u_{\varphi\varphi}(\partial_u + \gamma_{\varphi\varphi,u}), P_r]\varphi}_{\alpha} &= [A^u_{\varphi\varphi}, P_r] \underbrace{\partial_u \varphi}_{\nabla_u\varphi - \gamma_{\varphi\varphi,u}\varphi} + [A^u_{\varphi\varphi}\gamma_{\varphi\varphi,u}, P_r]\varphi \\ &= \underbrace{[A^u_{\varphi\varphi}, P_r]\big\{ (A^u_{\varphi\varphi})^{-1} \big[\underbrace{-A^B_{\varphi\varphi} \nabla_B \varphi - A^B_{\varphi\psi} \nabla_B \psi}_{-A^B_{\varphi\varphi}(\mathring{\nabla}_B + \Gamma_{\varphi\varphi,B})\varphi - A^B_{\varphi\psi}(\mathring{\nabla}_B + \Gamma_{\psi\psi,B})\psi} + G_\varphi \big] - \gamma_{\varphi\varphi,\mu}\varphi \big\}}_{\alpha_1} \\ &\quad + \underbrace{[A^u_{\varphi\varphi}\gamma_{\varphi\varphi,u}, P_r]\varphi}_{\alpha_3} =: \alpha_1 + \alpha_2 + \alpha_3, \end{aligned} \tag{2.26}$$

with α_2 defined by the last equality. Set

$$\begin{aligned} \tilde{A}^B_{\varphi\varphi,u} &:= \big(A^u_{\varphi\varphi}\big)^{-1} A^B_{\varphi\varphi}, \quad & \tilde{A}^B_{\varphi\psi,u} &:= \big(A^u_{\varphi\varphi}\big)^{-1} A^B_{\varphi\psi}, \quad & \tilde{G}_\varphi &= \big(A^u_{\varphi\varphi}\big)^{-1} G_\varphi, \\ \tilde{A}^B_{\psi\varphi,v} &:= \big(A^v_{\psi\psi}\big)^{-1} A^B_{\psi\varphi}, \quad & \tilde{A}^B_{\psi\psi,v} &:= \big(A^u_{\psi\psi}\big)^{-1} A^B_{\psi\psi}, \quad & \tilde{G}_\psi &= \big(A^v_{\psi\psi}\big)^{-1} G_\psi. \end{aligned}$$

By (2.17)–(2.18) we have the estimate

$$
\begin{aligned}
\|\alpha_1\|_{L^2(Y)} &\le C_M\big(\|A^u\|_{W^{1,\infty}(Y)}\|\nabla_u\varphi\|_{H^{k-1}(Y)} + \|A^u\|_{H^k(Y)}\|\nabla_u\varphi\|_{L^\infty(Y)}\big)\\
&\le C\big(Y,k,\|f\|_{W^{1,\infty}(Y)},\|A^u\|_{W^{1,\infty}(Y)},\|\tilde A\|_{L^\infty(Y)},\|\Gamma\|_{L^\infty(Y)},\|\tilde G_\varphi\|_{L^\infty(Y)}\big)\\
&\quad\times\big(\|f\|_{H^k(Y)}+\|A^u\|_{H^k(Y)}+\|\tilde A\|_{H^{k-1}(Y)}+\|\Gamma\|_{H^{k-1}(Y)}+\|\tilde G_\varphi\|_{H^{k-1}(Y)}\big).
\end{aligned}
\tag{2.27}
$$

Similarly,

$$
\begin{aligned}
\|\alpha_2\|_{L^2(Y)} &\le C\big(Y,k,\|\varphi\|_{L^\infty(Y)},\|A^u\|_{W^{1,\infty}(Y)},\|\gamma_{\varphi\varphi,u}\|_{L^\infty(Y)}\big)\\
&\quad\times\big(\|\varphi\|_{H^{k-1}(Y)}+\|A^u\|_{H^k(Y)}+\|\gamma_{\varphi\varphi,u}\|_{H^{k-1}(Y)}\big), \qquad (2.28)\\
\|\alpha_3\|_{L^2(Y)} &\le C\big(Y,k,\|\varphi\|_{L^\infty(Y)},\|A^u\|_{W^{1,\infty}(Y)},\|\gamma_{\varphi\varphi,u}\|_{W^{1,\infty}(Y)}\big)\\
&\quad\times\big(\|\varphi\|_{H^{k-1}(Y)}+\|A^u\|_{H^k(Y)}+\|\gamma_{\varphi\varphi,u}\|_{H^k(Y)}\big). \qquad (2.29)
\end{aligned}
$$

By symmetry we have a similar contribution from $[A^v_{\psi\psi}\nabla_v, P_r]\psi$. It follows that there exists a constant

$$
\begin{aligned}
\hat C_1 = C\big(&Y,k,\|f\|_{W^{1,\infty}(Y)},\|A\|_{W^{1,\infty}(Y)},\|\tilde A\|_{L^\infty(Y)},\|\gamma\|_{W^{1,\infty}},\\
&\|\Gamma\|_{W^{1,\infty}},\|G\|_{W^{1,\infty}},\|\tilde G\|_{L^\infty}\big)
\end{aligned}
\tag{2.30}
$$

such that (2.20) can be rewritten as

$$
\begin{aligned}
\int_{\mathcal U\times Y}\nabla_\mu(X^\mu(k))\,d\mu \le \int_{\mathcal U} e^{-\lambda(u+v)}\Big\{&\big(\|\nabla_\mu A^\mu\|_{L^\infty(Y)}-c\lambda\big)\|f\|^2_{H^k(Y)}\\
&+\hat C_1\|f\|_{H^k(Y)}\big(\|f\|_{H^k(Y)}+\|A\|_{H^k(Y)}+\|\tilde A\|_{H^{k-1}(Y)}\\
&+\|\Gamma\|_{H^k(Y)}+\|\gamma\|_{H^k(Y)}+\|G\|_{H^k(Y)}+\|\tilde G\|_{H^{k-1}(Y)}\big)\Big\}\,du\,dv.
\end{aligned}
\tag{2.31}
$$

3. The iterative scheme

3.1. Outline of the iteration argument. For the purpose of the arguments in this section, we let

$$\mathcal{N}^- := \{u = 0,\, v \in [0, b_0]\} \times Y, \qquad \mathcal{N}^+ := \{u \in [0, a_0],\, v = 0\} \times Y\,;$$

we will see later how to handle general initial characteristic hypersurfaces for systems arising from wave equations. The initial data $\overline{f} \equiv f|_{\mathcal{N}}$ will be given on

$$\mathcal{N} := \mathcal{N}^- \cup \mathcal{N}^+,$$

and will belong to a suitable Sobolev class. More precisely, we are free to prescribe $\overline{\varphi}(v) \equiv \varphi(0, v)$ on $\mathcal{N}^-$ and $\overline{\psi}(u) \equiv \psi(u, 0)$ on $\mathcal{N}^+$, and then the fields $\psi(0, v)$ on $\mathcal{N}^-$ and $\varphi(u, 0)$ on $\mathcal{N}^+$ can be calculated by solving transport equations. In this section we assume that these equations have global solutions on $\mathcal{N}^\pm$; this hypothesis will be relaxed later.

Throughout we use the convention that overlining a field denotes restriction to $\mathcal{N}$ (consistently with the last paragraph).

Our hypotheses will be symmetric with respect to the variables u and v, and therefore the result will also be symmetric. We will construct solutions on a neighborhood of $\mathcal{N}^-$ in

$$\Omega_{a_0,b_0} := \{u \in [0, a_0],\, v \in [0, b_0]\} \times Y,$$

and a neighborhood of $\mathcal{N}^+$ can then be obtained by applying the result to the system in which u is interchanged with v.

The method is to use a sequence $\overline{f}_i$ of smooth initial data approaching $\overline{f}$, and to solve a sequence of linear problems: We let f_0 be any smooth extension of $\overline{f}_0$ to Ω_{a_0,b_0}. Then, given f_i, the field f_{i+1} is defined as the solution of the linear system

$$L_i f_{i+1} = G_i, \tag{3.1}$$

where

$$L_i = A^\mu(f_i, \cdot)\nabla(i)_\mu, \qquad G_i = G(f_i, \cdot), \tag{3.2}$$

and where we have used the symbol $\nabla(i)$ to denote ∇, as determined by f_i. (The reader may wonder why we do not replace ∇ by an f-independent connection, putting all the dependence of ∇ upon f into the right-hand side of the equation. However, in some situations the new connection might not be compatible with the scalar product, which has been assumed in our calculations.) For smooth initial data and f_i, (3.1) always has a global smooth solution on Ω_{a_0,b_0} by [38].

By continuity, the f_i's will satisfy a certain set of inequalities, to be introduced shortly, on a subset

$$\Omega_i := \{u \in [0, a_i],\ v \in [0, b_0]\} \times Y.$$

We will show that there exists $a_* > 0$ such that $a_i \geq a_*$, so that there will be a common domain

$$\Omega_* := \{u \in [0, a_*],\ v \in [0, b_0]\} \times Y$$

on which the desired inequalities will be satisfied by all the f_i's. This will allow us to show convergence to a solution of the original problem defined on Ω_*.

We note that our system implies a system of nonlinear constraint equations on $\overline{f}$, sometimes called *transport equations*. The solutions of these constraints might blow up in finite time; see e.g. [1] for an example arising from a semilinear wave equation. It is part of our hypotheses that the constraints are satisfied throughout $\mathcal{N}$; in some situations this might require choosing a_0 and b_0 small enough so that a smooth solution of the constraint equations exists.

3.2. Bounds for the iterative scheme. In order to apply the energy identity of Section 2 we need to estimate the volume integrals appearing in (2.6). We could appeal to (2.31), but it is instructive to analyse (2.12) directly. All terms arising from I_r in (2.5) give a negative contribution, bounded above by

$$-\lambda c(K) \int_0^a \int_0^b e^{-\lambda(u+v)} \|f_{i+1}(u,v)\|^2_{H^k(Y)}\, du\, dv. \tag{3.3}$$

The terms arising from II_r give a contribution which, using obvious notation, is estimated by

$$\|(\nabla_\mu A^\mu)_i\|_{L^\infty} \int_0^a \int_0^b e^{-\lambda(u+v)} \|f_{i+1}(u,v)\|^2_{H^k(Y)}\, du\, dv. \tag{3.4}$$

The estimation of the terms arising from III_r requires care, as we need to control λ-dependence of the constants. One can proceed as follows:

$$\begin{aligned}
2\sum_r \int_0^a \int_0^b & \langle P_r f_{i+1}, L P_r f_{i+1}\rangle e^{-\lambda(u+v)}\, du\, dv\, d\mu_Y \\
&= 2\sum_r \int_0^a \int_0^b \langle P_r f_{i+1}, P_r G_i + [L_i, P_r] f_{i+1}\rangle e^{-\lambda(u+v)}\, du\, dv\, d\mu_Y \\
&\leq 2\int_0^a \int_0^b e^{-\lambda(u+v)} \|f_{i+1}(u,v)\|_{H^k(Y)} \\
&\qquad \times \Big(\underbrace{\|G_i(u,v)\|_{H^k(Y)}}_{\mathrm{III}_1} + \underbrace{\sum_r \|[L_i, P_r] f_{i+1}(u,v)\|_{L^2(Y)}}_{\mathrm{III}_2} \Big)\, du\, dv\, d\mu_Y .
\end{aligned}$$

The term III_1 can be estimated by the usual Moser inequality on Y,

$$\|G_i(u,v)\|_{H^k(Y)} \leq C\big(k, Y, \|f_i(u,v)\|_{L^\infty(Y)}\big)\big(\|\mathring{G}(u,v)\|_{H^k(Y)} + \|f_i(u,v)\|_{H^k(Y)}\big),$$

where $\mathring{G} = G(f = 0)$. Let $0 < \epsilon \leq 1$ be a constant which will be determined later. The inequality $ab \leq a^2/(4\epsilon) + \epsilon b^2$ then leads to a contribution of III_1 in (2.12) which is estimated by

$$
\begin{aligned}
&C_3\big(k, Y, \sup_{u,v} \|f_i(u,v)\|_{L^\infty(Y)}\big) \int_0^a \int_0^b e^{-\lambda(u+v)} \\
&\qquad \times \big(\|\mathring{G}(u,v)\|^2_{H^k(Y)} + \epsilon \|f_i(u,v)\|^2_{H^k(Y)} + c_1(\epsilon)\|f_{i+1}(u,v)\|^2_{H^k(Y)}\big)\, du\, dv, \qquad (3.5)
\end{aligned}
$$

with $c_1(\epsilon) \to \infty$ as $\epsilon \to 0$. The analysis of III_2 proceeds as in (2.21). Since the P_r's are u-independent we have

$$[A^u\partial_u, P_r]\varphi = [A^u, P_r]\partial_u\varphi,$$

and, calculating as in (2.26), we can use the equation satisfied by φ_{i+1} to replace $\partial_u\varphi_{i+1}$ by a first order differential operator in f_{i+1} tangential to Y (with coefficients that perhaps depend upon f_i); similarly for $A^v\partial_v\psi_{i+1}$. The Moser commutation inequality (2.17) on Y can then be used to obtain the following estimation for the corresponding contribution to (2.12):

$$
\begin{aligned}
&C_3\big(k, Y, \sup_{u,v} \|f_i(u,v)\|_{W^{1,\infty}(Y)}, \sup_{u,v} \|f_{i+1}(u,v)\|_{W^{1,\infty}(Y)}\big) \\
&\quad \times \int_0^a \int_0^b e^{-\lambda(u+v)} \big(\epsilon \|f_i(u,v)\|^2_{H^k(Y)} + c_2(\epsilon)\|f_{i+1}(u,v)\|^2_{H^k(Y)} + \|\mathring{A}\|^2_{H^k(Y)} \\
&\quad + \|\mathring{\tilde{A}}\|^2_{H^{k-1}(Y)} + \|\mathring{\Gamma}\|^2_{H^k(Y)} + \|\mathring{\gamma}\|^2_{H^k(Y)} + \|\mathring{G}\|^2_{H^k(Y)} + \|\mathring{\tilde{G}}\|^2_{H^{k-1}(Y)}\big)\, du\, dv, \qquad (3.6)
\end{aligned}
$$

where

$$\mathring{A}^\mu = A^\mu|_{f=0}, \quad \mathring{\Gamma}_\mu = \Gamma_\mu|_{f=0}, \quad \text{etc.},$$

with norms defined as in (2.25).

Define

$$C_0 = 1 + \sup_{i\in\mathbb{N},\, (u,v)\in([0,a_0]\times\{0\})\cup(\{0\}\times[0,b_0])} \|\overline{f}_i(u,v)\|_{W^{1,\infty}(Y)}.$$

We assume that C_0 is finite.

Let $\mathcal{K}$ be a compact neighborhood of the image of the initial data map $\overline{f}$. We will assume that the sequence $\overline{f}_i$ converges to $\overline{f}$ in $L^\infty(\mathcal{N})$, and similarly for first and second order derivatives. In particular we can assume that the image of $\overline{f}_i$ lies in $\mathcal{K}$.

Let

$$C_{\mathrm{div}} := \sup |\nabla_\mu A^\mu| + 1, \qquad (3.7)$$

where the supremum is taken over all points in $\mathcal{N}^+ \cup \mathcal{N}^-$ and over all $(\varphi, \psi, \nabla\varphi, \nabla\psi)$ satisfying

$$
\begin{aligned}
&(\varphi,\psi) \in \mathcal{K}, \quad |\mathring{\nabla}_B f(u,v)| \le 2C_0, \quad |\partial_u\psi| \le 2\sup_i \left\|\overline{\frac{\partial\psi_i}{\partial u}}\right\|_{L^\infty(\mathcal{N}^+\cup\mathcal{N}^-)} + 1, \\
&|\partial_v\varphi| \le 2\sup_i \left\|\overline{\frac{\partial\varphi_i}{\partial v}}\right\|_{L^\infty(\mathcal{N}^+\cup\mathcal{N}^-)} + 1. \qquad (3.8)
\end{aligned}
$$

(The suprema over i will be finite in view of our hypotheses on the sequence $\overline{f}_i$.) We note that

$$\nabla_\mu A^\mu = \partial_\psi A^\mu \partial_\mu\psi + \partial_\varphi A^\mu \partial_\mu\varphi + \text{terms independent of derivatives of } f, \qquad (3.9)$$

so that to control C_{div} one needs to control those derivatives of f which appear in $\partial_\psi A^\mu \partial_\mu \psi + \partial_\varphi A^\mu \partial_\mu \varphi$. Now, on the right-hand side of (3.9) the values $\partial_u \varphi$ and $\partial_v \psi$ can be algebraically determined in terms of other fields involved using the field equations: Indeed, using (2.1) we can view $\partial_u \varphi$ as a function, say F, of f and $\mathring{\nabla}_B f$. Then, when calculating C_{div}, we consider all values of F with $f \in \mathcal{K}$ and $|\mathring{\nabla}_B f| \le 2C_0$; similarly for $\partial_v \psi$.

Remark 3.1. It should be clear from (3.9) that the condition on $\partial_v \varphi$ in (3.8) is irrelevant if A^v does not depend upon φ. Similarly, the condition on $\partial_u \psi$ in (3.8) is irrelevant if A^u does not depend upon ψ.

Let a_i be the largest number in $(0, a_0]$ such that

$$\|(\nabla_\mu A^\mu)_i\|_{L^\infty(\Omega_{a_i,b_0})} \le C_{\mathrm{div}}, \tag{3.10a}$$

$$\sup_{(u,v)\in[0,a_i]\times[0,b_0]} \|f_i(u,v)\|_{W^{1,\infty}(Y)} \le 4C_0. \tag{3.10b}$$

For any $\epsilon > 0$ we can choose λ large enough, independent of i, so that the sum of (3.3), (3.4), and of the f_{i+1} contribution to (3.5) and (3.6), is negative on

$$\Omega_{\hat{a}_i,b_0}, \quad \text{where} \quad \hat{a}_i = \min(a_1, \ldots, a_{i+1}).$$

If we let $M_k(u,v)$ be any function satisfying

$$\begin{aligned} M_k(u,v) \ge \|\mathring{G}(u,v)\|^2_{H^k(Y)} + \|\dot{\mathring{G}}(u,v)\|^2_{H^{k-1}(Y)} + \|\mathring{A}\|^2_{H^k(Y)} \\ + \|\dot{\mathring{A}}(u,v)\|^2_{H^{k-1}(Y)} + \|\mathring{\gamma}(u,v)\|^2_{H^k(Y)} + \|\mathring{\Gamma}(u,v)\|^2_{H^k(Y)}, \end{aligned} \tag{3.11}$$

we conclude that:

Lemma 3.2. *Let $0 \le b \le b_0 < \infty$, and suppose that Y is compact and (3.10) holds. Then for every $0 < \epsilon \le 1$ there exist constants $\lambda_0(k, C_0, C_{\mathrm{div}}, Y, \epsilon)$ and $C_4(a_0, b_0, Y, k, C_0, C_{\mathrm{div}})$ such that for all $\lambda \ge \lambda_0$ and $0 \le a \le \hat{a}_i \le a_0$ we have*

$$\begin{aligned} E_{k,\lambda}[\varphi_{i+1}, a] + \mathcal{E}_{k,\lambda}[\psi_{i+1}, b] \le C_4 \Big\{ E_{k,\lambda}[\overline{\varphi}_{i+1}] + \mathcal{E}_{k,\lambda}[\overline{\psi}_{i+1}] \\ + \int_0^a \int_0^b e^{-\lambda(u+v)} \big(M_k(u,v) + \epsilon \|f_i(u,v)\|^2_{H^k(Y)}\big)\, du\, dv \Big\}. \ \blacksquare \end{aligned} \tag{3.12}$$

We need, next, to get rid of the i-dependent terms in the integrals on the right-hand side of (3.12). This can be done as follows: Set

$$\hat{C}(a,b) := C_4 \Big\{ \sup_{i\in\mathbb{N}} (E_{k,\lambda}[\overline{\varphi}_{i+1}] + \mathcal{E}_{k,\lambda}[\overline{\psi}_{i+1}]) + \int_0^a \int_0^b e^{-\lambda(u+v)} M_k(u,v)\, du\, dv \Big\}; \tag{3.13}$$

note that this depends only upon the initial data and the structure of the equations. Suppose that

$$\int_0^a \int_0^b e^{-\lambda(u+v)} \|f_i(u,v)\|^2_{H^k(Y)}\, du\, dv \le 2\hat{C}(a_0 + b_0). \tag{3.14}$$

We then have, using (3.12),

$$\int_0^a \int_0^b e^{-\lambda(u+v)} \|\varphi_{i+1}(u,v)\|^2_{H^k(Y)}\, du\, dv = \int_0^a E_{k,\lambda}[\varphi_{i+1}, u]\, du$$
$$\le \int_0^a (\hat C + 2\epsilon C_4 \hat C(a_0+b_0))\, du \le (\hat C + 2\epsilon C_4 \hat C(a_0+b_0))a_0 \le 2\hat C a_0,$$

if ϵ is chosen small enough. Similarly,

$$\int_0^a \int_0^b e^{-\lambda(u+v)} \|\psi_{i+1}(u,v)\|^2_{H^k(Y)}\, du\, dv = \int_0^b \mathcal{E}_{k,\lambda}[\psi_{i+1}, v]\, dv$$
$$\le (\hat C + 2\epsilon C_4 \hat C(a_0+b_0))b_0 \le 2\hat C b_0.$$

Adding, one obtains (3.14) with i replaced by $i+1$. Decreasing ϵ if necessary we obtain:

LEMMA 3.3. *Let $\hat C$ be defined by* (3.13)*. Under the hypotheses of Lemma* 3.2*, one can choose $\epsilon_0(a_0, b_0, Y, k, C_0, C_{\rm div})$ so that* (3.14)*is preserved under iteration for all* $0 \le b \le b_0$*, provided that* $0 \le a \le \hat a_i \le a_0$*, with the right-hand side of* (3.12) *being less than* $2\hat C(a_0, b_0)$ *for all $\lambda \ge \lambda_0(k, C_0, C_{\rm div}, Y, \epsilon_0)$.* ■

To continue, let us write

$$A^\mu = \begin{pmatrix} A^\mu_{\varphi\varphi} & A^\mu_{\varphi\psi} \\ A^\mu_{\psi\varphi} & A^\mu_{\psi\psi} \end{pmatrix}.$$

Since, by hypothesis, the only nonvanishing component of A^u is $A^u_{\varphi\varphi}$, on any level set of u the field ψ_{i+1} is a solution of the symmetric hyperbolic system

$$(A^\mu_{\psi\psi}\nabla_\mu)_i \psi_{i+1} \equiv A^\mu_{\psi\psi}(f_i, \cdot)\nabla_\mu(i)\psi_{i+1} = (\hat G_\psi)_i, \tag{3.15}$$

where

$$(\hat G_\psi)_i \equiv (G_\psi)_i - (A^\mu_{\psi\varphi}\nabla_\mu)_i \varphi_{i+1} := G_\psi(f_i, \cdot) - A^\mu_{\psi\varphi}(f_i, \cdot)\nabla_\mu(i)\varphi_{i+1}.$$

Set

$$C_{{\rm div},\psi\psi} = \sup |\nabla_\mu A^\mu_{\psi\psi}| \le C_{\rm div}, \tag{3.16}$$

where the sup is taken as in (3.7). A calculation similar to the one leading to the proof of Lemma 3.3 shows that for any $0 < \delta \le 1$ there exists $\lambda_1(k, C_0, C_{{\rm div},\psi\psi}, Y, \delta) < \infty$ such that for $\lambda \ge \lambda_1$ we obtain (recall that $\psi_{i+1}(u,0) = \overline{\psi}_{i+1}(u)$)

$$\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)} \le C_5(Y, k, C_0, C_{{\rm div},\psi\psi}) e^{\lambda v} \Big\{ \|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)}$$
$$+ \delta \int_0^v e^{-\lambda s}\big(\|f_i(u,s)\|^2_{H^{k-1}(Y)} + M_k(u,s) + \underbrace{\|(\hat G_\psi)_i(u,s)\|^2_{H^{k-1}(Y)}}_{*}\big)\, ds \Big\}. \tag{3.17}$$

The contribution of $*$ can be estimated as follows:

$$\int_0^v * \, ds = \int_0^v e^{-\lambda s} \|((G_\psi)_i - (A^B_{\psi\varphi}\nabla_B)_i \varphi_{i+1})(u,s)\|^2_{H^{k-1}(Y)}\, ds$$
$$\le 2\int_0^v e^{-\lambda s}\big(\|(G_\psi)_i(u,s)\|^2_{H^{k-1}(Y)} + \|(A^B_{\psi\varphi}\nabla_B)_i\varphi_{i+1}(u,s)\|^2_{H^{k-1}(Y)}\big)\, ds$$
$$\le C_6(Y,k,C_0)\int_0^v e^{-\lambda s}\big(\|\mathring{G}_\psi(u,s)\|^2_{H^{k-1}(Y)} + \|\mathring{A}\|^2_{H^{k-1}(Y)}$$
$$+ \|\mathring{\Gamma}\|^2_{H^{k-1}(Y)} + \|f_i(u,s)\|^2_{H^{k-1}(Y)} + \|\varphi_{i+1}(u,s)\|^2_{H^k(Y)}\big)\, ds$$

$$= C_6(Y,k,C_0)\Big\{\int_0^v e^{-\lambda s}\big(\|\mathring{G}_\psi(u,s)\|^2_{H^{k-1}(Y)} + \|\mathring{A}\|^2_{H^{k-1}(Y)} + \|\mathring{\Gamma}\|^2_{H^{k-1}(Y)} + \|f_i(u,s)\|^2_{H^{k-1}(Y)}\big)\,ds + e^{\lambda u}E_{k,\lambda}[\varphi_{i+1},u]\Big\},$$

with $\mathring{G}_\psi(\cdot) = G_\psi(f=0,\cdot)$. It follows that, for $0 \le u \le a \le \hat{a}_i \le a_0$,

$$e^{-\lambda v}\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)} \le C_7(Y,k,C_0,C_{\mathrm{div},\psi\psi})\Big\{\|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)} + \int_0^v e^{-\lambda s}\big(M_k(u,s) + \delta\|f_i(u,s)\|^2_{H^{k-1}(Y)}\big)\,ds + e^{\lambda u}E_{k,\lambda}[\varphi_{i+1},u]\Big\}.$$

By Lemma 3.3 the φ_i part of the f_i contribution can be estimated by $e^{\lambda u}E_{k-1,\lambda}[\varphi_i,u] \le 2e^{\lambda u}\hat{C}(u,v) \le 2e^{\lambda u}\hat{C}(u,b)$, so that

$$e^{-\lambda v}\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)} \le C_7(Y,k,C_0,C_{\mathrm{div},\psi\psi})\Big\{\|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)} + 2e^{\lambda u}\hat{C}(u,b) + \int_0^v e^{-\lambda s}\big(M_k(u,s) + \delta\|\psi_i(u,s)\|^2_{H^{k-1}(Y)}\big)\,ds + e^{\lambda u}E_{k,\lambda}[\varphi_{i+1},u]\Big\}. \tag{3.18}$$

Integrating in v one obtains, for $0 \le b \le b_0$,

$$\int_0^b e^{-\lambda v}\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)}\,dv \le \tilde{C}_\psi(u,b) + C_7(Y,k,C_0,C_{\mathrm{div},\psi\psi})\delta\int_0^b\int_0^v e^{-\lambda s}\|\psi_i(u,s)\|^2_{H^{k-1}(Y)}\,ds\,dv, \tag{3.19}$$

where

$$\tilde{C}_\psi(a,b) = C_7(Y,k,C_0,C_{\mathrm{div},\psi\psi})\sup_{i\in\mathbb{N},\,u\in[0,a]}\Big\{b\|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)} + \int_0^b\int_0^v e^{-\lambda s}M_k(u,s)\,ds\,dv + 2be^{\lambda u}\hat{C}(u,b) + be^{\lambda u}E_{k,\lambda}[\varphi_{i+1},u]\Big\}.$$

Suppose that there exists a constant $C_{\mathrm{div},\psi\psi}$ such that

$$\sup_i |(\nabla_\mu A^\mu_{\psi\psi})_i| \le C_{\mathrm{div},\psi\psi} \tag{3.20}$$

(note that we necessarily have $C_{\mathrm{div},\psi\psi} \le C_{\mathrm{div}}$), and that

$$\forall\ 0 \le v \le b, \qquad \int_0^v e^{-\lambda s}\|\psi_i(u,s)\|^2_{H^{k-1}(Y)}\,ds \le 2\tilde{C}_\psi(u,b). \tag{3.21}$$

Equation (3.19) shows that

$$\begin{aligned}\int_0^b e^{-\lambda v}\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)}\,dv &\le \tilde{C}_\psi(u,b) + C_7\delta\int_0^b\int_0^v e^{-\lambda s}\|\psi_i(u,s)\|^2_{H^{k-1}(Y)}\,ds\,dv\\ &\le \tilde{C}_\psi(u,b) + 2\delta b_0C_7\tilde{C}_\psi(u,b) \le 2\tilde{C}_\psi(u,b),\end{aligned} \tag{3.22}$$

if $\delta = \delta(b_0,\tilde{C}_\psi(a_0,b_0),C_7)$ is chosen small enough. It follows that (3.21) is preserved under the iteration scheme if (3.10) and (3.20) hold. With this choice of δ, (3.18) gives

$$e^{-\lambda v}\|\psi_{i+1}(u,v)\|^2_{H^{k-1}(Y)} \le C_7(Y,k,C_0,C_{\rm div})\Big\{\|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)} + 2e^{\lambda u}\hat{C}(u,b) + \int_0^v e^{-\lambda s}M_k(u,s)\,ds + 2\delta\tilde{C}_\psi(u,b) + e^{\lambda u}E_{k,\lambda}[\varphi_{i+1},u]\Big\}. \tag{3.23}$$

By an essentially identical argument using the symmetry of the equations under the interchange of u and v, but still working with $0 \le u \le \hat{a}_i$, $0 \le v \le b_0$, if we let $C_{{\rm div},\varphi\varphi}$ be a constant such that

$$\sup_i |(\nabla_\mu A^\mu_{\varphi\varphi})_i| \le C_{{\rm div},\varphi\varphi} \tag{3.24}$$

(note that $C_{{\rm div},\varphi\varphi} \le C_{\rm div}$), then the condition

$$\forall\, 0 \le u \le \hat{a}_i, \qquad \int_0^u e^{-\lambda s}\|\varphi_i(s,v)\|^2_{H^{k-1}(Y)}\,ds \le 2\tilde{C}_\varphi(a,v), \tag{3.25}$$

where

$$\tilde{C}_\varphi(a,b) = C_7(Y,k,C_0,C_{{\rm div},\varphi\varphi}) \sup_{i\in\mathbb{N},v\in[0,b]} \Big\{a\|\overline{\varphi}_{i+1}(v)\|^2_{H^{k-1}(Y)} + \int_0^a\int_0^u e^{-\lambda s}M_k(s,v)\,ds\,du + 2ae^{\lambda v}\hat{C}(a,v) + ae^{\lambda v}\mathcal{E}_{k,\lambda}[\psi_{i+1},v]\Big\},$$

is preserved under iteration, and we are led to:

Lemma 3.4. *Under the hypotheses of Lemma* 3.2*, the inequalities* (3.21) *and* (3.25) *are preserved under iteration, and there exist constants*

$$\lambda_2 = \lambda_2(k,C_0,C_{\rm div},Y,\tilde{C}_\psi(a_0,b_0),\tilde{C}_\varphi(a_0,b_0)),$$
$$C_7 = C_7(Y,k,C_0,C_{\rm div}),$$
$$C_8 = C_8(Y,k,C_0,C_{\rm div},a_0,b_0,\tilde{C}_\psi(a_0,b_0),\tilde{C}_\varphi(a_0,b_0))$$

such that for all $\lambda \ge \lambda_2$ we have, for $(u,v) \in [0,\hat{a}_i]\times[0,b_0]$,

$$\|f_{i+1}(u,v)\|^2_{H^{k-1}(Y)} \le C_7 e^{\lambda(a_0+b_0)}\Big\{\|\overline{\varphi}_{i+1}(v)\|^2_{H^{k-1}(Y)} + \|\overline{\psi}_{i+1}(u)\|^2_{H^{k-1}(Y)} + \int_0^{a_0} e^{-\lambda s}M_k(s,v)\,ds + \int_0^{b_0} e^{-\lambda s}M_k(u,s)\,ds + C_8\big(\hat{C}(a_0,b_0)+\tilde{C}_\psi(a_0,b_0)+\tilde{C}_\varphi(a_0,b_0)\big)\Big\}. \;\blacksquare \tag{3.26}$$

From now on, we will use the inequalities

(3.12), (3.17) and (3.26) with λ chosen to be the largest of λ_0, λ_1 and λ_2

regardless of the value of the parameter λ that might occur in the equation in which one of these inequalities is being used. In what follows, the letter C will denote a constant which depends perhaps upon C_0, $\hat{C}$, $\tilde{C}_\psi$, $\tilde{C}_\varphi$, Y, a_0, b_0 and k, and which may vary from line to line; similarly, the numbered constants C_n that follow may depend upon all those quantities, but not on i. We wish to show that we can choose $0 < a_* \le a_0$ small enough so that $\hat{a}_i \ge a_*$, hence for $0 \le u \le a_*$ the inequalities (3.10), (3.20) and (3.24) hold. Suppose

$$k_1 \text{ is the smallest integer such that } k_1 > (n-1)/2 + 3. \tag{3.27}$$

For $k \geq k_1$, from (3.26) with $k = k_1$ we obtain, by Sobolev's embedding, for $0 \leq u \leq \hat{a}_i$, $0 \leq v \leq b_0$,

$$\|f_{i+1}(u,v)\|_{C^2(Y)} \leq C. \tag{3.28}$$

It follows from the equations satisfied by f that

$$\|\partial_u \varphi_{i+1}(u,v)\|_{C^1(Y)} \leq C, \tag{3.29}$$

$$\|\partial_v \psi_{i+1}(u,v)\|_{C^1(Y)} \leq C. \tag{3.30}$$

Integrating (3.29) in u from $(0,v)$ to (u,v) we find that

$$\|\varphi_{i+1}(u,v)\|_{C^1(Y)} \leq C_0 + Cu \leq C_0 + Ca \leq 2C_0 \tag{3.31}$$

for a small enough, namely

$$0 \leq a \leq \min(\hat{a}_i, C_0 C^{-1}). \tag{3.32}$$

(Note that the bound is independent of k.) Further,

$$\|\varphi_{i+1}(u,v)\|_{C^1(Y)} \leq 2C_0 \quad \text{for } 0 \leq u \leq C_0 C^{-1}. \tag{3.33}$$

Next, we u-differentiate the equation satisfied by ψ_{i+1},

$$\begin{aligned}(A^\mu_{\psi\psi}\nabla_\mu)_i \frac{\partial \psi_{i+1}}{\partial u} &= -\partial_u\big((A^\mu_{\psi\psi}\nabla_\mu)_i\big)\psi_{i+1} - \partial_u\big((A^\mu_{\psi\varphi}\nabla_\mu)_i\varphi_{i+1} - (G_\psi)_i\big) \\ &=: (B_\psi)_i \frac{\partial \psi_i}{\partial u} + (b_\psi)_i,\end{aligned} \tag{3.34}$$

where, symbolically,

$$(B_\psi)_i := -\partial_\psi\big((A^\mu_{\psi\psi}\nabla_\mu)_i\big)\psi_{i+1} - \partial_\psi\big((A^\mu_{\psi\varphi}\nabla_\mu)_i\varphi_{i+1} - (G_\psi)_i\big).$$

The system (3.34) is again a symmetric hyperbolic system of first order, linear in $\partial_u\psi_i$ and $\partial_u\psi_{i+1}$, to which we apply (2.31) with $\mathcal{U} = \{u\} \times [0,v]$ and $0 \leq u \leq \hat{a}_i$. Note that, from the definition of $\hat{a}_i$ (see (3.10)) the relevant constant $\hat{C}_1$ there will be bounded from above by a finite constant, say, $\check{C}_1 \geq 1$ which is i-, λ-, and f_i-independent. Thus, by (2.31) with k there replaced by m,

$$\begin{aligned}&e^{-\lambda(u+v)}\|\partial_u\psi_{i+1}(u,v)\|^2_{H^m(Y)} \leq \check{C}_1\Big\{e^{-\lambda u}\|\partial_u\psi_{i+1}(u,0)\|^2_{H^m(Y)} \\ &\quad + \int_0^v e^{-\lambda(u+s)}\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)} - c\lambda\big)\|\partial_u\psi_{i+1}\|^2_{H^m(Y)} + \|\partial_u\psi_{i+1}\|_{H^m(Y)} \\ &\quad \times \Big(\|\partial_u\psi_{i+1}\|_{H^m(Y)} + \|(A)_i\|_{H^m(Y)} + \|(\tilde{A})_i\|_{H^{m-1}(Y)} + \|(\Gamma)_i\|_{H^m(Y)} + \|(\gamma)_i\|_{H^m(Y)} \\ &\quad + \Big\|(B_\psi)_i\frac{\partial\psi_i}{\partial u} + (b_\psi)_i\Big\|_{H^m(Y)} + \Big\|(\tilde{B}_\psi)_i\frac{\partial\psi_i}{\partial u} + (\tilde{b}_\psi)_i\Big\|_{H^{m-1}(Y)}\Big)\,ds\Big\},\end{aligned} \tag{3.35}$$

where $(\tilde{B}_\psi)_i = (A^v_{\psi\psi})^{-1}_i(\tilde{B}_\psi)_i$, $(\tilde{b}_\psi)_i = (A^v_{\psi\psi})^{-1}_i(\tilde{b}_\psi)_i$ and $(A)_i$, the value of the matrix A as determined by f_i. Again from (2.18) we have

$$\begin{aligned}&\|(A)_i\|_{H^m(Y)} + \|(\tilde{A})_i\|_{H^{m-1}(Y)} + \|(\Gamma)_i\|_{H^m(Y)} + \|(\gamma)_i\|_{H^m(Y)} \\ &\leq C(Y,m,\|f_i\|_{L^\infty})\big(\|f_i\|_{H^m} + \|\mathring{A}\|_{H^m(Y)} + \|\mathring{\tilde{A}}\|_{H^{m-1}(Y)} + \|\mathring{\Gamma}\|_{H^m(Y)} + \|\mathring{\gamma}\|_{H^m(Y)}\big) \\ &\leq C(Y,k,C_0)\big(\|f_i\|_{H^m(Y)} + \sqrt{M_k(u,s)}\big) \quad \text{for } m \leq k.\end{aligned} \tag{3.36}$$

Now, after eliminating $\partial_v\psi_{i+1}$ and $\nabla_B\partial_u\varphi_{i+1}$ using the equations, we have $(B_\psi)_i = B_\psi(f_i, f_{i+1}, \mathring{\nabla}_B f_{i+1})$ and

$$(b_\psi)_i = b_\psi\big(f_i, f_{i+1}, \mathring{\nabla}_B f_{i+1}, \mathring{\nabla}_B\mathring{\nabla}_C f_{i+1}, \partial_u\varphi_i\big),$$

which are affine functions of $\mathring{\nabla}_B f_{i+1}$ and $\mathring{\nabla}_B\mathring{\nabla}_C f_{i+1}$. Using again (2.18) we have

$$\begin{aligned}
&\left\|(B_\psi)_i\frac{\partial\psi_i}{\partial u}+(b_\psi)_i\right\|_{H^m(Y)}+\left\|(\tilde B_\psi)_i\frac{\partial\psi_i}{\partial u}+(\tilde b_\psi)_i\right\|_{H^{m-1}(Y)}\\
&\qquad\le C\big(Y,m,\|f_i\|_{L^\infty},\|f_{i+1}\|_{W^{2,\infty}},\|\partial_u\varphi_i\|_{L^\infty}\big)\\
&\qquad\quad\times\big(\|\partial_u\psi_i\|_{H^m(Y)}+\|f_i\|_{H^m(Y)}+\|f_{i+1}\|_{H^{m+2}(Y)}+\|\partial_u\varphi_i\|_{H^m(Y)}\\
&\qquad\qquad+\|\mathring{B}_\psi\|_{H^m(Y)}+\|\mathring{b}_\psi\|_{H^m(Y)}+\|\mathring{\tilde B}_\psi\|_{H^m(Y)}+\|\mathring{\tilde b}_\psi\|_{H^m(Y)}\big).
\end{aligned}\tag{3.37}$$

For $k\ge 3$ let $\hat M_k(u,v)$ be any function such that

$$\begin{aligned}
\hat M_k(u,v)\ge M_k(u,v)&+\|\mathring{B}_\psi(u,v)\|^2_{H^{k-3}(Y)}+\|\mathring{b}_\psi(u,v)\|^2_{H^{k-3}(Y)}\\
&+\|\mathring{\tilde B}_\psi(u,v)\|^2_{H^{k-3}(Y)}+\|\mathring{\tilde b}_\psi(u,v)\|^2_{H^{k-3}(Y)}.
\end{aligned}\tag{3.38}$$

Then, by using simultaneously the inequalities (3.10), and (3.26)–(3.29) with $i+1$ there replaced by i (note that $\hat a_i$ is decreasing by definition), for $m+2=k-1$ one obtains

$$\begin{aligned}
&\left\|(B_\psi)_i\frac{\partial\psi_i}{\partial u}+(b_\psi)_i\right\|_{H^{k-3}(Y)}+\left\|(\tilde B_\psi)_i\frac{\partial\psi_i}{\partial u}+(\tilde b_\psi)_i\right\|_{H^{k-3}(Y)}\\
&\qquad\qquad\le C\Big(\|\partial_u\psi_i\|_{H^{k-3}(Y)}+\sqrt{\hat M_k(u,s)}+C\Big).
\end{aligned}\tag{3.39}$$

Adding (3.36) and (3.39) we obtain

$$\begin{aligned}
e^{-\lambda(u+v)}\|\partial_u\psi_{i+1}(u,v)\|^2_{H^{k-3}(Y)}&\le C_9\Big\{e^{-\lambda u}\|\partial_u\overline{\psi}_{i+1}(u)\|^2_{H^{k-3}(Y)}\\
&+\int_0^v e^{-\lambda(u+s)}\Big[\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)}-c\lambda\big)\|\partial_u\psi_{i+1}\|^2_{H^{k-3}(Y)}\\
&+\|\partial_u\psi_{i+1}\|_{H^{k-3}(Y)}\Big(\|\partial_u\psi_{i+1}\|_{H^{k-3}(Y)}+\|\partial_u\psi_i\|_{H^{k-3}(Y)}+\sqrt{\hat M_k(u,s)}+C_{10}\Big)\Big]\,ds\Big\}\\
\le C_9\Big\{e^{-\lambda u}\|\partial_u\overline{\psi}_{i+1}(u)\|^2_{H^{k-3}(Y)}&+\int_0^v e^{-\lambda(u+s)}\big(\epsilon\|\partial_u\psi_i\|^2_{H^{k-3}(Y)}\\
&+\hat M_k(u,s)+C_{10}^2+\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)}-c\lambda+C(\epsilon)\big)\|\partial_u\psi_{i+1}\|^2_{H^{k-3}(Y)}\big)\,ds\Big\},
\end{aligned}\tag{3.40}$$

where in the last step we have used Cauchy–Schwarz with ϵ. It then follows from (3.20) that there exists a constant $\lambda_3=\lambda_3(Y,k,C_0,C_{\mathrm{div}})$ such that for all $\lambda\ge\lambda_3$,

$$\begin{aligned}
e^{-\lambda v}\|\partial_u\psi_{i+1}(u,v)\|^2_{H^{k-3}(Y)}\le C_9\Big\{&\|\partial_u\overline{\psi}_{i+1}(u)\|^2_{H^{k-3}(Y)}\\
&+\int_0^v e^{-\lambda s}\big(\hat M_k(u,s)+C_{10}^2+\epsilon\|\partial_u\psi_i\|^2_{H^{k-3}(Y)}\big)\,ds\Big\}.
\end{aligned}\tag{3.41}$$

Set

$$\check{C}_\psi = C_9\Big\{\sup_{i\in\mathbb{N}}\sup_{u\in[0,a_0]}\|\partial_u\overline{\psi}_i(u)\|^2_{H^{k-3}(Y)} + \int_0^{b_0} e^{-\lambda s}\big(\hat{M}_k(u,s)+C_{10}^2\big)\,ds\Big\}+1.$$

By an argument which should be standard by now, one can choose ϵ small enough such that the inequality

$$\|\partial_u\psi_{i+1}(u,v)\|^2_{H^{k-3}(Y)} \le 2e^{\lambda v}\check{C}_\psi \tag{3.42}$$

is *preserved under iteration* on $[0,\hat{a}_i]\times[0,b_0]\times Y$.

(This is not good enough yet for our purposes when A^u depends upon ψ, as we will then need (3.42) with $2C_0$ on the right-hand side to be able to make sure that the contribution from $(\nabla_\mu A^\mu)_i$ can be estimated by C_{div}; therefore some more work will have to be done in the general case.)

In any case, let

$$k_2 \text{ be the smallest integer larger than or equal to } (n+7)/2. \tag{3.43}$$

For $k\ge k_2$ we can use (3.42) with k replaced by k_2 there and the Sobolev embedding to obtain

$$\forall (u,v)\in[0,\hat{a}_i]\times[0,b_0],\qquad \|\partial_u\psi_{i+1}(u,v)\|_{C^1(Y)}\le C. \tag{3.44}$$

By integration in u we therefore find that

$$\|\psi_{i+1}(u,v)\|_{C^1(Y)}\le C_0+Cu\le 2C_0, \tag{3.45}$$

again in the range (3.32) (but note that the constant C there might have to be taken larger now, remaining independent of i and k).

Keeping in mind (3.33), we conclude that the condition

$$\|f_{i+1}(u,v)\|_{C^1(Y)}\le 4C_0 \qquad \text{for } 0\le u\le C_0C^{-1} \tag{3.46}$$

is stable under iteration.

Moreover, after replacing the bound C_0C^{-1} by a smaller i-independent number, say a_*, if necessary, integration in u shows that

$$f_{i+1}(u,v) \text{ cannot leave the neighborhood } \mathcal{K} \text{ for } 0\le u\le a_* \tag{3.47}$$

with $\mathcal{K}$ as in (3.8).

If A^v does not depend upon φ, and A^u does not depend upon ψ, then the conditions on $\partial_u\psi$ and $\partial_v\varphi$ in (3.8) are irrelevant for all the estimates so far, and so

$$\text{the bound (3.10b) cannot be violated for } 0\le u\le a_*. \tag{3.48}$$

Hence, by the definition of C_{div},

$$\text{the inequalities (3.10) cannot be violated for } 0\le u\le a_*. \tag{3.49}$$

Recall that $\hat{a}_i$ was defined as either a_0 or the first number at which the inequalities (3.10) fail for f_i or f_{i+1}. So, if we assume that (3.10) hold at the induction step i with $a_i\ge a_*$, we conclude that $a_{i+1}\ge a_*$ as well. Hence

$$\hat{a}_i\ge a_*.$$

The above implies that (3.20) and (3.24) hold for $0\le u\le a_*$. We have therefore obtained:

Proposition 3.5. *Let $k > (n+7)/2$, assume that A^v does not depend upon φ, and that A^u does not depend upon ψ. Suppose that there exists a constant $\mathcal{C}$ such that*

$$\sup_{\mathcal{N}^- \cup \mathcal{N}^+} \left\{ |\overline{\partial_v f}_i| + |\overline{\partial_u f}_i| + \|\overline{f}_i(u,v)\|_{H^k(Y)} + M_k(u,v) \right\} \leq \mathcal{C}. \tag{3.50}$$

There exists a constant $0 < a_* = a_*(a_0, b_0, \mathcal{C}, Y) \leq a_0$ such that all the fields f_i satisfy the hypotheses of Lemmata 3.2–3.4 on $[0, a_*] \times [0, b_0] \times Y$, as well as their conclusions with $\hat{a}_i$ replaced by a_*.

It remains to obtain the pointwise bounds (3.10a), (3.20) and (3.24) in the general case; these will follow from pointwise estimates on $\partial_v \varphi$, and improved estimates on $\partial_u \psi$.

We start by showing that the inequality

$$\sup_i \sup_{(u,v) \in [0,a_*] \times [0,b_0]} |\partial_u \psi_i(u,v)| \leq 2 \sup_i \sup_{v \in [0,b_0]} |\partial_u \psi_i(0,v)| + 1$$

is preserved under iteration, after reducing a_* if necessary.

We consider the restriction of the u-differentiated equation satisfied by ψ_{i+1} on $\mathcal{N}^-$, that is, for $u = 0$, which we write as

$$\overline{(A^\mu_{\psi\psi} \nabla_\mu)_i}\, \overline{\frac{\partial \psi_{i+1}}{\partial u}} = \overline{(B_\psi)_i}\, \overline{\frac{\partial \psi_i}{\partial u}} + \overline{(b_\psi)_i}. \tag{3.51}$$

Setting $\Psi_i = \frac{\partial \psi_i}{\partial u} - \overline{\frac{\partial \psi_i}{\partial u}}$ and subtracting (3.51) from (3.34) gives an equation of the form

$$(A^\mu_{\psi\psi} \nabla_\mu)_i \Psi_{i+1} = (B_\psi)_i \Psi_i + \mathcal{E}_i, \tag{3.52}$$

where

$$\mathcal{E}_i = \underbrace{-\big((A^\mu_{\psi\psi} \nabla_\mu)_i - \overline{(A^\mu_{\psi\psi} \nabla_\mu)_i}\big) \overline{\frac{\partial \psi_{i+1}}{\partial u}}}_{\Delta_1} + \underbrace{\big((B_\psi)_i - \overline{(B_\psi)_i}\big) \overline{\frac{\partial \psi_i}{\partial u}}}_{\Delta_2} + \underbrace{(b_\psi)_i - \overline{(b_\psi)_i}}_{\Delta_3}. \tag{3.53}$$

It is easy to see that both $(B_\psi)_i$ and $(b_\psi)_i$ are affine in $\mathring{\nabla}_B f_{i+1}$ and $\mathring{\nabla}_B \mathring{\nabla}_C f_{i+1}$ with coefficients depending upon f_i and f_{i+1}, thus if $k - 1 > (n-1)/2 + 3$ then by (3.26), (3.28), (2.16) and (2.18) we have

$$\|(B_\psi)_i\|_{H^{k-2}(Y)} + \|(b_\psi)_i\|_{H^{k-3}(Y)} + \|(B_\psi)_i\|_{W^{1,\infty}(Y)} + \|(b_\psi)_i\|_{W^{2,\infty}(Y)} < C. \tag{3.54}$$

Further,

$$(A^\mu_{\psi\psi} \nabla_\mu)_i \overline{\frac{\partial \psi_{i+1}}{\partial u}} = (A^v_{\psi\psi})_i \overline{\frac{\partial_u \partial \psi_{i+1}}{\partial v}} + \big((A^v_{\psi\psi} \gamma_{\psi\psi,v})_i + (A^B_{\psi\psi})_i \mathring{\nabla}_B + (A^B_{\psi\psi} \Gamma_{\psi\psi,B})_i\big) \overline{\frac{\partial \psi_{i+1}}{\partial u}}.$$

Hence we have the following contribution to the first term in (3.53):

$$\begin{aligned} &\big((A^v_{\psi\psi})_i - \overline{(A^v_{\psi\psi})_i}\big) \overline{\partial_u \partial_v \psi_{i+1}} \\ &\quad = \left\{ u \int_0^1 \frac{\partial (A^v_{\psi\psi_i})}{\partial u} \big(tu, v, t f_i(u,v) + (1-t) f_i(0,v)\big)\, dt + (f_i(u,v) - f_i(0,v)) \right. \\ &\qquad\qquad \left. \times \int_0^1 \frac{\partial (A^v_{\psi\psi_i})}{\partial f} \big(tu, v, t f_i(u,v) + (1-t) f_i(0,v)\big)\, dt \right\} \overline{\partial_u \partial_v \psi_{i+1}}. \end{aligned}$$

Now recall that by hypothesis $\sup_{v \in [0,b_0]} \sup_i \|\overline{\partial_v \partial_u \psi_i}\|_{H^k(Y)}$ is bounded and $\partial \psi_{i+1}/\partial v$ is an affine function of f_{i+1} and $\mathring{\nabla} f_{i+1}$ with coefficients depending upon f_i. Thus there exists a constant

$$C = C\Big(C_0, \sup_{v\in[0,b_0]} \sup_i \|\overline{\partial_v \partial_u \psi_i}\|_{H^k(Y)}\Big) > 0, \tag{3.55}$$

which is i-independent, such that for all $(u,v) \in [0,\hat{a}_i] \times [0,b_0]$,

$$\big\|\big((A^\mu_{\psi\psi}\nabla_\mu)_i - \overline{(A^\mu_{\psi\psi}\nabla_\mu)_i}\big)\overline{\partial_u\psi_{i+1}}\big\|_{L^2(Y)} \le C(u + \|f_i(u,v) - f_i(0,v)\|_{L^2(Y)}).$$

The L^2 norm of the remaining terms in the first term Δ_1 of (3.53) are estimated in the same way with $(A^\mu_{\psi\psi})_i$ replaced successively by $(A^v_{\psi\psi}\gamma_{\psi\psi,v})_i$, $(A^B_{\psi\psi})_i\mathring{\nabla}_B$, $(A^B_{\psi\psi}\Gamma_{\psi\psi,B})_i$, and $\overline{\partial_u\partial_v\psi_{i+1}}$ replaced by $\overline{\partial_u\psi_{i+1}}$, leading to the following estimate for Δ_1:

$$\forall (u,v) \in [0,\hat{a}_i] \times [0,b_0], \qquad \|\Delta_1(u,v)\|_{L^2(Y)} \le C\big(u + \|f_i(u,v) - f_i(0,v)\|_{L^2(Y)}\big). \tag{3.56}$$

We continue with the analysis of the second term Δ_2 of (3.53). The explicit expression of $(B_\psi)_i$ shows that $(B_\psi)_i$ is a collection of terms of the form $\Gamma_i P_r f_{i+1}$, $0 \le r \le 1$, where the Γ_i's are smooth functions depending upon the fields f_i. We order these terms in an arbitrary way and write

$$(B_\psi)_i = \sum_{m=1}^{p} \Gamma_{i,m} P_{r_m} f_{i+1},$$

where P_{r_m} is either the identity or $\mathring{\nabla}_B$. We have

$$\Delta_2 = \big((B_\psi)_i - \overline{(B_\psi)_i}\big)\overline{\partial_u\psi_i} = \sum_{m=1}^{p} \big(\Gamma_{i,m}P_{r_m}f_{i+1} - \overline{\Gamma_{i,m}P_{r_m}f_{i+1}}\big)\overline{\partial_u\psi_i}.$$

Thus,

$$\begin{aligned}
&(\Gamma_i P_r f_{i+1} - \overline{\Gamma_i P_r f_{i+1}})\overline{\partial_u\psi_i} \\
&\quad= \Gamma_i\big(P_r f_{i+1} - \overline{P_r f_{i+1}}\big)\overline{\partial_u\psi_i} + (\Gamma_i - \overline{\Gamma_i})\overline{P_r f_{i+1}\partial_u\psi_i} \\
&\quad= \Gamma_i\big(P_r f_{i+1} - \overline{P_r f_{i+1}}\big)\overline{\partial_u\psi_i} + u\left[\int_0^1 \frac{\partial\Gamma_i}{\partial u}(tu, tf_i(u,v) + (1-t)f_i(0,v))\,dt\right]\overline{P_r f_{i+1}\partial_u\psi_i} \\
&\qquad+ (f_i(u,v) - f(0,v))\left[\int_0^1 \frac{\partial\Gamma_i}{\partial f}(tu, tf_i(u,v) + (1-t)f_i(0,v))\,dt\right]\overline{P_r f_{i+1}\partial_u\psi_i}.
\end{aligned}$$

We then see that

$$\begin{aligned}
&\|(\Gamma_i P_r f_{i+1} - \overline{\Gamma_i P_r f_{i+1}})\overline{\partial_u\psi_i}\|_{L^2(Y)} \\
&\qquad\le C(C_0)\big(u + \|f_i(u,v) - f_i(0,v)\|_{L^2(Y)} + \|f_{i+1}(u,v) - f_{i+1}(0,v)\|_{H^1(Y)}\big),
\end{aligned}$$

which gives

$$\|\Delta_2\|_{L^2(Y)} \le C(C_0)\big(u + \|f_i(u,v) - f_i(0,v)\|_{L^2(Y)} + \|f_{i+1}(u,v) - f_{i+1}(0,v)\|_{H^1(Y)}\big). \tag{3.57}$$

As far as the last term Δ_3 of (3.53) is concerned, we note that $(b_\psi)_i$ is a sum of terms of the form

$$\tilde{\Gamma}_i \mathring{\nabla}_{r_1} \dots \mathring{\nabla}_{r_j} f_{i+1},$$

with $0 \le j \le 2$ and $\tilde{\Gamma}_i$ depending upon f_i and $\partial_u\varphi_i$. Thus, as in the previous case, we see that

$$\|\Delta_3\|_{L^2(Y)} \leq C(C_0)\big(u + \|f_i(u,v) - f_i(0,v)\|_{L^2(Y)} + \|\partial_u\varphi_i(u,v) - \partial_u\varphi_i(0,v)\|_{L^2(Y)} \\ + \|f_{i+1}(u,v) - f_{i+1}(0,v)\|_{H^2(Y)}\big). \quad (3.58)$$

Now from (3.29) and (3.44) we find (note that from these inequalities and the equation satisfied by φ_{i+1}, the L^∞ norm of $\partial_u^2\varphi_i$ is uniformly bounded) that

$$\forall (u,v) \in [0,\hat{a}_i] \times [0,b_0] \times Y \qquad \|\mathcal{E}_i(u,v)\|_{L^2(Y)} \leq Cu. \quad (3.59)$$

By (3.54), the L^2 norm of the right-hand side of (3.52) is estimated as follows:

$$\begin{aligned}\|(B_\psi)_i\Psi_i + \mathcal{E}_i\|_{L^2(Y)} &\leq \|(B_\psi)_i\Psi_i\|_{L^2(Y)} + \|\mathcal{E}_i\|_{L^2(Y)} \leq Cu + \|(B_\psi)_i\|_{L^\infty(Y)}\|\Psi_i\|_{L^2(Y)} \\ &\leq C(u + \|\Psi_i\|_{L^2(Y)}). \end{aligned} \quad (3.60)$$

Next, we write the energy estimate for the system (3.52). Consider the vector field (recall $w_r = e^{-\lambda(u+v)}$)

$$Z^\mu := w_r\langle\Psi_{i+1}, (A^\mu_{\psi\psi})_i\Psi_{i+1}\rangle, \quad (3.61)$$

so that

$$\nabla_\mu(Z^\mu) = \{-2\lambda\langle\Psi_{i+1}, (A^v_{\psi\psi})_i\Psi_{i+1}\rangle \\ + \langle\Psi_{i+1}, (\nabla_\mu A^\mu_{\psi\psi})_i\Psi_{i+1}\rangle + 2\langle\Psi_{i+1}, (A^\mu_{\psi\psi}\nabla_\mu)_i\Psi_{i+1}\rangle\}w_r.$$

We apply Stokes' theorem on the set $\{u\} \times [0,v] \times Y$ and obtain

$$\begin{aligned} e^{-\lambda v}\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)} &\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} + e^{\lambda u}\int_0^v \nabla_\mu(Z^\mu)(s,v)\,ds\,d\mu_Y\Big\} \\ &\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} - 2\lambda\int_0^v e^{-\lambda s}\big(\langle\Psi_{i+1}, (A^v_{\psi\psi})_i\Psi_{i+1}\rangle \\ &\quad + \langle\Psi_{i+1}, (\nabla_\mu A^\mu_{\psi\psi})_i\Psi_{i+1}\rangle + 2\langle\Psi_{i+1}, (A^\mu_{\psi\psi}\nabla_\mu)_i\Psi_{i+1}\rangle\big)\,ds\,d\mu_Y\Big\} \\ &\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} \\ &\quad + \int_0^v e^{-\lambda s}\big\{\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)} - 2c\lambda\big)\|\Psi_{i+1}(u,s)\|^2_{L^2(Y)} \\ &\quad + 2\|\Psi_{i+1}(u,s)\|_{L^2(Y)}\|((B_\psi)_i\Psi_i + \mathcal{E}_i)(u,s)\|_{L^2(Y)}\big\}\,ds\Big\}. \end{aligned}$$

Now from (3.60), we have

$$\begin{aligned} e^{-\lambda v}\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)} &\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} \\ &\quad + \int_0^v e^{-\lambda s}\big\{\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)} - 2c\lambda\big)\|\Psi_{i+1}(u,s)\|^2_{L^2(Y)} \\ &\quad + \|\Psi_{i+1}(u,s)\|_{L^2(Y)}\big(u + \|\Psi_i(u,s)\|_{L^2(Y)}\big)\big\}\,ds\Big\} \end{aligned}$$

$$\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)}$$
$$+\int_0^v e^{-\lambda s}\big\{\big(\|(\nabla_\mu A^\mu_{\psi\psi})_i\|_{L^\infty(Y)} - 2c\lambda + C(\epsilon)\big)\|\Psi_{i+1}(u,s)\|^2_{L^2(Y)}$$
$$+ (u^2 + \epsilon\|\Psi_i(u,s)\|^2_{L^2(Y)})\big\}\, ds\Big\}$$
$$\leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} + \int_0^v e^{-\lambda s}(u^2 + \epsilon\|\Psi_i(u,s)\|^2_{L^2(Y)})\, ds\Big\}$$

for λ large enough. Thus there exists $\lambda_\epsilon > 0$ such that for all $(u,v) \in \Omega_{\hat{a}_i}$,

$$e^{-\lambda_\epsilon v}\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)} \leq C\Big\{\|\Psi_{i+1}(u,0)\|^2_{L^2(Y)} + \int_0^v e^{-\lambda_\epsilon s}(u^2 + \epsilon\|\Psi_i(u,s)\|^2_{L^2(Y)})\, ds\Big\}.$$

Recall $\Psi_{i+1}(u,0) = \partial_u\psi(u,0) - \partial_u\psi(0,0)$; thus,

$$|\Psi_{i+1}(u,0)| \leq u \cdot \underbrace{\sup_{i\in\mathbb{N}}\sup_{u\in[0,a_0]}\|\partial_u^2\psi_i(u,0)\|_{L^\infty(Y)}}_{=:\hat{c}},$$

leading to

$$e^{-\lambda_\epsilon v}\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)}$$
$$\leq u^2\left(\hat{c}^2\mu_Y(Y) + C\int_0^v e^{-\lambda_\epsilon s}\, ds\right) + \epsilon C\int_0^v e^{-\lambda_\epsilon s}\|\Psi_i(u,s)\|^2_{L^2(Y)}\, ds.$$

Suppose now for the purpose of induction that (the constant $\overline{C}_0$ will be chosen shortly, see (3.64))

$$(u,v) \in [0,\hat{a}_i]\times[0,b_0], \qquad \|\Psi_i(u,v)\|^2_{L^2(Y)} \leq \overline{C}_0 e^{\lambda_\epsilon(v-b_0)}. \tag{3.62}$$

Then by the previous inequality,

$$e^{-\lambda_\epsilon(v-b_0)}\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)} \leq u^2\,\underbrace{e^{\lambda_\epsilon b_0}\big(\hat{c}^2\mu_Y(Y) + C/\lambda_\epsilon\big)}_{=:C(\lambda_\epsilon)} + \epsilon C\overline{C}_0 b_0.$$

We choose ϵ small enough such that $\epsilon C b_0 \leq 1/2$. Once this choice of ϵ is made (then $C(\lambda_\epsilon)$ is fixed) we see that $u^2 C(\lambda_\epsilon) \leq \overline{C}_0/2$ provided that

$$0 < u \leq \overline{C}_0\big(\sqrt{2C(\lambda_\epsilon)}\big)^{-1}. \tag{3.63}$$

We have thus proved that in the range of the u-variable given by (3.63),

$$\|\Psi_{i+1}(u,v)\|^2_{L^2(Y)} \leq \overline{C}_0 e^{\lambda_\epsilon(v-b_0)}.$$

This proves that (3.62) is preserved under iteration.

Now, recall from (3.42) that

$$e^{-\lambda v}\|\partial_u\psi_{i+1}(u,v)\|^2_{H^{k-3}(Y)} \leq 2\check{C}_\psi.$$

Note that the constant $\check{C}_\psi$ in (3.62) can be chosen independently of λ, and that the λ here is independent of λ_ϵ in the previous inequalities, but it is convenient to choose them to be equal, and we shall do so. Thus we can write

$$e^{-\lambda v/2}\|\Psi_{i+1}(u,v)\|_{H^{k-3}(Y)} \leq C_0 + (2\check{C}_\psi)^{1/2} =: C.$$

By interpolation, there exists a constant $c_m > 0$ such that, for all $m \in (0, k-3)$,

$$\|\Psi_{i+1}(u,v)\|_{H^m(Y)} \le c_m \|\Psi_{i+1}(u,v)\|^{\theta}_{H^{k-3}(Y)} \|\Psi_{i+1}(u,v)\|^{1-\theta}_{L^2(Y)},$$

with a certain constant $\theta \in (0,1)$. Then multiplying by $e^{-\lambda v/2}$ we obtain

$$\begin{aligned} e^{-\lambda v/2}\|\Psi_{i+1}(u,v)\|_{H^m(Y)} &\le c_m \|e^{-\lambda v/2}\Psi_{i+1}(u,v)\|^{\theta}_{H^{k-3}(Y)} \|e^{-\lambda v/2}\Psi_{i+1}(u,v)\|^{1-\theta}_{L^2(Y)} \\ &\le c_m C^{\theta} (\overline{C}_0 e^{-\lambda b_0})^{1/2(1-\theta)}, \end{aligned}$$

which can be rewritten as

$$e^{-\lambda(v-b_0)/2}\|\Psi_{i+1}(u,v)\|_{H^m(Y)} \le c_m C^{\theta} (\overline{C}_0)^{1/2(1-\theta)} (e^{\lambda b_0})^{\theta/2}.$$

For $m = k-4 > (n-1)/2$ (which is possible if $k > (n+7)/2$), from Sobolev's embedding theorem there exists a constant $C_S > 0$ such that

$$e^{-\lambda(v-b_0)/2}\|\Psi_{i+1}(u,v)\|_{L^\infty(Y)} \le C_S C^{\theta} (\overline{C}_0)^{1/2(1-\theta)} (e^{\lambda b_0})^{\theta/2}.$$

Finally, we choose $\overline{C}_0$ small enough so that

$$C_S C^{\theta} (\overline{C}_0)^{1/2(1-\theta)} (e^{\lambda b_0})^{\theta/2} < \sup_i \sup_{v\in[0,b_0]} |\partial_u \psi_i(0,v)| + 1, \tag{3.64}$$

and obtain

$$\|\Psi_{i+1}(u,v)\|_{L^\infty(Y)} \le e^{\lambda(v-b_0)/2}\Big(\sup_i \sup_{v\in[0,b_0]} |\partial_u \psi_i(0,v)| + 1\Big),$$

which leads to

$$\|\partial_u \psi_{i+1}(u,v)\|_{L^\infty(Y)} \le 2\sup_i \sup_{v\in[0,b_0]} |\partial_u \psi_i(0,v)| + 1 \tag{3.65}$$

for all $v \in [0, b_0]$ and all u in the range of (3.63), with $\overline{C}_0$ defined in (3.64). Thus we conclude, as after (3.49), that up to reducing a_* if necessary,

$$\sup_i \sup_{(u,v)\in[0,a_*]\times[0,b_0]} |\partial_u \psi_i(u,v)| \le 2\sup_i \sup_{v\in[0,b_0]} |\partial_u \psi_i(0,v)| + 1.$$

The estimate $(|\nabla_\mu A^\mu|)_i \le C_{\mathrm{div}}$ for all i follows when A^v does not depend upon φ.

When A^v depends upon φ it remains to obtain a pointwise estimate on $\partial_v\varphi$. We start by v-differentiating the equation satisfied by φ:

$$\begin{aligned} (A^\mu_{\varphi\varphi}\nabla_\mu)_i \frac{\partial \varphi_{i+1}}{\partial v} &= -\partial_v\big((A^\mu_{\varphi\varphi}\nabla_\mu)_i\big)\varphi_{i+1} - \partial_v\big((A^\mu_{\varphi\psi}\nabla_\mu)_i\psi_{i+1} - (G_\varphi)_i\big) \\ &=: (B_\varphi)_i \frac{\partial \varphi_i}{\partial v} + (b_\varphi)_i, \end{aligned} \tag{3.66}$$

where

$$(B_\varphi)_i := -\partial_\varphi\big((A^\mu_{\varphi\varphi}\nabla_\mu)_i\big)\varphi_{i+1} - \partial_\varphi\big((A^\mu_{\varphi\psi}\nabla_\mu)_i\psi_{i+1} - (G_\varphi)_i\big),$$

and with $(b_\varphi)_i$ containing all the remaining terms. After replacing v-derivatives of ψ_{i+1} using the field equations, $(B_\varphi)_i$ and $(b_\varphi)_i$ become affine in $\mathring{\nabla}_B f_{i+1}$ and $\mathring{\nabla}_B\mathring{\nabla}_C f_{i+1}$, with coefficients depending upon f_i.

Recall that k_2 has been defined in (3.43); for $k \ge k_2$ by (2.16) and (2.18) we have the estimate

$$\|(B_\varphi)_i\|_{H^{k-2}(Y)} + \|(b_\varphi)_i\|_{H^{k-3}(Y)} + \|(B_\varphi)_i\|_{W^{2,\infty}(Y)} + \|(b_\varphi)_i\|_{W^{1,\infty}(Y)} \le C_9. \tag{3.67}$$

Applying (2.31) with k replaced by $k-3$, with $\mathcal{U}=[0,u]\times\{v\}$, $f=\partial\varphi_{i+1}/\partial v$, etc., to (3.66), we obtain

$$\begin{aligned}
e^{-\lambda(u+v)}\left\|\frac{\partial\varphi_{i+1}}{\partial v}(u,v)\right\|^2_{H^{k-3}(Y)} &\le C_{10}\Bigg\{e^{-\lambda v}\underbrace{\left\|\frac{\partial\varphi_{i+1}}{\partial v}(0,v)\right\|^2_{H^{k-3}(Y)}}_{\left\|\frac{\partial\overline{\varphi}_{i+1}}{\partial v}(v)\right\|^2_{H^{k-3}(Y)}}\\
&+\int_0^u e^{-\lambda(s+v)}\Bigg\{\Big(\|(\nabla_\mu A^\mu_{\varphi\varphi})_i(s,v)\|_{L^\infty(Y)}-c\lambda\Big)\left\|\frac{\partial\varphi_{i+1}}{\partial v}(s,v)\right\|^2_{H^{k-3}(Y)}\\
&+C_{11}\left\|\frac{\partial\varphi_{i+1}}{\partial v}(s,v)\right\|_{H^{k-3}(Y)}\left(\left\|\frac{\partial\varphi_{i}}{\partial v}(s,v)\right\|_{H^{k-3}(Y)}+\left\|\frac{\partial\varphi_{i+1}}{\partial v}(s,v)\right\|_{H^{k-3}(Y)}+C_{12}\right)\Bigg\}\,ds\Bigg\}.
\end{aligned}\tag{3.68}$$

As before, using the inequality $ab\le a^2/(4\epsilon)+\epsilon b^2$, one is led to

$$\begin{aligned}
e^{-\lambda u}\left\|\frac{\partial\varphi_{i+1}}{\partial v}(u,v)\right\|^2_{H^{k-3}(Y)} &\le C_{10}\Bigg\{\left\|\frac{\partial\overline{\varphi}_{i+1}}{\partial v}(v)\right\|^2_{H^{k-3}(Y)}\\
&+\int_0^u e^{-\lambda s}\Bigg\{\left(\|(\nabla_\mu A^\mu_{\varphi\varphi})_i(s,v)\|_{L^\infty(Y)}+2C_{11}+\frac{C_{11}}{4\epsilon}-c\lambda\right)\left\|\frac{\partial\varphi_{i+1}}{\partial v}(s,v)\right\|^2_{H^{k-3}(Y)}\\
&+\epsilon C_{11}\left\|\frac{\partial\varphi_{i}}{\partial v}(s,v)\right\|^2_{H^{k-3}(Y)}+C_{11}C_{12}^2\Bigg\}\,ds\Bigg\}.
\end{aligned}\tag{3.69}$$

Since (see (3.10a))

$$|(\nabla_\mu A^\mu_{\varphi\varphi})_i|\le|(\nabla_\mu A^\mu)_i|\le C_{\mathrm{div}},\qquad\forall(u,v)\in[0,a_i]\times[0,b_0],$$

there exists a constant $\lambda_3=\lambda_3(C_{10},C_{\mathrm{div}},C_0,k)$ which does not depend on i such that, for all $\lambda\ge\lambda_3$, the previous inequality implies

$$\begin{aligned}
e^{-\lambda u}\left\|\frac{\partial\varphi_{i+1}}{\partial v}(u,v)\right\|^2_{H^{k-3}(Y)} &\le C_{10}\Bigg\{\left\|\frac{\partial\overline{\varphi}_{i+1}}{\partial v}\right\|^2_{H^{k-3}(Y)}\\
&+C_{11}\int_0^u e^{-\lambda s}\left\{\epsilon\left\|\frac{\partial\varphi_{i}}{\partial v}(s,v)\right\|^2_{H^{k-3}(Y)}+C_{12}^2\right\}ds\Bigg\}.
\end{aligned}\tag{3.70}$$

Integrating in u, for $0\le u\le\hat{a}_i\le a_0$, one obtains

$$\begin{aligned}
\int_0^u e^{-\lambda t}\left\|\frac{\partial\varphi_{i+1}}{\partial v}(t,v)\right\|^2_{H^{k-3}(Y)}dt &\le C_{10}\Bigg\{a_0\left\|\frac{\partial\overline{\varphi}_{i+1}}{\partial v}(v)\right\|^2_{H^{k-3}(Y)}\\
&+C_{11}\int_0^u\int_0^t e^{-\lambda s}\left\{\epsilon\left\|\frac{\partial\varphi_{i}}{\partial v}\right\|^2_{H^{k-3}(Y)}+C_{12}^2\right\}ds\,dt\Bigg\}.
\end{aligned}\tag{3.71}$$

Let

$$\mathring{C}_\varphi(u):=C_{10}\left\{\sup_{i\in\mathbb{N}}\sup_{v\in[0,b_0]}a_0\left\|\frac{\partial\overline{\varphi}_{i+1}}{\partial v}(v)\right\|^2_{H^{k-3}(Y)}+\int_0^u\int_0^t C_{11}C_{12}^2\,ds\,dt\right\}.\tag{3.72}$$

Proceeding as before, one gets rid of the $\partial\varphi_i/\partial v$ terms in the integral appearing in (3.70), for all $0 \le u \le \hat{a}_i$, as follows: suppose that

$$\forall 0 \le t \le u \le \hat{a}_i \le a_0 \qquad \int_0^t e^{-\lambda s} \left\| \frac{\partial \varphi_i}{\partial v}(s,v) \right\|^2_{H^{k-3}(Y)} ds \le 2\mathring{C}_\varphi(t); \tag{3.73}$$

then (3.71) gives

$$\int_0^u e^{-\lambda t} \left\| \frac{\partial \varphi_{i+1}}{\partial v}(t,v) \right\|^2_{H^{k-3}(Y)} dt \le \mathring{C}_\varphi(u) + 2a_0 \epsilon C_{10} C_{11} \mathring{C}_\varphi(u).$$

Thus, one can choose $\epsilon = \epsilon(C_{10}, C_{11}, C_{12}, C_{\mathrm{div}}, C_0, k, \lambda_3)$ small enough so that

$$\int_0^u e^{-\lambda t} \left\| \frac{\partial \varphi_{i+1}}{\partial v}(t,v) \right\|^2_{H^{k-3}(Y)} dt \le 2\mathring{C}_\varphi(u),$$

which shows that (3.73) is preserved under iteration.

For any $\lambda \ge \lambda_3|_{k=k_2}$ we deduce from (3.70) that

$$\left\| \frac{\partial \varphi_{i+1}}{\partial v}(u,v) \right\|^2_{H^{k_2-3}(Y)} \le C.$$

Now, Sobolev's embedding implies

$$\left\| \frac{\partial \varphi_{i+1}}{\partial v}(u,v) \right\|_{W^{1,\infty}(Y)} \le C.$$

As this holds for all i, (3.66) proves that

$$\left\| \frac{\partial^2 \varphi_{i+1}}{\partial u \partial v}(u,v) \right\|_{L^\infty(Y)} \le C. \tag{3.74}$$

By integration

$$\left| \frac{\partial \varphi_{i+1}}{\partial v}(u,v) \right| \le \left| \frac{\partial \varphi_{i+1}}{\partial v}(0,v) \right| + Cu \le 2 \sup_{i \in \mathbb{N}} \left\| \frac{\partial \overline{\varphi}_i}{\partial v} \right\|_{L^\infty(\mathcal{N}^+ \cup \mathcal{N}^-)},$$

provided that

$$0 \le u \le C^{-1} \left(\sup_{i \in \mathbb{N}} \left\| \frac{\partial \overline{\varphi}_i}{\partial v} \right\|_{L^\infty(\mathcal{N}^+ \cup \mathcal{N}^-)} \right). \tag{3.75}$$

Now, we choose a_* to be the smallest of a_0 and of the four constants appearing on the right-hand side of inequalities (3.33), (3.46), (3.63) and (3.75). Recall that a_i was defined as either a_0 or the first number at which the inequalities (3.10) fail for f_i or f_{i+1}. So, if we assume that the inequalities (3.10) hold at the induction step i with $a_i \ge a_*$, we conclude that $a_{i+1} \ge a_*$ as well. Hence $\hat{a}_i \ge a_*$ for all $i \in \mathbb{N}$. The above implies that (3.20) and (3.24) hold for $0 \le u \le a_*$. Since a_* is independent of k, we have obtained:

Proposition 3.6. *Let $\mathbb{N} \ni k > (n+7)/2$, and suppose that there exists a constant $\mathcal{C}$ such that for $(u,v) \in [0,a_0] \times [0,b_0]$ we have*

$$\sup_{\mathcal{N}^- \cup \mathcal{N}^+} \left\{ |\overline{\partial_v f}_i| + |\overline{\partial_u f}_i| + \|\overline{f}_i(u,v)\|_{H^k(Y)} + M_k(u,v) \right\} \le \mathcal{C}. \tag{3.76}$$

There exists a constant $0 < a_* = a_*(a_0, b_0, \mathcal{C}, Y) \le a_0$ such that the fields f_i satisfy the hypotheses of Lemma 3.2 on $[0,a_*] \times [0,b_0] \times Y$. As a consequence, there exists a constant

$C = C(a_0, b_0, \mathcal{C}, Y, k)$ *such that for* $(u,v) \in [0, a_*] \times [0, b_0]$ *we have*

$$\int_0^{a_*} \|\psi_i(s,v)\|^2_{H^k(Y)}\, ds + \int_0^{b_0} \|\varphi_i(u,s)\|^2_{H^k(Y)}\, ds + \|f_i(u,v)\|_{H^{k-1}(Y)} \\ + \|\partial_v \psi_i(u,v)\|_{H^{k-2}(Y)} + \|\partial_u \varphi_i(u,v)\|_{H^{k-2}(Y)} \\ + \|\partial_u \psi_i(u,v)\|_{H^{k-3}(Y)} + \|\partial_v \varphi_i(u,v)\|_{H^{k-3}(Y)} \le C. \tag{3.77}$$

REMARK 3.7. The result remains true for $k \in \mathbb{R}$; this can be established by commuting the equation with an appropriate pseudo-differential operator in the Y-variables. However, this will be of no concern to us here.

3.3. Convergence of the iterative sequence. To prove convergence of the sequence, we set

$$\delta f_{i+1} := f_{i+1} - f_i.$$

We have the equation

$$(A^\mu \nabla_\mu)_i \delta f_{i+1} = \delta G_i, \tag{3.78}$$

with

$$\delta G_i := G_i - G_{i-1} - ((A^\mu \nabla_\mu)_i - (A^\mu \nabla_\mu)_{i-1})\, f_i.$$

The standard identity

$$h(x) - h(y) = (x-y) \int_0^1 h'(tx + (1-t)y)\, dt,$$

applied both to $G_i - G_{i-1}$ and $(A^\mu \nabla_\mu)_i - (A^\mu \nabla_\mu)_{i-1}$, leads to the straightforward estimate, for all λ and $0 \le a \le a_*$,

$$\|e^{-\lambda(u+v)} \delta G_i\|_{L^2([0,a]\times[0,b_0]\times Y)} \le C_1 \|e^{-\lambda(u+v)} \delta f_i\|_{L^2([0,a]\times[0,b_0]\times Y)},$$

with a constant C_1 which depends upon $\sup_i \|f_i\|_{W^{1,\infty}}$, and which is independent of λ and of i. Here we reset the numbering of the constants, so that the constant C_1 of this section has nothing to do with the constant C_1 of the previous section, etc.

We apply the energy inequality (2.12) with $k = 0$; there are then no commutator terms in (2.20), leading to

$$\|e^{-\lambda(u+v)} \delta\varphi_{i+1}(u)\|_{L^2([0,b_0]\times Y)} + \|e^{-\lambda(u+v)} \delta\psi_{i+1}(v)\|_{L^2([0,a_*]\times Y)} \\ \le C_2 \big\{ \|e^{-\lambda v} \delta\overline{\varphi}_{i+1}\|_{L^2([0,b_0]\times Y)} + \|e^{-\lambda u} \delta\overline{\psi}_{i+1}\|_{L^2([0,a_*]\times Y)} \\ + (\|(A^\mu \nabla_\mu)_i\|_{L^\infty} - c\lambda) \|e^{-\lambda(u+v)} \delta f_{i+1}\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)} \\ + 2\|e^{-\lambda(u+v)} \delta f_{i+1}\|_{L^2([0,a_*]\times[0,b_0]\times Y)} \|e^{-\lambda(u+v)} \delta G_i\|_{L^2([0,a_*]\times[0,b_0]\times Y)} \big\} \\ \le C_2 \big\{ \|e^{-\lambda v} \delta\overline{\varphi}_{i+1}\|_{L^2([0,b_0]\times Y)} + \|e^{-\lambda u} \delta\overline{\psi}_{i+1}\|_{L^2([0,a_*]\times Y)} \\ + \big(\|(A^\mu \nabla_\mu)_i\|_{L^\infty} + C_1 - c\lambda\big) \|e^{-\lambda(u+v)} \delta f_{i+1}\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)} \big\} \\ + C_1 C_2 \|e^{-\lambda(u+v)} \delta f_i\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}. \tag{3.79}$$

Now, for the purpose of proving Theorem 3.9, the sequences $(\overline{\varphi}_i)_{i\in\mathbb{N}}$ and $(\overline{\psi}_i)_{i\in\mathbb{N}}$ are Cauchy sequences in the spaces $H^k([0,b_0]\times Y)$ and $H^k([0,a_0]\times Y)$ respectively, and thus in $L^2([0,b_0]\times Y)$ and $L^2([0,a_0]\times Y)$. Therefore, without loss of generality they can be replaced by subsequences, still denoted as $(\overline{\varphi}_i)_{i\in\mathbb{N}}$ and $(\overline{\psi}_i)_{i\in\mathbb{N}}$, such that

$$C_2\|\delta\overline{\varphi}_i\|_{L^2([0,b_0]\times Y)}\le\frac{1}{2^{i+1}}\quad\text{and}\quad C_2\|\delta\overline{\psi}_i\|_{L^2([0,a_0]\times Y)}\le\frac{1}{2^{i+1}}.\tag{3.80}$$

Assuming that (3.80) holds, we have

$$\begin{aligned}\|e^{-\lambda(u+v)}&\delta\varphi_{i+1}(u)\|_{L^2([0,b_0]\times Y)}+\|e^{-\lambda(u+v)}\delta\psi_{i+1}(v)\|_{L^2([0,a_*]\times Y)}\\&\le\frac{1}{2^i}+C_2\big(\|(A^\mu\nabla_\mu)_i\|_{L^\infty}+C_1-c\lambda\big)\|e^{-\lambda(u+v)}\delta f_{i+1}\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}\\&\quad+C_1C_2\|e^{-\lambda(u+v)}\delta f_i\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}.\end{aligned}\tag{3.81}$$

In particular, given any $0<\alpha<1/2$, for all λ sufficiently large and for all (u,v) in $[0,a_*]\times[0,b_0]$, we find that

$$\|e^{-\lambda(u+v)}\delta\varphi_{i+1}(u)\|^2_{L^2([0,b_0]\times Y)}\le\frac{1}{2^i}+C_1C_2\|e^{-\lambda(u+v)}\delta f_i\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)},\tag{3.82a}$$

$$\|e^{-\lambda(u+v)}\delta\psi_{i+1}(v)\|^2_{L^2([0,a_*]\times Y)}\le\frac{1}{2^i}+C_1C_2\|e^{-\lambda(u+v)}\delta f_i\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)},\tag{3.82b}$$

$$\|e^{-\lambda(u+v)}\delta f_{i+1}\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}\le\frac{1}{C_2\cdot 2^i}+\alpha\|e^{-\lambda(u+v)}\delta f_i\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}.\tag{3.82c}$$

Here λ has to be chosen so that

$$0<\frac{C_1}{c\lambda-\|(A^\mu\nabla_\mu)_i\|_{L^\infty}-C_1}<\alpha<\frac{1}{2}.\tag{3.83}$$

We can now make use of the elementary fact: If $(U_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers satisfying $U_{n+1}\le\alpha U_n+\beta/2^n$, then

$$U_n\le\alpha^nU_0+2\beta\bigg(\frac{(1/2)^n-\alpha^n}{1-2\alpha}\bigg).\tag{3.84}$$

Equations (3.82c)–(3.84) show that

$$\sum e^{-\lambda(u+v)}\delta f_i\ \text{converges in}\ L^2([0,a_*]\times[0,b_0]\times Y).$$

This implies that f_i converges in the same space to some function f. It further follows from (3.82a) that for all $0\le u\le a_*$ the sum $\sum_i e^{-\lambda(u+v)}\delta\varphi_i(u)$ converges in $L^2([0,b_0]\times Y)$, uniformly in u; this implies uniform convergence of $\varphi_i(u)$ to some function $\varphi(u)$ in that topology. Similarly for all $v\in[0,b_0]$ the sequence $\psi_i(v)$ converges, uniformly in v, to some function $\psi(v)$ in $L^2([0,a_*]\times Y)$.

For $k>(n+7)/2$ the estimates of the previous section apply and show that the sequence of derivatives ∇f_i is uniformly bounded so that, by Arzelà–Ascoli, a subsequence f_{i_j} can be chosen which converges uniformly to some function which is Lipschitz continuous in all variables on $[0,a_*]\times[0,b_0]\times Y$. It follows that f has a Lipschitz continuous representative; this representative will be chosen from now on. Similarly, f_{i_j+1} has a subsequence, still denoted by the same symbol, uniformly converging to some Lipschitz continuous function f'. Since f_{i_j+1} converges to f in L^2 we must have $f'=f$, thus f_{i_j+1} converges uniformly to f.

Now, by Proposition 3.6 the sequence $f_{i_j}(u,v)$ is bounded in $H^{k-1}(Y)$, and converges uniformly to the continuous function $f(u,v)$. By weak compactness

$$f(u,v) \equiv \big(\varphi(u,v),\psi(u,v)\big) \equiv \big(\varphi(u,v,\cdot),\psi(u,v,\cdot)\big) \in H^{k-1}(Y).$$

By interpolation, for every $s < k-1$ we have

$$f_{i_j}(u,v), f_{i_j+1}(u,v) \to f(u,v) \quad \text{ in } H^s(Y), \tag{3.85}$$

uniformly in u and v. In particular

$$f_{i_j}(u,v), f_{i_j+1}(u,v) \to f(u,v) \quad \text{ in } C^1(Y), \tag{3.86}$$

uniformly in u and v. Thus both φ and ψ are differentiable with respect to the x^A's.

In the notation of Section 2, (3.1) now shows that the sequence $\partial_u\varphi_{i_j+1}(u,v)$ converges uniformly to the Lipschitz continuous function

$$(*) := (A^u_{\varphi\varphi})^{-1}\big[-A^B_{\varphi\varphi}\nabla_B\varphi - A^B_{\varphi\psi}\nabla_B\psi + G_\varphi\big] - \gamma_{\varphi\varphi,u}\varphi.$$

Similarly, $\partial_v\psi_{i_j+1}(u,v)$ converges uniformly to a Lipschitz continuous function, as determined by the right-hand side of the equation involving $\partial_v\psi$. From

$$\underbrace{\varphi_{i_j+1}(u_2,\cdot)}_{\to\varphi(u_2,\cdot)} - \underbrace{\varphi_{i_j+1}(u_1,\cdot)}_{\to\varphi(u_1,\cdot)} = \int_{u_1}^{u_2} \underbrace{\partial_u\varphi_{i_j+1}(s,\cdot)}_{\to(*)}\, ds \tag{3.87}$$

one finds that φ is differentiable in u. Similarly ψ is differentiable in v, and (2.1) holds.

From what has been said we have

$$f \in L^\infty\big([0,a_*]\times[0,b_0]; H^{k-1}(Y)\big), \tag{3.88}$$

$$\partial_A f, \partial_u\varphi, \partial_v\psi \in L^\infty\big([0,a_*]\times[0,b_0]; H^{k-2}(Y)\big), \tag{3.89}$$

$$\partial_v\varphi, \partial_u\psi \in L^\infty\big([0,a_*]\times[0,b_0]; H^{k-3}(Y)\big). \tag{3.90}$$

Thus

$$f \in \bigcap_{0\le i\le 1} W^{i,\infty}\big([0,a_*]\times[0,b_0]; H^{k-2-i}(Y)\big) \subset C^{0,1}([0,a_*]\times[0,b_0]\times Y). \tag{3.91}$$

We note that the new field

$$f' = \begin{pmatrix}\varphi'\\ \psi'\end{pmatrix}, \quad \text{where} \quad \varphi' = \begin{pmatrix}\varphi\\ \partial_v\varphi\\ \partial_u\varphi\\ \partial_A\varphi\end{pmatrix} \quad \text{and} \quad \psi' = \begin{pmatrix}\psi\\ \partial_v\psi\\ \partial_u\psi\\ \partial_A\psi\end{pmatrix}, \tag{3.92}$$

is defined on $[0,a_*]\times[0,b_0]\times Y$ and solves a system of equations satisfying our structure conditions. By what has been said the initial data are of H^{k-3} differentiability class. So if $k-3 > (n+7)/2$, the argument leading to (3.91) applies to f' and gives

$$\begin{aligned} f \in\ & L^\infty\big([0,a_*]\times[0,b_0]; H^{k-1}(Y)\big) \cap \bigcap_{0<i\le 2} W^{i,\infty}\big([0,a_*]\times[0,b_0]; H^{k-3i}(Y)\big) \\ & \subset C^{1,1}([0,a_*]\times[0,b_0]\times Y). \end{aligned} \tag{3.93}$$

This argument can be applied k_1 times, where

$$k_1 \text{ is the largest number such that } k - 3k_1 > (n+7)/2. \tag{3.94}$$

Consequently,

$$\begin{aligned} f &\in L^\infty\big([0,a_*]\times[0,b_0];H^{k-1}(Y)\big)\cap \bigcap_{0<3i\le k-(n+7)/2} W^{i,\infty}\big([0,a_*]\times[0,b_0];H^{k-3i}(Y)\big)\\ &\subset C^{k_1-1,1}([0,a_*]\times[0,b_0]\times Y), \end{aligned} \tag{3.95}$$

where the last inclusion holds provided that $k_1 \ge 1$.

REMARK 3.8. For $k > 6+(n+7)/2$ the first line of (3.95) can be partly improved to

$$f\in C\big([0,a_*]\times[0,b_0];H^{k-1}(Y)\big)\cap \bigcap_{0<3i\le k-(n+7)/2-6} C^i\big([0,a_*]\times[0,b_0];H^{k-3i}(Y)\big). \tag{3.96}$$

To see this, note first that the map

$$(u,v)\mapsto \partial_u^i\partial_v^j f(u,v,\cdot)\in H^{k-3(i+j)}(Y) \tag{3.97}$$

is weakly continuous, being the limit of a bounded sequence of continuous maps. Using the equation satisfied by f and the trivial identities

$$\begin{aligned} \partial_u^i\partial_v^j\varphi(u,v) &= \partial_u^i\partial_v^j\varphi(0,v)+\int_0^u\left(\partial_u\partial_u^i\partial_v^j\varphi(s,0)+\int_0^v \partial_u\partial_v\partial_u^i\partial_v^j\varphi(s,t)\,dt\right)ds,\\ \partial_u^i\partial_v^j\psi(u,v) &= \partial_u^i\partial_v^j\psi(u,0)+\int_0^v\left(\partial_v\partial_u^i\partial_v^j\psi(0,t)+\int_0^u \partial_u\partial_v\partial_u^i\partial_v^j\psi(s,t)\,ds\right)dt, \end{aligned}$$

one sees that the function

$$(u,v)\mapsto \|\partial_u^i\partial_v^j f(u,v,\cdot)\|_{H^{k-3(i+j)}(Y)}$$

is continuous. This, together with standard arguments, implies that (3.97) is continuous, and (3.96) easily follows.

3.4. Existence and uniqueness. In order to complete the proof of the existence of a solution for the system (2.1), we need to initialize the iteration and make sure that condition (3.76) is fulfilled. Recall that in the current setting

$$\mathcal{N}^-=\{0\}\times[0,b_0]\times Y,\qquad \mathcal{N}^+=[0,a_0]\times\{0\}\times Y.$$

We have the following:

THEOREM 3.9. *Let Y be an $(n-1)$-dimensional compact manifold without boundary, let a_0 and b_0 two positive real numbers and set*

$$\Omega_0=[0,a_0]\times[0,b_0]\times Y.$$

Consider the symmetric hyperbolic system (2.1) *on Ω_0 with the splitting* (2.7) *and assume that* (2.8) *holds. Let $\overline{\varphi}$ and $\overline{\psi}$ be defined respectively on $\mathcal{N}^-$ and $\mathcal{N}^+$, providing Cauchy data for* (2.1)*:*

$$\begin{cases}\varphi=\overline{\varphi} & \text{on } \mathcal{N}^-,\\ \psi=\overline{\psi} & \text{on } \mathcal{N}^+.\end{cases} \tag{3.98}$$

Let $\ell\in\mathbb{N}$, $\ell>(n+9)/2$, and suppose that

$$\overline{\varphi}\in\bigcap_{0\le j\le \ell} C^j([0,b_0];H^{\ell-j}(Y)) \quad\text{and}\quad \overline{\psi}\in\bigcap_{0\le j\le \ell} C^j([0,a_0];H^{\ell-j}(Y)). \tag{3.99}$$

Assume that the transport equations

$$A^\mu_{\varphi\varphi}|_{v=0}\partial_\mu\varphi|_{v=0} = (-A^\mu_{\varphi\psi}\partial_\mu\psi + G_\varphi)|_{v=0}, \tag{3.100}$$

$$A^\mu_{\psi\psi}|_{u=0}\partial_\mu\psi|_{u=0} = (-A^\mu_{\psi\varphi}\partial_\mu\varphi + G_\psi)|_{u=0}, \tag{3.101}$$

with initial data

$$\varphi|_{u=v=0} = \overline{\varphi}|_{v=0} \quad \textit{and} \quad \psi|_{u=v=0} = \overline{\psi}|_{u=0},$$

have a global solution on $([0,a_0]\times Y)\cup([0,b_0]\times Y)$. *Then there exists an* ℓ*-independent constant* $a_* \in (0,a_0]$ *such that the Cauchy problem* (2.1), (3.98) *has a solution* f *defined on* $[0,a_*]\times[0,b_0]\times Y$ *satisfying* (3.88)–(3.90) *with* $k=\ell-1$. *If* $\ell > (n+12)/2$ *we further have*

$$\begin{aligned} f &\in L^\infty\big([0,a_*]\times[0,b_0];H^{\ell-2}(Y)\big)\cap\bigcap_{0<3i\le\ell-(n+9)/2} W^{i,\infty}\big([0,a_*]\times[0,b_0];H^{\ell-1-3i}(Y)\big)\\ &\subset C^{\ell_1-1,1}([0,a_*]\times[0,b_0]\times Y), \end{aligned}\tag{3.102}$$

where ℓ_1 *is the largest number such that* $\ell-3\ell_1 > (n+9)/2$. *The solution* f *is unique within the class of* C^1 *solutions, and is smooth if* $\overline{\varphi}$ *and* $\overline{\psi}$ *are.*

REMARK 3.10. Some remarks about the hypothesis that Y is *compact without boundary* are in order. First, our analysis applies to compact manifolds with boundary without further due when suitable boundary conditions are imposed on the boundary. For instance, in the case of systems obtained by rewriting the wave equation as in Section 4, Dirichlet, Neumann or maximally dissipative boundary conditions at ∂Y are suitable. Next, again for systems of wave equations, the case of noncompact Y's can be reduced to the compact one as follows: Let $p\in Y$; we replace Y by a small conditionally compact neighborhood of p with smooth boundary. We solve the equation on the new Y imposing e.g. Dirichlet conditions on $[0,a_0]\times[0,b_0]\times\partial Y$. Arguments based on uniqueness in domains of dependence show that there is a one-sided space-time neighborhood of the generators of $\mathcal{N}_\pm$ through p on which the solution is independent of the boundary conditions imposed. This provides the desired solution on the neighborhood. Returning to the original Y, the union of such neighborhoods with the corresponding solutions yields the desired solution.

Proof of Theorem 3.9. Let $(\overline{\varphi}_i)_{i\in\mathbb{N}}$ and $(\overline{\psi}_i)_{i\in\mathbb{N}}$ be any two sequences of smooth initial data which converge towards $\overline{\varphi}$ and $\overline{\psi}$ respectively in the spaces

$$\bigcap_{0\le j\le\ell} C^j([0,b_0];H^{\ell-j}(Y)) \quad\text{and}\quad \bigcap_{0\le j\le\ell} C^j([0,a_0];H^{\ell-j}(Y)).$$

Set $f_{-1}\equiv 0$, and for $i\in\mathbb{N}$ define $\overline{f}_i = (\overline{\varphi}_i,\overline{\psi}_i)$. Given f_i, we let f_{i+1} be the solution of the linear system (3.1) with Cauchy data

$$\begin{cases}\varphi_{i+1} = \overline{\varphi}_{i+1} & \text{on } \mathcal{N}^-,\\ \psi_{i+1} = \overline{\psi}_{i+1} & \text{on } \mathcal{N}^+.\end{cases}$$

We wish to apply Proposition 3.6 with $k=\ell-1$. For this we need to show that the constant $\mathcal{C}$ of (3.76) is finite. We start by noting that the sequence $(\overline{\psi}_i)_{i\in\mathbb{N}}$ has been chosen to converge in the space $\bigcap_{0\le j\le 2} C^j([0,a_0];H^{\ell-j}(Y))$, and since $\ell>(n+7)/2$ the continuous embedding

$$\bigcap_{0\le j\le 2} C^j([0,a_0];H^{\ell-j}(Y)) \hookrightarrow \bigcap_{0\le j\le 2} C^j([0,a_0];W^{1,\infty}(Y))$$

ensures that this convergence also holds in $\bigcap_{0\le j\le 2} C^j([0,a_0];W^{1,\infty}(Y))$. Since convergent sequences are bounded, we obtain

$$\sup_{i\in\mathbb{N},\, u\in[0,a_0]} \big(\|\overline{\psi}_i(u)\|_{W^{1,\infty}(Y)} + \|\partial_u\overline{\psi}_i(u)\|_{W^{1,\infty}(Y)} + \|\partial_u^2\overline{\psi}_i(u)\|_{W^{1,\infty}(Y)}\big) < \infty. \tag{3.103}$$

Similarly,

$$\sup_{i\in\mathbb{N},\, v\in[0,b_0]} \big(\|\overline{\varphi}_i(v)\|_{W^{1,\infty}(Y)} + \|\partial_v\overline{\varphi}_i(v)\|_{W^{1,\infty}(Y)} + \|\partial_v^2\overline{\varphi}_i(v)\|_{W^{1,\infty}(Y)}\big) < \infty. \tag{3.104}$$

By hypothesis, the transport equations with the initial data $(\overline{\varphi},\overline{\psi})$ have global solutions on $\mathcal{N}^\pm$. Continuous dependence of solutions of symmetric hyperbolic systems upon data implies that the transport equations with $(\overline{\varphi}_i,\overline{\psi}_i)$ will also have global solutions on $\mathcal{N}^\pm$ for all i large enough, bounded in $C^1(\mathcal{N})$ uniformly in i. We can thus use (3.23) at $u=0$ to obtain, for all $i\in\mathbb{N}$ and all λ sufficiently large,

$$\begin{aligned} e^{-\lambda v}\|\psi_i(0,v)\|^2_{H^{\ell-1}(Y)} \le C_7(Y,\ell,C_0,C_{\mathrm{div}})\Big\{ &\|\psi_i(0,0)\|^2_{H^{\ell-1}(Y)} \\ &+ 2\hat{C}(0,b) + \int_0^v e^{-\lambda s} M_\ell(0,s)\,ds + 2\delta\tilde{C}_\psi(0,b_0) + E_{\ell,\lambda}[\overline{\varphi}_i,0]\Big\}. \end{aligned}$$

The right-hand side is bounded uniformly in i and $v\in[0,b_0]$. Thus there exists a constant, which we denote again by C, such that

$$\forall i\in\mathbb{N}\ \forall v\in[0,b_0], \quad \|\psi_i(0,v)\|^2_{H^{\ell-1}(Y)} \le C.$$

We can repeat this process using the transport equation satisfied by $\partial_u\psi_{i+1}(0,v)$, which is obtained by u-differentiating the equation satisfied by ψ_{i+1}. This leads to the inequality (3.42) at $u=0$ for every $i\in\mathbb{N}$ with $k-3$ replaced by $\ell-2$; the gain of one derivative here, as compared to (3.42), is due to the fact that $\varphi|_{u=0}$ is directly given in terms of initial data, and hence is controlled in $H^\ell(Y)$, while in (3.42) we only had uniform control in $H^{k-1}(Y)$. That is, for all $i\in\mathbb{N}$,

$$\begin{aligned} &\|\partial_u\psi_i(0,v)\|^2_{H^{\ell-2}(Y)} \\ &\quad \le 2e^{\lambda v}\Big[C_9\Big\{\sup_{i\in\mathbb{N}}\sup_{u\in[0,a_0]} \|\partial_u\overline{\psi}_i(0)\|^2_{H^{\ell-2}(Y)} + \int_0^{b_0} e^{-\lambda s}\big(\hat{M}_\ell(0,s)+C_{10}^2\big)\,ds\Big\} + 1\Big]. \end{aligned}$$

An identical argument using (3.70) gives the desired control of $\varphi(u,0)$ and $\partial_v\varphi(u,0)$. This proves that the left-hand side of (3.76) is finite.

We can now appeal to Section 3.3 to conclude that the sequence $(f_i)_{i\in\mathbb{N}}$ converges towards a solution f of the Cauchy problem (2.1), (3.98) in a space as stated in the theorem. This requires choosing the sequences $(\varphi_i)_{i\in\mathbb{N}}$ and $(\psi_i)_{i\in\mathbb{N}}$ more carefully (see (3.80)), which is possible because the constant C_2 appearing in (3.80) is the same for all suitably bounded sequences, possibly after taking $i\ge i_0$ for some i_0 large enough.

Note that the neighborhood of $\mathcal{N}^-$ on which the solution has been constructed is independent of the Sobolev differentiability class of the data. This implies that smooth initial data lead to smooth solutions.

We continue with uniqueness of solutions. Let f_ℓ, $\ell = 1,2$, be two solutions of (2.1) with identical initial data (3.98). Setting $\delta f = f_1 - f_2$ leads to the equation

$$(A^\mu \nabla_\mu)_1 \delta f = (G)_1 - (G)_2 - \big((A^\mu \nabla_\mu)_1 - (A^\mu \nabla_\mu)_2\big) f_2, \tag{3.105}$$

with δf vanishing on $\mathcal{N}$. The calculation is now similar to that of Section 3.3. Equation (3.105) can be rewritten as (3.78) with δf_{i+1}, and δf_i there replaced by δf, $A^\mu(f_i)\nabla_\mu(f_i)$ replaced by $A^\mu(f_1)\nabla_\mu(f_1)$, and δG_i replaced by

$$\delta G := (G)_1 - (G)_2 - \big((A^\mu \nabla_\mu)_1 - (A^\mu \nabla_\mu)_2\big) f_2.$$

The current equivalent of (3.79) with $\overline{\delta\varphi} = \overline{\delta\psi} \equiv 0$ reads

$$\begin{aligned}
\|e^{-\lambda(u+v)}\delta\varphi(u)\|_{L^2([0,b_0]\times Y)} &+ \|e^{-\lambda(u+b)}\delta\psi(b)\|_{L^2([0,a_*]\times Y)} \\
&\le (\|(A^\mu(f_1)\nabla_\mu(f_1))\|_{L^\infty} - c\lambda)\|e^{-\lambda(u+v)}\delta f\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)} \\
&\quad + C_1\|e^{-\lambda(u+v)}\delta f\|^2_{L^2([0,a_*]\times[0,b_0]\times Y)}.
\end{aligned} \tag{3.106}$$

It then follows (compare with (3.82c)) that there exists $\alpha \in (0,1)$ such that

$$\|e^{-\lambda(u+v)}\delta f\|^2_{L^2([0,a]\times[0,b_0]\times Y)} \le \alpha\|e^{-\lambda(u+v)}\delta f\|^2_{L^2([0,a]\times[0,b_0]\times Y)}.$$

This means that $f_1 = f_2$ almost everywhere on $[0,a]\times[0,b_0]\times Y$, and since f_1 and f_2 are continuous, equality holds everywhere. ■

The symmetry of the problem under the interchange of u and v shows that our construction also provides a solution in a neighborhood of $\mathcal{N}^+$:

Corollary 3.11. *Under the hypotheses of Theorem* 3.9, *there exist constants* $0 < a_* \le a_0$ *and* $0 < b_* \le b_0$ *and a unique solution f of the Cauchy problem* (2.1), (3.98) *defined on the neighborhood*

$$\big([0,a_*]\times[0,b_0]\times Y\big) \cup \big([0,a_0]\times[0,b_*]\times Y\big)$$

of $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ *such that*

$$f \in L^\infty\big([0,a_*]\times[0,b_0]; H^{\ell-2}(Y)\big) \cap \bigcap_{0<3i\le \ell-(n+9)/2} W^{i,\infty}\big([0,a_*]\times[0,b_0]; H^{\ell-1-3i}(Y)\big),$$

and similarly on $[0,a_0]\times[0,b_*]$.

Remark 3.12. Theorem 3.9 can be used to obtain a solution of (2.1), (3.98) when the transport equations can be solved globally on the hypersurfaces $\widehat{\mathcal{N}}^- = \{0\}\times[0,\infty)\times Y$ and $\widehat{\mathcal{N}}^+ = [0,\infty)\times\{0\}\times Y$ as follows: Let a_0 and b_0 be arbitrary positive real numbers. Corollary 3.11 shows that there exist constants $0 < a_* \le a_0$ and $0 < b_* \le b_0$ and a unique continuous solution f of the Cauchy problem (2.1), (3.98) defined on

$$\mathcal{U}_{a_0,b_0} := \big([0,a_*]\times[0,b_0]\times Y\big) \cup \big([0,a_0]\times[0,b_*]\times Y\big).$$

Here a_* and b_* might depend upon a_0 and b_0. Uniqueness of solutions on each $\mathcal{U}_{a_0,b_0}$ shows that solutions defined on two such overlapping regions coincide on the overlap.

This allows one to define a solution on

$$\mathcal{U} = \bigcup_{a_0,b_0\in\mathbb{R}_+} \mathcal{U}_{a_0,b_0}$$

in an obvious way. We thus obtain a neighborhood of the entire initial data hypersurface $\widehat{\mathcal{N}} = \widehat{\mathcal{N}}^- \cup \widehat{\mathcal{N}}^+$. Note that the thickness of the neighborhood might shrink to zero when receding to infinity along $\widehat{\mathcal{N}}$.

3.5. Continuous dependence upon data. The aim of this section is to prove that the solutions obtained in Theorem 3.9 are stable under small perturbations of the Cauchy data. More precisely:

THEOREM 3.13. *Let f be a solution of* (2.1) *on* $[0,a_0]\times[0,b_0]\times Y$, *and let* $(f_i)_{i\in\mathbb{N}}$ *be a sequence of solutions on* $[0,a_0]\times[0,b_0]\times Y$ *such that the sequence of the associated initial data* $(\overline{f}_i)_{i\in\mathbb{N}}$ *converges to* $\overline{f}$ *in the topology determined by* (3.99) *with* $\ell \geq (n+15)/2$. *Then:*

(1) *There exists* $0 < a_* \leq a_0$ *such that*

$$\text{the sequence } f_i \text{ is bounded in } C^{1,1}([0,a_*]\times[0,b_0]\times Y).$$

(2) *Suppose that* $0 < a \leq a_0$ *is such that* $(f_i)_{i\in\mathbb{N}}$ *is bounded in* $C^{1,1}([0,a]\times[0,b]\times Y)$. *Then for any* $0 < s < \ell-(n+9)/2$ *the sequence* $(f_i)_{i\in\mathbb{N}}$ *converges to* f *in the topology of*

$$L^\infty\big([0,a_*]\times[0,b_0];H^{\ell-2}(Y)\big)\cap\bigcap_{0<3i\leq s} W^{i,\infty}\big([0,a_*]\times[0,b_0];H^{\ell-1-3i}(Y)\big). \tag{3.107}$$

REMARK 3.14. The sequence $(f_i)_{i\in\mathbb{N}}$ in (2) converges also in $C^1([0,a]\times[0,b_0]\times Y)$.

Proof of Theorem 3.13. Let us denote by $\|\overline{f}\|_\ell$ the norm associated to (3.99), and by $|||f|||_s$ the norm in the space (3.107). Let $(\overline{f}_{i,j})_{j\in\mathbb{N}}$ be a sequence of smooth initial data such that

$$\|\overline{f}_{i,j}-\overline{f}_i\|_\ell \leq 1/2^j.$$

Let $f_{i,j}$ be the (smooth) solution of (2.1) with initial data $\overline{f}_{i,j}$. By the estimates of Section 3.2 for all i,j large enough we can find $0\leq a_*\leq a_0$ such that all the $f_{i,j}$'s are defined on a common set $[0,a_*]\times[0,b_0]\times Y$, with a common bound in $C^{1,1}([0,a_*]\times[0,b_0]\times Y)$.

By Arzelà–Ascoli, when j tends to infinity the $f_{i,j}$'s converge to a solution of (2.1), say g_i, with initial data $\overline{f}_i$. By uniqueness $g_i = f_i$. This proves point (1).

Since $\overline{f}_{i,j}$ converges to $\overline{f}_i$ and $\overline{f}_i$ converges to $\overline{f}$, there exists a sequence $\overline{f}_{i,j(i)}$ which converges to $\overline{f}$ as i tends to infinity. By the argument just given, the associated solutions $f_{i,j(i)}$ of (2.1) converge, as i tends to infinity, to a solution g of (2.1). By uniqueness, $g=f$. Hence the $f_{i,j(i)}$'s converge to f.

Thus, for every $\epsilon>0$ there exists i_ϵ such that for $i\geq i_\epsilon$ and $j\geq j_0(i)$ we have

$$|||f_{i,j}-f|||_s \leq \tfrac{1}{2}\epsilon.$$

But for j large enough $|||f_{i,j}-f_i|||_s \leq \frac{1}{2}\epsilon$, which implies the claim. ∎

3.6. A continuation criterion. What has been said so far easily leads to the following *continuation criterion* for solutions with *smooth* initial data:

THEOREM 3.15. *Suppose that* (φ, ψ) *is a* C^1 *solution on* $[0, a) \times [0, b_0] \times Y$ *of the equations considered so far, for some* $a < a_0$*, with smooth initial data on* $\mathcal{N}$*. If* (φ, ψ) *is bounded in the* C^1 *norm on* $[0, a] \times [0, b_0] \times Y$*, then there exists* $\epsilon > 0$ *such that the solution can be extended to a smooth solution defined on* $[0, a + \epsilon] \times [0, b_0] \times Y$*.*

Indeed, for smooth data, if an a priori control of the C^1 norm of the fields is known, for any k one obtains the estimate for the kth order energy directly from (2.12), (2.31) and Gronwall's inequality, with no need to introduce the iterative scheme of Section 3. We emphasize that in the current case the constant $\hat{C}_1$ of equation (2.30) is controlled directly.

One would like to have a similar continuation criterion for solutions of finite differentiability class. However, due to the losses of differentiability occurring in our argument it is not clear whether such a result can be established. We have not attempted to investigate this issue any further.

4. Application to semilinear wave equations

4.1. Double-null coordinate systems. Let $(\mathcal{M}, g)$ be a smooth $(n+1)$-dimensional space-time, and let $\widehat{\mathcal{N}}^{\pm}$ be two null hypersurfaces in $\mathcal{M}$ emanating from a spacelike manifold Y of codimension two. We will denote by $\mathcal{N}^{\pm}$ the intersection of $\widehat{\mathcal{N}}^{\pm}$ with the causal future of Y.

In order to apply our results above to semilinear wave equations with initial data on $\mathcal{N}^{\pm}$ we need to construct local coordinate systems (u, v, x^A), where the x^A's are local coordinates on Y, near

$$\mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$$

so that

$$\mathcal{N}^- := \{u = 0\}, \quad \mathcal{N}^+ := \{v = 0\}. \tag{4.1}$$

We will further need

$$g(\nabla u, \nabla u) = 0 = g(\nabla v, \nabla v), \tag{4.2}$$

wherever defined. Such coordinates can be constructed in a standard way, but we give the details as specific parameterizations will be needed in the problem at hand.

Let ℓ_Y and ω_Y be any smooth null future pointing vector fields defined along Y and normal to Y such that ℓ_Y is tangent to $\mathcal{N}^+$ and ω_Y is tangent to $\mathcal{N}^-$. Then both $\widehat{\mathcal{N}}^+$ and $\mathcal{N}^+$ are threaded by the null geodesics issued from Y with initial tangent ℓ_Y at Y. These geodesics will be referred to as the *generators* of $\widehat{\mathcal{N}}^+$, respectively of $\mathcal{N}^+$. The associated field of tangents, normalized in any convenient way, will be denoted by ℓ^+. Let r_+ denote the corresponding parameter along the integral curves of ℓ^+, with $r_+ = 0$ at Y. We emphasize that the normalization of ℓ^+ is arbitrary at this stage, so that r^+ could e.g. be required to be affine, but we do *not* impose this condition. Similarly $\widehat{\mathcal{N}}^-$ and $\mathcal{N}^-$ are threaded by their null geodesic generators issued from Y, tangent to ω_Y at Y, with field of tangents ω^- and parameter r_-.

Let x^A_Y be any local coordinates on an open subset $\mathcal{O}$ of Y. They can be propagated to functions $x^A_\pm$ on $\mathcal{N}^{\pm}$ by requiring the $x^A_\pm$'s to be equal to x^A_Y along the corresponding null geodesic generators of $\mathcal{N}^{\pm}$. Then $(r_\pm, x^A_\pm)$ define local coordinates on $\mathcal{N}^{\pm}$ near each of the relevant generators.

On $\widehat{\mathcal{N}}^+$ we let ω^+ be any smooth field of null vectors transverse to $\widehat{\mathcal{N}}^+$ and normal to the level-sets of r_+ such that $\omega^+|_Y = \omega_Y$. The function u is defined by the requirement that u is constant along the null geodesics issued from $\widehat{\mathcal{N}}^+$ with initial tangent ω^+, equal to r_+ at $\widehat{\mathcal{N}}^+$. We denote by ω the field of tangents to those geodesics, normalized in any suitable way. Thus

$$\omega(u) = 0, \quad u|_{\mathcal{N}^-} = 0. \tag{4.3}$$

We claim that the level sets of u, say $\mathcal{N}_u^-$, are null hypersurfaces. To see this, consider a one-parameter family $\lambda \mapsto x(\lambda, s)$ of generators within $\mathcal{N}_u^-$. Then $X := \partial_\lambda x$ is tangent to $\mathcal{N}_u^-$ and solves the Jacobi equation along each of the generators $s \mapsto x(\lambda, s)$. Further, every vector tangent to $\mathcal{N}_u^-$ belongs to such a family of vectors. We have

$$\frac{d(g(X,\omega))}{ds} = g\left(\frac{DX}{ds},\omega\right) = g\left(\frac{D}{\partial s}\frac{\partial x}{\partial \lambda},\frac{\partial x}{\partial s}\right) = g\left(\frac{D}{\partial \lambda}\frac{\partial x}{\partial s},\frac{\partial x}{\partial s}\right) = \frac{1}{2}\partial_\lambda(g(\omega,\omega)) = 0. \tag{4.4}$$

Now, on $\mathcal{N}_u^- \cap \mathcal{N}^+$ the vector X can be decomposed as $X = X^\parallel + \alpha\omega$, where $X^\parallel$ is tangent to $\mathcal{N}_u^- \cap \mathcal{N}^+$ and $\alpha \in \mathbb{R}$. Both $X^\parallel$ and ω are orthogonal to ω, hence $g(X,\omega) = 0$ at the intersection. Equation (4.4) gives $g(X,\omega) \equiv 0$. This shows that all vectors tangent to $\mathcal{N}_u^-$ are orthogonal to ω, and since ω is also tangent to $\mathcal{N}_u^-$ we conclude that $T\mathcal{N}_u^-$ is null. Consequently, ∇u is proportional to the null vector ω, and thus

$$g(\nabla u, \nabla u) = 0.$$

Similarly, on $\mathcal{N}^-$ we let ℓ^- be any smooth field of null vectors transverse to $\mathcal{N}^-$ and normal to the level-sets of r_- such that $\ell^-|_Y = \ell_Y$. The function v is defined by the requirement that v is constant along the null geodesics issued from $\mathcal{N}^-$ with initial tangent ℓ^-, and with initial value r_- at $\mathcal{N}^-$. We denote by ℓ the field of tangents to those geodesics, normalized in any convenient way. Then

$$\ell(v) = 0, \qquad v|_{\mathcal{N}^+} = 0, \qquad g(\nabla v, \nabla v) = 0. \tag{4.5}$$

By construction we have

$$\ell|_{\mathcal{N}^\pm} = \ell^\pm, \qquad \omega|_{\mathcal{N}^\pm} = \omega^\pm. \tag{4.6}$$

So far the construction was completely symmetric; this symmetry will be broken now by defining the functions x^A through the requirement that the x^A's be constant along the null geodesics starting from $\mathcal{N}^-$ with initial tangent ℓ^-, and taking the values x_-^A at the intersection point.

The construction just given breaks down when the geodesics start intersecting. However, it always provides the desired coordinates in a neighborhood of $\mathcal{N}$. In particular, given two generators of $\mathcal{N}^\pm$ emanating from the same point on Y, there exists a neighborhood of those generators on which (u, v, x^A) form a coordinate system. We emphasize that

$$g(\omega,\omega) = g(\ell,\ell) = 0, \tag{4.7}$$

and that we also have

$$\ell^v = 0 = \ell^A \;\Leftrightarrow\; \ell = \ell^u\partial_u, \qquad \omega^u = 0 \;\Leftrightarrow\; \omega = \omega^v\partial_v + \omega^A\partial_A. \tag{4.8}$$

The first group of equations in (4.8) follows from the fact that both x^A and v are constant along the integral curves of ℓ, while the second is a consequence of the fact that u is constant along the integral curves of ω.

Finally, once the coordinates u and v have been constructed, for some purposes it might be convenient to rescale ℓ, or ω, or both, so that

$$g(\omega, \ell) = -1/2. \tag{4.9}$$

Such rescalings do not affect (4.7)–(4.8), which are the key properties of ℓ and ω for us. Equation (4.9) determines ℓ and ω up to one multiplicative strictly positive factor, $\ell \mapsto \alpha\ell$, $\omega \mapsto \alpha^{-1}\omega$.

4.1.1. $\mathbb{R}$-parameterizations. Let us finish this section by providing a construction in which the functions u and v run from zero to infinity on all generators of $\mathcal{N}^+$ and $\mathcal{N}^-$.

Let $\mathcal{U}^+ \subset \widehat{\mathcal{N}}^+ \times \mathbb{R}$ be the maximal domain of definition of the map, which we denote by

$$\Psi^+(p,s) : \mathcal{U}^+ \to \mathcal{M}, \qquad p \in \widehat{\mathcal{N}}^+,\ s \in \mathbb{R},$$

defined by following a null geodesic from $p \in \widehat{\mathcal{N}}^+$ with an affine parameter $s \in \mathbb{R}$ in the direction ℓ^+ at p. Let $\mathcal{V}^+ \subset \mathcal{U}^+$ be the domain of injectivity of Ψ^+. Then $\Psi^+(\mathcal{V}^+)$ is an open subset of $\mathcal{M}$ containing $\widehat{\mathcal{N}}^-$.

Let the set $\mathcal{U}^- \subset \widehat{\mathcal{N}}^- \times \mathbb{R}$, the map Ψ^-, and the set $\mathcal{V}^- \subset \mathcal{U}^-$ be the corresponding constructs on $\widehat{\mathcal{N}}^-$, using the integral curves of ω. Then $\Psi^-(\mathcal{V}^-)$ is an open subset of $\mathcal{M}$ containing $\mathcal{N}^+$.

Set

$$\mathcal{O} := \Psi^+(\mathcal{V}^+) \cap \Psi^-(\mathcal{V}^-) \supset \mathcal{N}^+ \cup \mathcal{N}^-.$$

Let h be any complete smooth Riemannian metric on $\mathcal{O}$. Rescale ℓ and ω to new vector fields on $\mathcal{O}$, still denoted by ℓ and ω, so that $h(\ell,\ell) = 1 = h(\omega,\omega)$. Then the integral curves of ℓ and ω are complete in $\mathcal{O}$. The corresponding parameters $r_\pm$ on $\widehat{\mathcal{N}}^\pm$ run over $\mathbb{R}$ for all generators of $\widehat{\mathcal{N}}^\pm$, as desired.

It should be pointed out that the above normalization of ℓ and ω has only been imposed for the sake of constructing u and v. Once we have the functions u and v on $\mathcal{O}$ we can revert to any other normalization of the fields ℓ and ω, in particular we can assume that (4.9) holds. It might then not be true anymore that $\ell(u) = 1$ on $\mathcal{N}^-$ and/or $\omega(v) = 1$ on $\mathcal{N}^+$, but these conditions are irrelevant for our purposes in this section. In fact condition (4.9) plays no essential role in what follows.

4.1.2. Regularity. Now, it is well known that coordinate systems obtained by shooting geodesics lead to a loss of differentiability of the metric. The aim of this section is to show that, in our context, the optical functions u, v are of the same differentiability class as the metric ([1]). As a result, after passing to a doubly-null coordinate system one loses one derivative of the metric. While unfortunate, this is not a serious problem for semilinear equations, as considered in this section. On the other hand, this leads to difficulties when attempting to apply our techniques to the harmonically reduced Einstein equations. This is why we will restrict ourselves to dimension four when analyzing the Einstein equations, as then a doubly-null formulation of Einstein equations is directly available, without having to pass to harmonic coordinates.

First, to avoid a conflict of notation, we will use the symbol x for the coordinate u of Section 3, and y for the coordinate v used there, *without assuming that x or y solve the eikonal equation.* Thus, we let (x, y, x^A) be any coordinate system such that

([1]) The argument here has been suggested to us by Hans Lindblad. We are grateful to Hans for useful discussions concerning this point.

$\mathcal{N}^- = \{x = 0\}$ and $\mathcal{N}^+ = \{y = 0\}$. We assume that these hypersurfaces are characteristic for the metric g. We have just seen how to construct solutions u and v to the eikonal equation, and we wish to analyze their differentiability properties.

To obtain the desired estimates, we start by differentiating the eikonal equation:

$$g^{\mu\nu}\partial_\mu v \partial_\nu v = 0 \;\Rightarrow\; g^{\mu\nu}\partial_\nu v \partial_\mu \partial_\alpha v = -\tfrac{1}{2}\partial_\alpha(g^{\mu\nu})\partial_\mu v\, \partial_\nu v. \tag{4.10}$$

Setting $f \equiv \varphi \equiv (\varphi_\alpha) := (\partial_\alpha v)$, we obtain a symmetric-hyperbolic evolution system

$$g^{\mu\nu}\varphi_\mu \partial_\nu \varphi_\alpha = -\tfrac{1}{2}\partial_\alpha(g^{\mu\nu})\varphi_\mu\varphi_\nu \;\Leftrightarrow\; A^\mu\partial_\mu\varphi = G, \tag{4.11}$$

with

$$A^\mu = (A^\mu{}_\alpha{}^\beta) = (-g^{\mu\nu}\varphi_\nu\delta^\beta_\alpha), \quad G = (G_\alpha) = \big(\tfrac{1}{2}\partial_\alpha(g^{\mu\nu})\varphi_\mu\varphi_\nu\big) \tag{4.12}$$

(the negative sign above is related to our convention $(- + \cdots +)$ for the signature of the metric, together with the requirement that ∇u and ∇v are both past pointing). The function v is required to vanish on $\mathcal{N}^+$.

An obvious corresponding equation can be derived for the second null coordinate u, which is required to vanish on $\mathcal{N}^-$.

We have:

Theorem 4.1. *Let $(\mathcal{M}, g)$ be a smooth space-time with a metric g with components*

$$g_{\mu\nu} \in \bigcap_{0\le j\le \ell} C^j([0,a_0]\times[0,b_0]; H^{\ell-j}(Y))$$

in the coordinate system above, with some $\ell \in \mathbb{N}$ satisfying $\ell > (n+6)/2$. Let $\overline{u}$, $\overline{v}$ be continuous functions on $\mathcal{N}$, with $\overline{u} \equiv 0$ on $\mathcal{N}^-$, differentiable on $\mathcal{N}^+$ and $\partial_x \overline{u}$ strictly positive there, and $\overline{v} \equiv 0$ on $\mathcal{N}^+$, differentiable on $\mathcal{N}^-$ and $\partial_y \overline{v}$ strictly positive there, with

$$\overline{u}|_{\mathcal{N}^+} \in \bigcap_{0\le j\le \ell} C^j([0,a_0]; H^{\ell-j}(Y)), \qquad \overline{v}|_{\mathcal{N}^-} \in \bigcap_{0\le j\le \ell} C^j([0,b_0]; H^{\ell-j}(Y)). \tag{4.13}$$

There exist ℓ-independent constants $\mu_* > 0$ and $a_* \in (0, a_0]$, with $b_0 - \mu_* a_* > 0$, such that the eikonal equations $g(\nabla u, \nabla u) = 0 = g(\nabla v, \nabla v)$ have unique solutions u and v, realising the initial data $\overline{u}$ and $\overline{v}$, defined on

$$\Omega_* := \{x \in [0, a_*],\, 0 \le y \le b_0 - \mu_* x\} \times Y \tag{4.14}$$

(see Figure 4.1), of differentiability class $C^3(\Omega_*)$, with ∇u and ∇v without zeros and linearly independent there, and satisfying

$$\begin{aligned} u \in\; & L^\infty([0, b_0 - \mu_* a_*]; H^\ell([0,a_*]\times Y)) \\ & \cap \bigcap_{1\le j\le \ell-1} C^j([0, b_0-\mu_* a_*]; H^{\ell-j}([0,a_*]\times Y)), \end{aligned} \tag{4.15}$$

$$\begin{aligned} v \in\; & L^\infty([0, a_*]; H^\ell([0,b_0-\mu_* a_*]\times Y)) \\ & \cap \bigcap_{1\le j\le \ell-1} C^j([0, a_*]; H^{\ell-j}([0,b_0-\mu_* a_*]\times Y)). \end{aligned} \tag{4.16}$$

The solutions u and v are smooth if the metric and the initial data $\overline{u}$ and $\overline{v}$ are.

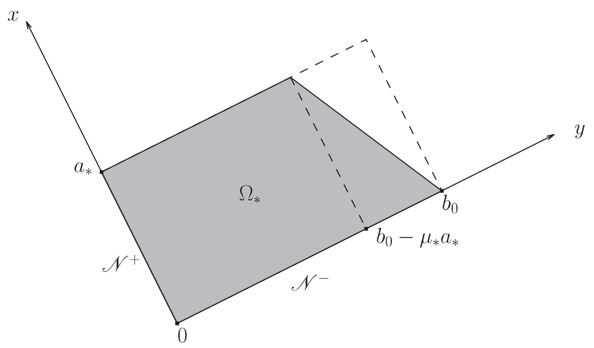

Fig. 4.1. The set Ω_*

REMARK 4.2. The constant μ_* is only needed for the function v, and can be set to zero if u only is considered. The functions u and v have differentiability properties similar to those in (4.15)–(4.16) on that part of Ω_* which is not covered by (4.15)–(4.16); we did not exhibit this because the result is somewhat cumbersome to write formally.

Proof of Theorem 4.1. Since the metric is C^2, existence follows from the arguments above, and we only need to justify the regularity properties. The result is established through a simplified version of the arguments from Section 3. Special care has to be taken in the proof to make sure that there are no unwanted contributions to the energy from some boundaries.

Let us start with the initial data for the function u. On $\mathscr{N}^+$ the inverse metric takes the form, for any function χ,

$$\overline{g(\nabla\chi,\nabla\chi)}|_{\mathscr{N}^+} = \overline{g}^{xx}(\partial_x\overline{\chi})^2 + 2\overline{g}^{xy}\partial_x\overline{\chi}\,\overline{\partial_y\chi} + 2\overline{g}^{xA}\partial_x\overline{\chi}\,\partial_A\overline{\chi} + \overline{g}^{AB}\partial_A\overline{\chi}\,\partial_A\overline{\chi} \tag{4.17}$$

(see e.g. [7, Appendix A]). Since neither $\overline{g}^{xy}$ nor $\partial_x\overline{u}$ has zeros, it follows from (4.17) that the equation

$$g(\nabla u,\nabla u)|_{\mathscr{N}^+} = 0$$

allows us to calculate

$$\overline{\partial_y u} \in \bigcap_{0\le j\le \ell-1} C^j([0,a_0];H^{\ell-j-1}(Y))$$

on $\mathscr{N}^+$ in terms of $\overline{g}$ and the tangential derivatives of $\overline{u}$, leading to

$$\overline{\psi}|_{\mathscr{N}^+} \equiv (\partial_\mu u)|_{\mathscr{N}^+} \in \bigcap_{0\le j\le \ell-1} C^j([0,a_0];H^{\ell-j-1}(Y)).$$

Next, we will need to control ∇u on $\mathscr{N}^-$. This proceeds as follows: On $\mathscr{N}^-$ we have, for any function $\overline{u}$,

$$\overline{g(\nabla u,\nabla u)} = \overline{g}^{yy}(\partial_y\overline{u})^2 + 2\overline{g}^{xy}\partial_y\overline{u}\,\overline{\partial_x u} + 2\overline{g}^{yA}\partial_y\overline{u}\,\partial_A\overline{u} + \overline{g}^{AB}\partial_A\overline{u}\,\partial_A\overline{u}. \tag{4.18}$$

Since $\overline{u}\equiv 0$ in our case, the equation $\overline{g(\nabla u,\nabla u)} = 0$ holds identically. We further have

$$\nabla u|_{\mathscr{N}^-} = \overline{g^{xy}\partial_x u}\partial_y, \tag{4.19}$$

and we need an equation for $\partial_x u|_{x=0}$. For this we can use the u-equivalent of (4.10),

$$g^{\mu\nu}\partial_\nu u\,\partial_\mu\partial_x u = -\tfrac{1}{2}\partial_x(g^{\mu\nu})\partial_\mu u\,\partial_\nu u, \tag{4.20}$$

which on $\mathcal{N}^-$ becomes

$$\overline{g}^{xy}\overline{\partial_x u}\,\partial_y(\overline{\partial_x u}) = -\tfrac{1}{2}\partial_x g^{xx}|_{\mathcal{N}^-}(\overline{\partial_x u})^2 \tag{4.21}$$

(note that g^{xx} vanishes on $\mathcal{N}^-$, but there is a priori no reason why $\partial_x g^{xx}|_{\mathcal{N}^-}$ should vanish as well). From this it is straightforward to obtain

$$\partial_x u|_{\mathcal{N}^-} \in \bigcap_{0\le j\le \ell-1} C^j([0,b_0];H^{\ell-j-1}(Y)). \tag{4.22}$$

Summarising:

$$\begin{aligned}\overline{\psi}|_{\mathcal{N}^+} \equiv (\partial_\mu u)|_{\mathcal{N}^+} &\in \bigcap_{0\le j\le \ell-1} C^j([0,a_0];H^{\ell-j-1}(Y)),\\ \overline{\psi}|_{\mathcal{N}^-} &\in \bigcap_{0\le j\le \ell-1} C^j([0,b_0];H^{\ell-j-1}(Y)).\end{aligned}$$

We continue with the energy inequality. Let $h = h_{\alpha\beta}dx^\alpha dx^\beta$ be any smooth Riemannian metric on $\Omega_{a_0,b_0}\times Y$. The L^2-energy-density vector associated with (4.11) can be defined as

$$E^\mu := h(\psi, A^\mu\psi) = -\psi^\mu h(\psi,\psi) = -h^{\alpha\beta}\partial_\alpha u\partial_\beta u\nabla^\mu u. \tag{4.23}$$

Similarly to Section 2, the energy inequality with $k=0$ is obtained by integrating the divergence of $e^{-\lambda y}E^\mu$ over a suitable set, say $\Omega_{a,b,\sigma}$, with $0\le a\le a_0$, $0\le b\le b_0$, where

$$\Omega_{a,b,\sigma} := \{0\le y\le b,\ 0\le \bar{x}\le a-\sigma y\}, \tag{4.24}$$

with $\bar{x}$ to be defined shortly, and where $0<\sigma<a_0/2b_0$ is a small constant which will also be determined shortly.

Indeed, for further purposes we will need to have good control of the causal character of the level sets of x. This is achieved by modifying x so that, after suitable redefinitions, $\partial_x g^{xx}|_{\mathcal{N}^-}=0$. For this, let us pass to a new coordinate system

$$\bar{x} = \chi(x,y,x^A)x,\ \bar{y}=y,\ \bar{x}^A = x^A \ \Rightarrow\ \partial_x = \partial_x(x\chi)\partial_{\bar{x}},$$

with a function χ which is determined as follows: We have

$$\begin{aligned}g^{\bar{x}\bar{x}} &= g^{\mu\nu}\left(x\frac{\partial\chi}{\partial x^\mu}+\frac{\partial x}{\partial x^\mu}\chi\right)\left(x\frac{\partial\chi}{\partial x^\nu}+\frac{\partial x}{\partial x^\nu}\chi\right)\\ &= x^2 g^{\mu\nu}\frac{\partial\chi}{\partial x^\mu}\frac{\partial\chi}{\partial x^\nu} + 2xg^{\mu x}\frac{\partial\chi}{\partial x^\mu}\chi + g^{xx}\chi^2,\end{aligned}$$

leading to

$$\partial_{\bar{x}} g^{\bar{x}\bar{x}}\big|_{x=0} = \frac{1}{\chi}\partial_x g^{\bar{x}\bar{x}}\big|_{x=0} = 2g^{yx}\big|_{\bar{x}=0}\partial_y\chi + \partial_x g^{xx}\big|_{x=0}\chi.$$

This will vanish if we set

$$\begin{aligned}\chi(0,y,x^A) &= \exp\left(-\frac{1}{2}\int_0^y \frac{\partial_x g^{xx}}{g^{xy}}\bigg|_{x=0}(s,x^A)\,dx\right)\times\frac{\partial\overline{u}}{\partial x}(0,0,x^A)\\ &\in \bigcap_{0\le j\le \ell-1} C^j([0,b_0];H^{\ell-j-1}(Y)).\end{aligned} \tag{4.25}$$

We let $\chi(x, y, x^A)$ be any extension of $\chi(0, y, x^A)$ which is smooth in all its arguments for $x > 0$; the existence of such extensions is standard. We pass to the new coordinate system, and change the notation $(\bar{x}, \bar{y}, \bar{x}^A)$ back to (x, y, x^A) for the new coordinates. The factor $\frac{\partial \overline{u}}{\partial x}(0, 0, x^A)$ in (4.25) has been chosen to obtain

$$\partial_x u(x = 0, y = 0, x^A) = 1. \tag{4.26}$$

In the new coordinates, from (4.21) we find

$$\partial_x u(x = 0, y, x^A) = 1. \tag{4.27}$$

Now, $\partial\Omega_{a,b,\sigma}$ takes the form

$$\begin{aligned}\partial\Omega_{a,b,\sigma} = \mathcal{N}^- \cup \mathcal{N}^+ \cup \underbrace{\big(\{y \in [0, b], x = a - \sigma y\} \times Y\big)}_{=:\mathrm{II}_\sigma} \\ \cup \underbrace{\big(\{y = b, 0 \le x \le a - \sigma b\} \times Y\big)}_{=:\mathrm{I}_\sigma}\end{aligned} \tag{4.28}$$

(see Figure 4.2). Before analyzing the boundary terms arising, recall that we wish to

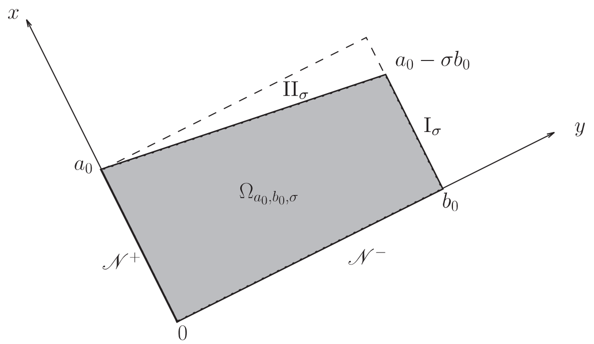

Fig. 4.2. The set $\Omega_{a_0,b_0,\sigma}$

obtain estimates on various norms of the field. This will be achieved by repeating the inductive scheme of Section 3, but now using Sobolev spaces associated with the level sets of y instead of $H^k(Y)$. For this we let $\overline{u}_i$ be a sequence of smooth functions on $\mathcal{N}^+$ converging to $\overline{u}$, with $\overline{u}_i = 0$ on $\mathcal{N}^-$, each u_i solving a linear equation as done in Section 3 with coefficients determined by u_{i-1}. Let c_1 and C_1 be any positive constants such that

$$\sup_{i\in\mathbb{N}} \sup_{\mathcal{N}} \big(|\partial u_i| + |\partial_x \partial u_i| + |\partial_x^2 \partial u_i|\big) \le C_1, \qquad \inf_{i\in\mathbb{N}} \inf_{\mathcal{N}} \partial_x u_i \ge c_1 > 0. \tag{4.29}$$

Note that a pair of such constants can be determined purely in terms of the initial data for u on $\mathcal{N}$.

To the definition of the sequence $0 < a_i$, given just before (3.10), we add the requirement that $a_i \le 1$, and that

$$\inf_{\Omega_{a_i,b_0,\sigma_i}} \partial_x u_i \ge \tfrac{1}{2} c_1, \tag{4.30}$$

$$\sup_{\Omega_{a_i,b_0,\sigma_i}} \big(|\partial u_i| + |\partial_x \partial u_i| + |\partial_x^2 \partial u_i|\big) \le C_1 + 1. \tag{4.31}$$

Consider the L^2-energy identity on $\Omega_{a_i,b_0,\sigma_i}$ associated with the equation satisfied by u_i,

$$\int_{\partial\Omega_{a_i,b_0,\sigma}} e^{-\lambda y} E^\mu n_\mu = \int_{\Omega_{a_i,b_0,\sigma}} \nabla_\mu (e^{-\lambda y} E^\mu), \tag{4.32}$$

with E^μ given by

$$E^\mu := h(\psi_i, A^\mu(\psi_{i-1})\psi_i) = -\psi^\mu_{i-1} h(\psi_i,\psi_i) = -h^{\alpha\beta}\partial_\alpha u_i\,\partial_\beta u_i\,\nabla^\mu u_{i-1}. \tag{4.33}$$

On $\mathcal{N}^- = \{x=0\}$ the conormal $n_\mu dx^\mu$ satisfies $n_y = n_A = 0$, so by (4.19) the boundary integrand vanishes:

$$h(\psi_i,\psi_i) n_\mu \nabla^\mu u_{i-1} = h(\psi_i,\psi_i) n_x \underbrace{\nabla^x u_{i-1}}_{=0} = 0 \quad \text{on } \mathcal{N}^-.$$

On $\mathcal{N}^+$ the conormal $n_\mu dx^\mu$ satisfies $n_x = n_A = 0$, so by (4.19) the boundary integrand satisfies

$$\begin{aligned} h(\psi_i,\psi_i)\nabla^\mu u_{i-1}\, n_\mu|_{\mathcal{N}^+} &= h(\psi_i,\psi_i)\nabla^y u_{i-1}\, n_y|_{\mathcal{N}^+} = h(\psi_i,\psi_i) g^{xy}\partial_x u_{i-1}\, n_y|_{\mathcal{N}^+} \\ &\sim h(\psi_i,\psi_i), \end{aligned}$$

where "$f \sim g$" means that the functions f and g are bounded by positive constant multiples of each other.

On II_σ the conormal $n = n_\mu dx^\mu$ is proportional to $dx + \sigma dy$. Differentiability of the metric implies that there exists a constant C_2 such that, for $x > 0$,

$$|g_{\mu\nu}(x,\cdot) - g_{\mu\nu}(0,\cdot)| \le C_2 x.$$

By definition of a_i, on $\Omega_{a_i,b_0,0}$ we have

$$|\partial_x^2 \partial u_i| \le 1 + C_1. \tag{4.34}$$

Since u_i vanishes on $\mathcal{N}^-$, so do $\partial_y u_i$ and $\partial_A u_i$. Further, from (4.27), $\partial_x\partial_y u_i$ and $\partial_x\partial_A u_i$ vanish on $\mathcal{N}^-$ as well and we obtain

$$|\partial_A u_i| + |\partial_y u_i| \le (1+C_1)x^2. \tag{4.35}$$

If we write the conormal n_μ to the level sets of II_{σ_i} as $n_\mu dx^\mu = n_x(dx + \sigma_i dy)$, the above gives, with $n_x \ge \delta > 0$ and $0 < \sigma_i < 1$, for any i,

$$\begin{aligned} g^{\mu\nu} n_\nu \partial_\mu u_i &= n_x (g^{\mu x} + \sigma_i g^{\mu y})\partial_\mu u_i \\ &= \underbrace{n_x}_{\ge\delta}\Big(\big(\underbrace{g^{xx}}_{O(x^2)\ge -CC_3x^2} + \sigma_i \underbrace{g^{xy}}_{\ge c}\big)\underbrace{\partial_x u_i}_{1+O(x)\ge 1-(1+C_1)x\ge 1/2} \\ &\qquad + \underbrace{(g^{yx} + \sigma_i g^{yy})\partial_y u_i + (g^{Ax} + \sigma_i g^{Ay})\partial_A u_i}_{\ge -C(1+C_1)x^2}\Big) \\ &\ge \frac{\delta}{2}\big(\underbrace{c\sigma_i - \underbrace{CC_3x^2}_{\le CC_3 a_i x}}_{\ge c\sigma_i/2} - \underbrace{2C(1+C_1)x^2}_{\le 2C(1+C_1)xa_i}\big) \ge 0 \end{aligned} \tag{4.36}$$

for

$$x \le \min\left(\frac{1}{2(1+C_1)}, \frac{c}{2CC_3}\times\frac{\sigma_i}{a_i}, \frac{c}{4C(1+C_1)}\times\frac{\sigma_i}{a_i}\right), \tag{4.37}$$

where C_1 is as in (4.34) and

$$C_3 = \sup |\partial_x^2 g^{\mu\nu}|. \tag{4.38}$$

Choosing

$$\sigma_i = \frac{a_i}{2b_0} \tag{4.39}$$

leads to an i-independent bound in (4.37).

The II_{σ_i} boundary integral now gives a contribution to the energy identity which we simply discard, replacing equality by an inequality.

Note that with this choice we have

$$\left[0, \tfrac{1}{2}a_i\right] \times [0, b_0] \times Y \subset \Omega_{a_i, b_0, \sigma_i} \subset [0, a_i] \times [0, b_0] \times Y. \tag{4.40}$$

On I_{σ_i} the conormal $n_\mu dx^\mu$ takes the form $n_y dy$, and from (4.18) the boundary integrand takes the form

$$e^{-\lambda b_0} h(\psi_i, \psi_i) \nabla^\mu u_{i-1}\, n_\mu|_{I_{\sigma_i}} = e^{-\lambda b_0} h(\psi_i, \psi_i)\big(g^{xy}\partial_x u_{i-1}\, n_y + O(x)\big)|_{y=b_0} \sim h(\psi_i, \psi_i).$$

As a result we obtain

$$\forall 0 \le b \le b_0 \quad \mathcal{E}_{0,\lambda}[\psi_i, b] \le C\mathcal{E}_{0,\lambda}[\psi_i, 0] + \int_{\Omega_{a_i,b,\sigma_i}} \nabla_\mu(e^{-\lambda y} E^\mu), \tag{4.41}$$

and similarly for higher-order energy inequalities, with

$$\begin{aligned} \mathcal{E}_{k,\lambda}[\psi_i, b] &= e^{-\lambda b} \sum_{0 \le j+\ell \le k} \int_{[0, a_i - \sigma_i b] \times Y} |\mathring{\nabla}_{q_{r_1}} \dots \mathring{\nabla}_{q_{r_j}} \partial_x^\ell \psi_i|^2 \, dx \, d\mu_Y \\ &= e^{-\lambda b} \sum_{0 \le \ell \le k} \int_0^{a_i - \sigma_i b} \|\partial_x^\ell \psi(x, b)\|^2_{H^{k-\ell}(Y)} \, dx. \end{aligned} \tag{4.42}$$

A simpler version of the arguments of Section 3 gives the result.

The estimates for v are essentially standard, as we only need to solve for a short-time in the evolving direction. Should one want to use an iterative argument as in Section 3, we note that given $\mu_* > 0$ as in the statement of the theorem we can impose an i-independent upper bound on the a_i's so that the boundary

$$\{x \in [0, a_*],\, 0 \le y \le b_0 - \mu_* x\} \times Y$$

gives a nonnegative contribution to the energy identity, and hence is harmless when considering energy estimates. ■

4.2. The wave equation in doubly-null coordinates. We are now ready to pass to the PDE problem. Let W be a vector bundle over $\mathcal{M}$. We will be seeking a section h of W, defined on a neighborhood of $\mathcal{N}^-$ and of differentiability class at least C^2 there, such that the following hold:

$$\Box_g h = H(h, \nabla h, \cdot) \quad \text{on } I^+(\mathcal{N}^+ \cup \mathcal{N}^-), \tag{4.43a}$$

$$h = h^+ \quad \text{on } \mathcal{N}^+, \tag{4.43b}$$

$$h = h^- \quad \text{on } \mathcal{N}^-, \tag{4.43c}$$

with the prescribed fields $h^{\pm}$, for some map H, allowed to depend upon the coordinates. For simplicity we assume H to be smooth in all its arguments, though the results here apply to maps of finite, sufficiently large, order of differentiability in h and ∇h, and of Sobolev differentiability in the coordinates: the resulting thresholds can easily be read off from the conditions set forth in Section 2.

Let (u, v, x^A) be a coordinate system as in Section 4.1, and let ω and ℓ be the vector fields defined there, with

$$g(\omega, \omega) = g(\ell, \ell) = 0, \qquad g(\omega, \ell) = -2. \tag{4.44}$$

As already pointed out, ℓ and ω are determined up to one multiplicative strictly positive factor,

$$\ell \mapsto \alpha\ell, \quad \omega \mapsto \alpha^{-1}\omega, \qquad \alpha = \alpha(u, v, x^A) > 0. \tag{4.45}$$

Now, every vector orthogonal to ℓ is tangent to the level sets of v. Similarly, a vector orthogonal to ω is tangent to the level sets of u. Hence vectors orthogonal to both have no u- and v-components in the coordinate system above. We can thus write $(\mathrm{Vect}\{\omega, \ell\})^{\perp} = \mathrm{Vect}\{e_B : B = 1, \ldots, n-1\}$, where the e_B's form an ON-basis of TY. Thus

$$g(e_A, e_B) = \delta^A_B, \quad \text{and} \quad e_A = e_A{}^B \partial_B \ \Leftrightarrow\ e_A{}^u = 0 = e_A{}^v .$$

For further purposes, we note that the e_A's are determined up to an $O(n-1)$ rotation:

$$e_A \mapsto \omega_A{}^B e_B, \qquad \omega_A{}^B = \omega_A{}^B(u, v, x^C) \in O(n-1)\,; \tag{4.46}$$

this freedom can be used to impose constraints on the projection on $\mathrm{Vect}\{e_B : B = 1, \ldots, n-1\}$ of $\nabla_\ell e_A$ or $\nabla_\omega e_A$.

The inverse metric in terms of this frame reads

$$g^\sharp = -\tfrac{1}{2}(\ell \otimes \omega + \omega \otimes \ell) + \sum_B e_B \otimes e_B ,$$

so that the wave operator takes the form

$$-\tfrac{1}{2}\nabla_\omega \nabla_\ell - \tfrac{1}{2}\nabla_\ell \nabla_\omega + \sum_C \nabla_{e_C}\nabla_{e_C} + \cdots ,$$

where "$\cdots$" denotes first- and zero-derivative terms arising from the precise nature of the field h. This can be rewritten as

$$-\nabla_\omega \nabla_\ell + \sum_C \nabla_{e_C}\nabla_{e_C} - \tfrac{1}{2}[\nabla_\ell, \nabla_\omega] + \cdots ,$$

or

$$-\nabla_\ell \nabla_\omega + \sum_C \nabla_{e_C}\nabla_{e_C} - \tfrac{1}{2}[\nabla_\omega, \nabla_\ell] + \cdots$$

(where the commutator terms can be absorbed in "$\cdots$" in any case). Setting

$$\varphi_0 = \psi_0 = h, \qquad \varphi_A = \psi_A = e_A(h), \qquad \varphi_+ = \omega(h), \qquad \psi_- = \ell(h) \tag{4.47}$$

leads to the following set of equations:

$$\ell(\varphi_0) = \psi_0,$$

$$\ell(\varphi_+) - \sum_C e_C(\psi_C) = H_{\varphi_+}, \tag{4.48}$$

$$\ell(\varphi_C) - e_C(\psi_-) = H_{\varphi_C},$$

$$\omega(\psi_-) - \sum_C e_C(\varphi_C) = H_{\psi_-}, \tag{4.49}$$

$$\omega(\psi_C) - e_C(\varphi_+) = H_{\psi_C},$$

$$\omega(\psi_0) = \varphi_0, \tag{4.50}$$

where H_{φ_+} etc. contain H and all remaining terms that do not involve second derivatives of h.

This is a first-order system of PDEs in the unknown

$$f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad \text{with} \quad \varphi = \begin{pmatrix} \varphi_0 \\ \varphi_+ \\ \varphi_A \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \psi_0 \\ \psi_- \\ \psi_A \end{pmatrix}.$$

Let us check that it is symmetric hyperbolic, of the form considered in Section 2. We have

$$A^\mu \nabla_\mu f = G(f),$$

or equivalently

$$\begin{pmatrix} A^\mu_{\varphi\varphi} & A^\mu_{\varphi\psi} \\ A^\mu_{\psi\varphi} & A^\mu_{\psi\psi} \end{pmatrix} \nabla_\mu \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} G_\varphi \\ G_\psi \end{pmatrix}, \tag{4.51}$$

with

$$A^u_{\varphi\varphi} = \ell^u \cdot \mathrm{Id}, \qquad A^u_{\varphi\psi} = A^u_{\psi\varphi} = A^u_{\psi\psi} = 0, \tag{4.52}$$

$$A^v_{\psi\psi} = \omega^v \cdot \mathrm{Id}, \qquad A^v_{\varphi\psi} = A^v_{\psi\varphi} = A^v_{\varphi\varphi} = 0, \tag{4.53}$$

$$A^B_{\varphi\psi} = A^B_{\psi\varphi} = - \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \delta^B_1 & \dots & \delta^B_{n-1} \\ 0 & \delta^B_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \delta^B_{n-1} & 0 & \dots & 0 \end{pmatrix}, \tag{4.54}$$

$$A^B_{\varphi\varphi} = 0, \qquad A^B_{\psi\psi} = \omega^B \cdot \mathrm{Id}, \tag{4.55}$$

$$G_\varphi(\varphi, \psi) = \begin{pmatrix} \psi_0 \\ H_{\varphi_+} \\ H_{\varphi_C} \end{pmatrix}, \qquad G_\psi(\varphi, \psi) = \begin{pmatrix} H_{\psi_-} \\ H_{\psi_C} \\ \varphi_0 \end{pmatrix}. \tag{4.56}$$

4.3. The existence theorem. We denote by $\overline{\phi}^-$ the restriction of a map ϕ to $\mathcal{N}^-$ and by $\overline{\phi}^+$ to $\mathcal{N}^+$.

In order to apply the results of the previous sections to the Cauchy problem (4.43) we need to show, given smooth data h^+ on $\mathcal{N}^+$ and h^- on $\mathcal{N}^-$, how to determine the initial data for f on a suitable subset of $\mathcal{N}^+ \cap \mathcal{N}^-$, and that these fields are in the right spaces. We recall that

$$T\mathcal{N}^+ = \mathrm{Vect}\{\ell, e_1, \dots, e_{n-1}\} \quad \text{and} \quad T\mathcal{N}^- = \mathrm{Vect}\{\omega, e_1, \dots, e_{n-1}\},$$

which implies

$$\overline{\omega(h)}^- = \omega(h^-), \qquad \overline{\ell(h)}^+ = \ell(h^+), \qquad \overline{e_B(h)}^\pm = e_B(h^\pm).$$

The remaining restrictions $\overline{\ell(h)}^-$ and $\overline{\omega(h)}^+$ will be determined using the wave equation: Indeed, considering the restriction of (4.48) to $\mathcal{N}^+$ and the restriction of (4.49) to $\mathcal{N}^-$ leads to the following, in general nonlinear, *transport equations* for $\overline{\ell(h)}^-$ and $\overline{\omega(h)}^+$:

$$\begin{aligned} -\omega(\overline{\ell(h)}^-) + \overline{g^{BC}}^- \nabla_{e_B}\nabla_{e_C} h^- &= H_{\psi^-}(h^-, \partial h^-, \overline{\ell(h)}^-, \cdot), \\ \overline{\ell(h)}^-|_{\mathcal{N}^+\cap\mathcal{N}^-} &= \ell(h^+)|_{\mathcal{N}^+\cap\mathcal{N}^-}, \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} -\ell(\overline{\omega(h)}^+) + \overline{g^{BC}}^- \nabla_{e_B}\nabla_{e_C} h^+ &= H_{\varphi^+}(h^+, \partial h^+, \overline{\omega(h)}^+, \cdot), \\ \overline{\omega(h)}^+|_{\mathcal{N}^+\cap\mathcal{N}^-} &= \omega(h^-)|_{\mathcal{N}^+\cap\mathcal{N}^-}. \end{aligned} \tag{4.58}$$

These are ODEs along the integral curves of the vector fields ω and ℓ.

For every generator, say Γ, of $\mathcal{N}^-$ let Γ_0 be the maximal interval of existence of the solution of the transport equation (4.58). Thus the set

$$\mathcal{N}_0^- = \bigcup_\Gamma \Gamma_0 \subset \mathcal{N}^-$$

is the largest subset of $\mathcal{N}^-$ on which the solution of the transport equation, with the required data on $\mathcal{N}^- \cap \mathcal{N}^+$, exists. By lower semicontinuity of the existence time of solutions of ODEs the set $\mathcal{N}_0^-$ is an open subset of $\mathcal{N}^-$.

The set $\mathcal{N}_0^+$ is defined analogously.

Applying the construction of Section 4.1.1 to $\mathcal{N}_0^- \cup \mathcal{N}_0^+$ instead of $\mathcal{N}^- \cup \mathcal{N}^+$, we obtain a double-null coordinate system (u, v, x^A) near $\mathcal{N}_0^+ \cup \mathcal{N}_0^-$ in which the function v runs from 0 to ∞ along all generators of $\mathcal{N}_0^-$, and the function u runs from 0 to ∞ along all generators of $\mathcal{N}_0^+$. Theorem 3.9 and Remark 3.12 apply, leading to:

THEOREM 4.3. *Let $\ell \geq (n+11)/2$. Consider the Cauchy problem* (4.43) *for a semilinear system of wave equations, with $H = H(h, \nabla h, \cdot)$ of C^ℓ differentiability class in all arguments. Without loss of generality we can parameterize $\mathcal{N}^\pm$ by $[0,\infty) \times Y$, with the level sets of the first coordinate transverse to the generators of $\mathcal{N}^\pm$. Given the initial data*

$$h^\pm \in \bigcap_{j=0}^{\ell} C^j([0,\infty); H^{\ell-j}(Y)) \tag{4.59}$$

denote by

$$\mathcal{N}_0 = \mathcal{N}_0^+ \cup \mathcal{N}_0^- \subset \mathcal{N}^+ \cup \mathcal{N}^-$$

the maximal domain of existence on $\mathcal{N}^- \cup \mathcal{N}^+$ of the transport equations (4.57)–(4.58). *There exists a neighborhood $\mathcal{V}$ of $\mathcal{N}_0$ and a unique solution h defined there with the following properties: Reparameterizing the generators of $\mathcal{N}_0^\pm$ if necessary, we can obtain*

$\mathcal{N}_0^\pm \approx [0,\infty) \times Y$. Then for every $i \in \mathbb{N}$ there exist $a_i, b_i > 0$ such that the set (see Figure 4.3)

$$\mathcal{V}_i := \Big(\underbrace{\big([0,a_i] \times [0,i]\big) \cup \big([0,i] \times [0,b_i]\big)}_{=:\mathcal{U}_i} \Big) \times Y$$

is included in $\mathcal{V}$, and we have

$$\begin{aligned} h \in L^\infty\big(\mathcal{U}_i; H^{\ell-2}(Y)\big) \cap W^{1,\infty}\big(\mathcal{U}_i; H^{\ell-3}(Y)\big) \\ \cap \bigcap_{0<3j\le \ell-(n+11)/2} W^{j+1,\infty}\big(\mathcal{U}_i; H^{\ell-2-3j}(Y)\big) \subset C^{\ell_1-1,1}(\mathcal{U}_i \times Y), \end{aligned} \tag{4.60}$$

with the last inclusion holding provided that $\ell > (n+17)/2$, with $\ell_1 \ge 1$ being the largest integer such that $\ell - 3\ell_1 > (n+11)/2$. The solution depends continuously on the initial data, and is smooth if the initial data are.

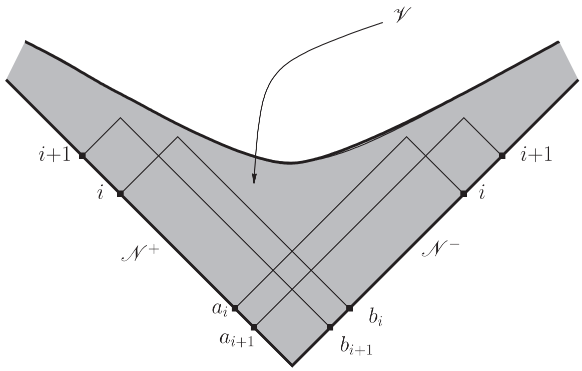

Fig. 4.3. The neighborhood $\mathcal{V}$ of $\mathcal{N}$

Remark 4.4. Condition (4.59) will hold for $h^\pm \in C^\ell([0,\infty) \times Y)$.

Remark 4.5. An obvious analogue of Remark 3.8 concerning further regularity of h applies.

5. Einstein equations

In this section we will show that our existence theorems above can be used (in a somewhat indirect manner) to establish neighborhood theorems for both the Einstein equations with suitable sources and the Friedrich conformal vacuum Einstein equations.

One could try to analyse whether the harmonic coordinate reduction of Einstein equations leads to equations with a nonlinearity structure to which Theorem 3.9 applies. Here a problem arises, because our iteration scheme requires a doubly-null decomposition of the principal symbol of the wave equation, which is the wave operator. This in turn requires going to harmonic coordinates, but those lead to a loss of derivative of the coefficients. It is conceivable that this can be overcome, but it appears simpler to work directly in a formalism where the doubly-null decomposition of the equation is built-in from the outset, namely the Newman–Penrose–Friedrich–Christodoulou–Klainerman–Nicolò equations. We will show that this decomposition fits indeed in our set-up.

We use the conventions and notation of [11]. For the convenience of the reader we include in Appendix A.1 a shortened version of a section in [11] which introduces the relevant formalism.

5.1. The Einstein vacuum equations. We start with the vacuum Einstein equations, which we write as a set of equations for a tetrad $e_q = e_q{}^\mu\partial_\mu$, for the related connection coefficients defined as

$$\nabla_i e_j = \Gamma_i{}^k{}_j e_k, \tag{5.1}$$

and for the tetrad components $d^i{}_{jk\ell}$ of the Weyl tensor. We assume that the scalar products $g_{ij} := g(e_i, e_j)$ are point-independent, with the matrix g_{ij} having Lorentzian signature. We require that ∇ is g-compatible, which is equivalent to

$$\Gamma_{ijk} = -\Gamma_{ikj}, \quad \text{where} \quad \Gamma_{ijk} := g_{j\ell}\Gamma_i{}^\ell{}_k. \tag{5.2}$$

Consider the set of equations due to Friedrich (see [26] and references therein)

$$[e_p, e_q] = (\Gamma_p{}^l{}_q - \Gamma_q{}^l{}_p)\, e_l, \tag{5.3a}$$

$$e_p(\Gamma_q{}^i{}_j) - e_q(\Gamma_p{}^i{}_j) - 2\,\Gamma_k{}^i{}_j\,\Gamma_{[p}{}^k{}_{q]} + 2\,\Gamma_{[p}{}^i{}_{|k|}\Gamma_{q]}{}^k{}_j$$
$$= d^i{}_{jpq} + \delta^i_{[p}R_{q]j} - g_{j[p}R_{q]}{}^i + \frac{R}{3}g_{j[p}\delta^i_{q]}, \tag{5.3b}$$

$$D_i d^i{}_{jkl} = J_{jkl}. \tag{5.3c}$$

Equation (5.3a) says that Γ has no torsion. Recall that we have assumed that the Γ_{ijk}'s are anti-symmetric in the last two indices; together with (5.3a) this implies that Γ is the Levi-Civita connection of g. We will assume that d_{ijkl} has the symmetries of the Weyl tensor; then the left-hand side of (5.3b) is simply the definition of the curvature tensor of the connection Γ, with d_{ijkl} being the Weyl tensor, R_{ij} being the Ricci tensor and R the Ricci scalar.

In vacuum ($R_{ij} \equiv \lambda g_{ij}$ for some constant λ), (5.3c) with $J_{jkl} \equiv 0$ follows from the Bianchi identities for the curvature tensor.

As shown by Friedrich [21, Theorem 1] (compare [35]), every solution of (5.3) with $J_{jkl} \equiv 0$ satisfying suitable constraint equations on the initial data surface is a solution of the vacuum Einstein equations.

In Section 5.2 below we will consider a class of nonvacuum Einstein equations, in which case we will complement the above with equations for further fields satisfying wave equations, and then R_{ij} and J_{jkl} in (5.3) will be viewed as prescribed functions of the remaining fields, their first derivatives, the tetrad, the Christoffel coefficients, and the d_{ijkl}'s, as determined from the energy-momentum tensor of the matter fields.

We would like to apply Theorem 3.9 to the problem at hand. The first step is to show that we can bring a subset of (5.3) to the form needed there. This will be done using the frame formalism of Christodoulou and Klainerman, as described in Appendix A.1.

Before pursuing, we will need to reduce the gauge freedom available. For this we need to understand what conditions can be imposed on coordinates and frames without losing generality.

Given a metric g, we have seen in Section 4 how to construct a coordinate system (u, v, x^A) and vector fields e_i, $i = 1, \dots, 4$, with e_3 proportional to the vector field ℓ constructed there, and e_4 proportional to ω there, so that the metric takes the form (A.1) below, with

$$e_3 = \partial_u, \qquad e_4 = e_4{}^v \partial_v + e_4{}^A \partial_A. \tag{5.4}$$

With this choice of tetrads $e_i = e_i{}^\mu \partial_\mu$, (5.3a) becomes an evolution equation for the tetrad coefficients $e_i{}^\mu$:

$$[e_3, e_i] = \partial_u e_i{}^\mu \partial_\mu = (\Gamma_3{}^l{}_i - \Gamma_i{}^l{}_3)\, e_l. \tag{5.5}$$

By construction, the $\partial_u e_a$'s have no u and v components, which gives the identities

$$0 = \Gamma_3{}^3{}_a - \Gamma_a{}^3{}_3 = \Gamma_3{}^4{}_a - \underbrace{\Gamma_a{}^4{}_3}_{=0}. \tag{5.6}$$

Equivalently, in the notation of Appendix A.1,

$$\eta_a = \zeta_a, \qquad \underline{\xi}_a = 0. \tag{5.7}$$

Similarly, $\partial_u e_4$ has no u component, which implies

$$0 = \underbrace{\Gamma_3{}^3{}_4}_{=0} - \Gamma_4{}^3{}_3 \;\Leftrightarrow\; \underline{\upsilon} = 0. \tag{5.8}$$

Next, the vector fields e_a, $a = 1, 2$, are determined up to rotations in the planes $\mathrm{Vect}\{e_1, e_2\}$, and we can get rid of this freedom by imposing

$$\Gamma_3{}^a{}_b = 0. \tag{5.9}$$

By construction, the integral curves of the vector fields e_3 and e_4 are null geodesics, though not necessarily affinely parameterized:

$$\nabla_{e_3} e_3 \sim e_3, \qquad \nabla_{e_4} e_4 \sim e_4. \tag{5.10}$$

In this gauge, using the notation of Appendix A.1 (see (A.8f)), we have

$$\Gamma_3{}^a{}_3 = 0 = \Gamma_4{}^a{}_4 \;\Leftrightarrow\; \underline{\xi}^a = 0 = \xi^a. \tag{5.11}$$

The vanishing of the rotation coefficients just listed allows us to get rid of the second term in some of the combinations

$$e_3(\Gamma_q{}^i{}_j) - e_q(\Gamma_3{}^i{}_j)$$

appearing in (5.3b). In this way, we can algebraically determine

$$\partial_u \Gamma_q{}^a{}_3 \quad \text{and} \quad \partial_u \Gamma_q{}^a{}_b$$

in terms of the remaining fields appearing in (5.3b). Similarly,

$$e_4(\Gamma_q{}^a{}_4) - e_q(\Gamma_4{}^a{}_4) = e_4(\Gamma_q{}^a{}_4), \qquad e_4(\Gamma_q{}^3{}_3) - e_q(\Gamma_4{}^3{}_3) = e_4(\Gamma_q{}^3{}_3),$$

which gives equations for $e_4(\Gamma_q{}^a{}_4)$ and $e_4(\Gamma_q{}^3{}_3)$.

In view of (5.7) and the symmetries of the $\Gamma_i{}^j{}_k$'s, all the nonvanishing connection coefficients satisfy ODEs along the integral curves of $e_3 = \partial_u$ or of e_4.

The analysis of the divergence equation (5.3c) in Appendix A.1 leads in vacuum to the following two collections of fields:

$$\varphi = (e_i, \Gamma_i{}^a{}_b, \Gamma_i{}^a{}_3, \alpha, \mathring{\underline{\beta}}, \rho, \sigma, \mathring{\beta}), \tag{5.12}$$

$$\psi = (\Gamma_i{}^a{}_4, \Gamma_i{}^3{}_3, \underline{\beta}, \mathring{\sigma}, \mathring{\rho}, \beta, \underline{\alpha}), \tag{5.13}$$

with the gauge conditions just given,

$$\begin{gathered} e_3{}^u = 1, \qquad 0 = e_3{}^v = e_3{}^A = e_4{}^u = e_a{}^u = e_a{}^v, \\ \Gamma_3{}^3{}_a = \Gamma_a{}^3{}_3, \qquad 0 = \Gamma_3{}^a{}_3 = \Gamma_4{}^i{}_4 = \Gamma_3{}^a{}_b, \end{gathered} \tag{5.14}$$

to which Theorem 3.9 and Remark 3.12 apply. This will be used to establish our main result for the vacuum Einstein equations. However, before stating the theorem, an overview of some initial value problems for the vacuum Einstein equations is in order.

As discussed in detail in [12], the characteristic initial data for the vacuum Einstein equations on each of the hypersurfaces $\mathcal{N}^\pm$ consist of a symmetric tensor field $\tilde{g}$ with signature $(0, +, \dots, +)$, so that the integral curves of the kernel of $\tilde{g}$ describe the generators of $\mathcal{N}^\pm$. To the tensor field $\tilde{g}$ one needs to add a connection κ on the bundle of tangents to the generators. In a coordinate system (r, x^A) on $\mathcal{N}$ such that ∂_r is tangent to the generators we have $\nabla_{\partial_r}\partial_r = \kappa\partial_r$. The fields $\tilde{g}$ and κ are not arbitrary, but are subject to a constraint, the Raychaudhuri equation. If we write

$$\tilde{g} = \overline{g}_{AB}(r, x^C)dx^A dx^B \tag{5.15}$$

and, in dimension $n+1$, we set

$$\tau = \tfrac{1}{2}\overline{g}^{AB}\partial_r \overline{g}_{AB}, \qquad \sigma_{AB} = \tfrac{1}{2}\partial_r \overline{g}_{AB} - \tfrac{1}{2}\tau \overline{g}_{AB}, \tag{5.16}$$

then, in vacuum, the Raychaudhuri constraint equation reads

$$\partial_r \tau - \kappa\tau + |\sigma|^2 + \frac{\tau^2}{n-1} = 0. \tag{5.17}$$

Here it is appropriate to mention the alternative approach of Rendall [38], where one prescribes the conformal class of $\tilde{g}$ and one solves (5.17) for the conformal factor, after adding the requirement that κ vanishes identically. Thus, in Rendall's scheme the starting point is an initial data symmetric tensor field $\gamma = \gamma_{AB}(r, x^C)dx^A dx^B$ which is assumed to form a one-parameter family of Riemannian metrics $r \mapsto \gamma(r, x^A)$ on the level sets of r, all assumed to be diffeomorphic to a fixed $(n-1)$-dimensional manifold Y. The conformal factor Ω relating $\tilde{g}$ and the initial data γ, $\overline{g}_{AB} = \Omega^2 \gamma_{AB}$ can be written as

$$\Omega = \varphi \left(\frac{\det s}{\det \gamma} \right)^{1/(2n-2)}, \tag{5.18}$$

where $s = s_{AB}(x^C)dx^A dx^B$ is any r-independent convenient auxiliary metric on the surfaces $r = \text{const}$. Note that the field σ_{AB} defined in (5.16) is independent of φ, thus is defined uniquely by the representative γ of the conformal class of $\tilde{g}$. One has

$$\tau = (n-1)\partial_r \log \varphi, \tag{5.19}$$

which allows one to rewrite (5.17) as a second-order *linear* ODE:

$$\partial_r^2 \varphi - \kappa \partial_r \varphi + \frac{|\sigma|^2}{n-1}\varphi = 0. \tag{5.20}$$

In this case, after solving (5.20), *one has to replace the initial hypersurface by its subset on which* $\varphi > 0$.

Recall next that, again in the approach of Rendall (compare [7]), the remaining metric functions on $\mathcal{N}$ are obtained by solving linear ODEs along the generators of $\mathcal{N}$. One could then worry that the requirement that the resulting tensor has Lorentzian signature might lead to the need of passing to a further subset of $\mathcal{N}$. This is indeed the case in the original formulation of [38], but the problem disappears when handled appropriately, as it can be reformulated in such a way that the remaining metric functions are freely prescribable [12].

We finally note that the characteristic data on each of $\mathcal{N}^\pm$ have to be complemented by certain data on $\mathcal{N}^+ \cap \mathcal{N}^-$, the precise description of which is irrelevant here; the reader is referred to [7, 12, 38] for details.

To continue, it is useful to summarize some known results about Cauchy problems for the Einstein equations:

THEOREM 5.1 (Rendall). *Given smooth characteristic vacuum initial data on* $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$, *complemented by suitable data on* $Y := \mathcal{N}^+ \cap \mathcal{N}^-$, *there exists a unique, up to isometry, vacuum metric defined in a future neighborhood of* $\mathcal{N}^+ \cap \mathcal{N}^-$, *as shown in Figure* 5.1.

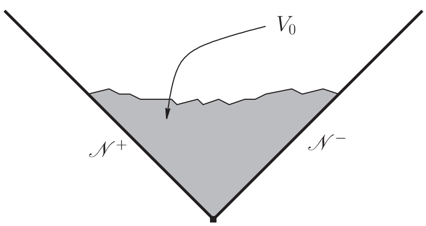

Fig. 5.1. The guaranteed domain of existence of the solution in Rendall's theorem

The following result is standard:

THEOREM 5.2. *Given smooth vacuum initial data on a spacelike hypersurface Σ with nonempty boundary $\partial\Sigma$ there exists a unique, up to isometry, vacuum metric defined in a future neighborhood of Σ, bounded near $\partial\Sigma$ by smooth null "ingoing" hypersurfaces orthogonal to $\partial\Sigma$, as shown in Figure* 5.2.

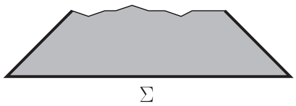

Fig. 5.2. The guaranteed future domain of existence of the solution with initial data on a hypersurface with boundary

One has of course a similar domain of existence to the past of Σ, but this is irrelevant for our purposes.

From Theorems 5.1 and 5.2 one easily obtains existence of solutions of the mixed Cauchy problems illustrated in Figures 5.3 and 5.4.

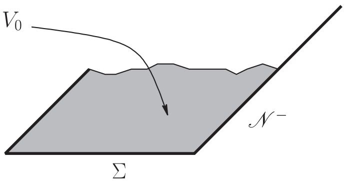

Fig. 5.3. The guaranteed domain of existence of solutions of a mixed Cauchy problem with a "left" boundary and a characteristic initial data hypersurface emanating normally from the "right" boundary

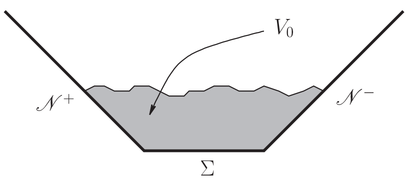

Fig. 5.4. The guaranteed domain of existence of solutions of a mixed Cauchy problem with characteristic initial data hypersurfaces emanating normally from the boundaries of a spacelike hypersurface

We are now ready to turn to our main result, which for simplicity we state for smooth metrics. The interested reader can chase the losses of differentiability which arise at various steps of the proof to obtain the corresponding theorem with initial data of finite Sobolev differentiability (cf. [33]):

THEOREM 5.3. *For any set of smooth characteristic initial data for the vacuum Einstein equations on two transversely intersecting null hypersurfaces $\mathscr{N} := \mathscr{N}^+ \cup \mathscr{N}^-$ there exists* *a smooth vacuum metric defined in a future neighborhood $\mathcal{U}$ of $\mathcal{N}$. The solution is unique up to diffeomorphism when $\mathcal{U}$ is appropriately chosen and when appropriate initial data on $\mathcal{N}^+ \cap \mathcal{N}^-$ are given.*

Proof. As shown in Section 4.1.1, without loss of generality we can parameterize each of $\mathcal{N}^\pm$ as $[0,\infty) \times Y$. Symmetry under interchange of u and v, together with the argument presented in Remark 3.12, shows that it suffices to establish that given $b_0 > 0$ there exists $a_* > 0$ and a solution of the vacuum Einstein equations defined in a doubly-null coordinate system covering the set $[0, a_*] \times [0, b_0] \times Y$.

Theorem 3.9 shows indeed that there exists such a constant a_* and a set of fields (5.12)–(5.13) solving the equations described above with the initial data determined from the general relativistic initial data by a standard procedure. The theorem would immediately follow if one knew that every resulting set of fields (5.12)–(5.13) provides a solution of the Einstein equations. While we believe that this is the case, such a direct proof would require a considerable amount of work. Fortunately one can proceed in a less work-intensive manner, adapting the idea of Luk [33] to use the function $u + v$ as a tool to "build up" the solution:

Let $\mathcal{U}$ be any maximal domain of existence of a solution of the vacuum Einstein equations assuming the given initial data. (Note that the question whether a *unique* such maximal domain exists is irrelevant for our purposes.) As explained in Section 4.1, there exists a neighborhood $\mathcal{V}_0$ of $\mathcal{N}$ in $\mathcal{U}$ on which we can introduce a coordinate system (u, v, x^A) comprising a pair of null coordinates u and v. On $\mathcal{V}_0$ define

$$t := u + v; \tag{5.21}$$

then ∇t is timelike, and hence the level sets of t are spacelike.

Define

$$t_* := \sup \big\{ t : \text{the coordinates } u \text{ and } v \text{ cover the set} \\ ([0, a_*] \times [0, b_0]) \cap \{u + v < t\} \big\}. \tag{5.22}$$

It follows from Theorem 5.1 that $t_* > 0$.

On the set

$$\big(([0, a_*] \times [0, b_0]) \cap \{u + v < t_*\}\big) \times Y \tag{5.23}$$

we have a solution of the vacuum Einstein equations, and therefore corresponding fields (φ, ψ) as in (5.12)–(5.13) calculated from the vacuum metric, with $\mathring{\beta} = \beta$, $\mathring{\underline{\beta}} = \underline{\beta}$, $\mathring{\sigma} = \sigma$, and $\mathring{\rho} = \rho$. Let us denote those fields by (φ_E, ψ_E). But on this set we also have a smooth solution (φ, ψ) of the equations described in Appendix A.1, with initial data calculated from the solution of the Einstein equation. Since both fields satisfy the same system of equations and have identical initial data, uniqueness gives

$$(\varphi, \psi) = (\varphi_E, \psi_E).$$

Suppose that $t_* < a_*$, as shown in Figure 5.5. Since (φ, ψ) extend smoothly to the boundary $t = t_*$, so do (φ_E, ψ_E). The pair (φ_E, ψ_E) at $t = t_*$ can be used to determine smooth Cauchy data for the vacuum Einstein equations for a Cauchy problem as shown in Figure 5.4. The solution of this Cauchy problem allows us to extend the solution beyond $t = t_*$, contradicting the fact that t_* was maximal.

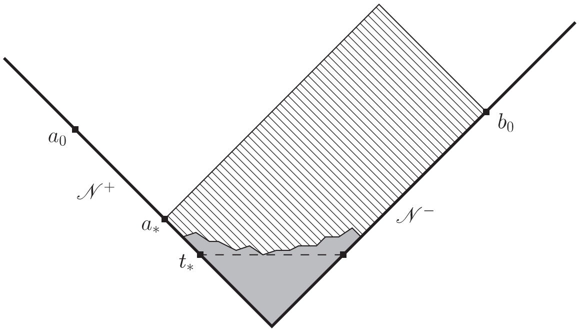

Fig. 5.5. The case $a_* > t_*$

The hypothesis that $a_* \leq t_* \leq b_0$ leads to a contradiction by an identical argument, using instead the Cauchy problem illustrated in Figure 5.3; see Figures 5.6 and 5.7.

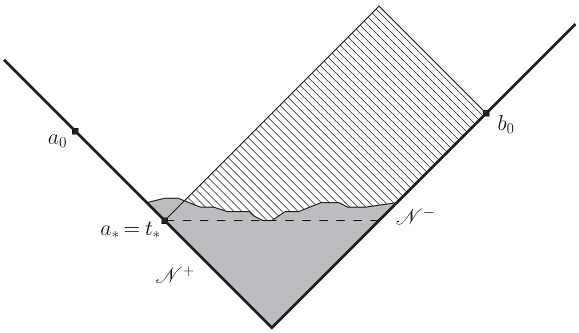

Fig. 5.6. The case $a_* = t_*$

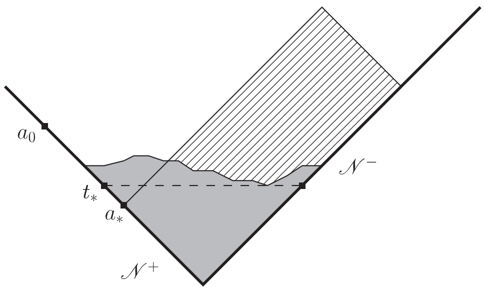

Fig. 5.7. The case $a_* < t_* < b_0$

The hypothesis that $b_0 \leq t_* < a_* + b_0$ (see Figure 5.8) leads to a contradiction by an identical argument, using Theorem 5.2; compare Figure 5.2.

Hence $t_* = a_* + b_0$, and the result is established. ∎

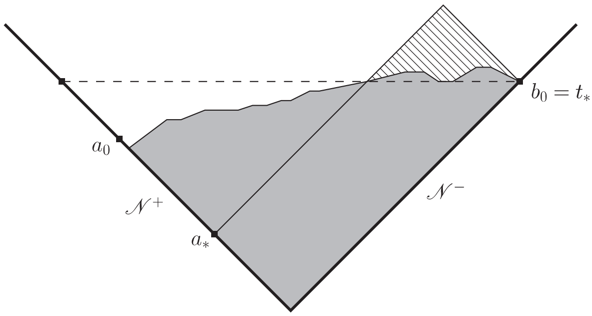

Fig. 5.8. The case $a_* < t_* = b_0$

5.2. Einstein equations with sources satisfying wave equations. The analysis of the previous section generalizes immediately to Einstein equations with matter fields satisfying wave equations, such as the Einstein-scalar field system or the Einstein–Yang–Mills–Higgs equations. More generally, consider a system of equations of the form

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = T_{\mu\nu}, \qquad T_{\mu\nu} = T_{\mu\nu}(\Phi, \partial\Phi, g, \partial g), \tag{5.24}$$

with $\nabla_\mu T^\mu{}_\nu = 0$ whenever the matter fields Φ satisfy a set of wave equations of the form

$$\Box_g \Phi = F(\Phi, \partial\Phi, g, \partial g). \tag{5.25}$$

As explained in Section 4.1, one can obtain a doubly-null system of equations from (5.25). The Einstein equations (5.24) are treated as in the vacuum case, with nonzero source terms J_{ijk} in the Bianchi equations determined by the matter fields. This leads to an obvious equivalent of Theorem 5.3; the reader should have no difficulties formulating a precise statement.

5.3. Friedrich's conformal equations. Let $\tilde{g}$ be the physical space-time metric (not to be confused with the initial data tensor field of (5.15)), let Ω be a function and let $g = \Omega^2 \tilde{g}$ be the unphysical conformally rescaled counterpart of $\tilde{g}$. (To make easier reference to [21, 22, 24, 25], throughout this section the symbol g denotes the *unphysical* metric.) Consider any frame field $e_k = e^\mu{}_k \, \partial_{x^\mu}$ such that the $g(e_i, e_k) \equiv g_{ik}$'s are constants, with i, k, etc. running from 0 to 3. Using the Einstein vacuum field equations, Friedrich [21,22] has derived a set of equations for the fields

$$e^\mu{}_k, \qquad \Gamma_i{}^j{}_k, \qquad d^i{}_{jkl} = \Omega^{-1} C^i{}_{jkl}, \qquad L_{ij} = \tfrac{1}{2} R_{ij} - \tfrac{1}{12} R g_{ij},$$
$$\Omega, \qquad s = \tfrac{1}{4} \nabla_i \nabla^i \Omega + \tfrac{1}{24} R \, \Omega,$$

where $\Gamma_i{}^j{}_k$ denotes the Levi-Civita connection coefficients in the frame e_k, $\nabla_i e_k = \Gamma_i{}^j{}_k e_j$, while $C^i{}_{jkl}$, R_{ij}, and R stand respectively for the Weyl tensor, the Ricci tensor, and the Ricci scalar of g. Friedrich's "conformal field equations" read

$$[e_p, e_q] = (\Gamma_p{}^l{}_q - \Gamma_q{}^l{}_p)\, e_l, \tag{5.26a}$$

$$\begin{aligned} e_p(\Gamma_q{}^i{}_j) - e_q(\Gamma_p{}^i{}_j) - 2\,\Gamma_k{}^i{}_j\,\Gamma_{[p}{}^k{}_{q]} + 2\,\Gamma_{[p}{}^i{}_{|k|}\Gamma_{q]}{}^k{}_j \\ = 2\,g^i{}_{[p}\,L_{q]j} - 2\,g^{ik}\,g_{j[p}\,L_{q]k} + \Omega\, d^i{}_{jpq}, \end{aligned} \tag{5.26b}$$

$$\nabla_i d^i{}_{jkl} = 0, \tag{5.26c}$$

$$\nabla_i L_{jk} - \nabla_j L_{ik} = \nabla_l \Omega\, d^l{}_{kij}, \tag{5.26d}$$

$$\nabla_i \nabla_j \Omega = -\Omega\, L_{ij} + s g_{ij}, \tag{5.26e}$$

$$\nabla_i s = -L_{ij} \nabla^j \Omega, \tag{5.26f}$$

$$6\Omega s - 3\nabla_j \Omega \nabla^j \Omega = 0. \tag{5.26g}$$

The first equation expresses the fact that the Levi-Civita connection is torsion-free; the second is the definition of the Riemann tensor; the third is the Bianchi identity assuming that $\tilde{g}$ is Ricci flat. The remaining equations are obtained by algebraic manipulations from the vacuum Einstein equations, using the conformal transformation laws for the various objects at hand. In regions where $\Omega > 0$ the system is equivalent to the vacuum Einstein equations [21, 22].

We have seen in Section 5.1 how to bring (5.26a)–(5.26c) to a form to which Theorem 3.9 applies. It remains to provide equations for the fields L_{ij}, s and Ω. For this we can use a subset of the wave equations derived in [35]:

$$\Box_g L_{ij} = 4L_{ik}L_j{}^k - g_{ij}|L|^2 - 2\Omega d_{imj}{}^\ell L_\ell{}^m + \tfrac{1}{6}\nabla_i\nabla_j R, \tag{5.27}$$

$$\Box_g s = \Omega|L|^2 - \tfrac{1}{6}\nabla_k R\, \nabla^k \Omega - \tfrac{1}{6} s R, \tag{5.28}$$

$$\Box_g \Omega = 4s - \tfrac{1}{6}\Omega R, \tag{5.29}$$

with the conformal gauge $R = 0$. In order to control the first derivatives of the Christoffel symbols that appear in $\Box_g L_{ij}$ we add to the above set of equations the set of equations obtained by differentiating (5.26a)–(5.26c) with respect to all coordinates. This collection of fields will be referred to as *Friedrich's fields.*

The wave equations (5.27)–(5.29) are rewritten as a doubly-null system as in Section 4, upon noting that the inverse metric $g^{\mu\nu} = g^{ij} e^\mu{}_i e^\nu{}_j$ is directly in a doubly-null form by construction. This leads to a system of equations to which Theorem 3.9 applies provided that the initial data have the properties required there.

For this, we will assume that the characteristic initial data on two transversely intersecting null hypersurfaces $\mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$ are smoothly conformally extendable across a boundary at infinity. The reader is referred to [14, 36] for a detailed description of this class of initial data.

Given such initial data, we can use Theorem 5.1 to solve the Einstein equations to the future of $\mathcal{N}$. The solution can be used to provide the initial data for Friedrich's collection of fields just described on $\mathcal{N}$. We can then extend the resulting initial data to a hypersurface which extends beyond the conformal boundary at infinity. Theorem 3.9 guarantees the existence of a uniform neighborhood of the extended hypersurface and a smooth solution of the Friedrich fields there. An argument identical to the one in the proof of Theorem 5.1 shows that the solution of the Einstein equations exists on a uniform neighborhood of $\mathcal{N}$ in the region where $\Omega > 0$. This leads to:

THEOREM 5.4. *For any set of characteristic initial data for the vacuum Einstein equations on two transversely intersecting null hypersurfaces $\mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$ which are smoothly conformally extendable across a boundary at infinity, there exists a smooth vacuum metric defined in a future neighborhood $\mathcal{U}$ of $\mathcal{N}$ such that the resulting space-time has a smooth nonempty conformal boundary at null infinity. The solution is unique up to diffeomorphism when $\mathcal{U}$ is appropriately chosen and when appropriate initial data on $\mathcal{N}^+ \cap \mathcal{N}^-$ are given.*

An identical theorem applies to initial data given on a null cone. When the initial data are sufficiently near to the Minkowskian ones, all causal geodesics will be future complete in the resulting vacuum space-time, with the null geodesics acquiring an end point on a conformal boundary at null infinity.

Appendix
Doubly-null decompositions of the vacuum Einstein equations

The material in this appendix follows closely the presentation in [11].

A.1. Connection coefficients in a doubly-null frame. Consider any field of vectors e_i, $i = 1, \dots, 4$, such that

$$(g_{ij}) := (g(e_i, e_j)) = \begin{pmatrix} \delta^a_b & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \tag{A.1}$$

where indices i, j etc. run from 1 to 4, while indices a, b etc. run from 1 to 2. One therefore has

$$(g^{ij}) := g(\theta^i, \theta^j) = \begin{pmatrix} \delta^a_b & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix},$$

where θ^i is a basis of $T^*\mathcal{M}$ dual to e_i. If α_i, $i = 1, \dots, 4$, is a usual Lorentzian orthonormal basis of $T\mathcal{M}$,

$$g(\alpha_i, \alpha_j) = \eta_{ij} = \mathrm{diag}(+1, +1, +1, -1),$$

then a basis e_i as above can be constructed by setting

$$e_a = \alpha_a, \qquad e_3 = \alpha_3 + \alpha_4, \qquad e_4 = \alpha_4 - \alpha_3.$$

Let Vol_g be the Lorentzian volume element of g, with the associated completely anti-symmetric tensor ϵ_{ijkl}:

$$\mathrm{Vol}_g = \beta^1 \wedge \beta^2 \wedge \beta^3 \wedge \beta^4 = \frac{1}{4!} \epsilon_{ijkl}\, \beta^i \wedge \beta^j \wedge \beta^k \wedge \beta^l,$$

where β^i is a dual basis to α_j. We have $\theta^3 = (\beta^3 + \beta^4)/2$, $\theta^4 = (\beta^4 - \beta^3)/2$, $\beta^3 = \theta^3 - \theta^4$, $\beta^4 = \theta^3 + \theta^4$, hence

$$\mathrm{Vol}_g = 2\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = \frac{1}{4!} \epsilon_{ijkl}\, \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l.$$

It follows that in the basis e_i the entries of the ϵ tensor are zeros, twos, and their negatives:

$$\epsilon_{1234} = 2. \tag{A.2}$$

We let

$$\mathcal{S} = \mathrm{Vect}(\{e_1, e_2\}),$$

where $\mathrm{Vect}(X)$ denotes the vector space spanned by the elements of the set X.

For any connection D we define the connection coefficients $\Gamma_i{}^j{}_k$ by

$$\Gamma_i{}^j{}_k := \theta^j(D_{e_i}e_k),$$

so that

$$D_{e_i}e_k = \Gamma_i{}^j{}_k e_j.$$

The connection D has no torsion if and only if

$$D_{e_i}e_k - D_{e_k}e_i = [e_i, e_k],$$

and it is metric compatible if and only if

$$D_i g_{jk} \equiv (D_{e_i}g)(e_j, e_k) = -\Gamma_{ijk} - \Gamma_{ikj} = 0. \tag{A.3}$$

Here and elsewhere,

$$\Gamma_{ijk} := g_{jm}\Gamma_i{}^m{}_k.$$

The *null second fundamental forms* of a codimension two submanifold S are the two symmetric tensors on S defined as (1)

$$\chi(X,Y) = g(D_X e_4, Y), \qquad \underline{\chi}(X,Y) = g(D_X e_3, Y), \tag{A.4}$$

where D is the Levi-Civita connection of $(\mathcal{M}, g)$, while X, Y are tangent to S. The *torsion* of S is a 1-form on S, defined for vector fields X tangent to S by

$$\zeta(X) = -\tfrac{1}{2}g(D_X e_3, e_4) = \tfrac{1}{2}g(D_X e_4, e_3). \tag{A.5}$$

In the definitions above it is also assumed that e_3 and e_4 are normal to S, so that $\mathcal{S}$ coincides, over S, with the distribution TS of the planes tangent to S. (Throughout, the indices are raised and lowered with the metric g.)

Following (2) Klainerman and Nicolò, we use the following labeling of the remaining Newman–Penrose coefficients associated with the frame fields e_i:

$$\xi_a = \tfrac{1}{2}g(\widehat{D}_{e_4}e_4, e_a), \tag{A.6a}$$

$$\underline{\xi}_a = \tfrac{1}{2}g(\widehat{D}_{e_3}e_3, e_a), \tag{A.6b}$$

$$\eta_a = -\tfrac{1}{2}g(\widehat{D}_{e_3}e_a, e_4) = \tfrac{1}{2}g(\widehat{D}_{e_3}e_4, e_a), \tag{A.6c}$$

$$\underline{\eta}_a = -\tfrac{1}{2}g(\widehat{D}_{e_4}e_a, e_3) = \tfrac{1}{2}g(\widehat{D}_{e_4}e_3, e_a), \tag{A.6d}$$

$$2\omega = -\tfrac{1}{2}g(\widehat{D}_{e_4}e_3, e_4), \tag{A.6e}$$

$$2\underline{\omega} = -\tfrac{1}{2}g(\widehat{D}_{e_3}e_4, e_3), \tag{A.6f}$$

$$2\upsilon = -\tfrac{1}{2}g(\widehat{D}_{e_3}e_3, e_4), \tag{A.6g}$$

$$2\underline{\upsilon} = -\tfrac{1}{2}g(\widehat{D}_{e_4}e_4, e_3). \tag{A.6h}$$

(1) Those objects are only defined up to an overall multiplicative function, related to the possibility of rescaling the null vector fields e_3 and e_4; some definite choices of this scale will be made later.

(2) We are grateful to S. Klainerman and F. Nicolò for making their TEX files available to us.

(The principle that determines which symbols are underlined, and which are not, should be clear from equations (A.8) below: all the terms *on the right-hand sides* of those equations have a coefficient of e_4 which is underlined.) The above definitions, together with the properties of the connection coefficients Γ_{ijk}, imply the following:

$$\chi_{ab} = \Gamma_{ab4} = -\Gamma_{a4b} = 2\Gamma_a{}^3{}_b = -2\Gamma_{ab}{}^3, \tag{A.7a}$$

$$\underline{\chi}_{ab} = \Gamma_{ab3} = -\Gamma_{a3b} = 2\Gamma_a{}^4{}_b = -2\Gamma_{ab}{}^4, \tag{A.7b}$$

$$\zeta_a = \Gamma_a{}^3{}_3 = -\tfrac{1}{2}\Gamma_{a43} = \Gamma_{a4}{}^4, \tag{A.7c}$$

$$\underline{\zeta}_a = \Gamma_a{}^4{}_4 = -\tfrac{1}{2}\Gamma_{a34} = -\Gamma_{a3}{}^3, \tag{A.7d}$$

$$\xi_a = \Gamma_4{}^3{}_a = -\Gamma_{4a}{}^3 = \tfrac{1}{2}\Gamma_{4a4} = -\tfrac{1}{2}\Gamma_{44a}, \tag{A.7e}$$

$$\underline{\xi}_a = \Gamma_3{}^4{}_a = -\Gamma_{3a}{}^4 = \tfrac{1}{2}\Gamma_{3a3} = -\tfrac{1}{2}\Gamma_{33a}, \tag{A.7f}$$

$$\eta_a = \Gamma_3{}^3{}_a = -\tfrac{1}{2}\Gamma_{34a} = \tfrac{1}{2}\Gamma_{3a4} = -\Gamma_{3a}{}^3, \tag{A.7g}$$

$$\underline{\eta}_a = \Gamma_4{}^4{}_a = -\tfrac{1}{2}\Gamma_{43a} = \tfrac{1}{2}\Gamma_{4a3} = -\Gamma_{4a}{}^4, \tag{A.7h}$$

$$2\omega = \Gamma_4{}^3{}_3 = -\tfrac{1}{2}\Gamma_{443} = \Gamma_{44}{}^4, \tag{A.7i}$$

$$2\underline{\omega} = \Gamma_3{}^4{}_4 = -\tfrac{1}{2}\Gamma_{334} = \Gamma_{33}{}^3, \tag{A.7j}$$

$$2\upsilon = \Gamma_3{}^3{}_3 = -\tfrac{1}{2}\Gamma_{343} = \Gamma_{34}{}^4, \tag{A.7k}$$

$$2\underline{\upsilon} = \Gamma_4{}^4{}_4 = -\tfrac{1}{2}\Gamma_{434} = \Gamma_{43}{}^3. \tag{A.7l}$$

This leads to

$$D_a e_b = \slashed{\nabla}_a e_b + \tfrac{1}{2}\chi_{ab} e_3 + \tfrac{1}{2}\underline{\chi}_{ab} e_4, \tag{A.8a}$$

$$D_3 e_a = \slashed{\nabla}_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4, \tag{A.8b}$$

$$D_4 e_a = \slashed{\nabla}_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3, \tag{A.8c}$$

$$D_a e_3 = \underline{\chi}_a{}^b e_b + \zeta_a e_3, \tag{A.8d}$$

$$D_a e_4 = \chi_a{}^b e_b + \underline{\zeta}_a e_4, \tag{A.8e}$$

$$D_3 e_3 = 2\underline{\xi}^a e_a + 2\upsilon e_3, \tag{A.8f}$$

$$D_4 e_4 = 2\xi^a e_a + 2\underline{\upsilon} e_4, \tag{A.8g}$$

$$D_4 e_3 = 2\underline{\eta}^b e_b + 2\omega e_3, \tag{A.8h}$$

$$D_3 e_4 = 2\eta^b e_b + 2\underline{\omega} e_4. \tag{A.8i}$$

Here and elsewhere, $\slashed{\nabla}_a e_b$, $\slashed{\nabla}_3 e_a$ and $\slashed{\nabla}_4 e_a$ are defined as the orthogonal projection of the left-hand side of the corresponding equation to $\mathcal{S}$. We stress that no simplifying assumptions have been made concerning the nature of the vector fields e_a, except for the orthonormality relations (A.1).

A.2. The double-null decomposition of Weyl-type tensors. Let $d^i{}_{jkl}$ be any tensor field with the symmetries of the Weyl tensor,

$$d_{ijkl} = d_{klij}, \quad d_{ijkl} = -d_{jikl}, \quad g^{jk} d_{ijkl} = 0, \quad d_{i[jkl]} = 0. \tag{A.9}$$

We decompose $d^i{}_{jkl}$ into its null components, relative to the null pair $\{e_3, e_4\}$, as follows:

$$\underline{\alpha}(d)(X,Y) = d(X,e_3,Y,e_3), \qquad \alpha(d)(X,Y) = d(X,e_4,Y,e_4), \tag{A.10a}$$

$$\underline{\beta}(d)(X) = \tfrac{1}{2}d(X,e_3,e_3,e_4), \qquad \beta(d)(X) = \tfrac{1}{2}d(X,e_4,e_3,e_4), \tag{A.10b}$$

$$\rho(d) = \tfrac{1}{4}d(e_3,e_4,e_3,e_4), \qquad \sigma(d) := \rho({}^{\star_g}d) = \tfrac{1}{4}{}^{\star_g}d(e_3,e_4,e_3,e_4), \tag{A.10c}$$

where X, Y are arbitrary vector fields orthogonal to e_3 and e_4, while ^{*_g} denotes the space-time Hodge dual with respect to the first two indices of d_{ijkl}:

$${}^{\star_g}d_{ijkl} = \tfrac{1}{2}\epsilon_{ij}{}^{mn}d_{mnkl}. \tag{A.11}$$

The fields α and $\underline{\alpha}$ are symmetric and traceless. From equations (A.10) one finds

$$d_{a3b3} = \underline{\alpha}_{ab}, \qquad d_{a4b4} = \alpha_{ab}, \tag{A.12a}$$

$$d_{a334} = 2\underline{\beta}_a, \qquad d_{a434} = 2\beta_a, \tag{A.12b}$$

$$d_{3434} = 4\rho, \qquad d_{ab34} = 2\sigma\epsilon_{ab}, \tag{A.12c}$$

$$d_{abc3} = \epsilon_{ab}\,{}^{\star}\underline{\beta}_c, \qquad d_{abc4} = -\epsilon_{ab}\,{}^{\star}\beta_c, \tag{A.12d}$$

$$d^a{}_{3b4} = -\rho\delta^a_b + \sigma\epsilon^a{}_b, \qquad d_{abcd} = -\rho\epsilon_{ab}\epsilon_{cd}, \tag{A.12e}$$

where

$$\epsilon_{12} = -\epsilon_{21} = 1, \qquad \epsilon_{11} = \epsilon_{22} = 0. \tag{A.13}$$

Further, * denotes the Hodge dual on $\mathcal{S}$ with respect to the metric induced by g on $\mathcal{S}$:

$${}^{\star}\beta_a = \epsilon_a{}^b\beta_b. \tag{A.14}$$

A.3. The double-null decomposition of the Bianchi equations. Recall the second Bianchi identity for the Levi-Civita connection D,

$$D_iR_{jk\ell m} + D_jR_{ki\ell m} + D_kR_{ij\ell m} = 0. \tag{A.15}$$

Contracting i with m one obtains

$$D_iR_{jk\ell}{}^i + D_jR_{k\ell} - D_kR_{j\ell} = 0. \tag{A.16}$$

Inserting into this equation the expression for the Riemann tensor in terms of the Weyl and Ricci tensors,

$$R_{jk\ell}{}^i[g] = W_{jk\ell}{}^i + 2(g_{\ell[j}L_{k]}{}^i - \delta^i_{[j}L_{k]\ell}), \tag{A.17}$$

where

$$L_{ij} := \tfrac{1}{2}R_{ij} - \tfrac{1}{12}Rg_{ij}, \tag{A.18}$$

we obtain

$$D_iW^i{}_{jk\ell} = J_{jk\ell}, \tag{A.19}$$

where

$$J_{jkl} = D_{[j}R_{k]\ell} - \tfrac{1}{6}g_{\ell[k}D_{j]}R. \tag{A.20}$$

Here and elsewhere, square brackets around a set of ℓ indices denote antisymmetrization with a multiplicative factor $1/\ell!$.

Recall that the dual ${}^{\star_g}W^i{}_{jk\ell}$ of $W^i{}_{jk\ell}$ is defined as

$$ {}^{\star_g}W_{ijk\ell} := \tfrac{1}{2}\epsilon_{ijmn}W^{mn}{}_{k\ell}. $$

The well-known identity

$$ \epsilon_{ijmn}W^{mn}{}_{k\ell} = \epsilon_{k\ell mn}W^{mn}{}_{ij}, $$

together with (A.19), leads to

$$ D_i{}^{\star_g}W^i{}_{jk\ell} = {}^{\star_g}J_{jk\ell}, \tag{A.21} $$

where

$$ {}^{\star_g}J_{jmn} := \tfrac{1}{2}\epsilon_{mn}{}^{k\ell}J_{jk\ell} = \tfrac{1}{2}\epsilon_{mn}{}^{k\ell}(D_{[j}R_{k]\ell} - \tfrac{1}{6}g_{\ell[k}D_{j]}R). \tag{A.22} $$

Equations (A.19) and (A.21) are often referred to as the *Bianchi equations.*

We use, as in Section A.2, the symbol d_{ijkl} for the Weyl tensor W_{ijkl}. In vacuum, (A.19) becomes

$$ D_i(g^{im}d_{mjkl}) = g^{im}D_i d_{mjkl} = 0. \tag{A.23} $$

Equation (A.23) with $k = 3$ and $k = 4$ implies

$$ D_3 d_{43kl} = 2h^{ab}D_a d_{b3kl} - 2J_{3kl}, \tag{A.24a} $$

$$ D_4 d_{34kl} = 2h^{ab}D_a d_{b4kl} - 2J_{4kl}, \tag{A.24b} $$

which will give equations for β, $\underline{\beta}$, σ and ρ; we use the symbol h to denote the metric induced on $\mathcal{S}$ by g: for all $X, Y \in T\mathcal{M}$,

$$ h(X,Y) = g(X,Y) + \tfrac{1}{2}g(e_3,X)g(e_4,Y) + \tfrac{1}{2}g(e_4,X)g(e_3,Y). \tag{A.25} $$

The equations for α_{ab} and $\underline{\alpha}_{ab}$ can be obtained from

$$ D_i d^i{}_{ab4} = J_{ab4}. \tag{A.26} $$

For any tensor field T_{ab} we denote by $\overline{T_{ab}}$ the symmetric traceless part of T_{ab}, and by $\operatorname{tr} T$ its trace. As already pointed out, we set

$$ \nabla\!\!\!/_3\beta_a := e_3(\beta_a) - \Gamma_3{}^b{}_a\beta_b, \tag{A.27} $$

$$ \nabla\!\!\!/_3\alpha_{ab} := e_3(\alpha_{ab}) - \Gamma_3{}^c{}_a\alpha_{cb} - \Gamma_3{}^c{}_b\alpha_{ac}. \tag{A.28} $$

Following Christodoulou and Klainerman [8], we use the notation $\eta \mathbin{\overline{\otimes}_s} \beta$ for *twice* the trace-free symmetric tensor product of vectors,

$$ (X \mathbin{\overline{\otimes}_s} Y)^{ab} = X^aY^b + X^bY^a - g^{ab}X_cY^c, \tag{A.29} $$

and similarly for covectors. We let $\nabla\!\!\!/$ be the orthogonal projection on $\mathcal{S}$ of the relevant covariant derivatives in directions tangent to $\mathcal{S}$, e.g.

$$ \nabla\!\!\!/_a e_b = \Gamma_a{}^c{}_b e_c. \tag{A.30} $$

Tedious but otherwise straightforward calculations allow one to obtain the equations satisfied by the tensor field d, listed out as (A.34) below. A useful symmetry principle, which allows one to reduce the number of calculations by half, is to note that under the interchange of e_3 with e_4 the underlined rotation coefficients in (A.7) are exchanged with the nonunderlined ones. On the other hand, the null components of the tensor d transform as follows:

$$\alpha \leftrightarrow \underline{\alpha}, \qquad \rho \leftrightarrow \rho, \qquad \beta \leftrightarrow -\underline{\beta}, \qquad \sigma \leftrightarrow -\sigma. \tag{A.31}$$

A convenient identity in the relevant manipulations is

$$\slashed{\nabla}_c \epsilon_{ab} = -2 f_c \epsilon_{ab} = -(\zeta_c + \underline{\zeta}_c)\epsilon_{ab}, \tag{A.32}$$

as well as

$$\slashed{\nabla}_c {\epsilon_a}^b = 0. \tag{A.33}$$

The dynamical equations obtained by the doubly-null decomposition of equation (A.19) read ([3]):

$$\begin{aligned}
\slashed{\nabla}_4 \underline{\alpha} &= -\tfrac{1}{2}\operatorname{tr}\chi \underline{\alpha} - \slashed{\nabla}\, \overline{\otimes}_s\, \underline{\beta} + (2\omega - 2\underline{\upsilon})\underline{\alpha} \\
&\quad - 3(\underline{\overline{\chi}}\rho - {}^\star\underline{\overline{\chi}}\sigma) - (4\underline{\eta} - \zeta)\, \overline{\otimes}_s\, \underline{\beta} + 2\overline{J(\cdot,\cdot,e_3)}, &\text{(A.34a)}\\
\slashed{\nabla}_3 \underline{\beta} &= -2\operatorname{tr}\underline{\chi}\underline{\beta} - \slashed{\mathrm{div}}\,\underline{\alpha} + 2\upsilon\underline{\beta} - \underline{\alpha}\cdot(\eta - 2\zeta) + 3(-\underline{\xi}\rho + {}^\star\underline{\xi}\sigma) \\
&\quad - J(e_3,\cdot,e_3), &\text{(A.34b)}\\
\slashed{\nabla}_4 \underline{\beta} &= -\operatorname{tr}\chi\underline{\beta} - \slashed{\nabla}\rho + {}^\star\slashed{\nabla}\sigma + 2\underline{\overline{\chi}}\cdot\beta + 2\omega\underline{\beta} + 3(-\underline{\eta}\rho + {}^\star\underline{\eta}\sigma) \\
&\quad + (\zeta + \underline{\zeta})\rho - ({}^\star\zeta + {}^\star\underline{\zeta})\sigma - \xi\cdot\underline{\alpha} + J(e_4,e_3,\cdot), &\text{(A.34c)}\\
\widehat{D}_3\rho &= -\tfrac{3}{2}\operatorname{tr}\underline{\chi}\rho - \slashed{\mathrm{div}}\,\underline{\beta} - \tfrac{1}{2}\overline{\chi}\cdot\underline{\alpha} + (2\zeta + \underline{\zeta} - 2\eta)\cdot\underline{\beta} \\
&\quad + 2\underline{\xi}\cdot\beta + 4(\upsilon + \underline{\omega})\rho + \tfrac{1}{2}J_{334}, &\text{(A.34d)}\\
\widehat{D}_4\rho &= -\tfrac{3}{2}\operatorname{tr}\chi\rho + \slashed{\mathrm{div}}\,\beta - \tfrac{1}{2}\underline{\overline{\chi}}\cdot\alpha - (2\underline{\zeta} + \zeta - 2\underline{\eta})\cdot\beta \\
&\quad - 2\xi\cdot\underline{\beta} + 4(\underline{\upsilon} + \omega)\rho + \tfrac{1}{2}J_{443}, &\text{(A.34e)}\\
\widehat{D}_3\sigma &= -\tfrac{3}{2}\operatorname{tr}\underline{\chi}\sigma - \slashed{\mathrm{div}}\,{}^\star\underline{\beta} + 2(\underline{\omega} + \upsilon)\sigma - \tfrac{1}{2}{}^t\chi\cdot{}^\star\underline{\alpha} - 2\underline{\xi}\cdot{}^\star\beta \\
&\quad + (\underline{\zeta} + 2\zeta - 2\eta)\cdot{}^\star\underline{\beta} - \tfrac{1}{2}a(J(e_3,\cdot,\cdot)), &\text{(A.34f)}\\
\widehat{D}_4\sigma &= -\tfrac{3}{2}\operatorname{tr}\chi\sigma - \slashed{\mathrm{div}}\,{}^\star\beta + 2(\omega + \underline{\upsilon})\sigma + \tfrac{1}{2}{}^t\underline{\chi}\cdot{}^\star\alpha - 2\xi\cdot{}^\star\underline{\beta} \\
&\quad + (\zeta + 2\underline{\zeta} - 2\underline{\eta})\cdot{}^\star\beta - \tfrac{1}{2}\epsilon^{ab}J_{4ab}, &\text{(A.34g)}\\
\slashed{\nabla}_3\beta &= -\operatorname{tr}\underline{\chi}\beta + \slashed{\nabla}\rho + {}^\star\slashed{\nabla}\sigma + 2\overline{\chi}\cdot\underline{\beta} + 2\underline{\omega}\beta + 3(\eta\rho + {}^\star\eta\sigma) \\
&\quad - (\zeta + \underline{\zeta})\rho - ({}^\star\zeta + {}^\star\underline{\zeta})\sigma + \underline{\xi}\cdot\alpha - J(e_3,e_4,\cdot), &\text{(A.34h)}\\
\slashed{\nabla}_4\beta &= -2\operatorname{tr}\chi\beta + \slashed{\mathrm{div}}\,\alpha + 2\underline{\upsilon}\beta + \alpha\cdot(\underline{\eta} - 2\underline{\zeta}) + 3(\xi\rho + {}^\star\xi\sigma) \\
&\quad - J(e_4,\cdot,e_4), &\text{(A.34i)}\\
\slashed{\nabla}_3\alpha &= -\tfrac{1}{2}\operatorname{tr}\underline{\chi}\alpha + \slashed{\nabla}\, \overline{\otimes}_s\, \beta + (2\underline{\omega} - 2\upsilon)\alpha \\
&\quad - 3(\overline{\chi}\rho + {}^\star\overline{\chi}\sigma) + (4\eta - \underline{\zeta})\, \overline{\otimes}_s\, \beta + 2\overline{J(\cdot,\cdot,e_4)}. &\text{(A.34j)}
\end{aligned}$$

([3]) Equations (A.34) are essentially a subset of the Newman–Penrose equations written out in a tensor formalism. The equations in [8] or in [31] can be obtained from (A.34) by specialization and straightforward changes of notation. We have corrected some inessential misprints in the equations of [31].

For the convenience of the reader we give a summary of the notation used: The operators $\nabla\!\!\!/_4$ and $\nabla\!\!\!/_3$ are defined as the orthogonal projections on $\mathcal{S}$ of the D-covariant derivatives along the null directions e_3 and e_4, e.g.

$$\nabla\!\!\!/_3 e_a = \Gamma_3{}^b{}_a e_b, \qquad \nabla\!\!\!/_4 e_a = \Gamma_4{}^b{}_a e_b.$$

In particular

$$\nabla\!\!\!/_3 \rho = \widehat{D}_3 \rho = e_3(\rho), \qquad \nabla\!\!\!/_3 \sigma = \widehat{D}_3 \sigma = e_3(\sigma),$$

etc., with $\nabla\!\!\!/_3 \beta$ and $\nabla\!\!\!/_3 \alpha_{ab}$ written out explicitly in equations (A.27) and (A.28). Next, the $\nabla\!\!\!/_a$'s are differential operators in directions tangent to $\mathcal{S}$ defined as the orthogonal projection on $\mathcal{S}$ of the relevant covariant derivatives in directions tangent to $\mathcal{S}$ (cf. (A.30)). We use the symbol $\mathrm{d}\!\!/\mathrm{iv}$ to denote the "$\mathcal{S}$-divergence" operator: if $X = X^a e_a$ and $Y = Y^{ab} e_a \otimes e_b$ then

$$\mathrm{d}\!\!/\mathrm{iv}\, X = \nabla\!\!\!/_a X^a, \qquad \mathrm{d}\!\!/\mathrm{iv}\, Y = (\nabla\!\!\!/_a Y^{ab}) e_b.$$

We have also set

$${}^t\chi^{ab} = \chi^{ba}.$$

Next, a bar over a valence-two tensor denotes its *symmetric traceless part*, e.g.

$$\overline{\underline{\chi}}_{ab} = \tfrac{1}{2}\{\underline{\chi}_{ab} + \underline{\chi}_{ba} - g^{cd}\underline{\chi}_{cd} g_{ab}\},$$

while, for any two-index tensor χ_{ab},

$$a(\chi) = \varepsilon^{ab}\chi_{ab}.$$

To avoid ambiguities, we emphasize that in equations (A.34) the free slot in J, whenever occurring, refers to vectors in $\mathcal{S}$, in particular

$$a(J(e_4,\cdot,\cdot)) := \epsilon^{ab} J_{4ab}, \qquad a(J(e_3,\cdot,\cdot)) := \epsilon^{ab} J_{3ab}.$$

Finally the symbol $\overline{\otimes}_s$ has been defined in (A.29).

A.4. Bianchi equations and symmetric hyperbolic systems. Let us now turn to a specific null reformulation of the equations at hand. Let α, β, etc. be the null components of d, and for reasons which will become apparent below introduce

$$\mathring{\beta} := \beta, \quad \mathring{\underline{\beta}} := \underline{\beta}, \tag{A.35a}$$

$$\mathring{\sigma} := \sigma, \quad \mathring{\rho} := \rho. \tag{A.35b}$$

A convenient doubly-null form of equations (A.34) is obtained, in vacuum, by rewriting (A.34) using (A.35) as follows ([4]):

$$\begin{aligned} \nabla\!\!\!/_4 \underline{\alpha} + \tfrac{1}{2}\operatorname{tr}\chi\underline{\alpha} &= -\nabla\!\!\!/\,\overline{\otimes}_s\, \underline{\beta} + (2\omega - 2\underline{\upsilon})\underline{\alpha} - 3(\overline{\underline{\chi}}\rho - {}^\star\overline{\underline{\chi}}\sigma) \\ &\quad - (4\underline{\eta} - \zeta)\,\overline{\otimes}_s\, \underline{\beta} + 2\overline{J(\cdot,\cdot,e_3)}, \end{aligned} \tag{A.36a}$$

$$\begin{aligned} \nabla\!\!\!/_3 \underline{\beta} + 2\operatorname{tr}\underline{\chi}\underline{\beta} &= -\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha} + 2\upsilon\underline{\beta} - \underline{\alpha}\cdot(\eta - 2\zeta) + 3(-\underline{\xi}\rho + {}^\star\underline{\xi}\sigma) \\ &\quad - J(e_3,\cdot,e_3), \end{aligned} \tag{A.36b}$$

([4]) There is a certain amount of freedom which undifferentiated terms on the right should be decorated with "∘"'s, which is irrelevant for our purposes in this work.

$$
\begin{aligned}
\not\nabla_4\mathring{\underline{\beta}} + \operatorname{tr}\chi\mathring{\underline{\beta}} &= -\not\nabla\mathring{\rho} + {}^\star\not\nabla\mathring{\sigma} + 2\overline{\underline{\chi}}\cdot\beta + 2\omega\mathring{\underline{\beta}} + 3(-\underline{\eta}\mathring{\rho} + {}^\star\underline{\eta}\mathring{\sigma}) \\
&\quad + (\zeta+\underline{\zeta})\mathring{\rho} - ({}^\star\zeta + {}^\star\underline{\zeta})\mathring{\sigma} - \xi\cdot\underline{\alpha} + J(e_4,e_3,\cdot), && \text{(A.37a)} \\
D_3\mathring{\sigma} + \tfrac{3}{2}\operatorname{tr}\underline{\chi}\mathring{\sigma} &= -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\mathring{\underline{\beta}} + 2(\underline{\omega}+\upsilon)\mathring{\sigma} - \tfrac{1}{2}{}^t\chi\cdot{}^\star\underline{\alpha} - 2\underline{\xi}\cdot{}^\star\beta \\
&\quad + (\underline{\zeta} + 2\zeta - 2\eta)\cdot{}^\star\mathring{\underline{\beta}} - \tfrac{1}{2}a(J(e_3,\cdot,\cdot)), && \text{(A.37b)} \\
D_3\mathring{\rho} + \tfrac{3}{2}\operatorname{tr}\underline{\chi}\mathring{\rho} &= -\not{\mathrm{d}}\mathrm{iv}\,\mathring{\underline{\beta}} - \tfrac{1}{2}\overline{\chi}\cdot\underline{\alpha} + (2\zeta + \underline{\zeta} - 2\eta)\cdot\mathring{\underline{\beta}} \\
&\quad + 2\underline{\xi}\cdot\beta + 4(\upsilon+\underline{\omega})\mathring{\rho} + \tfrac{1}{2}J_{334}, && \text{(A.37c)} \\
D_4\rho + \tfrac{3}{2}\operatorname{tr}\chi\rho &= \not{\mathrm{d}}\mathrm{iv}\,\beta - \tfrac{1}{2}\overline{\underline{\chi}}\cdot\alpha - (2\underline{\zeta} + \zeta - 2\underline{\eta})\cdot\beta \\
&\quad - 2\xi\cdot\underline{\beta} + 4(\underline{\upsilon}+\omega)\rho + \tfrac{1}{2}J_{443}, && \text{(A.38a)} \\
D_4\sigma + \tfrac{3}{2}\operatorname{tr}\chi\sigma &= -\not{\mathrm{d}}\mathrm{iv}\,{}^\star\beta + 2(\omega+\underline{\upsilon})\sigma + \tfrac{1}{2}{}^t\underline{\chi}\cdot{}^\star\alpha - 2\xi\cdot{}^\star\underline{\beta} \\
&\quad + (\zeta + 2\underline{\zeta} - 2\underline{\eta})\cdot{}^\star\beta - \tfrac{1}{2}\epsilon^{ab}J_{4ab}, && \text{(A.38b)} \\
\not\nabla_3\beta + \operatorname{tr}\underline{\chi}\beta &= \not\nabla\rho + {}^\star\not\nabla\sigma + 2\overline{\chi}\cdot\underline{\beta} + 2\underline{\omega}\beta + 3(\eta\rho + {}^\star\eta\sigma) \\
&\quad - (\zeta+\underline{\zeta})\rho - ({}^\star\zeta + {}^\star\underline{\zeta})\sigma + \underline{\xi}\cdot\alpha - J(e_3,e_4,\cdot), && \text{(A.38c)} \\
\not\nabla_4\mathring{\beta} + 2\operatorname{tr}\chi\mathring{\beta} &= \not{\mathrm{d}}\mathrm{iv}\,\alpha + 2\underline{\upsilon}\mathring{\beta} + \alpha\cdot(\underline{\eta} - 2\underline{\zeta}) + 3(\xi\rho + {}^\star\xi\sigma) \\
&\quad - J(e_4,\cdot,e_4), && \text{(A.39a)} \\
\not\nabla_3\alpha + \tfrac{1}{2}\operatorname{tr}\underline{\chi}\alpha &= \not\nabla\,\overline{\otimes}_s\,\mathring{\beta} + (2\underline{\omega} - 2\upsilon)\alpha - 3(\overline{\chi}\rho + {}^\star\overline{\chi}\sigma) \\
&\quad + (4\eta - \underline{\zeta})\,\overline{\otimes}_s\,\beta + 2\overline{J(\cdot,\cdot,e_4)}. && \text{(A.39b)}
\end{aligned}
$$

We have kept the source terms J for future reference; however, in vacuum, which is of interest here, we have $J \equiv 0$.

Let us show that the principal part of each of the systems (A.36)–(A.39) is symmetric hyperbolic, and of the form required in our analysis, when the scalar products are appropriately chosen.

1. *The* $(\underline{\alpha},\underline{\beta})$ *equations* (A.36). We have $\underline{\alpha}_{12} = \underline{\alpha}_{21}$ and $\underline{\alpha}_{11} = -\underline{\alpha}_{22}$, hence the pair $(\underline{\alpha},\underline{\beta})$ can be parameterized by $f = (\underline{\alpha}_{11},\underline{\alpha}_{12},\underline{\beta}_1,\underline{\beta}_2)$. Equations (A.36) can be rewritten as

$$A^\mu\partial_\mu f + Af = F \tag{A.40}$$

with

$$A^\mu\partial_\mu = \begin{pmatrix} e_4 & 0 & e_1 & -e_2 \\ 0 & e_4 & e_2 & e_1 \\ e_1 & e_2 & e_3 & 0 \\ -e_2 & e_1 & 0 & e_3 \end{pmatrix}, \tag{A.41}$$

which is obviously symmetric with respect to the scalar product

$$
\begin{aligned}
\langle f,f\rangle &= \underline{\alpha}_{11}^2 + \underline{\alpha}_{12}^2 + \underline{\beta}_1^2 + \underline{\beta}_2^2 && \text{(A.42a)} \\
&= \tfrac{1}{2}h^{ac}h^{bd}\underline{\alpha}_{ab}\underline{\alpha}_{cd} + h^{ab}\underline{\beta}_a\underline{\beta}_b. && \text{(A.42b)}
\end{aligned}
$$

2. *The* $(\mathring{\underline{\beta}},(\mathring{\sigma},\mathring{\rho}))$ *equations* (A.37). The analysis is obtained by obvious renamings and permutations from that of (A.38), leading to a system with identical principal part.

3. *The* $((\rho, \sigma), \beta)$ *equations* (A.38). We set $f = ((\rho, \sigma), \beta) = (\rho, \sigma, \beta_1, \beta_2)$. Equation (A.38) can be rewritten in the form (A.40) with

$$A^\mu \partial_\mu = \begin{pmatrix} e_4 & 0 & -e_1 & -e_2 \\ 0 & e_4 & -e_2 & e_1 \\ -e_1 & -e_2 & e_3 & 0 \\ -e_2 & e_1 & 0 & e_3 \end{pmatrix}, \tag{A.43}$$

which is obviously symmetric with respect to the scalar product

$$\begin{aligned} \langle f, f \rangle &= \rho^2 + \sigma^2 + \beta_1^2 + \beta_2^2 \\ &= \rho^2 + \sigma^2 + h^{ab}\beta_a\beta_b. \end{aligned}$$

4. *The* $(\mathring{\beta}, \alpha)$ *equations* (A.39). The analysis here is obtained by obvious renamings and permutations from that of (A.36), done above.

References

[1] A. Cabet, *Local existence of a solution of a semi-linear wave equation in a neighborhood of initial characteristic hypersurfaces*, Ann. Fac. Sci. Toulouse Math. (6) 12 (2003), 47–102.

[2] A. Cabet, *Local existence of a solution of a semilinear wave equation with gradient in a neighborhood of initial characteristic hypersurfaces of a Lorentzian manifold*, Comm. Partial Differential Equations 33 (2008), 2105–2156.

[3] G. Caciotta and F. Nicolò, *Global characteristic problem for Einstein vacuum equations with small initial data. I. The initial data constraints*, J. Hyperbolic Differential Equations 2 (2005), 201–277; arXiv:gr-qc/0409028.

[4] G. Caciotta and F. Nicolò, *On a class of global characteristic problems for the Einstein vacuum equations with small initial data*, J. Math. Phys. 51 (2010), 102503, 21 pp.; arXiv: gr-qc/0608038.

[5] F. Cagnac, *Problème de Cauchy sur les hypersurfaces caractéristiques des équations d'Einstein du vide*, C. R. Acad. Sci. Paris Sér. A-B 262 (1966), A1488–A1491.

[6] F. Cagnac, *Problème de Cauchy sur les hypersurfaces caractéristiques des équations d'Einstein du vide*, C. R. Acad. Sci. Paris Sér. A-B 262 (1966), A1356–A1359.

[7] Y. Choquet-Bruhat, P. T. Chruściel, and J. M. Martín-García, *The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions*, Ann. Henri Poincaré 12 (2011), 419–482; arXiv:1006.4467.

[8] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, Princeton Univ. Press, Princeton, NJ, 1993.

[9] D. Christodoulou et H. Müller zum Hagen, *Problème de valeur initiale caractéristique pour des systèmes quasi linéaires du second ordre*, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), 39–42.

[10] P. T. Chruściel, *The existence theorem for the general relativistic Cauchy problem on the light-cone*, Forum Math. Sigma 2 (2014), e10, 50 pp.; arXiv:1209.1971 [gr-qc].

[11] P. T. Chruściel and O. Lengard, *Solutions of Einstein equations polyhomogeneous at Scri*, in preparation.

[12] P. T. Chruściel and T.-T. Paetz, *The many ways of the characteristic Cauchy problem*, Classical Quantum Gravity 29 (2012), 145006, 27 pp.; arXiv:1203.4534 [gr-qc].

[13] P. T. Chruściel and T.-T. Paetz, *Solutions of the vacuum Einstein equations with initial data on past null infinity*, Classical Quantum Gravity 30 (2013), 235037, 14 pp.; arXiv:1307.0321 [gr-qc].

[14] P. T. Chruściel and T.-T. Paetz, *Characteristic initial data and smoothness of Scri. I. Framework and results*, Ann. Henri Poincaré 16 (2015), 2131–2162; arXiv:1403.3558 [gr-qc].

[15] T. Damour and B. Schmidt, *Reliability of perturbation theory in general relativity*, J. Math. Phys. 31 (1990), 2441–2453.

[16] G. Dautcourt, *Zum charakteristischen Anfangswertproblem der Einsteinschen Feldgleichungen*, Ann. Physik (7) 12 (1963), 302–324.

[17] M. Dossa, *Problèmes de Cauchy sur un conoïde caractéristique pour les équations d'Einstein (conformes) du vide et pour les équations de Yang–Mills–Higgs*, Ann. Henri Poincaré 4 (2003), 385–411.

[18] M. Dossa et S. Bah, *Solutions de problèmes de Cauchy semi-linéaires hyperboliques sur un conoïde caractéristique*, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 179–184.

[19] M. Dossa and R. Tagne Wafo, *Solutions with a uniform time of existence of a class of characteristic semi-linear wave equations near S^+*, arXiv:1307.6412 [math.AP] (2013).

[20] M. Dossa et F. Touadera, *Solutions globales de systèmes hyperboliques non linéaires sur un cône caractéristique*, C. R. Math. Acad. Sci. Paris 341 (2005), 409–414.

[21] H. Friedrich, *The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system*, Proc. Roy. Soc. London Ser. A 378 (1981), 401–421.

[22] H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. Roy. Soc. London Ser. A 375 (1981), 169–184.

[23] H. Friedrich, *On purely radiative space-times*, Comm. Math. Phys. 103 (1986), 35–65.

[24] H. Friedrich, *Einstein equations and conformal structure: Existence of anti-de-Sitter-type space-times*, J. Geom. Phys. 17 (1995), 125–184.

[25] H. Friedrich, *Gravitational fields near space-like and null infinity*, J. Geom. Phys. 24 (1998), 83–163.

[26] H. Friedrich, *Smoothness at null infinity and the structure of initial data*, in: The Einstein Equations and the Large Scale Behavior of Gravitational Fields, P. T. Chruściel and H. Friedrich (eds.), Birkhäuser, Basel, 2004, 121–203; arXiv:gr-qc/0304003.

[27] H. Friedrich, *The Taylor expansion at past time-like infinity*, Comm. Math. Phys. 324 (2013), 263–300; arXiv:1306.5626 [gr-qc].

[28] S. A. Hayward, *The general solution to the Einstein equations on a null surface*, Classical Quantum Gravity 10 (1993), 773–778.

[29] D. E. Houpa Danga et M. Dossa, *Problèmes de Goursat pour des systèmes semi-linéaires hyperboliques*, C. R. Math. Acad. Sci. Paris 341 (2005), 15–20.

[30] D. E. Houpa Danga et M. Dossa, *Solutions semi-globales de problèmes de Goursat semi-linéaires hyperboliques. I. Solutions semi-globales du système intégral des formules de Kirchhoff associées*, IMHOTEP J. Afr. Math. Pures Appl. 7 (2007), 15–100.

[31] S. Klainerman and F. Nicolò, *The Evolution Problem in General Relativity*, Progr. Math. Phys. 25, Birkhäuser, Boston, MA, 2003.

[32] J. Li and X.-P. Zhu, *On the local extension of the future null infinity*, arXiv:1406.0048 (2014).

[33] J. Luk, *On the local existence for the characteristic initial value problem in general relativity*, Int. Math. Res. Notices 2012, 4625–4678.

[34] H. Müller zum Hagen and H.-J. Seifert, *On characteristic initial-value and mixed problems*, Gen. Relativity Gravitation 8 (1977), 259–301.

[35] T.-T. Paetz, *Conformally covariant systems of wave equations and their equivalence to Einstein's field equations*, Ann. Henri Poincaré 16 (2015), 2059–2129; arXiv:1306.6204 [gr-qc].

[36] T.-T. Paetz, *Characteristic initial data and smoothness of Scri. II. Asymptotic expansions and construction of conformally smooth data sets*, J. Math. Phys. 55 (2014), 102503, 46 pp.; arXiv:1403.3560 [gr-qc].

[37] R. Penrose, *Null hypersurface initial data for classical fields of arbitrary spin and for general relativity*, Gen. Relativity Gravitation 12 (1980), 225–264.

[38] A. D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations*, Proc. Roy. Soc. London Ser. A 427 (1990), 221–239.

[39] R. K. Sachs, *On the characteristic initial value problem in gravitational theory*, J. Math. Phys. 3 (1962), 908–914.