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COMPONENT CLUSTERS FOR ACYCLIC QUIVERS

BY

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Abstract. The theory of Caldero–Chapoton algebras of Cerulli Irelli, Labardini-Fragoso and Schröer (2015) leads to a refinement of the notions of cluster variables and clusters, via so-called component clusters. We compare component clusters to classical clusters for the cluster algebra of an acyclic quiver. We propose a definition of mutation between component clusters and determine the mutation relations of component clusters for affine quivers. In the case of a wild quiver, we provide bounds for the size of component clusters.

1. Introduction. In [7] Cerulli Irelli, Labardini-Fragoso and Schröer propose a broad generalization of the theory of cluster algebras [12]. They give a recipe to attach to any basic algebra Λ a subalgebra $\mathcal{A}_\Lambda$ of a ring of rational functions: $\mathcal{A}_\Lambda$ is the *Caldero–Chapoton algebra* of Λ. Similarly to cluster algebras, Caldero–Chapoton algebras come with an interesting collection of sets of generators which are called *CC-clusters*. In this paper, we investigate various properties of Caldero–Chapoton algebras and CC-clusters in the special case when Λ is the path algebra of an acyclic quiver.

Note that if Q is an acyclic quiver and $\Lambda = kQ$ is its path algebra, the Caldero–Chapoton algebra of Λ is equal to the ordinary cluster algebra $\mathcal{A}_Q$ of Q.

However even in this case the set of generators of $\mathcal{A}_Q$ that we obtain by viewing it as a Caldero–Chapoton algebra is larger than the set of classical cluster variables. Further, contrary to classical clusters, CC-clusters can have smaller cardinality than the vertex set of Q: the classical clusters of $\mathcal{A}_Q$ coincide with the CC-clusters of maximal size.

The construction of the CC-clusters in [7] builds on work of Caldero, Chapoton and Keller on the cluster character [5], [6]. The authors first introduce *component clusters*, which are families of irreducible components of the representation varieties of Λ having some special properties. CC-clusters

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are then obtained by applying the Caldero–Chapoton map to the component clusters (see [7] for a fuller explanation).

In this paper we study the structure of component clusters when $\Lambda = kQ$, and Q is an acyclic quiver. We show that, as a consequence of Kac's theorem [15], component clusters are in bijection with sets of pairwise ext-orthogonal distinct Schur roots. Hence component clusters are closely linked with generic decompositions of dimension vectors [15], which have been studied also by Schofield [20] and more recently by Derksen and Weyman [9].

When Q is affine, we give a complete description of component clusters: they are either of size n or $n-1$, where n is the number of vertices of Q. Component clusters are of size $n-1$ if and only if they contain the unique positive isotropic Schur root. The situation is considerably more complicated when Q is of wild type. However, in this case, we are able to obtain an optimal upper bound for the number of imaginary Schur roots appearing in a component cluster. We also show that, if Q is of wild type, we always have an infinite number of component clusters of size one. Further, motivated by the exchange relations between cluster variables, we give a definition of exchange relations between component clusters. For affine quivers, we explicitly determine these exchange relations.

The paper is structured as follows: In Section 2, we recall Kac's generic decomposition theorem and classical results on root systems of quivers. We introduce negative Schur roots in order to define generic decompositions for any vector in $\mathbb{Z}^n$. In Section 3, we determine the cluster components for affine quivers. In Section 4, we study the sizes of component clusters if Q is of wild type. Finally, in Section 5, we define mutations of component clusters and give an interpretation of exchange relations between two component clusters that are connected by a mutation. We work out the exact exchange relations for affine quivers.

2. Generalized generic decompositions and cluster components of quivers

2.1. Back to the roots. In this section, we introduce notation and recall some basic facts on root systems of acyclic quivers. We refer to [1], [21] and [22] for a complete introduction to the representation theory of finite, affine and wild quivers and the related background.

Throughout this paper, Q is a finite quiver without oriented cycles, and k is a field. We denote the set of vertices by Q_0. We assume that the vertices are equipped with a total order and we denote them by $1, \dots, n$. We denote by Q_1 the set of arrows. Furthermore, $s, t : Q_1 \to Q_0$ are the maps which send an arrow to its source and to its target respectively.

To a dimension vector $d : Q_0 \to \mathbb{N}$ we associate the variety of representations

$$\operatorname{rep}_d Q := \bigoplus_{a \in Q_1} \operatorname{Hom}(k^{d(s(a))}, k^{d(t(a))}).$$

It is a finite-dimensional vector space, hence an irreducible affine variety. There is a canonical action of $\prod_{i \in Q_0} \operatorname{Gl}_{d(i)}(k)$ on $\operatorname{rep}_d Q$ having the property that the G_d orbits are in bijection with the isomorphism classes of kQ-modules of dimension vector d.

The *support* of a dimension vector $d \in \mathbb{N}^n$ is the subset

$$\operatorname{supp} d := \{i \in Q_0 \mid d(i) \neq 0\}$$

of Q_0. We say that the support is *connected* if the full subquiver of Q generated by the vertices belonging to $\operatorname{supp} d$ is connected. A dimension vector d is called a *root* if $\operatorname{rep}_d Q$ contains an indecomposable representation. A *Schur root* is a dimension vector of a representation whose endomorphism ring is isomorphic to k. Such a representation is necessarily indecomposable and is called a *Schurian representation.*

Let d and b be dimension vectors. The functions

$$\begin{aligned} hom(-,-) : \operatorname{rep}_d Q \times \operatorname{rep}_b Q \to \mathbb{N}, &\quad (M,N) \mapsto \dim hom_Q(M,N), \\ ext(-,-) : \operatorname{rep}_d Q \times \operatorname{rep}_b Q \to \mathbb{N}, &\quad (M,N) \mapsto \dim \operatorname{Ext}^1_Q(M,N), \\ end(-) : \operatorname{rep}_d Q \to \mathbb{N}, &\quad N \mapsto \dim \operatorname{End}_Q(N), \end{aligned}$$

are upper semicontinuous. Hence there are open subsets in $\operatorname{rep}_d Q \times \operatorname{rep}_b Q$ on which ext and hom are constant of minimal value, and there is an open subset of $\operatorname{rep}_d Q$ on which end is constant of minimal value. We set $ext(d,b)$, $hom(d,b)$ and $end(d)$ to be the minimal value of these functions. Note that all open subsets of an irreducible variety are dense.

We call two dimension vectors a and b *ext-orthogonal* if $ext(a,b)$ and $ext(b,a)$ vanish. It also follows from upper semicontinuity that for any Schur root d there is a dense open subset of Schurian representations in $\operatorname{rep}_d Q$. Let n be the cardinality of Q_0. The *Euler form* is the bilinear form

$$\langle -,- \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}, \quad (a,b) \mapsto \sum_{i \in Q_0} a_i b_i - \sum_{f \in Q_1} a_{s(f)} b_{t(f)}.$$

By [20], the Euler form can alternatively be described by the formula

$$\langle a,b \rangle = \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}(N,M)$$

for any $M \in \operatorname{rep}_a Q$ and any $N \in \operatorname{rep}_b Q$. As two open sets intersect non-trivially in an irreducible variety, we also have the identity

$$\langle a,b \rangle = hom(a,b) - ext(a,b).$$

The *symmetrized Euler form* is the bilinear form on $\mathbb{Z}^n \times \mathbb{Z}^n$ given by

$$(a,b) \mapsto \langle a,b \rangle + \langle b,a \rangle,$$

and the *Tits form* is the quadratic form $q(a) := \langle a, a\rangle$, for all $a, b \in \mathbb{Z}^n$. Roots are classified by their Tits form into three types. We refer to a root d as *real* if $q(d) = 1$, *imaginary* if $q(d) \leq 0$, and *isotropic* if $q(d) = 0$. A representation M in $\operatorname{rep}_d Q$ is *rigid* if $\operatorname{Ext}^1(M, M)$ vanishes. This is the case if and only if the representations isomorphic to M form an open subset of $\operatorname{rep}_d Q$. If d is a root and $\operatorname{rep}_d Q$ contains a rigid representation, then d is a real Schur root. Conversely, if d is a real Schur root then $\operatorname{rep}_d Q$ contains a rigid representation M, which is necessarily Schurian. Further, all Schurian representations of dimension vector d are isomorphic to M.

Kac's generic decomposition theorem shows that Schur roots play an important role in understanding the variety of representations of a quiver:

Theorem 2.1 ([15]).

(1) *Every dimension vector d has a unique decomposition*
$$d = d_1 \oplus \cdots \oplus d_s$$
as a sum of Schur roots d_i such that the image of the natural embedding
$$\coprod_{i=1}^{s} \operatorname{rep}_{d_i} Q \to \operatorname{rep}_d Q, \qquad (M_1, \dots, M_s) \mapsto \bigoplus_{i=1}^{s} M_i,$$
is an open set. In this case the generic extensions $ext(d_i, d_j)$ vanish for all $i \neq j$.

(2) *Conversely, every decomposition of d into a sum of Schur roots d_i such that the generic extensions $ext(d_i, d_j)$ vanish for all $i \neq j$ gives rise to an open embedding of $\coprod_{i=1}^{s} \operatorname{rep}_{d_i} Q$ into $\operatorname{rep}_d Q$.*

This unique decomposition of a dimension vector into a sum of Schur roots is called the *generic decomposition.*

We will use the following standard notation. We denote by P_i, I_i and S_i respectively the projective indecomposable, injective indecomposable and simple module associated to the vertex $i \in Q_0$.

2.2. The cluster category. Here we briefly summarize the relations between quiver representations and the theory of cluster algebras.

We refer to [17] for a fuller account. We assume that the ground field k has characteristic 0. Let $\mathcal{D}_Q$ denote the bounded derived category of kQ-modules. It is a triangulated category and we denote its suspension functor by $\Sigma : \mathcal{D}_Q \to \mathcal{D}_Q$. As kQ has finite global dimension, Auslander–Reiten triangles exist in $\mathcal{D}_Q$ by [13, Theorem 1.4]. We denote the Auslander–Reiten translation of $\mathcal{D}_Q$ by τ. On non-projective modules, it coincides with the Auslander–Reiten translation of $\operatorname{mod} kQ$. The *cluster category* [4]
$$\mathcal{C}_Q = \mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$$

is the orbit category of $\mathcal{D}_Q$ under the action of the cyclic group generated by $\tau^{-1}\Sigma$. One can show [16] that $\mathcal{C}_Q$ admits a canonical structure of triangulated category such that the projection functor $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a functor of triangulated categories.

We refer to [6] for the definition of the cluster character $L \mapsto X_L$ from the set of isomorphism classes of objects L of $\mathcal{C}_Q$ to the ring of Laurent polynomials $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We have $X_{\tau P_i} = x_i$ for all vertices i of Q, and $X_{M \oplus N} = X_M X_N$ for all objects M and N of $\mathcal{C}_Q$. We call an object M in $\mathcal{C}_Q$ *rigid* if it has no self-extensions, that is, $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, M)$ vanishes. The next theorem explains in which way the cluster character allows us to view the cluster category as a categorification of $\mathcal{A}_Q$.

THEOREM 2.2 ([6]).

(a) *The map $L \mapsto X_L$ induces a bijection from the set of isomorphism classes of rigid indecomposable objects of the cluster category $\mathcal{C}_Q$ onto the set of cluster variables of the cluster algebra $\mathcal{A}_Q$.*

(b) *If L and M are indecomposables and $\mathrm{Ext}^1_{\mathcal{C}_Q}(L, M)$ is one-dimensional, then we have a generalized exchange relation*

$$X_L X_M = X_E + X_{E'} \tag{2.1}$$

where E and E' are the middle terms of the 'unique' non-split triangles

$$L \to E \to M \to \Sigma L \quad \textit{and} \quad M \to E' \to L \to \Sigma M \tag{2.2}$$

Let L and M be two indecomposable objects in the cluster category such that $\mathrm{Ext}^1_{\mathcal{C}_Q}(M, L)$ is one-dimensional. If both L and M are rigid, then so are E and E', and the sequence Theorem 2.1 is an exchange relation of the cluster algebra $\mathcal{A}_Q$. For this reason, in this case, we call the triangles in (2.2) *exchange triangles*. If L or M is not rigid, we call them *generalized exchange triangles*.

For all dimension vectors d the cluster character is a constructible function on $\mathrm{rep}_d\, Q$. Hence it takes a constant value X_d on an open subset of $\mathrm{rep}_d\, Q$. We call X_d the *generic cluster character* of d. The generic cluster characters have been conjectured to be a basis of the cluster algebra, called the *generic basis* (we refer to [7, Section 1.2] for further details on this conjecture).

2.3. Generalized Schur roots and generic decompositions. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. It will be useful to consider also the *negative Schur roots*, $-e_i$ for $i \in \mathbb{Q}_0$. Indeed, all indecomposable objects in the cluster category $\mathcal{C}_Q$ are isomorphic either to a stalk complex of an indecomposable representation of Q or to ΣP_i. We will interpret the negative Schur root $-e_i$ as the dimension vector of ΣP_i.

An alternative point of view on negative Schur roots is given by the decorated representations introduced in [10]. Decorated representations yield a combinatorial construction of the representations associated to negative Schur roots. In this article we rely instead on the categorical setup of cluster categories that was described in the previous paragraph.

Negative Schur roots allow us to define generic decompositions for any dimension vector with integer values. They are real Schur roots, as ΣP_i has no self-extensions in the cluster category. We define a negative Schur root $-e_i$ to be *ext-orthogonal* to a positive Schur root d if there is an M in $\operatorname{rep}_d Q$ such that $\operatorname{Ext}^1_{\mathcal{C}_Q}(M, \Sigma P_i)$ vanishes. This is the case if and only if d_i vanishes. As $\operatorname{Ext}^1_{\mathcal{C}_Q}(\Sigma P_i, \Sigma P_j)$ and $\operatorname{Ext}^1_{\mathcal{C}_Q}(\Sigma P_j, \Sigma P_i)$ vanish for all $i \neq j$, all negative Schur roots are pairwise ext-orthogonal.

We say that a general dimension vector $d : Q_0 \to \mathbb{Z}$ has a *generic decomposition* $d = d_1 \oplus \cdots \oplus d_s$ into generalized Schur roots d_i if the d_i are pairwise ext-orthogonal. A generic decomposition of d always exists and is unique. If d is non-negative, then it coincides with Kac's generic decomposition.

2.4. Component cluster. Let Λ be a basic algebra. Component clusters for Λ have first been introduced in [7, Section 6]. They are maximal collections of indecomposable strongly reduced components of the representation variety of Λ with pairwise vanishing generic extensions. We refer to [7] for additional details.

In the case where $\Lambda = kQ$ is the path algebra of Q, the representation varieties $\operatorname{rep}_v Q$, for a fixed dimension vector v, are vector spaces and thus in particular irreducible. Furthermore, they are strongly reduced. Also, by Kac's generic decomposition theorem, $\operatorname{rep}_v Q$ is indecomposable if and only if v is a Schur root.

Hence, we can give an alternative definition of component clusters for $\Lambda = kQ$ in terms of roots. The *component graph* of Q is a graph with vertices corresponding to the Schur roots and arrows connecting two Schur roots b and d if and only if $b \neq d$ and they are ext-orthogonal. The maximal complete subgraphs are called *component clusters*.

The component clusters consisting only of real Schur roots correspond to the classical clusters and are in bijection with the cluster-tilting objects of the cluster category. These objects have been studied extensively in the context of categorification of cluster algebras. In fact the cluster character establishes a bijection between the cluster-tilting objects and the clusters of $\mathcal{A}_Q$ (see Theorem 2.2). We know by [14] that these component clusters have size n. Their mutations can be entirely described using cluster combinatorics (see [6]).

In this paper we will consider all component clusters, and not just the component clusters corresponding to cluster-tilting objects; we will study their structure, calculate their size, and provide an interpretation of their mutations.

3. Component cluster for affine quivers. We first determine the size and composition of component clusters of affine quivers. We refer to [21, Chapter XIII] for a complete introduction to the representation theory of affine quivers.

The roots of affine quivers are either real or isotropic. Let δ denote the smallest positive isotropic root. All other isotropic roots are $\mathbb{Z}$-multiples of δ.

Considering the Auslander–Reiten component of Q gives a natural classification of indecomposable representations into three types:

- The *preprojective component* consists of τ^{-1}-orbits of projective indecomposable modules. The roots x associated to these representations are real Schur roots. Furthermore, $\langle x, \delta\rangle > 0$.
- The *preinjective component* consists of τ-orbits of injective indecomposable modules. The roots x associated to these representations are also real Schur roots and satisfy $\langle x, \delta\rangle < 0$.
- Finally, the third type of representations are the *regular* indecomposable modules appearing in tubes. They are τ-periodic representations. The roots x associated to these representations satisfy $\langle x, \delta\rangle = 0$.

We distinguish two types of tubes in the Auslander–Reiten quiver: the *exceptional tubes*, which are of size greater than one and form a finite set, and the *homogeneous tubes*, which are parametrized by the projective line. By [8] an indecomposable regular representation is Schurian if and only if its dimension vector is smaller than or equal to δ (that is, all the entries of the dimension vector are smaller than or equal to the entries of δ).

These are exactly the dimension vectors of the regular representations that lie in the first p rows of a tube of rank p. The dimension vectors of the regular representations in the first $p-1$ rows are real Schur roots. The dimension vector of the regular representation in the row p is always the isotropic root δ. Hence there are infinitely many isomorphism classes of indecomposable modules with dimension vector δ. It follows that $hom(\delta, \delta)$ vanishes and as a consequence the generic extension $ext(\delta, \delta)$ vanishes, even though every indecomposable module of dimension vector δ has non-vanishing self-extension.

The additive category of regular modules appearing in one tube is abelian and closed under extensions. Its simple objects are called *regular simple* and the number of isomorphism classes of regular simple modules equals the rank of the tube. The indecomposable regular modules are uniserial with respect to the regular simple modules appearing in the same tube. The maximal rigid objects in the exceptional tubes have been described by

Buan and Krause [2], [3]. From [2, Corollary 3.8] and [3, Corollary 2.4 and Theorem 5.2] we derive the next result.

THEOREM 3.1. *A maximal basic rigid object in a tube of rank p has $p-1$ pairwise non-isomorphic indecomposable direct summands, each of which has at most $p-1$ regular simples in its regular composition series.*

We determine next which Schur roots appearing in tubes are ext-orthogonal. We say that a Schur root *belongs to a tube* if it is the dimension vector of a regular representation.

LEMMA 3.2. *Two Schur roots belonging to different tubes are ext-orthogonal. The isotropic Schur root δ is ext-orthogonal to a Schur root α if and only if α is regular.*

Proof. It is well-known that two indecomposable regular representations A and B lying in different tubes have no extension. As there exists a Schurian representation of dimension vector δ that does not appear in an exceptional tube, δ is ext-orthogonal to all regular roots.

Let d be a preinjective or preprojective root. In the preinjective case $\langle d, \delta\rangle$ is negative, and in the preprojective case $\langle \delta, d\rangle$ is negative. It follows that either $ext(d, \delta)$ or $ext(\delta, d)$ is non-zero. ∎

We can now determine the component clusters.

THEOREM 3.3. *The component clusters are of size either n or $n-1$. They are of size $n-1$ if and only if they contain δ.*

Proof. If δ is not contained in a component cluster, then the component cluster corresponds to a cluster-tilting object, hence it is of size n. Suppose now that δ is contained in a component cluster. Then all other Schur roots in the component cluster belong to tubes and are real. In a tube of rank $p > 1$ the maximal number of pairwise ext-orthogonal real Schur roots is $p-1$ by Theorem 3.1. By Lemma 3.2 all Schur roots appearing in different tubes are ext-orthogonal. So a component cluster containing δ will also contain $p-1$ Schur roots for each exceptional tube of the Auslander–Reiten quiver. As the sum over all ranks minus 1 is equal to $n-2$ by [8], the component clusters containing δ are of size $n-1$. ∎

Note that, as there are only finitely many regular Schur roots, there are only finitely many component clusters of size $n-1$ but infinitely many component clusters of size n.

LEMMA 3.4. *The $\mathbb{Z}$-span of Schur roots appearing in a component cluster of size $n-1$ forms a pure sublattice of $\mathbb{Z}^n$ of rank $n-1$.*

Proof. For n pairwise ext-orthogonal real Schur roots, their $\mathbb{Z}$-span is the entire lattice $\mathbb{Z}^n$. If $\delta, \alpha_1, \dots, \alpha_{n-2}$ is a component cluster, then there is a dimension vector of a representation $\tau^{-l}P_e$ which is ext-orthogonal to

$\alpha_1, \ldots, \alpha_{n-2}$. This is equivalent to the fact that $\tau^{l+1}(\alpha_1 + \cdots + \alpha_{n-2})$ does not have support in e. Hence the $\mathbb{Z}$-span of $\tau^{l+1}\alpha_1, \ldots, \tau^{l+1}\alpha_{n-2}$ is the lattice $0 \times U$, where U is a pure sublattice of rank $n-2$. It follows that $\tau^{l+1}\delta = \delta, \tau^{l+1}\alpha_1, \ldots, \tau^{l+1}\alpha_{n-2}$ span a pure sublattice of rank $n-1$. As τ is a bijective integral linear form on $\mathbb{Z}^n$, the $\mathbb{Z}$-span of $\delta, \alpha_1, \ldots, \alpha_{n-2}$ is a pure sublattice of rank $n-1$. ∎

4. Component clusters for wild quivers. In this section we obtain an optimal bound for the maximal number of imaginary Schur roots appearing in a component cluster.

The *fundamental domain*

$$\mathcal{F} := \{d \in \mathbb{Z}^m \mid (d, e_i) \leq 0 \text{ for all } i \in \{1, \ldots, m\} \text{ and } \operatorname{supp}(d) \text{ is connected}\}$$

is a subset of the positive imaginary roots. We call these roots *fundamental*. The set of positive imaginary roots is given by the image of the Weyl group action on $\mathcal{F}$. Note that the symmetrized Euler form is invariant under the Weyl group W, that is,

$$(\alpha, \beta) = (w\alpha, w\beta)$$

for all $w \in W$. The set of positive imaginary roots is invariant under the action of W, but the set of real roots is not. Indeed, if α is a real Schur root, $w\alpha$ will not be positive in general. Furthermore, the Weyl group action does not map Schur roots to Schur roots and does not preserve ext-orthogonality.

LEMMA 4.1. *Let α be a fundamental root. Then either α is isotropic and $\alpha = \bigoplus_{i=1}^{n} \beta$, where $n \in \mathbb{N}$ and β is an isotropic fundamental Schur root, or α is a Schur root.*

Proof. If α is isotropic and fundamental, then its support is an affine quiver. As every affine quiver has a unique positive isotropic non-divisible positive root β, we have $\alpha = \bigoplus_{i=1}^{n} \beta$ for some $n \in \mathbb{N}$. Suppose that α is not a Schur root and is not isotropic; then by [20, Theorem 6.2] it contains at least one real Schur root β in its decomposition, and (α, β) is positive. But this contradicts the fact that α is fundamental. ∎

For any dimension vector α, its *null-cone* is given by

$$N_\alpha := \{i \in Q_0 \mid (e_i, \alpha) = 0\}.$$

We say that a dimension vector α is *sincere* if all its entries are positive integers.

LEMMA 4.2. *Suppose that α lies in the fundamental domain and is sincere. Then either Q is an affine quiver and α is isotropic, or the full subquiver on the set of vertices N_α is a union of Dynkin quivers.*

In the previous section we have determined the component clusters of affine quivers.

LEMMA 4.3. *Assume that either*

- *α and β are positive imaginary ext-orthogonal roots, or*
- *α is an imaginary fundamental root and β real and ext-orthogonal to α.*

Then $hom(\alpha, \beta)$, $hom(\beta, \alpha)$ and (α, β) vanish.

Proof. In the first case, we can consider a Weyl group element w such that $w\alpha$ lies in the fundamental domain. Then $w\beta$ is a positive root and we have $(\alpha, \beta) = (w\alpha, w\beta) \leq 0$. In the second case, since α is fundamental, $(\alpha, \beta) \leq 0$. As α and β are ext-orthogonal, we also have $0 = (\alpha, \beta) = hom(\alpha, \beta) + hom(\beta, \alpha)$. ∎

Note that the previous lemma does not hold if, in the second part, we replace the assumption that α is fundamental with the assumption that α is imaginary.

LEMMA 4.4. *Let α and β be two positive imaginary roots which are ext-orthogonal. Suppose that α lies in the fundamental domain. Then the support of β is totally disconnected from the support of α.*

Proof. If β and α are ext-orthogonal, then $(\alpha, \beta) = 0$. Therefore the support of β is totally disconnected from the support of α or it is contained in $N_\alpha \cap \operatorname{supp} \alpha$. Note that the quiver generated by the vertices of $N_\alpha \cap \operatorname{supp} \alpha$ is a Dynkin quiver. Thus, since β is an imaginary root, its support cannot be contained in $N_\alpha \cap \operatorname{supp} \alpha$. ∎

Let α be an imaginary Schur root which is fundamental and not sincere. Then a component cluster contains α if and only if it contains all the negative Schur roots corresponding to the vertices connected to the support of α.

LEMMA 4.5. *Let $\alpha_1, \dots, \alpha_n$ be imaginary Schur roots appearing in the same component cluster. Then there exists a Weyl group element w such that the $w\alpha_i$ are all fundamental and the supports of $w\alpha_i$ and $w\alpha_j$ are totally disconnected for all $i \neq j$. Also, there is a component cluster containing $w\alpha_1, \dots, w\alpha_n$.*

Proof. There is a Weyl group element w_1 such that $w_1\alpha_1$ is fundamental. Then all $w_1\alpha_i$ are positive imaginary roots satisfying $(w_1\alpha_i, w_1\alpha_1) = 0$. Hence the support of $w_1\alpha_i$ is totally disconnected from the support of $w_1\alpha_1$ for all $i \neq 1$.

If we restrict $w_1\alpha_2$ to the quiver Q_2 generated by its support, then $w_1\alpha_2$ is a positive imaginary root for that quiver. Hence there is a Weyl group element w_2 which is a product of simple reflections on vertices of Q_2 such that $w_2w_1\alpha_2$ is fundamental in Q_2. Then $w_1w_2\alpha_2$ is also fundamental in Q with support contained in Q_2 and $w_2w_1\alpha_1 = w_1\alpha_1$. It follows that the support of $w_2w_1\alpha_i$ is totally disconnected from the support of $w_2w_1\alpha_1$ and $w_2w_1\alpha_2$ for all $i \neq 1, 2$. By induction on n, there is an element $w := w_n \cdots w_1$

such that $w\alpha_i$ are all fundamental roots with pairwise totally disconnected supports.

Roots with totally disconnected supports are always ext-orthogonal and fundamental roots are always Schur by Lemma 4.1. Hence there is a component cluster containing $w\alpha_1, \ldots, w\alpha_n$. ∎

Corollary 4.6. *The maximal number of imaginary Schur roots that can appear in a component cluster is given by the maximal number of totally disconnected subgraphs of wild or tame type.*

Proof. By Lemma 4.5, we can assume without loss of generality that the imaginary Schur roots in a cluster are fundamental and have totally disjoint supports. The supports of the roots are quivers of tame or wild type. ∎

Note that by [9, Corollary 21] the number of real Schur roots in a component cluster is bounded by the number of vertices of Q minus twice the number of imaginary Schur roots appearing in the component cluster.

Lemma 4.7. *Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots \beta_s$ be a component cluster such that $\alpha_1, \ldots, \alpha_k$ are imaginary non-isotropic Schur roots. Then for all $n \in \mathbb{N}$, $n\alpha_1, \ldots, n\alpha_k, \beta_1, \ldots \beta_s$ is also a component cluster.*

Proof. By [20, Theorem 3.7] the $\mathbb{N}$-multiple of an imaginary non-isotropic Schur root α_i is also a Schur root. Let α and β be two positive roots, and let $n \in \mathbb{N}$. We show that they are ext-orthogonal if and only if $n\alpha$ and β are ext-ortogonal. If α and β are ext-orthogonal there are representations A and B of dimension vector α and β respectively such that $\mathrm{Ext}^1(A, B)$ and $\mathrm{Ext}^1(B, A)$ vanish. Note that this holds if and only if $\mathrm{Ext}^1(\bigoplus_{i=1}^n A, B)$ and $\mathrm{Ext}^1(B, \bigoplus_{i=1}^n A)$ vanish. Thus α and β are ext-orthogonal if and only if $n\alpha$ and β are ext-orthogonal. ∎

Remark 4.8. For any wild quiver there is a sincere fundamental imaginary root α such that N_α is empty. Clearly, this Schur root appears as the only element of a component cluster. As the null-cone of a root and the null-cone of its positive multiples coincide, we conclude that wild quivers always have infinitely many component clusters of size one.

The next example shows that the size of component clusters also depends on the orientation of the quiver: suppose α is a Schur roots for two quivers Q and Q' with isomorphic underlying (non-oriented) graphs. Then the maximal size of component clusters containing α may be different for Q and Q'.

Example 4.9. Let α be given by

$$\begin{tikzcd}
 & 1 \arrow[d] \arrow[dr] & & \\
1 & 2 \arrow[l] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

It is a fundamental root, and hence Schur.

We change the orientation of one arrow and consider the fundamental Schur root β:

$$\begin{tikzcd} & 1 \arrow[d] & & \\ 1 & 2 \arrow[l] & 2 \arrow[l] \arrow[ul] & 1 \arrow[l] \end{tikzcd}$$

Direct computation shows that the component cluster containing α has exactly two elements α and α', where α' is given by

$$\begin{tikzcd} & 0 \arrow[d] \arrow[dr] & & \\ 0 & 1 \arrow[l] & 1 \arrow[l] & 1 \arrow[l] \end{tikzcd}$$

On the other hand, β appears alone in a component cluster. Note also that by Corollary 4.6 the number of imaginary roots appearing in the same component cluster is at most one.

Remark 4.10. As the orientation of the quiver affects the size of component clusters but the Tits form is independent of it, we cannot hope for an exact upper bound involving the Tits form of a root. Another way to see this is as follows. Start with a fundamental sincere root α of a quiver Q and add a vertex x to Q and $n > 2$ arrows from x to y, where y is a vertex of Q which is totally disconnected from N_α. Let us denote the new quiver by Q', and let α' be a new root with $\alpha'(z) = \alpha(z)$ for all $z \in Q_0$ and $\alpha'(x) = 1$. Then the root α' is fundamental and sincere, and there is a canonical bijection between the component clusters containing α and the component clusters containing α', but $q(\alpha')$ can be made arbitrarily small by increasing n.

Lemma 4.11. *Let α be a fundamental non-divisible isotropic root. Then α appears in a component cluster of size $|Q_0| - 1$.*

Proof. The support of α is an affine quiver and we know by Theorem 3.3 that α is *ext*-orthogonal to $|\operatorname{supp}\alpha| - 1$ real Schur roots that have supports contained in $\operatorname{supp}\alpha$. Now for every vertex connected to $\operatorname{supp}\alpha$ we add the negative Schur root $-e_j$ to this collection. We can now complete the *ext*-orthogonal collection by real Schur roots with supports in the vertices totally disconnected from $\operatorname{supp}\alpha$. ■

Note that the previous lemma is false in general if we drop the hypothesis that α is fundamental. Also, non-divisible isotropic roots are not necessarily Schur (see [9, Example 27]).

It is clear by the uniqueness of the generic decomposition that n Schur roots appearing in the same component cluster are linearly independent.

5. Mutation of component clusters. Motivated by the cluster mutations and exchange relations which appear in the definition of cluster algebras, we propose a definition of mutations and exchange relations of component clusters.

DEFINITION 5.1. Two component clusters C_1 and C_2 are connected by a *mutation* if their intersection has cardinality $\min(|C_1|,|C_2|)-1$.

If a component cluster consists of real Schur roots, then they correspond uniquely to clusters of the cluster algebra $\mathcal{A}_Q$, and the above definition recovers the ordinary definition of cluster mutation.

PROPOSITION 5.2. *The mutation graph is connected.*

Proof. Clearly, by the Bongartz completion, there is a path of length at most n from a component cluster to a classical cluster. By [14] the full subgraph consisting of classical clusters is connected by mutation. Hence all component clusters are connected by mutation. ∎

In order to define exchange relations, we recall a few preliminary results.

LEMMA 5.3. *Let N and M be kQ-modules. Then we have a canonical isomorphism* $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,N) \cong \operatorname{Ext}^1_{kQ}(M,N) \oplus D\operatorname{Ext}^1_{kQ}(N,M)$.

Proof. This is [4, Proposition 1.7(c)]. ∎

Let C_1 and C_2 be two clusters connected by mutation. Then there exist unique roots α,α' such that $\{\alpha\} = C_1 - C_2$ and $\{\alpha'\} = C_2 - C_1$. An *exchange relation* between C_1 and C_2 is a polynomial equation in the cluster algebra $\mathcal{A}_Q$ relating the cluster characters X_α, $X_{\alpha'}$ and X_d for $d \in C_1 \cap C_2$. Here we are working one categorical level up, on the level of roots. Hence for us an exchange relation will be given by a generic decomposition of $\alpha+\alpha'$, where $\alpha \in C_2 - C_1$ and $\alpha' \in C_1 - C_2$. We will show that the generic decomposition involves roots which are ext-orthogonal to all roots in $C_1 \cap C_2$.

LEMMA 5.4. *Let α and β be two roots which are ext-orthogonal to a root d and suppose that $ext(\alpha,\beta) \neq 0$. Then there are open subsets U_α, U_β and U_d of $\operatorname{rep}_\alpha Q$, $\operatorname{rep}_\beta Q$ and $\operatorname{rep}_d Q$ respectively such that for all $A \in U_\alpha$ and $C \in U_\beta$ and all non-split triangles*

$$A \to B \to C \to \Sigma A \quad \textit{and} \quad C \to B' \to A \to \Sigma C$$

the spaces $\operatorname{Ext}^1_{\mathcal{C}_Q}(B,D)$ and $\operatorname{Ext}^1_{\mathcal{C}_Q}(B',D)$ vanish for all $D \in U_d$.

Proof. By the irreducibility of the varieties of representations, there are open subsets U_α, U_d and U_β of $\operatorname{rep}_\alpha Q$, $\operatorname{rep}_d Q$ and $\operatorname{rep}_\beta Q$ respectively such that $\operatorname{Ext}^1_{\mathcal{C}_Q}(A,D) = \operatorname{Ext}^1_{\mathcal{C}_Q}(C,D) = 0$ by Lemma 5.3 and $\operatorname{Ext}^1_{\mathcal{C}_Q}(A,C) \neq 0$ for all $A \in U_a$, $C \in U_c$ and $D \in U_d$. So by the 2-Calabi–Yau property we obtain the existence of two non-split triangles

$$A \to B \to C \to \Sigma C \quad \text{and} \quad C \to B' \to A \to \Sigma A$$

in $\mathcal{C}_Q$. Applying $\mathrm{Hom}(-,D)$ to the distinguished triangles allows us to conclude that $\mathrm{Ext}^1_{\mathcal{C}_Q}(B,D)$ and $\mathrm{Ext}^1_{\mathcal{C}_Q}(B',D)$ vanish for all $D \in U_d$. ∎

Proposition 5.5. *Let $\alpha_1,\dots,\alpha_n$ be a collection of ext-rthogonal Schur roots. Suppose that $\alpha_1' \neq \alpha_1$ is a Schur root ext-orthogonal to $\alpha_2,\dots,\alpha_n$ and $ext(\alpha_1,\alpha_1')$ does not vanish. Then the generic decomposition of $\alpha_1+\alpha_1'$ involves only Schur roots that are different from both α_1 and α_1' and are ext-orthogonal to $\alpha_2,\dots,\alpha_n$.*

Proof. As $ext(\alpha_1,\alpha_1')$ does not vanish, there are open sets U_{α_1} and $U_{\alpha_1'}$ such that, for all $A_1 \in U_{\alpha_1}$ and $A_1' \in U_{\alpha_1'}$, the space $\mathrm{Ext}^1(A_1,A_1')$ does not vanish. Hence there is a non-split exact sequence

$$0 \to A_1 \to A \to A_1' \to 0$$

and A has dimension vector $\alpha_1+\alpha_1'$. By Lemma 5.4, for all $i=2,\dots,n$ there is a representation A_i with dimension vector α_i such that $\mathrm{Ext}^1_{\mathcal{C}_Q}(A,A_i)$ vanishes. Let $d_1\oplus\cdots\oplus d_s$ be a generic decomposition of $\alpha_1+\alpha_1'$. Then the d_i's and the α_j's are all pairwise ext-orthogonal. Further, the d_i's are different from both α_1 and α_1' by [20, Theorem 3.3]. ∎

The next statement follows immediately from the previous two.

Theorem 5.6. *Let C_1 and C_2 be two component clusters that are related by mutation. Then for every pair (α,α') with $\alpha \in C_1-C_2$ and $\alpha' \in C_2-C_1$ there is a component cluster C_3 containing $C_1\cap C_2$ such that all Schur roots in the generic decomposition of $\alpha+\alpha'$ are contained in C_3.*

Hence we obtain an *exchange relation* between two component clusters which are related by mutation for every pair $\alpha \in C_1-C_2$ and $\alpha' \in C_2-C_1$ in terms of a third cluster C_3 containing $C_1\cap C_2$ and all Schur roots appearing in the generic decomposition of $\alpha+\alpha'$. In the case of classical clusters containing only real Schur roots, we have a more precise result: the Schur roots in a decomposition of $\alpha+\alpha'$ are contained in the intersection $C_1\cap C_2$. In the next section we will see that this result cannot be extended to component clusters, as it fails for affine quivers.

5.1. Exchange relations for affine quivers. In the case of an affine quiver, we have concrete descriptions of the component clusters. We will use these to obtain exchange relations between the cluster characters of Schur roots appearing in component clusters which are related by mutation.

In the first part, we work out the exchange relations arising from mutation between a component cluster of size n and a component cluster of size $n-1$. In the second part we will consider exchange relations arising from mutation between two component clusters of size $n-1$.

Lemma 5.7 ([11, Theorem 3.14]). *Let N and M be two regular simple kQ-modules whose dimension vectors equal δ. Then X_M equals X_N.*

The regular simple modules of dimension vector δ form an open subset of $\operatorname{rep}_\delta Q$. Hence the generic cluster character X_δ equals X_M for any regular simple module M with dimension vector δ.

Recall that a vertex e of Q is *extending* if $\delta_e = 1$.

LEMMA 5.8. *Let $\delta, \alpha_1, \ldots, \alpha_{n-2}$ be a component cluster. Then there exists a positive Schur root $\beta \neq \delta$ such that β is ext-orthogonal to $\alpha_1, \ldots, \alpha_{n-2}$. In this case β is either the dimension vector of the preprojective module $\tau^{-l} P_e$ or the dimension vector of the preinjective module $\tau^l I_e$, where $l \in \mathbb{N}$ and e is an extending vertex.*

Proof. The existence of β is clear by [14]. As β is a real root it is either preprojective or preinjective. So β is the dimension vector of either the preprojective module $\tau^{-l} P_e$ or the preinjective module $\tau^l I_e$, for some positive integer l and some vertex e of Q.

By the Auslander formula, the ext-vanishing condition is equivalent to the vanishing of $hom(\underline{\dim}\, \tau^{-l} P_e, \tau\alpha_i)$ or $hom(\tau^{-1}\alpha_i, \underline{\dim}\, \tau^l I_e)$ respectively for all $1 \le i \le n-2$. Both conditions are equivalent to the fact that

$$\tau^{(l+1)}(\alpha_1 + \cdots + \alpha_{n-2})$$

has no support in e. It remains to show that e is an extending vertex. If Q is an orientation of a Kronecker quiver or of $\tilde{A}_n$, there is nothing to show, as every vertex is extending.

In the remaining cases the Auslander–Reiten quiver contains at least one exceptional tube of size 2. Let α and $\beta := \tau(\alpha)$ denote the dimension vectors of the regular simples in such a tube. We assume without loss of generality that α belongs to $\alpha_1, \ldots, \alpha_{n-2}$. Then α has no support in e by the first part of the proof. As α and β are roots, their supports have to be connected. As $\alpha + \beta = \delta$ by [8] and δ is sincere, we know that e is contained in the support of β. So the supports of α and β are disconnected and they are linked by one arrow $e' \to e$ with $\alpha(e')$ non-zero. As α is a real Schur root, we have

$$hom(\alpha, \beta) = ext(\alpha, \alpha) = 0 \quad \text{and} \quad ext(\alpha, \beta) = hom(\alpha, \alpha) = 1.$$

Suppose e is not extending. Then

$$2 \le \delta(e)\delta(e') = -\langle \alpha, \beta \rangle = ext(\alpha, \beta) - hom(\alpha, \beta) = 1,$$

a contradiction. Hence e has to be an extending vertex. ∎

We denote by g the smallest common multiple of the tube ranks.

COROLLARY 5.9. *Let $\alpha_1, \ldots, \alpha_{n-2}$ be a collection of pairwise ext-orthogonal exceptional Schur roots, and let β be a preprojective Schur root which is ext-orthogonal to this collection. Then, for all $m \in \mathbb{N}$, $\tau^{-mg}\beta$ is also ext-orthogonal to $\alpha_1, \ldots, \alpha_{n-2}$.*

Proof. Clearly, τ preserves ext-orthogonality and τ^g acts as the identity on regular modules. Hence, for all $m \in \mathbb{N}$, $\tau^{-mg}\beta$ is also ext-orthogonal to $\alpha_1, \ldots, \alpha_{n-2}$. ■

From this result it follows immediately that there are infinitely many clusters which are connected by mutation to a component cluster containing δ. The next theorem gives the exchange relations between a component cluster and a cluster. In this case, we also obtain exchange relations of generic cluster characters.

THEOREM 5.10. *Let $\delta, \alpha_1, \ldots, \alpha_{n-2}$ be a component cluster. Let β be a preprojective Schur root such that $\beta, \alpha_1, \ldots, \alpha_{n-2}$ is a collection of pairwise ext-orthogonal Schur roots. Then there are exactly two completions $\beta, \beta_1, \alpha_1, \ldots, \alpha_{n-2}$ and $\beta, \beta_1', \alpha_1, \ldots, \alpha_{n-2}$ to clusters satisfying:*

- *$\delta + \beta = \beta_1$ and β_1 is the dimension vector of a preprojective module;*
- *either β_1' is the dimension vector of a preinjective module, or β_1' is the dimension vector of a preprojective module and $\delta + \beta_1' = \beta$;*
- *the generic cluster characters satisfy $X_\delta X_\beta = X_{\beta_1} + X_{\beta_1'}$.*

Proof. By Lemma 5.8 the real Schur root β is the dimension vector of a module in the τ-orbit of the projective indecomposable module associated with an extending vertex e. Therefore $\langle \delta, \beta \rangle = \delta_e = 1$ and for every indecomposable regular simple representation $C \in \operatorname{rep}_\delta Q$ and every indecomposable representation $A \in \operatorname{rep}_\beta Q$, there is a non-split exact sequence

$$0 \to A \to B \to C \to 0.$$

By [8] the module B is preprojective and indecomposable and therefore its dimension vector is a Schur root, which we denote by β_1. As $ext(\beta, \beta) = ext(\beta, \delta) = 0$, we also have $ext(\beta, \beta_1) = 0$.

From $1 = \langle \beta, \delta \rangle + \langle \delta, \delta \rangle = \langle \beta_1, \delta \rangle$ and $1 = \langle \beta_1, \beta_1 \rangle = \langle \beta_1, \delta \rangle + \langle \beta_1, \beta \rangle$, we deduce that $\langle \beta_1, \beta \rangle$ vanishes. We conclude from the vanishing of $ext(\beta, \beta_1)$ that every non-zero map in $hom(\beta_1, \beta)$ has to be surjective. As β is a Schur root, $hom(\beta_1, \beta)$ vanishes and so does $ext(\beta_1, \beta)$. Therefore β and β_1 are *ext*-orthogonal. We conclude by Lemma 5.6 that $\beta_1, \beta, \alpha_1, \ldots, \alpha_{m-2}$ is a cluster.

By the 2-Calabi–Yau property of the cluster category, there exists a non-split triangle

$$C \to B' \to A \xrightarrow{f} \Sigma C.$$

In the cluster category, the object ΣC is isomorphic to $\tau(C) \cong C$. Hence we have $\operatorname{Hom}_{\mathcal{C}_Q}(A, C) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{D^b(Q)}(A, \Sigma^i C)$, and as $\operatorname{Ext}^1(A, C)$ vanishes, we can view f as a morphism of modules $f : A \to C$. Then the object B' splits into $B_1' \oplus \Sigma^{-1} B_2'$, where B_1' is isomorphic to the kernel of f and B_2' is isomorphic to the cokernel of f.

We assume first that B'_1 does not vanish. Clearly, B'_1 is preprojective and indecomposable as $\langle \underline{\dim} B'_1, \delta\rangle = 1$. The dimension vector of B'_1 is therefore a Schur root, which we denote by β'_1. If B'_1 does not vanish, the image of f is a regular module and hence B'_2 is also a regular module of dimension vector smaller than δ. As the generic hom-space between δ and any exceptional Schur root vanishes, B'_2 vanishes and f is surjective.

Now, β'_1 is ext-orthogonal to β, as can be seen by applying $\langle \beta, -\rangle$ to the exact sequence $0 \to \ker f \to A \to \operatorname{Im} f \to 0$. Then $ext(\beta, \beta_1) = \langle \beta, \beta_1\rangle = \langle \beta, \beta\rangle - \langle \beta, \delta\rangle = 0$. Furthermore we have $ext(\beta'_1, \beta) \le ext(\beta'_1, \beta'_1)$ $+\, ext(\beta'_1, \underline{\dim} \operatorname{Im} f)$ and the last term vanishes as $\operatorname{Im} f$ is regular. Hence $\beta, \beta'_1, \alpha_1, \dots, \alpha_{n-2}$ is a cluster and $\beta'_1 + \delta = \beta$.

If $\ker f$ vanishes, then the cokernel of f satisfies $\langle \underline{\dim} B'_2, \delta\rangle = -1$, hence it has a preinjective direct summand. Applying $hom(-, B'_2)$ induces the exact sequence

$$0 \to \operatorname{Hom}(B'_2, B'_2) \to \operatorname{Hom}(C, B'_2) \to \operatorname{Hom}(A, B'_2) = 0.$$

As $hom(\delta, \underline{\dim} B'_2) = \langle \delta, \underline{\dim} B'_2\rangle = 1$, the module B'_2 is indecomposable and its dimension vector is a real Schur root. If B'_2 is not injective, then the Schur root of $\tau^{-1} B'_2$ extends the ext-orthogonal collection $\alpha_1, \dots, \alpha_{n-2}$. Furthermore, $hom(\beta, \underline{\dim} B'_2) = \langle \beta, \beta'_2\rangle = \langle \beta, \delta\rangle - \langle \beta, \beta\rangle = 0$. Hence $\tau^{-1} B'_2$ is ext-orthogonal to β and its Schur root β'_1 completes $\beta, \alpha_1, \dots, \alpha_{n-2}$ to a cluster.

If B'_2 is the injective module associated to the vertex i, then the roots $\beta, \alpha_1, \dots, \alpha_{n-2}$ have vanishing supports in i. Hence the Schur root $-e_i$ associated with the decorated representation ΣP_i completes $\beta, \alpha_1, \dots, \alpha_{n-2}$ to a cluster.

Finally, the multiplication formula yields the relation

$$X_C X_B = X_{B_1} + X_{B'_1}.$$

As B, B_1 and B'_1 are indecomposable and rigid, their cluster characters equal the generic cluster character of their Schur roots. By Lemma 5.7, we also have $X_C = X_\delta$. ∎

Remark 5.11. Note that if β'_1 is preinjective we obtain similar exchange relations: there is a unique preinjective root β''_1 such that $\beta'_1, \beta''_1, \alpha_1, \dots, \alpha_{n-2}$ is a cluster and $\delta + \beta'_1 = \beta''_1$. The proof is similar.

Next we study the exchange relations between two component clusters of size $n-1$. It is useful to restrict first to ext-orthogonal collections of Schur roots appearing in the same exceptional tube $\mathcal{T}$. Let $\mathcal{T}$ be of rank m and let $S \in \mathcal{T}$ be a regular simple module.

In order to study exchange relations between two regular clusters, we need to introduce the combinatorics as in [3, Appendix A]. We consider the intervals $[i, j] := \{i, i+1, \dots, j\} \bmod m+1$ for $i, j \in \{0, \dots, m\}$ with

$i \neq j$. Let $\mathcal{I}(m)$ denote the set of all these intervals. We call two intervals *compatible* if as sets either they are disjoint or one is a subset of the other. Then there is a bijection between the Schur roots of $\mathcal{T}$ and $\mathcal{I}(m)$ sending $[i, j]$ to the Schur root of the indecomposable representation with regular composition series $\tau^{-i}S, \dots, \tau^{-j+1}S$. Then the Schur root δ corresponds to the interval $[0, m]$. The proof of the following fact is elementary.

LEMMA 5.12. *Two Schur roots in $\mathcal{T}$ are ext-orthogonal if and only if the corresponding intervals are compatible. Every set of compatible intervals can be completed to a set of m compatible intervals.*

We next consider the set $\mathcal{B}$ of maximal sets of compatible intervals containing $[0, m]$.

LEMMA 5.13. *Let $\alpha_1, \dots, \alpha_m$ be a maximal set of ext-orthogonal Schur roots in $\mathcal{T}$. Then there is exactly one Schur root $\alpha_1' \neq \alpha_1$ in $\mathcal{T}$ such that $\alpha_1', \alpha_2, \dots, \alpha_m$ is a maximal ext-orthogonal collection.*

Furthermore, there are at most two distinct Schur roots

$$\alpha, \beta \in \{\delta, \alpha_2, \dots, \alpha_m\}$$

such that there is up to isomorphism exactly one non-split exact sequence

$$0 \to A_1 \to A \to A_1' \to 0$$

where A_1 and A_1' are indecomposable regular representations with dimension vectors α_1 and α_1', and A is the direct sum of two indecomposable regular representations of $\mathcal{T}$ with dimension vectors α and β.

Proof. Let I be the interval corresponding to α_1, and let $Z \in \mathcal{B}$ be the set of maximal compatible intervals containing the m intervals associated to $\alpha_1, \dots, \alpha_m$. Without loss of generality, we can assume that $\inf I = 0$. Then there is an interval I^+ in $Z - \{I\}$ such that either $\inf I = \inf I^+$ or $\sup I = \sup I^+$. We assume without loss of generality that the first case holds and pick the smallest interval I^+ with that property. By compatibility, we also assume that I is a subset of I^+. The converse case of I^+ being contained in I can be treated similarly. Set $I' := [i, j]$ where $j := \sup I^+$ and $i := \min\{\inf S \mid S \in Z - \{I\}, \sup S = \sup I\}$ or $i = \sup I$ if the set is empty. Then I' is the unique interval different from I and compatible with $Z - \{I\}$. We can see this as follows. Assume that there is another interval $I'' \neq I'$ compatible with $Z - \{I\}$. Then I'' is not compatible with I and I'. Furthermore, by the compatibility with $Z - \{I\}$, we see that I'' is contained in I^+. Hence

$$\inf I < \inf I'' < \inf I' \leq \sup I < \sup I'' \leq \sup I'.$$

Then $A := [\inf I'', \sup I']$ is compatible with $Z - \{I\} \cup \{I'\}$. But A does not lie in $Z - \{I\}$ as it is not compatible with I. Hence we obtain a contradiction to the assumption that Z is maximal.

We have $I^+ = I \cup I'$, and $I^- := I \cap I'$ is also contained in Z, as it is compatible by construction with all intervals in $Z - \{I\}$. If I^- consists of only one point, then it is not an element of $\mathcal{I}$ and we ignore it.

Let α_1', α and β be the Schur roots corresponding to I', I^+ and I^-. Furthermore, let A_1' and A_1 be the Schurian representations associated to α_1 and α_1', and let A be the direct sum of the two indecomposable representations associated to the roots α and β. Then there exists a non-split exact sequence

$$0 \to A_1 \to A \to A_1' \to 0.$$

It is uniquely determined up to isomorphism, as extensions between two Schurian representations in a tube are at most one-dimensional. The second case follows analogously. ■

Note that it is not clear whether we can obtain exchange relations on the level of generic cluster characters. Indeed, A could have an indecomposable direct summand C of dimension vector δ. Then C would not be a regular simple representation, and in this case it is not known wether X_C is equal to the generic cluster character X_δ.

THEOREM 5.14. *Let $\delta, \alpha_1, \ldots, \alpha_{n-2}$ be a component cluster ordered in such a way that $\alpha_1, \ldots, \alpha_m$ belong to the same tube. Then there is exactly one Schur root $\alpha_1' \neq \alpha_1$ such that $\delta, \alpha_1', \ldots, \alpha_{n-2}$ is a component cluster.*

In this case α_1 and α_1' belong to the same tube, and $\alpha_1 + \alpha_1'$ has a generic decomposition as a direct sum of either one or two Schur roots in $\{\delta, \alpha_2, \ldots, \alpha_m\}$.

Proof. By Lemma 5.13, there is exactly one Schur root α_1' different from α_1 which belongs to the same tube of α_1, and which completes $\delta, \alpha_2, \ldots, \alpha_m$ to a component cluster. The second part follows immediately from the previous lemma. ■

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